Name ________________________________ Date ____________________

Math 10250 Activity 19: The Chain Rule (Section 3.7)

GOAL: To learn how to compute the derivative of a composition of two functions.

Q1: Which rule would you use to compute the following derivatives?

(a) $\left[\dfrac{x^2+1}{x+1}\right]'$ *quotient*

(b) $\left[\dfrac{x^2+1}{e+1}\right]'$ *quotient*

(c) $[x^2+e^x]'$ *sum*

(d) $[(x^2+1)\cdot 2^x]'$ *product*

(e) $[\ln(x^2+1)]'$ *none*

▶ **The Composite Function.** A function $h(x)$ is said to be a composite function of $g(x)$ followed by $f(x)$ if $h(x) = f(g(x))$. We may write: $h : x \overset{g}{\longmapsto} g(x) \overset{f}{\longmapsto} f(g(x))$

Example 1 Find the functions $f(x)$ and $g(x)$, for unequal x, such that $h(x) = f(g(x))$:

(a) $h(x) = (x^4+2x^2+7)^{21}$ $\qquad h : x \overset{g}{\longmapsto} g(x) = x^4+2x^2+7 \overset{f}{\longmapsto} f(g(x)) = (x^4+2x^2+7)^{21}$

Ans: $f(x) \overset{?}{=} x^{21}$ and $g(x) \overset{?}{=} x^4+2x^2+7$

(b) $h(x) = e^{x^2+1}$ $\qquad h : x \overset{g}{\longmapsto} x^2+1 \overset{f}{\longmapsto} f(g(x)) = e^{x^2+1}$

Ans: $f(x) \overset{?}{=} e^x$ and $g(x) \overset{?}{=} x^2+1$

▶ **The Chain Rule**

Q2: In a SMS (short message service) competition for the title of "Fastest SMS Thumbs", it is observed that Competitor A inputs text three times faster than B and Competitor B inputs text two times faster than C. How much faster is Competitor A than Competitor C? Why?

A — — $\overset{3}{\text{Rate of } A \text{ relative to } B}$ — — B — — $\overset{2}{\text{Rate of } B \text{ relative to } C}$ — — C

$\dfrac{\Delta A}{\Delta B} = 3$ $\qquad \dfrac{\Delta B}{\Delta C} = 2$ $\qquad \Longrightarrow \dfrac{\Delta A}{\Delta B}\cdot\dfrac{\Delta B}{\Delta C} = \dfrac{\Delta A}{\Delta C}$

— — — — — — — — — — ? Rate of A relative to C ? — — — — — — — — — —

Suppose $y = f(g(x))$. To find a formula for $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))]$, we set $u = g(x)$ then $y = f(u)$.

y — — $\overset{\frac{dy}{du}}{\text{Rate of } y \text{ relative to } u}$ — — u — — $\overset{\frac{du}{dx}}{\text{Rate of } u \text{ relative to } x}$ — — x

$\dfrac{dy}{dx} \overset{?}{=}$

— — — — — — — — — ? Rate of y relative to x ? — — — — — — — — — —

So expect $\boxed{\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx}}$

Our guess is in fact correct, and the formula for $\dfrac{dy}{dx}$ is called the **Chain Rule** (in Leibniz notation).

But $\dfrac{dy}{dx} = \dfrac{d}{dx}[f(g(x))] = [f(g(x))]'$, $\dfrac{dy}{du} = f'(u) = f'(g(x))$ and $\dfrac{du}{dx} = g'(x)$. Thus we also have:

$$\boxed{\frac{d}{dx}[f(g(x))] = [f(g(x))]' = f'(g(x))\cdot g'(x)}$$

Example 2 Find the derivatives:

(a) $\left[\ln(x^2+1)\right]' \stackrel{?}{=} \frac{1}{u}\frac{du}{dx}$

$= \frac{1}{x^2+1} \cdot 2x$

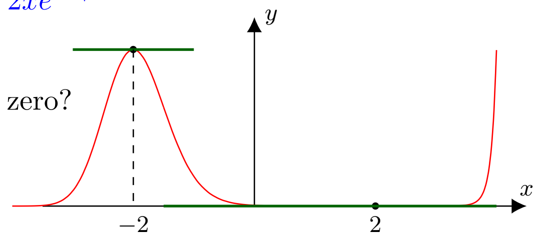
(b) $\left[(x^4+2x^2+7)^{21}\right]' \stackrel{?}{=}$

$= 21(x^4+2x^2+7)^{20}(4x^3+4x)$

(c) $[x\ln(2+e^x)]' \stackrel{?}{=} 1 \cdot \ln(2+e^x) + x \cdot \frac{1}{2+e^x}(0+e^x)$

(d) $\left[e^{x^2+1}\right]' \stackrel{?}{=} e^{x^2+1}(2x+0) = 2xe^{x^2+1}$

Example 3 For what x does the graph of $y = e^{\frac{1}{3}x^3-4x}$ have slope zero?



$$\frac{dy}{dx} = e^{\frac{1}{3}x^3-4x} \cdot (x^2-4) = 0 \implies x^2-4=0 \implies \boxed{x=\pm 2}$$

Example 4 Let $f(x) = \dfrac{g(x^2)}{\sqrt{x+1}}$. Find the slope of the graph of $f(x)$ at $x=3$.

x	$g(x)$	$g'(x)$
3	5	2
4	0	7
9	−2	3

$$f'(x) = \frac{g'(x^2)\cdot 2x \cdot \sqrt{x+1} - g(x^2)\cdot \frac{1}{2}(x+1)^{-\frac{1}{2}}}{x+1} = \frac{3\cdot 2\cdot 3\cdot 2 - (-2)\cdot \frac{1}{2}\cdot\frac{1}{2}}{4}$$

Example 5 Let $A(x) = g(f(x))$ and $B(x) = g(g(x))$. Use the graph of $f(x)$ and $g(x)$ to compute each of the following derivatives if it exists. If it does not exist, explain why.

Ans: One of them does not exist. Why?

(a) $A'(1) \stackrel{?}{=}$

$A'(x) = g'(f(x))\cdot f'(x)$

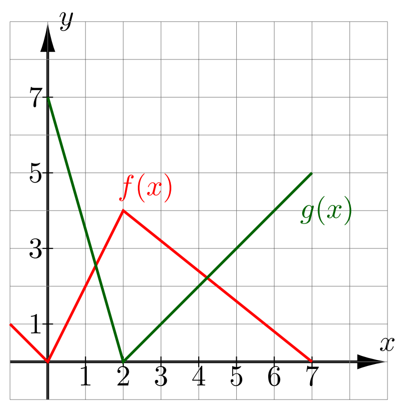
$A'(1) = g'(f(1))\cdot f'(1) = g'(2)\cdot f'(1)$

does not exist since $g'(2)$ *does not exist.*

(b) $B'(1) \stackrel{?}{=}$

$B'(x) = g'(g(x))\cdot g'(x)$

$B'(1) = g'(g(1))\cdot g'(1) = g'(3)\cdot g'(1) = 1\cdot \frac{-7}{2} = \boxed{-\frac{7}{2}}$



$f'(1) = 2$

$g'(1) = -\frac{7}{2}$

$g'(2)$ *does not exist*

$g'(3) = 1$

Example 6 Diatoms are microscopic algae surrounded by a silica shell that are found both in salt and fresh water, and they are a major source of atmospheric oxygen. The size of a diatom colony depends on many factors, including temperature. Suppose that samples taken in a Midwestern lake showed that the concentration of diatoms was modeled as a function of the temperature by the equation

$$C = 1.4 - e^{-0.001h^2} \quad \text{for } 0 < h < 40,$$

where C is the concentration of diatoms (in million per cubic centimeter) and h is the temperature of the water (in degrees Celsius).

(a) $\frac{dC}{dh} \stackrel{?}{=} 0 - e^{-0.001h^2}(-0.002h) = \boxed{0.002he^{-0.001h^2}}$

(b) Suppose the temperature of the lake is 10°C and falling at the rate of 2 degrees per hour. At what rate is the concentration of diatoms changing with respect to time? Ans: $-0.04e^{0.1}$

$$\frac{dc}{dt} = \frac{dc}{dh}\cdot\frac{dh}{dt} = 0.002he^{-0.001h^2}\cdot\frac{dh}{dt} = -0.002\cdot 10e^{-0.001\cdot 100}\cdot 2 = \boxed{-0.04e^{-0.1}}$$