Name ______________________________ Date ____________________

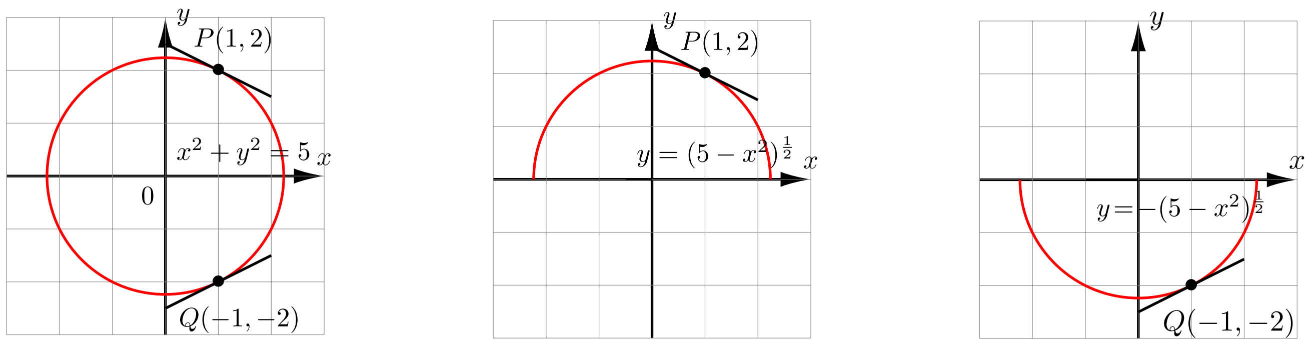
Math 10250 Activity 20: Implicit Differentiation and Related Rates (Section 3.8)

GOALS: To learn implicit differentiation and use it to find the slope of the graph of a given relation between x and y. Apply implicit differentiation to quantities changing with related rates.

▶ Implicit differentiation

Q1: How can we find the slope at the points $P(1,2)$ and $Q(1,-2)$ on the circle $x^2+y^2=5$?

A1: Write y as two **explicit** functions of x.

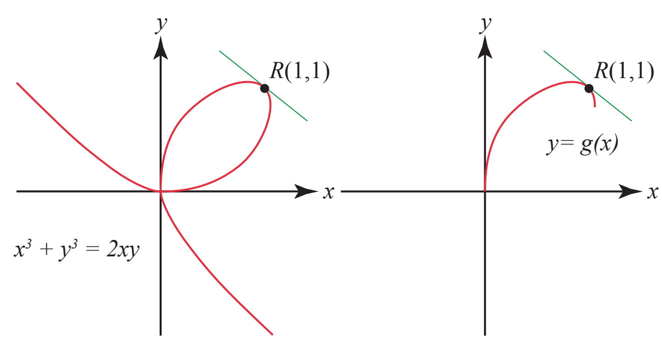


To find the slope at $P(1,2)$, we use the *upper* half of the circle: $y=\sqrt{5-x^2}=(5-x^2)^{1/2}$. Therefore $\dfrac{dy}{dx}=\dfrac{1}{2}(5-x^2)^{-\frac{1}{2}}(0-2x)$. So, the slope at $P=\left.\dfrac{dy}{dx}\right|_{x=1}=\dfrac{1}{2}(5-1)^{-\frac{1}{2}}\cdot(-2)=-\dfrac{1}{2}$.

To find the slope at $Q(1,-2)$, we use the *lower* half of the circle: $y=-\sqrt{5-x^2}=-(5-x^2)^{1/2}$. Therefore $\dfrac{dy}{dx}=-\dfrac{1}{2}(5-x^2)^{-\frac{1}{2}}(0-2x)$. So, the slope at $Q=\left.\dfrac{dy}{dx}\right|_{x=1}=-\dfrac{1}{2}(5-1)^{-\frac{1}{2}}\cdot(-2)=\dfrac{1}{2}$.

Remark: For a general relation between x and y, it is difficult to write y as a function of x. For example, $x^3+y^3=2xy$. To find the slope at $R(1,1)$ on the curve using the above method, we need to find **explicitly** $g(x)$. This is very hard!!

We say that y is an implicit function of x. To find $\dfrac{dy}{dx}$ in such situation we employ a powerful method called **implicit differentiation**.



Example 1 Use implicit differentiation to find the slope at $Q(1,-2)$ on $x^2+y^2=5$.

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(5)$$
$$2x+2y\frac{dy}{dx}=0$$

$$\Longrightarrow 2y\frac{dy}{dx}=-2x$$
$$\Longrightarrow \frac{dy}{dx}=\left.\frac{-x}{y}\right|_{x=1,y=-2}=\frac{-1}{-2}=\frac{1}{2}$$

Method for Implicit Differentiation

- Differentiate both sides of the equation with respect to x.
- Apply the chain rule whenever you meet an expression involving y.
- Move all terms involving $\frac{dy}{dx}$ to the left side of the eq. and all the rest of the terms to the right.
- Factor out $\frac{dy}{dx}$ on the left.
- Solve for $\frac{dy}{dx}$.

Example 2 Find the equation of the tangent line to the curve $x^3 + y^3 = 2xy$ at $(1, 1)$.

$$\begin{aligned}\frac{d}{dx}(x^3+y^3) &= \frac{d}{dx}(2xy)\\ 3x^2+3y^2\frac{dy}{dx} &= 2y+2x\frac{dy}{dx}\\ (3y^2-2x)\frac{dy}{dx} &= 2y-3x^2\end{aligned}$$

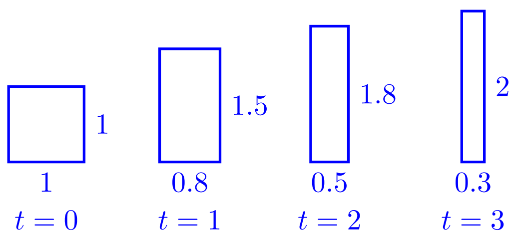
$$\frac{dy}{dx} = \left.\frac{2y-3x^2}{3y^2-2x}\right|_{x=1,y=1} = \frac{2-3}{3-2} = -1$$

equation of the tangent line: $y - 1 = -1(x-1)$.

► **Related rates** Using implicit differentiation, we can find a relation between the rates of change of two variables (which are both functions of a third variable, such as time t). We say the rates are **related**, and we can find one just by knowing the other.

Example 3 The area of a rectangle is a function of its height and width. The height h in inches and width w in inches of a rectangle changing with respect to time t (seconds) is given by the table below.

(a) Sketch the rectangles for each of the times below:



t	0	1	2	3
h	1	1.5	1.8	2
w	1	0.8	0.5	0.3

(b) Let A be the area of the rectangle. Find $\frac{dA}{dt}$, where A is the area and t is time. Your answer will involve h, w, and their rates.

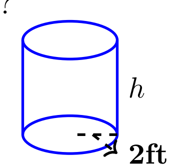
$$A = h\cdot w \Longrightarrow \frac{dA}{dt} = \frac{dh}{dt}\cdot w + h\cdot\frac{dw}{dt}$$

(c) Suppose that when the height is 1.5 inches, it is growing at 0.2 inches per second, and when the width is 0.8 inches, it is decreasing at 0.3 inches per second. How fast is the area of the rectangle changing at the given instant? State units and whether the area is increasing or decreasing.

$$\frac{dA}{dt} = 0.2\cdot 0.8\cdot + 1.5\cdot(-0.3) = 0.16 - 0.45 = -0.29 \quad \textit{sq. in/sec. decreasing?}$$

Example 4 Suppose a cylindrical tank, whose base is a circle of radius 2 feet, is filling with water at the rate of 0.3 cubic feet per minute. How fast is the water level rising?

Step 1: Sketch a picture labeling important dimensions. What do you know about each dimension?

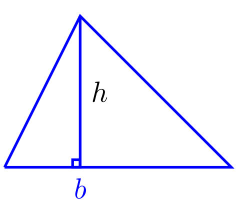


Step 2: Write down an equation that relates the variables. Equation: $V = \pi\cdot 2^2\cdot h$

Step 3: Differentiate to obtain the required rate.

$$V = 4\pi h \Longrightarrow \frac{dV}{dt} = 4\pi\frac{dh}{dt} \Longrightarrow 0.3 = 4\pi\frac{dh}{dt} \Longrightarrow \frac{dh}{dt} = \frac{0.3}{4\pi} \quad \textit{sq. ft/min.}$$

Example 5 A triangle with **fixed** area of 50 cm^2 has its height decreasing at a rate of 0.1 cm/sec. At what rate is the base of the triangle changing at the instant when its height is 5 cm? (Ans: 0.4 cm/sec)



$$A = \frac{1}{2}b\cdot h$$
$$50 = \frac{1}{2}b\cdot h$$
$$b\cdot h = 100$$

$b'h + bh' = 0$

when $h = 5, b = 20$

Also $h' = -0.1$

$b'5 + 20\cdot(-0.1) = 0$

$b' = \frac{2}{5} = 0.4$ *cm/sec*