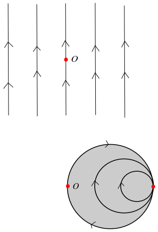

Tame flows

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University of Notre Dame

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1. Basic constructions
2. Properties of tame flows
3. Morse-like tame flows
4. Combinatorial Morse theory
5. Flip-flops and Morse like flows

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A tame flow is a pair (X, Φ) such that

- X is a tame set.
- Φ is a continous tame map $\mathbb{R} \times X \to X$, $(t, x) \mapsto \Phi^t(x)$ such that

$$\Phi^{t+s}(x) = \Phi^t\big(\Phi^s(x)\big) \quad \forall t, s \in \mathbb{R}, \; x \in X.$$

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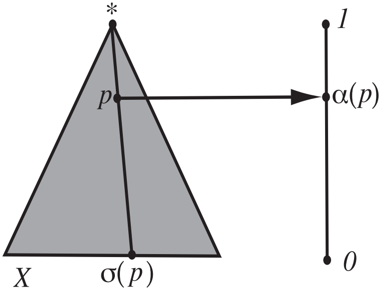

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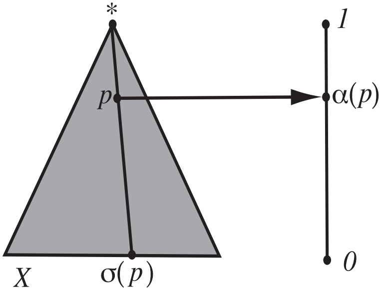

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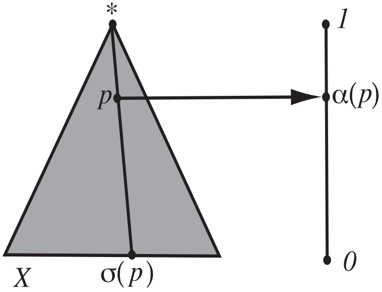

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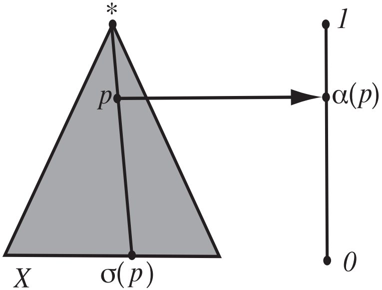

The shadow $\sigma(p)$ is governed by the flow Φ on X. The altitude $\alpha(p)$ changes according to the canonical flow on $[0, 1]$. The vertex $*$ is a stationary repelling point.

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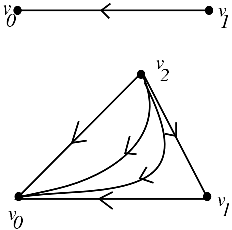

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- Thus, tame flows exist on any compact tame set.

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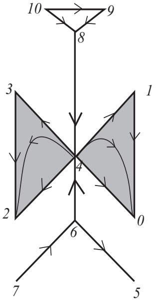

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- The tame flow associated to this dynamical orientation is called the *barycentric flow*.

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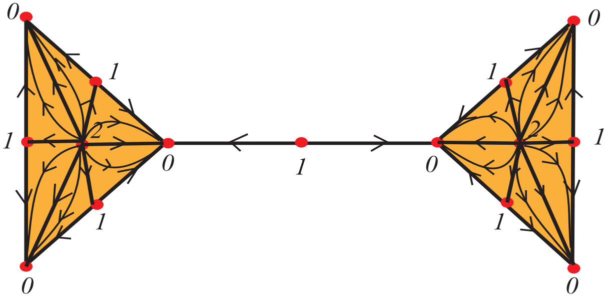

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Then the flow generated by $\nabla^g f$ is a tame flow.

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- One can show that if $\nu > \frac{m-2}{2}$, then almost all vectors $\vec{\mu} \in \mathbb{R}^m$ satisfy the (C, ν)-condition for some $C > 0$.

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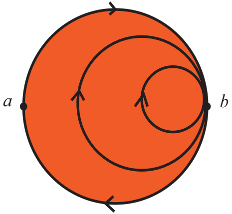
  The (un)stable varieties are tame sets.

A tame flow with lots of homoclinic trajectories



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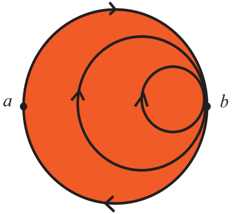

The above flow has two stationary points, a, b, and the overlap $W^{+}(b) \cap W^{-}(b)$ is the interior of the disk. This overlap is filled by homoclinic trajectories.

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is called the cell decomposition dual to the triangulation $\mathcal{S}$. It plays a central role in the classical proof of the Poincaré duality theorem.

Barycentric flows on manifolds

- If X is an oriented smooth, tame manifold, of dimension m equipped with a *piecewise linear* triangulation $\mathcal{S}$. Denote by Φ the associated barycentric flow.
- For any $\sigma \in \mathcal{S}$, the stable variety of b_σ is tamely homeomorphic to an open disk of dimension $m - \dim \sigma$ and intersects the face σ "transversally" at one point, the barycenter b_σ.
- The resulting cell decomposition

$$X = \bigcup_{\sigma \in \mathcal{S}} W^+(b_\sigma)$$

  is called the cell decomposition dual to the triangulation $\mathcal{S}$. It plays a central role in the classical proof of the Poincaré duality theorem.
- If we fix an orientation o_σ on each face $\sigma \in \mathcal{S}$, then we have an induced orientation $o_\sigma^\perp$ on $W^+(b_\sigma)$ uniquely determined by the equality $o_\sigma^\perp \wedge o_\sigma =$ orientation of M.

Tame flows on manifolds

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Then the volume of Γ_t is bounded from above by a constant independent of t and as $t \to \infty$ the current $[\Gamma_t]$ converges weakly (and in the flat distance) to an integral subanalytic current $[\Gamma_\infty]$. The current $[\Gamma_\infty]$ is deformation of the diagonal, i.e., it is homologous in the sense of integral subanalytic currents with the current supported by the diagonal $[\Gamma_0]$.

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Suppose that M is a compact, oriented tame smooth manifold and Φ is a tame flow on M. Then for any smooth form α on M the pullbacks $(\Phi^t)^*\alpha$ have a limit in the sense of currents as $t \to \infty$.

Barycentric flows on manifolds

Corollary

Suppose that M is a tame, compact oriented smooth manifold of dimension m equipped with a piecewise linear triangulation $\mathcal{S}$*. Fix orientations* o_σ *on each of the faces* $\sigma \in \mathcal{S}$ *and denote by* Φ *the barycentric flow on M determined by* $\mathcal{S}$*. Then the following hold.*

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(a) For any smooth, k-form α on M, the pullback $(\Phi^t)^*\alpha$ comverges in the sense of currents as $t \to -\infty$ to the $(m-k)$-dimensional current

$$(\Phi^{-\infty})^*\alpha := \sum_{\dim \sigma = m-k} (-1)^{k(m-k)} \left(\int_{(W^+(b_\sigma), o_\sigma^\perp)} \alpha \right) [W^-(b_\sigma), o_\sigma],$$

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(b) If α is a closed k-form on M, then the current $(\Phi^{-\infty})^*\alpha$ is a $(m-k)$-dimensional simplicial cycle whose homology class is Poincaré dual to the cohomology class of α.

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A triplet (X, Φ, f) with the above properties is called a *Morse triplet*.

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- The cone of a Morse-like tame flow is a Morse-like tame flow.

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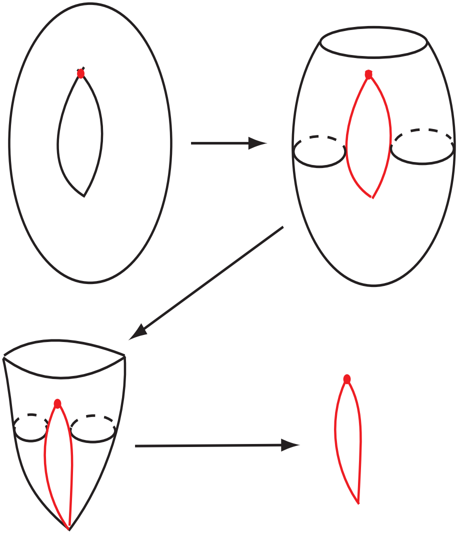

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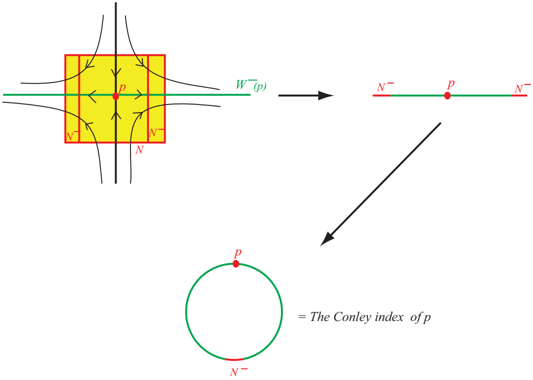

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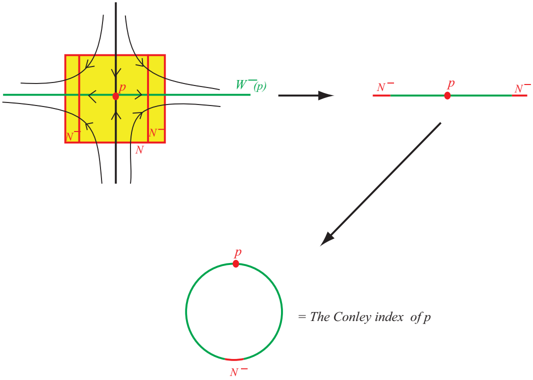

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Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$ we have

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(c) For a combinatorial Morse function ω on $\mathcal{S}$ we denote by Φ_ω the associated simplicial flow. The stationary points of this flow are the barycenters b_σ of the faces $\sigma \in \mathcal{S}$. We denote by $\mathcal{I}_\omega(\sigma)$ the Conley index of b_σ with respect to the flow Φ_ω.

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$$\mathcal{S} \ni \sigma \mapsto \dim \sigma$$

is a combinatorial Morse function on $\mathcal{S}$.

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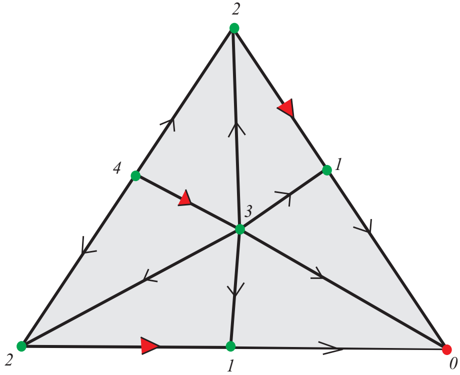

Green points correspond to regular faces while red points correspond to critical faces. The red arrow indicate the dynamical violations.

The Conley indices of combinatorial Morse functions

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(a) *If σ is a regular face, then the Conley index $\mathcal{I}_\omega(b_\sigma)$ is trivial.*

(b) *If σ is a critical face, then the Conley index $\mathcal{I}_\omega(b_\sigma)$ is homotopic to a sphere of dimension $\dim \sigma$.*

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- A *weight* for a tame blowdown map $\beta : X \to Y$ is a tame continuous function $w : Y \to [0, \infty)$ such that

  $$L_\beta = w^{-1}(0).$$

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- *Similar results are true for the maps* β_i^-.

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- We obtain a string of weighted blowdown maps

$$X_{c_n} \stackrel{\beta_n^-}{\leftarrow} X_{r_n} \stackrel{\beta_n^+}{\rightarrow} X_{c_{n-1}} \stackrel{\beta_{n-1}^-}{\leftarrow} \cdots \stackrel{\beta_2^+}{\rightarrow} X_{c_1} \stackrel{\beta_1^-}{\leftarrow} X_{r_1} \stackrel{\beta_n^+}{\rightarrow} X_{c_0},$$

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  where X_{c_n} and X_{c_0} are finite.
- Moreover, X is tamely homeomorphic to the space

$$M_{\beta_n^-} \cup_{X_{r_n}} M_{\beta_n^+} \cup_{X_{c_{n-1}}} \cdots \cup_{X_{r_1}} M_{\beta_1^+}.$$

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- A *flip-flop* is a sequence $\mathcal{F}$ of blowdown maps

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- We see that to any Morse triplet (X, Φ, f) we have associated a canonical weighted flip-flop $\mathcal{F}(X, \Phi, f)$ with total space X.

Characterization of Morse-like tame flows

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- *For every weighted flip-flop* $\mathcal{F}$ *there exists a canonical tame Morse-like flow* $\Phi = \Phi_{\mathcal{F}}$ *on the total space* $X = M(\mathcal{F})$ *and a tame Lyapunov function* $f_{\mathcal{F}}$ *such that the flip-flop associated to the Morse triplet* $(M(\mathcal{F}), \Phi_{\mathcal{F}}, f_{\mathcal{F}})$ *coincides with* $\mathcal{F}$*.*

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- *Conversely, if* $\mathcal{F}$ *is the flip-flop associated to a Morse-triplet* (X, Φ, f), *then* $\Phi = \Phi_{\mathcal{F}}$.