Name ______________________________ Date ____________________

Math 10250 Activity 9: Compound Interest and the Number e (Section 2.2)

GOAL: Understand compounding in continuous time as the limiting case of n-times per year compounding.

Last time: Let $A(t)$ be the balance at time t (in years) of a bank account earning interest at an annual rate r (in decimals) compounded n times a year. Then we have:

$$\boxed{A(t) = P\left(1+\frac{r}{n}\right)^{tn}, \text{where } P \text{ is the principal; i.e. } A(0) = P}$$

Example 1 The balance $M(t)$ of a retirement account with interest compounded daily is given by the formula $M(t) = 30000(1.00022)^{365t}$. What is the principal and the annual interest rate?

$$30000(1.00022)^{365t} = P\left(1+\frac{r}{n}\right)^{tn} \Longrightarrow P = 3,000 \textit{ and } (1.00022)^{365t} = \left(1+\frac{r}{365}\right)^{365t}$$

$$\Longrightarrow 1.00022 = 1 + \frac{r}{365} \Longrightarrow \frac{r}{365} = 1.00022 - 1 = 0.00022 \Longrightarrow \boxed{r = 0.08} \quad \textit{(Ans: P=30000; r=8\%)}$$

Next, we want to consider the balance of an account where interest is compounded continuously; i.e., we are earning interest every instant the money is with the bank. (Good deal?)

▶ The number e

In the general formula above, if $P = 1, r = 1$ and $t = 1$ then $A(1) = \left(1+\frac{1}{n}\right)^n$.

Letting n go to ∞ we obtain that:

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \stackrel{?}{=} e.} \leftarrow \text{balance at end of 1 yr. of an investment of \$1 at an annual interest rate of 100\% compounded continuously}$$

Example 2 Estimate e by completing the table:

n	1	2	10	100	1000
$(1+\frac{1}{n})^n$	2	2.25	2.59	2.70	2.716

$\Longrightarrow e \approx 2.71828182845\cdots$ (irrational number! like π)

Continuously compounded interest

Compute the limit:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \underbrace{\lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}\cdot r}}_{\text{letting } m = n/r, \text{ so that } n = mr} = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m\cdot r} = \underbrace{\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^r}_{\text{by definition of } e} = e^r.$$

Setting: As above except now $n \to \infty$.

The amount after t years with **continuously compounded interest** is:

$$A(t) = \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{tn} = P\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{tn} = P\lim_{n\to\infty}\left[\left(1+\frac{r}{n}\right)^n\right]^t = P\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n\right]^t = P(e^r)^t = Pe^{rt}.$$

<u>General formula</u>: $\boxed{A(t) = Pe^{rt}}$ ←(rate)(time in years)

↑ amount in account at end of t years

↖ initial amount invested (principal)

$\boxed{\textbf{Example 3}}$ If you open an account paying 9% interest, compounded continuously, then how much should you deposit to insure that there will be \$60,000 in 15 years? Ans. $60,000e^{-1.35}$

$P = PV = ?$ —— $r = 0.09$ —— 15: 60000

$$Pe^{0.09\cdot 15} = 60,000 \Longrightarrow P = \frac{60,000}{e^{1.35}} \approx 15554.42$$

$\boxed{\textbf{Example 4}}$ $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{\underbrace{2n}_{=m}}\right)^{3n} \stackrel{?}{=} \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m\cdot\frac{3}{2}} = e^{\frac{3}{2}}$ Ans. $e^{3/2}$

$\boxed{\textbf{Example 5}}$ Suppose you put \$5000 in an account paying 4% annual interest, and you leave it there without adding or withdrawing anything. How much will you have at the end of 3 years if the interest is compounded:

(a) 6 times a year? $A(t) = 5000\left(1+\frac{0.04}{6}\right)^{3\times 6} \approx 5635.24$ Ans. $\$5,635.24$

(b) 24 times a year? $A(t) = 5000\left(1+\frac{0.04}{24}\right)^{3\times 24} \approx 5636.92$ $PV = 5000$ —— $r = 0.04$ —— 3: $A(t) = FV = ?$ Ans. $\$5,636.92$

(c) continuously? $A(t) = 5000e^{0.04\cdot 3} = 5000e^{0.12} \approx 5,637.48$ Ans. $\$5,637.48$

Remark: What could you conclude from the answers obtained in Example 5?

I will have the most money when the interest is compounded continuous.

▶ The natural exponential function

Recall: The exponential function is $f(x) = b^x$, where b is a positive constant. The most **popular** b is e.

Definition: The **natural exponential function** is $f(x) = e^x$.

$\boxed{\textbf{Example 6}}$ Graph the natural exponential function and its inverse. Write down all intercepts and asymptotes of the natural exponential function. Also, recall the laws of exponents with basis $b = e$.

- *Continuous and increasing*
- $e^0 = 1$
- $\lim_{x\to\infty} e^x = \infty$ *and* $\lim_{x\to-\infty} e^x = 0$

Laws of exponents for with basis e:

- $e^{x_1+x_2} = e^{x_1}\cdot e^{x_2}$
- $e^{x_1-x_2} = \frac{e^{x_1}}{e^{x_2}}$
- $(e^x)^r = e^{rx}$
- $e^{-x} = \frac{1}{e^x}$

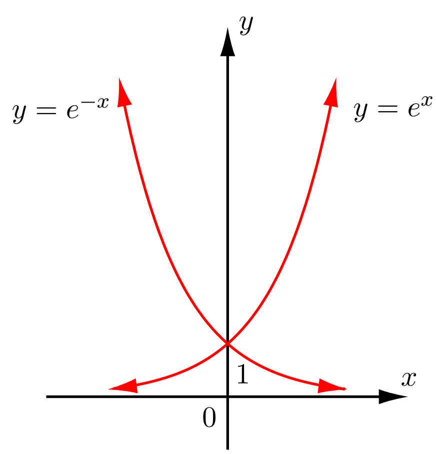

Figure 1