Name ______________________________ Date ____________________

Math 10250 Activity 8: Exponential Functions (Section 2.1)

GOAL: Learn exponential functions with different bases and use them to model real-world situations.

▶ **Exponential functions** are of the form : $\boxed{f(x) = b^x}$, where $b > 0$ is called the **base**, like $f(x) = 2^x$.

Q1: Where do they appear?

A1: Everywhere! For example, if we put \$1 in an account paying 5% interest, compounded annually, then t years later it will become $f(t) = (1.05)^t$, which is an **exponential function** with base $b = 1.05$.

▶ **The laws of exponents.** For $b > 0$ and u and v any numbers, we have

(1) $b^{u+v} \stackrel{?}{=} b^u \cdot b^v$; e.g., $2^{3+2} \stackrel{?}{=} 2^3 \cdot 2^2$ and $2^3 \cdot 2^2 \stackrel{?}{=} 2^{3+2}$

(2) $b^{u-v} \stackrel{?}{=} \frac{b^u}{b^v}$; e.g., $2^{3-2} \stackrel{?}{=} \frac{2^3}{2^2}$ and $\frac{2^3}{2^2} \stackrel{?}{=} 2^{3-2} = 2$

(3) $b^{ru} \stackrel{?}{=} (b^u)^r$ for any real number r; e.g., $2^{3\cdot 2} \stackrel{?}{=} (2^3)^2$ and $(2^2)^3 \stackrel{?}{=} 2^{2\cdot 3}$

(4) $b^0 \stackrel{?}{=} 1$

(5) $b^{-v} \stackrel{?}{=} \frac{1}{b^v}$; e.g., $2^{-2} \stackrel{?}{=} \frac{1}{2^2} = \frac{1}{4}$

Example 1 If $b^u = 2$ and $b^v = 3$ then $b^{u-v} \stackrel{?}{=} \frac{b^u}{b^v} = \frac{2}{3}$

▶ **Graph of** $y = b^x$

Case 1: $b > 1$

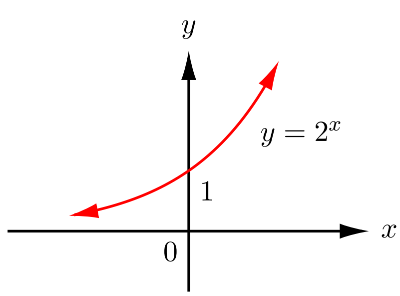
For example, $y = 2^x$.

(i) Complete the table below:

x	-1	-0.5	0	0.5	1
2^x	0.5	0.71	1	1.41	2

Truncate answers to 2 decimal places

(ii) Plot the points and sketch graph:



(iii) **Properties of b^x when $b > 1$:**

- $b^0 \stackrel{?}{=} 1$
- domain $\stackrel{?}{=}$ *all numbers* range $\stackrel{?}{=}$ *all positive numbers*
- $\lim_{x\to-\infty} b^x \stackrel{?}{=} 0$ $\lim_{x\to\infty} b^x \stackrel{?}{=} \infty$
- Asymptote: $y = 0$

Case 2: $0 < b < 1$

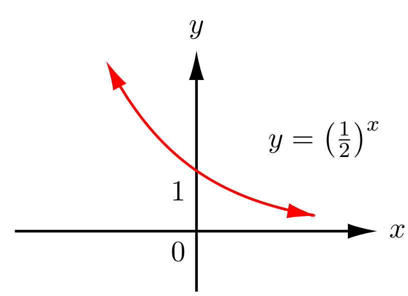
For example, $y = (1/2)^x$.

(i) Complete the table below:

x	-1	-0.5	0	0.5	1
$(1/2)^x$	2	1.41	1	0.71	0.5

Truncate answers to 2 decimal places

(ii) Plot the points and sketch graph:



(iii) **Properties of b^x when $0 < b < 1$:**

- $b^0 \stackrel{?}{=} 1$
- domain $\stackrel{?}{=}$ *all numbers* range $\stackrel{?}{=}$ *all pos. numb.*
- $\lim_{x\to-\infty} b^x \stackrel{?}{=} \infty$ $\lim_{x\to\infty} b^x \stackrel{?}{=} 0$
- Asymptote: $y = 0$

▶ Three applications of the exponential function

1 Compound interest

Example 1 If \$1,000 is invested in an account paying 5% interest, how much will it grow in 10 years if the interest is compounded monthly?

- Annual rate $= r \overset{?}{=} 0.05$ (in **decimals**)
- Compounding per year $= n \overset{?}{=} 12$
- Compounding rate $= \frac{r}{n} \overset{?}{=} \frac{0.05}{12}$
- Time $= t \overset{?}{=} 10$ (in **years**)

At the end of 1st period have: $A_1 = 1000 + 1000\frac{0.05}{12} = 1000\left(1+\frac{0.05}{12}\right)^1$

At the end of 2nd period have: $A_2 = A_1 + A_1 \cdot \frac{0.05}{12} = A_1\left(1+\frac{0.05}{12}\right)^1 = 1000\left(1+\frac{0.05}{12}\right)^2$

At the end of 3th period have: $A_3 = A_2 + A_2 \cdot \frac{0.05}{12} = A_2\left(1+\frac{0.05}{12}\right)^1 = 1000\left(1+\frac{0.05}{12}\right)^3$

⋮

At the end of 12th period have: $A_{12} = A_{11} + A_{11} \cdot \frac{0.05}{12} = A_{11}\left(1+\frac{0.05}{12}\right)^1 = 1000\left(1+\frac{0.05}{12}\right)^{12}$

Interest compounded 12 times a year over t **years**

At the end of 1 year (12 periods) have: $A(1) = 1000\left(1+\frac{0.05}{12}\right)^{12\cdot 1}$

At the end of 2 years (24 periods) have: $A(2) = A(1) + A(1)\left(1+\frac{0.05}{12}\right)^{12\cdot 1} = 1000\left(1+\frac{0.05}{12}\right)^{12\cdot 2}$

⋮

At the end of t years have: $A(t) = 1000\left(1+\frac{0.05}{12}\right)^{12\cdot t}$

- *annually* ⟶ $n = 1$
- *quarterly* ⟶ $n = 4$
- *monthly* ⟶ $n = 12$
- *weekly* ⟶ $n = 52$
- *daily* ⟶ $n = 365$

General formula: $A(t) = P\left(1+\frac{r}{n}\right)^{tn}$ or $FV = PV\left(1+\frac{r}{n}\right)^{tn}$

Example 2 If \$8,000 is invested in an account paying 3% interest, how much will it grow in 15 years if the interest is compounded quarterly?

$A(15) = 8000\left(1+\frac{0.03}{4}\right)^{15\cdot 4} \approx 12,525$

2 Population Growth (with unlimited resources) $P(t) = P_0 b^t$

Example 3 A certain bacteria culture grows exponentially. In 1 hour the population grows from 300,000 to 500,000. Write a formula expressing the population P as a function of the time t in hours.

Ans. $P(t) = 300,000(\frac{5}{3})^t$

$P(t) = 300,000b^t$

$P(1) = 300,000b^1 = 500,000$

$\Longrightarrow b = \frac{5}{3}$

$\Longrightarrow P(t) = 300,000\left(\frac{5}{3}\right)^t$

3 Decay of radioactive substances: $y = y_0 b^t$

Example 4 Radon gas decays according to the formula $y = y_0(0.835)^t$, where t is measured in days. If there are 500 cubic centimeters left after 7 days, how much was there to begin with?

$y_0 = 500(0.835)^{-7}$

$y(7) = y_0(0.835)^7 = 500 \Longrightarrow y_0 = 500(0.835)^{-7}$