Name ______________________________ Date ____________________

Math 10250 Activity 5: One-sided and Infinite Limits (Sec. 1.1 continued & Sec. 1.2)

GOAL: To learn about the limit of a function $f(x)$ as x approaches a number a from one side (left or right), get an understanding of infinite limits and relate them to vertical asymptotes.

▶ One-sided limits

Example 1 For the function $y = f(x)$ whose graph is shown in Figure 1, find (by visual inspection) the indicated **one-sided limits** (if they exist) and determine whether the limit of $f(x)$ exists at the given values of x.

(i) $\lim_{x\to -1^-} f(x) \stackrel{?}{=} 1$ (Left-hand limit) $\quad \lim_{x\to -1^+} f(x) \stackrel{?}{=} -1$ (Right-hand limit) $\quad f(-1) \stackrel{?}{=} 3$ *no limit*

(ii) $x = 0$ $\lim_{x\to 0^-} f(x) = 0,\ \lim_{x\to 0^+} f(x) = 0,$ *f has limit*

(iii) $x = 1$ $\lim_{x\to 1^-} f(x) = 1,\ \lim_{x\to 1^+} f(x) = 2,$ *no limit*

(iv) $x = 3$ $\lim_{x\to 3^-} f(x)$ *does not exist*, $\lim_{x\to 3^+} f(x) = 1,$ *no limit*

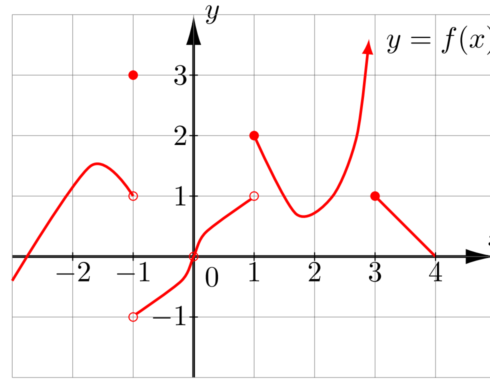

Figure 1

- **Definition of one-sided limits**

  ⋆ $\lim_{x\longrightarrow a^+} f(x) = L \iff x \underset{x>a}{\approx} a \Longrightarrow f(x) \approx L$

  ⋆ $\lim_{x\longrightarrow a^-} f(x) = L \iff x \underset{x<a}{\approx} a \Longrightarrow f(x) \approx L$

- **Fact:** $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

- **Rules of one-sided limits**. They are the **same** as the ones for usual (double-sided) limits.

Example 2 Find $\lim_{t\to 1^+} \frac{t^2-1}{\sqrt{t-1}} = \lim_{t\to 1^+} \frac{(t+1)(t-1)}{(t-1)^{\frac{1}{2}}} = \lim_{t\to 1^+} (t+1)(t-1)^{\frac{1}{2}} = 2\cdot 0 = 0$

Example 3 If $f(x)$ is the function in Example 1 and $g(x) = 8x - 1$, then find the following one-sided limits:

(i) $\lim_{x\to 1^+} [f(x)\cdot g(x)] \stackrel{?}{=} \left(\lim_{x\to 1^+} f(x)\right)\cdot\left(\lim_{x\to 1^+} g(x)\right) = 2\cdot(8\cdot 1 - 1) = 14.$

(ii) $\lim_{x\to 1^-} \frac{f(x)}{g(x)} \stackrel{?}{=} \frac{\lim_{x\to 1^-} f(x)}{\lim_{x\to 1^-} g(x)} = \frac{1}{8\cdot 1 - 1} = \frac{1}{7}.$

▶ Explain the meaning of the **infinite limits:**

- $\lim_{x\to a} f(x) = \infty \iff f(x) \approx \textit{Huge for all } x \textit{ near } a \textit{ (but } x \neq a)$ *or* $x \underset{x\neq a}{\approx} a \implies f(x) \approx +\textit{Huge}$

- $\lim_{x\to a} f(x) = -\infty \iff x \underset{x\neq a}{\approx} a \implies f(x) \approx -\textit{Huge}$

- $\lim_{x\to a^+} f(x) = \infty$ (or $-\infty$) $\iff x \underset{x>a}{\approx} a \implies f(x) \approx +\textit{Huge}$ (*or* $-$ *Huge*)

- $\lim_{x\to a^-} f(x) = \infty$ (or $-\infty$) $\iff x \underset{x<a}{\approx} a \implies f(x) \approx +\textit{Huge}$ (*or* $-$ *Huge*)

Example 4 For the function whose graph is shown in Figure 2 determine its limiting behavior as x approaches each of the points:

(i) $x = -2$ (ii) $x = 0$ (iii) $x = 2$ (iv) $x = 4$,

and find its **vertical asymptotes** (v.a.).

i) $\lim_{x\to -2^-} f(x) = \infty$, $\lim_{x\to -2^+} f(x) = -\infty \implies x = -2$ *is v.a.*

ii) $\lim_{x\to 0^\pm} f(x) = \infty = \lim_{x\to 0} f(x) \implies x = 0$ *is v.a.*

iii) $\lim_{x\to 2^-} f(x) = 0$, $\lim_{x\to 2^+} f(x) = -\infty \implies x = 2$ *is v.a.*

iv) $\lim_{x\to 4^\pm} f(x) = -\infty = \lim_{x\to 4} f(x) \implies x = 4$ *is v.a.*

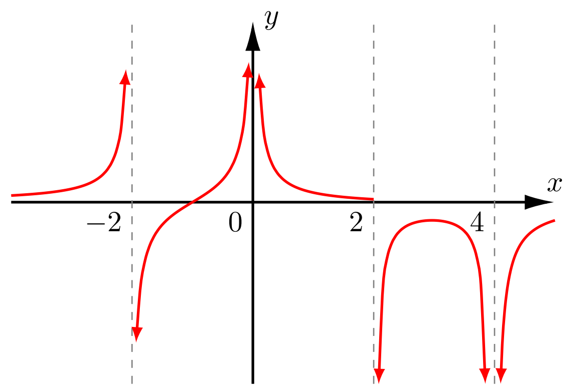

Figure 2

Example 5 $\lim_{x\to 3} \frac{1}{(x-3)^2} \overset{?}{=} \frac{1}{(3 \pm \textit{small} - 3)^2} = \frac{1}{(\pm \textit{small})^2} \approx \frac{1}{(\textit{small})^2} \simeq \infty$

Ans. ∞

Example 6 $\lim_{x\to 3} \frac{x}{x^2-9} \overset{?}{=} \lim_{x\to 3} \frac{x}{x+3} \cdot \frac{1}{x-3}$

Hint. Check both left and right hand limits.

- $\lim_{x\to 3^-} \frac{x}{x+3}\frac{1}{x-3} \approx \frac{3}{3+3} \cdot \frac{1}{(3-\textit{small})-3} \approx \frac{1}{2} \cdot \frac{1}{-\textit{small}} \approx -\textit{HUGE}$

- $\lim_{x\to 3^+} \frac{x}{x+3}\frac{1}{x-3} \approx \frac{3}{3+3} \cdot \frac{1}{(3+\textit{small})-3} \approx \frac{1}{2} \cdot \frac{1}{\textit{small}} \approx \textit{HUGE}$

Ans. DNE