Name ______________________________ Date ____________________

Math 10250 Activity 23: Second Derivative Tests (Section 4.2)

GOAL: To study how the graph of a given $f(x)$ "bends", and how these features of the graph are described by $f'(x)$ and $f''(x)$.

▶ The second derivative test for concavity

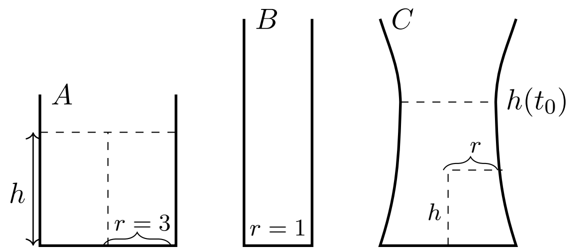
Example 1 Water is filling up each of the following cylindar vessels at a constant rate of 1 cm^3/sec.

Sln. *Recall that the volume of a cylinder vessels is*

$$V = \text{area of base} \cdot \text{height} = \pi r^2 \cdot h(t).$$

Also the volume of water in the vessel at time t is $V = 1 \cdot t$. So we have

$$\pi r^2 \cdot h(t) = t \Longrightarrow h(t) = \boxed{\frac{1}{\pi r^2} t}.$$



$h_A(t) = \frac{1}{9\pi} t$ $\quad h_B(t) = \frac{1}{\pi} t$

Let h be the height of the water level in the vessel at time t.

a. Sketch the graphs of h versus t for Vessels A and B in the axes for Figure 1. Indicate which graph belongs to A and which to B.

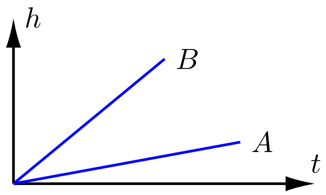


Figure 1

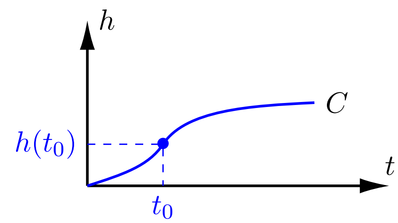


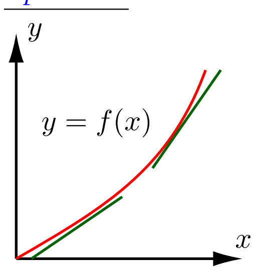
Figure 2

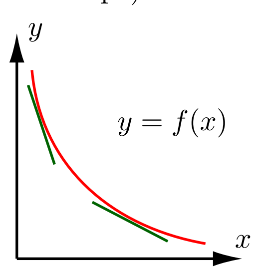
b. Sketch the graph of h versus time t for Vessel C in the axes for Figure 2.

c. Comment on how the "bending" (up or down) of the graph changes with $h'(t)$. Mark on the graph where the "bending" changes.

We now introduce terminologies that describe the "bending" of a graph.

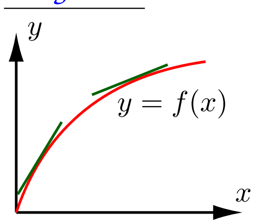
Case 1: For $a < x < b$, the slope of the graph $f(x)$ is **increasing** as x increases (i.e., $f'(x)$ is increasing). So $f''(x)$ is __*positive*__ for $a < x < b$ (portions of u-shape).

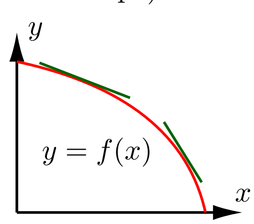




We say that the graph of $f(x)$ is __*concave up*__ for $a < x < b$.

Case 2: For $a < x < b$, the slope of the graph $f(x)$ is **decreasing** as x increases (i.e., $f'(x)$ is decreasing). So $f''(x)$ is __*negative*__ for $a < x < b$ (portions of n-shape).





We say that the graph of $f(x)$ is __*concave down*__ for $a < x < b$.

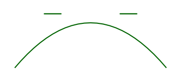
The above gives us:

The Second derivative test for concavity

Let $f(x)$ be a function that has a second derivative in an interval.

- If $\boldsymbol{f''(x) > 0}$ for all x then its graph is <u>*concave up*</u>. (like $f(x) = x^2$)

- If $\boldsymbol{f''(x) < 0}$ for all x then its graph is <u>*concave down*</u>. (like $f(x) = -x^2$)



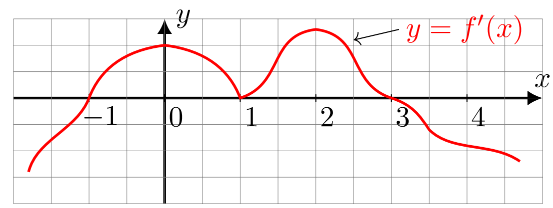
Note: The places where the graph of $f(x)$ changes its concavity are called <u>*inflection points*</u>.

Example 2 Using the graph of the derivative of $f(x)$ below, determine the concavity of $f(x)$.

Concave up: *when* • $x < 0$ • $1 < x < 2$

Concave down: • $0 < x < 1$ • $2 < x$

Inflection points: • $x = 0$ • $x = 1$ • $x = 2$



Q1: Where can $f''(x)$ change signs (i.e., $f(x)$ changes concavity)?

A1: At the points where (i) $f'' = 0$, or (ii) f'' is undefined (e.g., f' has a sharp corner).

Example 3 The position of an object moving on a straight line is given by $s(t) = 2t^3+3t^2-36t+7$. Determine (a) where the graph of $s(t)$ is concave up, (b) where it is concave down, and (c) where there are inflection points, if any. Give physical interpretations for each of (a), (b), and (c).

$s'(t) = 6t^2 + 6t - 36$

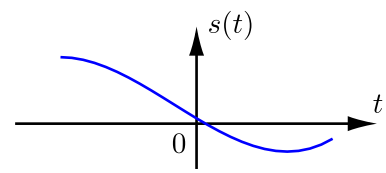
$s''(t) = 12t + 6 = 0 \Longrightarrow t = -\frac{1}{2}$

(a) Object is accelerating.

(b) It is decelerating.

(c) Object from decelerating changes to accelerating.

t		$-1/2$	
sign of $s''(t)$	$-$	0	$+$
info about $s(t)$	*(b)*	*ip (c)*	*(a)*

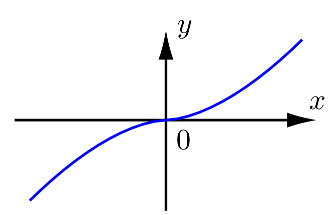


Example 4 Determine where the graph of $f(x) = x^{5/3}$ is concave up, where it is concave down, and where there are inflection points, if any. Sketch the graph of $f(x)$.

$f'(x) = \dfrac{5}{3}x^{\frac{2}{3}}$

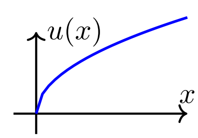
$f''(x) = \dfrac{10}{9}x^{-\frac{1}{3}}$

x		0	
sign of $f''(x)$	$-$	*D.N.E.*	$+$
info about $f(x)$		*ip*	



Application in Economics: Utility functions $u(x)$ are

- increasing $\iff u'(x) > 0$
- concave down $\iff u''(x) < 0$. (Like $u(x) = \sqrt{x}$)



Your turn (*Application to Population/Pandemics Model*): For the solution $y = y(t)$ of the logistic model below, show that its concavity changes when $y(t) = K/2$ (as the picture indicates).

$$\frac{dy}{dt} = ry\Big(1 - \frac{y}{K}\Big), \quad y(0) = y_0.$$