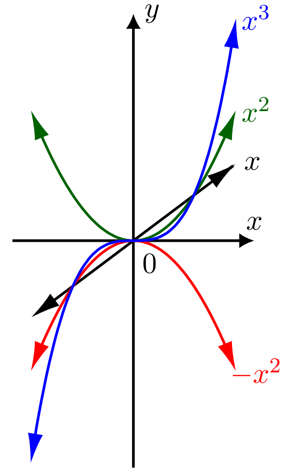
Name ______________________________ Date ___________________

## Math 10250 Activity 3: Polynomial, Rational, and Power Functions (Section 0.6)

**GOAL:** To learn the basic properties and behavior of polynomial, rational, and power functions.

**Q1:** Write an example of a function which is:

- linear: $f(x) = x$ *or* $2x$ *or* $2x + 1$
- quadratic: $f(x) = x^2$ *or* $3x^2$ *or* $3x^2 - 5x + 8$
- cubic: $f(x) = x^3$ *or* $2x^3$ *or* $-x^3 + 5x^2 + x - 3$
- a $4^{th}$ degree polynomial: $f(x) = x^4$ *or* $2x^4 - 3x^2 + 5x - 4$




An $n$th-degree **polynomial** function is of the form:

$$f(x) = \underset{\substack{\uparrow \\ \textbf{leading coefficient } a_n \neq 0}}{a_n} x^n + \cdots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are constants.

### ▶ Behavior of polynomials

**Q2:** How does the graph of $f(x) = a_nx^n + \cdots + a_2x^2 + a_1x + a_0$ behave as $x$ moves to the left and right?

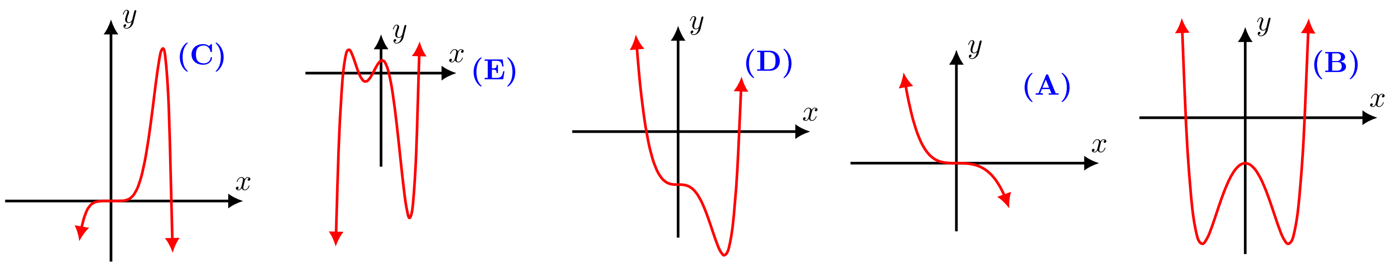
***Tool:*** If $|x|$ is very large then $a_nx^n + \cdots + a_2x^2 + a_1x + a_0 \underset{\substack{\uparrow \\ \text{approximately}}}{\approx} a_nx_n$ .

**Example 1** Match the graphs to the given polynomials $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$, and $f_5(x)$. The behavior at infinity is indicated by arrows in the graphs.

(A) $f_1(x)$ is a $5^{th}$ degree polynomial with negative leading coefficient.

(B) $f_2(x) = 3x^6 - 4x^2 - 1$ (C) $f_3(x) = 15x^5 - 10x^8$ (D) $f_4(x) = -1 - 4x^3 + 3x^6$

(E) $f_5(x) = x + 3 - 1.04x^3 + 0.04x^5 + 0.12x^4 - 3.12x^2$




### ▶ Arithmetic with polynomials

- polynomial + polynomial = polynomial: Ex. $(2x^2 + 3x) + (x^5 + 1) \stackrel{?}{=} x^5 + 2x^2 + 3x + 1$

- polynomial × polynomial = polynomial: Ex. $(x^2 + 1)(x^3 - 5x) \stackrel{?}{=} x^5 - 5x^3 + x^3 - 5x = x^5 - 4x^3 - 5x$

**Q3:** What do we call a $\dfrac{\text{polynomial}}{\text{polynomial}}$?

**A3:** $\dfrac{\text{polynomial}}{\text{polynomial}} = \textit{rational function}.$

**Example 2** Give the natural domain and the **equations** of all vertical asymptotes of the given rational functions:

**a.** $\dfrac{x-2}{x^2+3x+2} = \dfrac{x-2}{(x+1)(x+2)}$

$x^2+3x+2 = (x+1)(x+2)$

*Natural domain* $x \neq -1, -2$

*Vert. asympt.* $x = -1$ *and* $x = -2$

**b.** $\dfrac{x-4}{x^2-5x+4} = \dfrac{x-4}{(x-1)(x-4)} = \dfrac{1}{x-1}$

$x^2-5x+4 = (x-1)(x-4)$

*Natural domain* $x \neq 1, 4$

*Vert. asympt.* $x = 1$

**Q4:** Where are rational functions defined?

**A4:** $x \neq$ *zero of denominator*

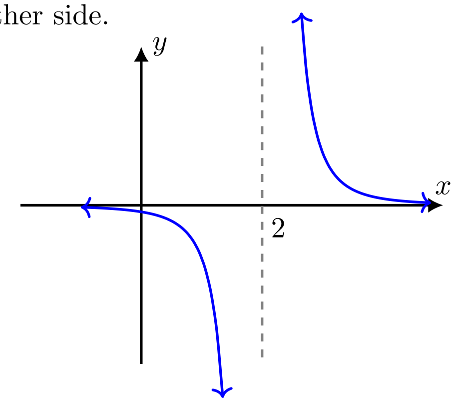
**Q5:** Where do rational functions have **vertical asymptotes**?

**A5:** *At the zeros of denominator that are not zeros of numerator*

**Example 3** Find a vertical asymptote of $f(x) = \dfrac{1}{(x-2)^3}$, if any. If there is a vertical asymptote, determine how the graph behaves (i.e., rises or falls) as it approaches the asymptote from either side by checking whether the function is positive or negative close to the asymptote on either side.

*v.a.* $(x-2)^3 = 0 \Longrightarrow x = 2$

| | | | | |
|---|---|---|---|---|
| $x$ | $2-\frac{1}{10}$ | $2-\frac{1}{100}$ | $2+\frac{1}{100}$ | $2+\frac{1}{10}$ |
| $\frac{1}{(x-2)^3}$ | $-10^3$ | $-10^6$ | $10^6$ | $10^3$ |




## ▶ Power functions

**Q6:** What do **power functions** look like?

**A6:** $\boxed{y = Cx^m,}$ where $C$ and $m$ are fixed numbers. Ex. $y = 2x^5$, $y = 8x^{\frac{4}{3}}$

**Q7:** How do we interpret $x^m$ when $m = \frac{p}{q}$ for positive integers $p$ and $q$?

**A7:**

$$\boxed{x^{p/q} = (\sqrt[q]{x})^p = \phantom{xxxx}} \qquad \boxed{x^{-p/q} = \frac{1}{(\sqrt[q]{x})^p}.}$$

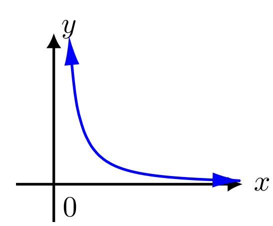
**Example 4** Compute $(27)^{2/3}$ without using a calculator.

$$(\sqrt[3]{27})^2 = 3^2 = 9$$

**Example 5** Find the natural domain of the power function $f(x) = x^{-3/2}$ and sketch its graph.

$f(x) = x^{-\frac{3}{2}} = \frac{1}{(\sqrt{x})^3}$

*Natural domain* $x > 0$




**Application** In economics, $U(x) = x^\alpha$, $0 < \alpha < 1$, is called power utility function, where $x$ is the level of consumption and $U(x)$ is satisfaction from $x$.

*Ex.* $U(x) = x^{\frac{1}{2}}$.