Name ______________________________ Date ____________________

# Math 10250 Activity 7: Continuity (Sec. 1.3)

**GOAL:** Understand the concept of continuity and its basic properties, including the intermediate value theorem.

**Idea of Continuity:** A function is **continuous** if you never have to lift your pencil while drawing its graph. The **discontinuities** are where you have to lift your pencil, i.e, at places where there are **gaps** or **holes**.

**Example 1** Referring to the function $f$, whose graph is shown in Figure 1, find all the discontinuities of $f$ in the interval $[-2, 5]$.

$x = -1, 0, 1, 2, 3, 4$

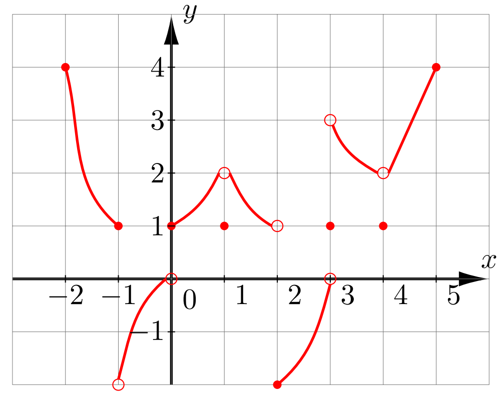



Figure 1

**Definition of continuity**

A function $f(x)$ is continuous at a point $a$ **in** its **domain** if

1. $\lim_{x\to a} f(x)$ *exists as a finite number (no gap!)*
2. $\lim_{x\to a} f(x) = f(a)$ *(no hole!)*

**Fact:** $f(x) = x^m$ is continuous everywhere

**Theorem** (Continuity Rules): If $f$ and $g$ are continuous functions at $a$ then

$$\underset{\uparrow \text{constant}}{c}f(x), \quad f(x)+g(x), \quad f(x)\cdot g(x) \quad \text{and} \quad \frac{f(x)}{g(x)} \text{ where } g(a)\neq 0 \text{ are continuous at } a.$$

From this fact and the theorem we get:

1. Polynomials are continuous everywhere.
2. $\dfrac{\text{polynomial}}{\text{polynomial}}$ (↑ rational function) is continuous *except at the zeros of denominator*.

**Example 2** Determine where the following functions are continuous.

(a) $f(x) = 2x^5 - 3x^2 + 4x - 15$ *For all numbers $x$*

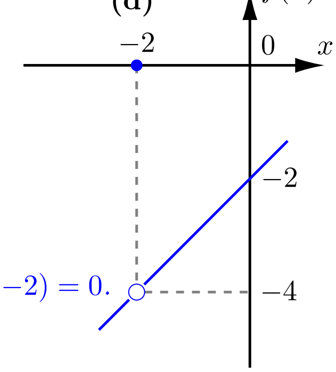
(b) $f(x) = \dfrac{x^3+1}{x^2+25}$ *For all numbers $x$*

(c) $f(x) = \dfrac{x^3+1}{x^2-25}$ *For all numbers $x$ except for $x = \pm 5$*

(d) $f(x) = \begin{cases} \dfrac{x^2-4}{x+2}, & \text{if } x \neq -2 \\ 0, & \text{if } x = -2 \end{cases}$

- *Discontinuous at $x = -2$ because* $\lim_{x\to -2} \frac{x^2-4}{x+2} = \lim_{x\to -2}(x-2) = -4$ *but* $f(-2) = 0$.
- *Continuous at all other $x$.*

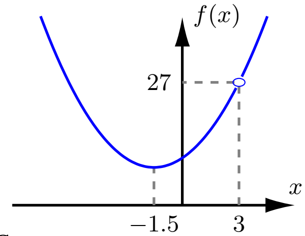
**(d)**



**Example 3** Find the number $c$ that makes $f(x) = \begin{cases} \dfrac{x^3-27}{x-3}, & \text{if } x \neq 3 \\ c, & \text{if } x = 3 \end{cases}$ continuous for every $x$.

*For $f(x)$ to be continuous at $x=3$, we must have:*

$$c = f(3) = \lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3}$$
$$= \lim_{x\to 3}(x^2+3x+9) = 3^2 + 3\cdot 3 + 9 = \boxed{27}$$

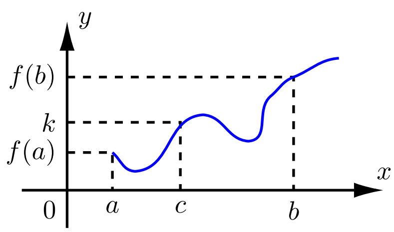



Ans. $c = 27$

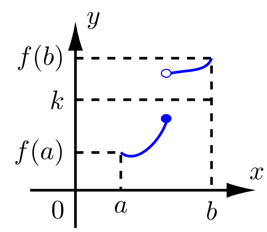
## ▸ The intermediate value theorem and zeros of functions

**Intermediate Value Theorem (IVT):** If $f$ is continuous on $[a,b]$ and $k$ is any number between $f(a)$ and $f(b)$ then there is at least one number $c$ in $[a,b]$ such that $f(c) = k$.

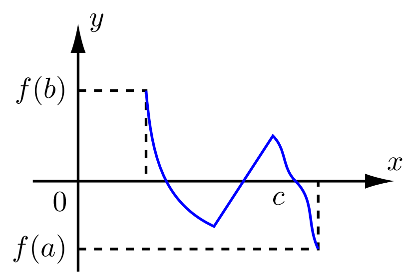
**Pictures:** Good for IVT, since $f(x)$ is continuous!




Not good for IVT, since $f(x)$ is not continuous!




**Existence of Zeros Theorem:** Take the above situation where $f(a)$ and $f(b)$ have opposite signs.
Then by IVT, there is at least one number $c$ in $(a,b)$ such that $f(c) = 0$. This helps us find zeros of functions (i.e roots).




**Example 4** Suppose a continuous function $f(x)$ satisfies the following table of values:

| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | $-2$ | $-3$ | $-2$ | $-1$ | $1$ | $2$ | $1$ | $-1$ | $-2$ |

How many roots can you be sure of $f(x)$ having on the interval $(-4,4)$, and where they are located.

*Two roots, one inside the interval $(-1,0)$ and another inside $(2,3)$.*

**Example 5** Does the equation $x^4 + 8x^3 - x^2 - 4x - 1 = 0$ have a root inside the interval $(0,1)$?

*Since $f(0) = -1 < 0$, $f(1) = 1+8-1-4-1 = 3 > 0$, the equation has a root in $(0,1)$.*

**Problem** Explain why there was a time between the day you were born and today when your height in inches (say 21) was equal to your weight in pounds (say 7). *(At 18 $h = 6ft = 72in$, $w = 150lb$)*

- $f(t) = h(t) - w(t)$
- $f(0) = 21 - 7 = 14$
- $f(18) = 72 - 150 < 0$

*Since height and weight change continuously.*

*There is $t^*$, $0 < t^* < 18$ such that $h(t^*) - w(t^*) = 0$ or $h(t^*) = w(t^*)$*

**Question** Is temperature at ND changing continuously? What about the Dow Jones Industrial Average, interest rates, or prices of products?

*Yes!  Yes!  No!  No!  (Think of more situations)*