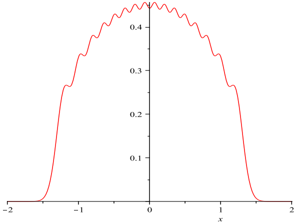

Critical Sets of Random Smooth Functions on Compact Manifolds

Liviu I. Nicolaescu

A baby case

A baby case

- Denote by $\mathcal{P}_d \subset C^\infty(S^2)$ the space consisting of the restrictions to S^2 of the polynomials in 3-variables of degree $\leq d$. ($\dim \mathcal{P}_d = d^2$).

A baby case

- Denote by $\mathcal{P}_d \subset C^\infty(S^2)$ the space consisting of the restrictions to S^2 of the polynomials in 3-variables of degree $\leq d$. ($\dim \mathcal{P}_d = d^2$). $S(\mathcal{P}_d) :=$ the unit sphere in $\mathcal{P}_d$ with respect to the L^2-metric.

A baby case

- Denote by $\mathcal{P}_d \subset C^\infty(S^2)$ the space consisting of the restrictions to S^2 of the polynomials in 3-variables of degree $\leq d$. ($\dim \mathcal{P}_d = d^2$). $S(\mathcal{P}_d) :=$ the unit sphere in $\mathcal{P}_d$ with respect to the L^2-metric.
- For $d \gg 1$ almost any function $p \in \mathcal{P}_d$ has a finite number $\mathcal{N}(p)$ of critical points.

A baby case

- Denote by $\mathcal{P}_d \subset C^\infty(S^2)$ the space consisting of the restrictions to S^2 of the polynomials in 3-variables of degree $\leq d$. ($\dim \mathcal{P}_d = d^2$). $S(\mathcal{P}_d) :=$ the unit sphere in $\mathcal{P}_d$ with respect to the L^2-metric.
- For $d \gg 1$ almost any function $p \in \mathcal{P}_d$ has a finite number $\mathcal{N}(p)$ of critical points.
- Main Question: What is the behavior of

$$\boldsymbol{N}_d := \frac{1}{\text{area}\, S(\mathcal{P}_d)} \int_{S(\mathcal{P}_d)} \mathcal{N}(p)\, dA(p),$$

as $d \to \infty$.

A baby case

- Denote by $\mathcal{P}_d \subset C^\infty(S^2)$ the space consisting of the restrictions to S^2 of the polynomials in 3-variables of degree $\leq d$. ($\dim \mathcal{P}_d = d^2$). $S(\mathcal{P}_d) :=$ the unit sphere in $\mathcal{P}_d$ with respect to the L^2-metric.
- For $d \gg 1$ almost any function $p \in \mathcal{P}_d$ has a finite number $\mathcal{N}(p)$ of critical points.
- Main Question: What is the behavior of

$$\boldsymbol{N}_d := \frac{1}{\operatorname{area} S(\mathcal{P}_d)} \int_{S(\mathcal{P}_d)} \mathcal{N}(p) dA(p),$$

  as $d \to \infty$. $\boldsymbol{N}_d$ is the expected number of critical points of the restriction to S^2 of a random polynomial of degree $\leq d$.

A baby case

- Denote by $\mathcal{P}_d \subset C^\infty(S^2)$ the space consisting of the restrictions to S^2 of the polynomials in 3-variables of degree $\leq d$. ($\dim \mathcal{P}_d = d^2$). $S(\mathcal{P}_d) :=$ the unit sphere in $\mathcal{P}_d$ with respect to the L^2-metric.
- For $d \gg 1$ almost any function $p \in \mathcal{P}_d$ has a finite number $\mathcal{N}(p)$ of critical points.
- Main Question: What is the behavior of

$$\boldsymbol{N}_d := \frac{1}{\operatorname{area} S(\mathcal{P}_d)} \int_{S(\mathcal{P}_d)} \mathcal{N}(p)dA(p),$$

  as $d \to \infty$. $\boldsymbol{N}_d$ is the expected number of critical points of the restriction to S^2 of a random polynomial of degree $\leq d$.
- Answer: $\boldsymbol{N}_d \sim \frac{8\pi}{3\sqrt{3}} d^2$ as $d \to \infty$.

Mental health break

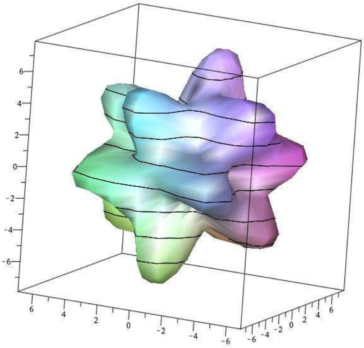

Figure: Random degree 6 polynomial on S^2. More

The general setup

Input

- A smooth compact, connected Riemann manifold (M, g) of dimension $m \geq 1$.

The general setup

Input

- A smooth compact, connected Riemann manifold (M, g) of dimension $m \geq 1$.
- For any $L > 0$ we set

$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$

The general setup

Input

- A smooth compact, connected Riemann manifold (M, g) of dimension $m \geq 1$.
- For any $L > 0$ we set

$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$

- Denote by $d\gamma_L$ the Gaussian probability measure on $\boldsymbol{U}_L$

$$d\gamma_L(\boldsymbol{u}) := (2\pi)^{-\frac{\dim \boldsymbol{U}_L}{2}} e^{-\frac{1}{2}\|\boldsymbol{u}\|^2} |du|, \;\; \|\boldsymbol{u}\|^2 := \int_M u^2 |dV_g|.$$

The general setup

Input

- A smooth compact, connected Riemann manifold (M, g) of dimension $m \geq 1$.
- For any $L > 0$ we set

$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$

- Denote by $d\gamma_L$ the Gaussian probability measure on $\boldsymbol{U}_L$

$$d\gamma_L(\boldsymbol{u}) := (2\pi)^{-\frac{\dim \boldsymbol{U}_L}{2}} e^{-\frac{1}{2}\|\boldsymbol{u}\|^2} |du|, \quad \|\boldsymbol{u}\|^2 := \int_M u^2 |dV_g|.$$

- For $L \gg 0$ almost all functions $\boldsymbol{u} \in \boldsymbol{U}_L$ are Morse. Denote by $\mathcal{N}_L(\boldsymbol{u})$ the number of critical points of $\boldsymbol{u} \in \boldsymbol{U}_L$ and define the expected number of critical points of a random function $\boldsymbol{u} \in \boldsymbol{U}_L$ to be

$$\boldsymbol{N}_L(M, g) := \int_{\boldsymbol{U}_L} \mathcal{N}_L(\boldsymbol{u})\, d\gamma_L(\boldsymbol{u}).$$

Theorem (Asymptotics of $\boldsymbol{N}_L(M, g)$, Nic., 2011)

Theorem (Asymptotics of $\boldsymbol{N}_L(M,g)$, Nic., 2011)

For any $m \geq 1$ *there exists a positive constant* C_m *such that, for any compact, connected Riemann manifold* (M,g) *of dimension* m *we have*

Theorem (Asymptotics of $\boldsymbol{N}_L(M,g)$, Nic., 2011)

For any $m \geq 1$ there exists a positive constant C_m such that, for any compact, connected Riemann manifold (M,g) of dimension m we have

$$\boldsymbol{N}_L(M,g) \sim C_m \dim \boldsymbol{U}_L \sim C'_m \mathrm{vol}_g(M) L^{\frac{m}{2}} \text{ as } L \to \infty,$$

where

Theorem (Asymptotics of $\boldsymbol{N}_L(M,g)$, Nic., 2011)

For any $m \geq 1$ there exists a positive constant C_m such that, for any compact, connected Riemann manifold (M,g) of dimension m we have

$$\boldsymbol{N}_L(M,g) \sim C_m \dim \boldsymbol{U}_L \sim C_m' \mathrm{vol}_g(M) L^{\frac{m}{2}} \;\; \text{as} \;\; L \to \infty,$$

where

$$C_m' = \frac{C_m \boldsymbol{\omega}_m}{(2\pi)^m},$$

$\boldsymbol{\omega}_m =$ *the volume of the m-dimensional unit Euclidean ball.*

The explicit description of C_m

Denote by $\mathcal{S}_m$ the space of real, symmetric $m \times m$ matrices.

The explicit description of C_m

Denote by $\mathcal{S}_m$ the space of real, symmetric $m \times m$ matrices.
Then

$$C_m = \left(\frac{4\pi}{m+4}\right)^{\frac{m}{2}} \Gamma\left(1+\frac{m}{2}\right) \underbrace{\int_{\mathcal{S}_m} |\det X|\, d\gamma_*(X)}_{=:I_m},$$

where $d\gamma_*$ is the Gaussian probability measure on $\mathcal{S}_m$ described by

The explicit description of C_m

Denote by $\mathcal{S}_m$ the space of real, symmetric $m \times m$ matrices.
Then

$$C_m = \left(\frac{4\pi}{m+4}\right)^{\frac{m}{2}} \Gamma\Big(1+\frac{m}{2}\Big) \underbrace{\int_{\mathcal{S}_m} |\det X|\, d\gamma_*(X)}_{=:I_m},$$

where $d\gamma_*$ is the Gaussian probability measure on $\mathcal{S}_m$ described by

$$d\gamma_*(X) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}\sqrt{\mu_m}} \cdot e^{-\frac{1}{4}\left(\operatorname{tr} X^2 - \frac{1}{m+2}(\operatorname{tr} X)^2\right)} 2^{\frac{1}{2}\binom{m}{2}} \prod_{i\leq j} dx_{ij},$$

$$\mu_m = 2^{\binom{m}{2}+1}(m+2)^{m-1}.$$

Theorem (The asymptotics of C_m, Nic,2011)

Theorem (The asymptotics of C_m, Nic,2011)

$$\log C_m \sim \frac{m}{2} \log m \;\; as \;\; m \to \infty.$$

Idea of proof

Idea of proof

Step 1. We prove a key formula that describes a smooth function $\rho_L : M \to \mathbb{R}$ such that

$$\boldsymbol{N}_L(M,g) = \int_M \rho_L(\boldsymbol{x}) |dV_g(\boldsymbol{x})|.$$

Idea of proof

Step 1. We prove a key formula that describes a smooth function $\rho_L : M \to \mathbb{R}$ such that

$$\boldsymbol{N}_L(M,g) = \int_M \rho_L(\boldsymbol{x})|dV_g(\boldsymbol{x})|.$$

Step 2. We use probabilistic ideas to express ρ_L in terms of the spectral function of the Laplacian of g, i.e., the Schwartz kernel of the orthogonal projection onto $\boldsymbol{U}_L$.

Idea of proof

Step 3.Using results of L. Hörmander (1968) and X. Bin (2004) about the spectral function we show that

$$L^{-\frac{m}{2}}\rho_L(\boldsymbol{x}) \to const_m \ \text{ as } L \to \infty,$$

uniformly in $\boldsymbol{x}$. This yields the estimate of $\boldsymbol{N}_L(m,g)$.

Idea of proof

Step 3.Using results of L. Hörmander (1968) and X. Bin (2004) about the spectral function we show that

$$L^{-\frac{m}{2}}\rho_L(\boldsymbol{x}) \to const_m \ \text{ as } L \to \infty,$$

uniformly in $\boldsymbol{x}$. This yields the estimate of $\boldsymbol{N}_L(m,g)$.

Step 4. We reduce the estimate of C_m to Wigner's semi-circle theorem in random matrix theory.

Corollaries

Corollaries

Corollary (Asymptotic equidistribution of critical points)

The expected number $\boldsymbol{N}_L(\mathcal{O}, g)$ *of critical points points inside an open subset* $\mathcal{O} \subset M$ *of a random function in* $\boldsymbol{U}_L$ *satisfies the asymptotic estimate*

$$\boldsymbol{N}_L(\mathcal{O}, g) = \int_{\mathcal{O}} \rho_L(\boldsymbol{x})|dV_g(\boldsymbol{x})| \sim const_m L^{\frac{m}{2}} \mathrm{vol}_g(\mathcal{O}) \ \text{ as } \ L \to \infty.$$

Corollaries

Corollaries

Corollary

Consider the evaluation map

$$\mathbf{ev}_L : M \to \boldsymbol{U}_L{}^{\vee} := \operatorname{Hom}(\boldsymbol{U}_L, \mathbb{R}), \;\; \boldsymbol{x} \mapsto \mathbf{ev}_L(\boldsymbol{x}),$$

where

$$\mathbf{ev}_L(\boldsymbol{x})(\boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{x}), \;\; \forall \boldsymbol{u} \in \boldsymbol{U}_L.$$

For L large $\mathbf{ev}_L$ *is an embedding and we denote by* g_L *the pull back by* $\mathbf{ev}_L$ *of the Euclidean metric on* $\boldsymbol{U}_L{}^{\vee}$*. Then*

$$L^{-\frac{m+2}{2}} g_L \xrightarrow{C^0} const_m \cdot g \;\; as \;\; L \to \infty.$$

Corollaries

Corollaries

Remark (R. Adler & J. Taylor)

The sectional curvature of the metric g_L has a simple and elegant probabilitic description.

The Laplacian on S^2.

- The eigenvalues of the Laplacian Δ_d on S^2 are $\lambda_n(d) = n(n+1)$, $n \geq 0$.

The Laplacian on S^2.

- The eigenvalues of the Laplacian Δ_d on S^2 are $\lambda_n(d) = n(n+1)$, $n \geq 0$.
- We set $\mathcal{Y}_{n,d} := \ker(\lambda_n(d) - \Delta_d)$.

The Laplacian on S^2.

- The eigenvalues of the Laplacian Δ_d on S^2 are $\lambda_n(d) = n(n+1)$, $n \geq 0$.
- We set $\mathcal{Y}_{n,d} := \ker(\lambda_n(d) - \Delta_d)$.
- $u \in \mathcal{Y}_n$ iff u is the restriction of a harmonic, homogeneous polynomial of degree n in 3 variables.

The Laplacian on S^2.

- The eigenvalues of the Laplacian Δ_d on S^2 are $\lambda_n(d) = n(n+1)$, $n \geq 0$.
- We set $\mathcal{Y}_{n,d} := \ker(\lambda_n(d) - \Delta_d)$.
- $u \in \mathcal{Y}_n$ iff u is the restriction of a harmonic, homogeneous polynomial of degree n in 3 variables.
- $\dim \mathcal{Y}_n = 2n + 1$.

The Laplacian on S^2.

- The eigenvalues of the Laplacian Δ_d on S^2 are $\lambda_n(d) = n(n+1)$, $n \geq 0$.
- We set $\mathcal{Y}_{n,d} := \ker(\lambda_n(d) - \Delta_d)$.
- $u \in \mathcal{Y}_n$ iff u is the restriction of a harmonic, homogeneous polynomial of degree n in 3 variables.
- $\dim \mathcal{Y}_n = 2n + 1$.
- Equip $\mathcal{Y}_n$ with the standard Gaussian measure induced by the natural L^2-inner products. We denote by μ_n the expected number of critical points of a random harmonic polynomial $P \in \mathcal{Y}_n$).

The Laplacian on S^2.

- The eigenvalues of the Laplacian Δ_d on S^2 are $\lambda_n(d) = n(n+1)$, $n \geq 0$.
- We set $\mathcal{Y}_{n,d} := \ker(\lambda_n(d) - \Delta_d)$.
- $u \in \mathcal{Y}_n$ iff u is the restriction of a harmonic, homogeneous polynomial of degree n in 3 variables.
- $\dim \mathcal{Y}_n = 2n+1$.
- Equip $\mathcal{Y}_n$ with the standard Gaussian measure induced by the natural L^2-inner products. We denote by μ_n the expected number of critical points of a random harmonic polynomial $P \in \mathcal{Y}_n$).

Theorem (Nic. 2010)

$$\mu_n \sim \frac{n^2}{\sqrt{3}} \text{ as } n \to \infty.$$

Zonal domains of random spherical harmonics

Zonal domains of random spherical harmonics

Denote by ζ_n the expected number of zonal domains of a random spherical harmonic of degree n on S^2.

Zonal domains of random spherical harmonics

Denote by ζ_n the expected number of zonal domains of a random spherical harmonic of degree n on S^2.

Corollary

$$\forall \varepsilon > 0, \;\; \zeta_n < \left(\frac{1}{2\sqrt{3}} + \varepsilon\right) n^2, \;\; \forall n \gg 0.$$

(Nic. 2010)

Zonal domains of random spherical harmonics

Denote by ζ_n the expected number of zonal domains of a random spherical harmonic of degree n on S^2.

Corollary

$$\forall \varepsilon > 0, \;\; \zeta_n < \left(\frac{1}{2\sqrt{3}} + \varepsilon \right) n^2, \;\; \forall n \gg 0. \qquad \text{(Nic. 2010)}$$

- F. Nazarov, M. Sodin (2009) have proved that

$$\exists a \in (0,1): \;\; \zeta_n \sim a n^2 \;\text{ as }\; n \to \infty.$$

The above corollary implies that $a \leq \frac{1}{2\sqrt{3}} \approx 0.288$

Zonal domains of random spherical harmonics

Denote by ζ_n the expected number of zonal domains of a random spherical harmonic of degree n on S^2.

Corollary

$$\forall \varepsilon > 0, \;\; \zeta_n < \left(\frac{1}{2\sqrt{3}} + \varepsilon \right) n^2, \;\; \forall n \gg 0. \qquad \text{(Nic. 2010)}$$

- F. Nazarov, M. Sodin (2009) have proved that

  $$\exists a \in (0,1) : \;\; \zeta_n \sim a n^2 \;\text{ as }\; n \to \infty.$$

  The above corollary implies that $a \leq \frac{1}{2\sqrt{3}} \approx 0.288$
- Previous work of Pleijel, Peetre, Bérard-Meyer implies that $a \leq \frac{4}{j_0^2} \approx 0.692$, where j_0 is the first positive zero of the Bessel function J_0.

Water and ice

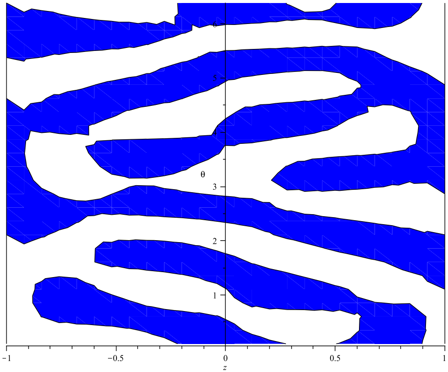

Figure: Zonal domains of a random harmonic polynomial of degree 7.

The key formula: input

The key formula: input

- Fix a smooth, compact, connected manifold M of dimension m.

The key formula: input

- Fix a smooth, compact, connected manifold M of dimension m.
- Fix a sample space on M, i.e., a pair $(\boldsymbol{U}, h)$, where $\boldsymbol{U}$ is a finite dimensional subspace $\boldsymbol{U} \subset C^{\infty}(M)$ and h is an Euclidean metric on $\boldsymbol{U}$. Set $N := \dim \boldsymbol{U}$, and denote by $S(\boldsymbol{U})$ the unit sphere in $\boldsymbol{U}$.

The key formula: input

- Fix a smooth, compact, connected manifold M of dimension m.
- Fix a sample space on M, i.e., a pair $(\boldsymbol{U}, h)$, where $\boldsymbol{U}$ is a finite dimensional subspace $\boldsymbol{U} \subset C^\infty(M)$ and h is an Euclidean metric on $\boldsymbol{U}$. Set $N := \dim \boldsymbol{U}$, and denote by $S(\boldsymbol{U})$ the unit sphere in $\boldsymbol{U}$.
- Denote by $d\gamma_h(\boldsymbol{u})$ the associated Gaussian measure on $\boldsymbol{U}$

$$d\gamma_h(\boldsymbol{u}) = (2\pi)^{-\frac{N}{2}} e^{-\frac{1}{2}|\boldsymbol{u}|_h^2} |dV_h(\boldsymbol{u})|.$$

The key formula: input

- Fix a smooth, compact, connected manifold M of dimension m.
- Fix a sample space on M, i.e., a pair $(\boldsymbol{U}, h)$, where $\boldsymbol{U}$ is a finite dimensional subspace $\boldsymbol{U} \subset C^\infty(M)$ and h is an Euclidean metric on $\boldsymbol{U}$. Set $N := \dim \boldsymbol{U}$, and denote by $S(\boldsymbol{U})$ the unit sphere in $\boldsymbol{U}$.
- Denote by $d\gamma_h(\boldsymbol{u})$ the associated Gaussian measure on $\boldsymbol{U}$

$$d\gamma_h(\boldsymbol{u}) = (2\pi)^{-\frac{N}{2}} e^{-\frac{1}{2}|\boldsymbol{u}|_h^2} |dV_h(\boldsymbol{u})|.$$

- Denote by $\mathcal{N}(\boldsymbol{u})$ the number of critical points of $\boldsymbol{u} \in \boldsymbol{U}$.

The key formula: input

- Fix a smooth, compact, connected manifold M of dimension m.
- Fix a sample space on M, i.e., a pair $(\boldsymbol{U}, h)$, where $\boldsymbol{U}$ is a finite dimensional subspace $\boldsymbol{U} \subset C^\infty(M)$ and h is an Euclidean metric on $\boldsymbol{U}$. Set $N := \dim \boldsymbol{U}$, and denote by $S(\boldsymbol{U})$ the unit sphere in $\boldsymbol{U}$.
- Denote by $d\gamma_h(\boldsymbol{u})$ the associated Gaussian measure on $\boldsymbol{U}$

$$d\gamma_h(\boldsymbol{u}) = (2\pi)^{-\frac{N}{2}} e^{-\frac{1}{2}|\boldsymbol{u}|_h^2} |dV_h(\boldsymbol{u})|.$$

- Denote by $\mathcal{N}(\boldsymbol{u})$ the number of critical points of $\boldsymbol{u} \in \boldsymbol{U}$.
- Set

$$\boldsymbol{N}(\boldsymbol{U}, h) := \int_{\boldsymbol{U}} \mathcal{N}(\boldsymbol{u}) d\gamma_h(\boldsymbol{u}) = \frac{1}{\operatorname{area} S(\boldsymbol{U})} \int_{S(\boldsymbol{U})} \mathcal{N}(\boldsymbol{u}) dA(\boldsymbol{u}).$$

$\boldsymbol{N}(\boldsymbol{U}, h)$ is the expected number of critical points of a random function $\boldsymbol{u} \in \boldsymbol{U}$.

The key formula: goal

The key formula: goal

Find a density (measure) $\rho = \rho_{\boldsymbol{U},h}$ on M such that

$$\boldsymbol{N}(\boldsymbol{U}, h) = \int_M \rho_{\boldsymbol{U},h}.$$

Ampleness

Ampleness

- A sample space $(\boldsymbol{U}, h)$ on M is said to be k-ample if, for any $\boldsymbol{p} \in M$, and any $f \in C^{\infty}(M)$, there exists a function $\boldsymbol{u} \in \boldsymbol{U}$ whose k-jet at $\boldsymbol{x}$ is equal to the k-jet of f at $\boldsymbol{x}$.

Ampleness

- A sample space $(\boldsymbol{U}, h)$ on M is said to be k-ample if, for any $\boldsymbol{p} \in M$, and any $f \in C^\infty(M)$, there exists a function $\boldsymbol{u} \in \boldsymbol{U}$ whose k-jet at $\boldsymbol{x}$ is equal to the k-jet of f at $\boldsymbol{x}$.
- To the sample space $(\boldsymbol{U}, h)$ we associate the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}^\vee \cong_h \boldsymbol{U}$

$$M \ni \boldsymbol{p} \mapsto \mathbf{ev}_{\boldsymbol{p}} \in \operatorname{Hom}(\boldsymbol{U}, \mathbb{R}), \;\; \mathbf{ev}_{\boldsymbol{p}}(\boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{p}).$$

Ampleness

- A sample space $(\boldsymbol{U}, h)$ on M is said to be k-ample if, for any $\boldsymbol{p} \in M$, and any $f \in C^\infty(M)$, there exists a function $\boldsymbol{u} \in \boldsymbol{U}$ whose k-jet at $\boldsymbol{x}$ is equal to the k-jet of f at $\boldsymbol{x}$.
- To the sample space $(\boldsymbol{U}, h)$ we associate the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}^\vee \cong_h \boldsymbol{U}$

$$M \ni \boldsymbol{p} \mapsto \mathbf{ev}_{\boldsymbol{p}} \in \operatorname{Hom}(\boldsymbol{U}, \mathbb{R}), \;\; \mathbf{ev}_{\boldsymbol{p}}(\boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{p}).$$

Then

Ampleness

- A sample space $(\boldsymbol{U}, h)$ on M is said to be k-ample if, for any $\boldsymbol{p} \in M$, and any $f \in C^\infty(M)$, there exists a function $\boldsymbol{u} \in \boldsymbol{U}$ whose k-jet at $\boldsymbol{x}$ is equal to the k-jet of f at $\boldsymbol{x}$.
- To the sample space $(\boldsymbol{U}, h)$ we associate the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}^\vee \cong_h \boldsymbol{U}$

$$M \ni \boldsymbol{p} \mapsto \mathbf{ev}_{\boldsymbol{p}} \in \operatorname{Hom}(\boldsymbol{U}, \mathbb{R}), \;\; \mathbf{ev}_{\boldsymbol{p}}(\boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{p}).$$

Then

1. $\boldsymbol{U}$ is 1-ample $\Longleftrightarrow$ $\mathbf{ev}$ is an immersion.

Ampleness

- A sample space $(\boldsymbol{U}, h)$ on M is said to be k-ample if, for any $\boldsymbol{p} \in M$, and any $f \in C^\infty(M)$, there exists a function $\boldsymbol{u} \in \boldsymbol{U}$ whose k-jet at $\boldsymbol{x}$ is equal to the k-jet of f at $\boldsymbol{x}$.
- To the sample space $(\boldsymbol{U}, h)$ we associate the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}^\vee \cong_h \boldsymbol{U}$

$$M \ni \boldsymbol{p} \mapsto \mathbf{ev}_{\boldsymbol{p}} \in \operatorname{Hom}(\boldsymbol{U}, \mathbb{R}), \;\; \mathbf{ev}_{\boldsymbol{p}}(\boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{p}).$$

Then

1. $\boldsymbol{U}$ is 1-ample $\Longleftrightarrow$ $\mathbf{ev}$ is an immersion.
2. $\boldsymbol{U}$ is 1-ample $\Rightarrow$ the typical function $\boldsymbol{u} \in \boldsymbol{U}$ is Morse.

The ingredients of the key formula

The ingredients of the key formula

- Let $(\boldsymbol{U}, h)$ be a 1-ample sample space on M.

The ingredients of the key formula

- Let $(\boldsymbol{U}, h)$ be a 1-ample sample space on M.
- For $\boldsymbol{p} \in M$ we define the adjunction map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$ to be the differential at p of the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}$.

The ingredients of the key formula

- Let $(\boldsymbol{U}, h)$ be a 1-ample sample space on M.
- For $\boldsymbol{p} \in M$ we define the adjunction map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$ to be the differential at p of the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}$.
- Fix metric g on M (needed only for ease of presentation.)

The ingredients of the key formula

- Let $(\boldsymbol{U}, h)$ be a 1-ample sample space on M.
- For $\boldsymbol{p} \in M$ we define the adjunction map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$ to be the differential at p of the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}$.
- Fix metric g on M (needed only for ease of presentation.)
- $J_g(\boldsymbol{p}) := \sqrt{\det \mathcal{A}_{\boldsymbol{p}}^{\dagger}\mathcal{A}_{\boldsymbol{p}}}$, $\mathcal{A}_{\boldsymbol{p}}^{\dagger}$=transpose of $\mathcal{A}_{\boldsymbol{p}}$.

The ingredients of the key formula

- Let $(\boldsymbol{U}, h)$ be a 1-ample sample space on M.
- For $\boldsymbol{p} \in M$ we define the adjunction map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$ to be the differential at p of the evaluation map $\mathbf{ev} : M \to \boldsymbol{U}$.
- Fix metric g on M (needed only for ease of presentation.)
- $J_g(\boldsymbol{p}) := \sqrt{\det \mathcal{A}_{\boldsymbol{p}}^{\dagger}\mathcal{A}_{\boldsymbol{p}}}$, $\mathcal{A}_{\boldsymbol{p}}^{\dagger}$=transpose of $\mathcal{A}_{\boldsymbol{p}}$.
- For $\boldsymbol{p} \in M$, $\boldsymbol{U}_{\boldsymbol{p}}^{0}$:= the space of functions in $\boldsymbol{U}$ that admit $\boldsymbol{p}$ as critical point.

The ingredients of the key formula

- Let $(\boldsymbol{U}, h)$ be a 1-ample sample space on M.
- For $\boldsymbol{p} \in M$ we define the adjunction map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$ to be the differential at p of the evaluation map **ev** $: M \to \boldsymbol{U}$.
- Fix metric g on M (needed only for ease of presentation.)
- $J_g(\boldsymbol{p}) := \sqrt{\det \mathcal{A}_{\boldsymbol{p}}^{\dagger}\mathcal{A}_{\boldsymbol{p}}}$, $\mathcal{A}_{\boldsymbol{p}}^{\dagger}$=transpose of $\mathcal{A}_{\boldsymbol{p}}$.
- For $\boldsymbol{p} \in M$, $\boldsymbol{U}_{\boldsymbol{p}}^{0}$:= the space of functions in $\boldsymbol{U}$ that admit $\boldsymbol{p}$ as critical point.($\boldsymbol{U}$ is1-ample $\Rightarrow \dim \boldsymbol{U}_{\boldsymbol{p}}^{0} = \dim \boldsymbol{U} - \dim M$, $\forall \boldsymbol{p}$.)

The ingredients of the key formula

- Let $(\boldsymbol{U}, h)$ be a 1-ample sample space on M.
- For $\boldsymbol{p} \in M$ we define the adjunction map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$ to be the differential at p of the evaluation map **ev** $: M \to \boldsymbol{U}$.
- Fix metric g on M (needed only for ease of presentation.)
- $J_g(\boldsymbol{p}) := \sqrt{\det \mathcal{A}_{\boldsymbol{p}}^{\dagger} \mathcal{A}_{\boldsymbol{p}}}$, $\mathcal{A}_{\boldsymbol{p}}^{\dagger}$=transpose of $\mathcal{A}_{\boldsymbol{p}}$.
- For $\boldsymbol{p} \in M$, $\boldsymbol{U}_{\boldsymbol{p}}^0$:= the space of functions in $\boldsymbol{U}$ that admit $\boldsymbol{p}$ as critical point.($\boldsymbol{U}$ is1-ample $\Rightarrow \dim \boldsymbol{U}_{\boldsymbol{p}}^0 = \dim \boldsymbol{U} - \dim M$, $\forall \boldsymbol{p}$.)
- For $\boldsymbol{p} \in M$ and $\boldsymbol{v} \in \boldsymbol{U}_{\boldsymbol{p}}^0$, $\mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{v})$:= the Hessian of $\boldsymbol{v}$ at $\boldsymbol{p}$.

Theorem (The key formula)

Suppose that $(\boldsymbol{U}, h)$ *is a* 1*-ample sample space on* M*. Then,*

Theorem (The key formula)

Suppose that $(\boldsymbol{U}, h)$ *is a* 1*-ample sample space on* M*. Then,*

$$\boldsymbol{N}(\boldsymbol{U}, h) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M \rho_g(\boldsymbol{p}) |dV_g(\boldsymbol{p})|, \qquad \text{(Nic. 2010)}$$

where

Theorem (The key formula)

Suppose that $(\boldsymbol{U}, h)$ *is a* 1*-ample sample space on* M*. Then,*

$$\boldsymbol{N}(\boldsymbol{U}, h) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M \rho_g(\boldsymbol{p}) |dV_g(\boldsymbol{p})|, \qquad \text{(Nic. 2010)}$$

where

- $$\rho_g(\boldsymbol{p}) = \frac{1}{J_g(\boldsymbol{p})} \int_{\boldsymbol{U}^0_{\boldsymbol{p}}} |\det \mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u})| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{N-m}{2}}} |dV_h(\boldsymbol{u})|,$$

Theorem (The key formula)

Suppose that $(\boldsymbol{U}, h)$ *is a* 1*-ample sample space on* M*. Then,*

$$\boldsymbol{N}(\boldsymbol{U}, h) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M \rho_g(\boldsymbol{p}) |dV_g(\boldsymbol{p})|, \qquad \text{(Nic. 2010)}$$

where

- $$\rho_g(\boldsymbol{p}) = \frac{1}{J_g(\boldsymbol{p})} \int_{\boldsymbol{U}_{\boldsymbol{p}}^0} |\det \mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u})| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{N-m}{2}}} |dV_h(\boldsymbol{u})|,$$
- *we recall that* $\dim \boldsymbol{U}_{\boldsymbol{p}}^0 = \dim \boldsymbol{U} - \dim M = N - m$*, and*

Theorem (The key formula)

Suppose that $(\boldsymbol{U}, h)$ *is a* 1*-ample sample space on* M*. Then,*

$$\boldsymbol{N}(\boldsymbol{U}, h) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M \rho_g(\boldsymbol{p}) |dV_g(\boldsymbol{p})|, \qquad \text{(Nic. 2010)}$$

where

- $$\rho_g(\boldsymbol{p}) = \frac{1}{J_g(\boldsymbol{p})} \int_{\boldsymbol{U}^0_{\boldsymbol{p}}} |\det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{N-m}{2}}} |dV_h(\boldsymbol{u})|,$$
- *we recall that* $\dim \boldsymbol{U}^0_{\boldsymbol{p}} = \dim \boldsymbol{U} - \dim M = N - m$*, and*
- *we identify the bilinear form* $\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})$ *with a linear operator* $T_{\boldsymbol{p}}M \to T_{\boldsymbol{p}}M$ *using the metric* g*.*

Remarks about the key formula

Remarks about the key formula

- The proof use the ubiquitous double-fibration trick in integral geometry applied to the double “fibration”

$$M \leftarrow \mathcal{I} \rightarrow S(\boldsymbol{U}),$$

where $\mathcal{I}$ is the incidence variety

$$\mathcal{I} := \big\{ (p, \boldsymbol{u}) \in M \times S(\boldsymbol{U});\ \ d\boldsymbol{u}(p) = 0 \,\big\}.$$

Remarks about the key formula

- The proof use the ubiquitous double-fibration trick in integral geometry applied to the double "fibration"

  $$M \leftarrow \mathcal{I} \rightarrow S(\boldsymbol{U}),$$

  where $\mathcal{I}$ is the incidence variety

  $$\mathcal{I} := \big\{ (p, \boldsymbol{u}) \in M \times S(\boldsymbol{U});\;\; d\boldsymbol{u}(p) = 0 \,\big\}.$$

- The key formula generalizes the Chern-Lashof formula.

Remarks about the key formula

- The proof use the ubiquitous double-fibration trick in integral geometry applied to the double "fibration"

  $$M \leftarrow \mathcal{I} \rightarrow S(\boldsymbol{U}),$$

  where $\mathcal{I}$ is the incidence variety

  $$\mathcal{I} := \big\{ (p, \boldsymbol{u}) \in M \times S(\boldsymbol{U});\ \ d\boldsymbol{u}(p) = 0 \,\big\}.$$

- The key formula generalizes the Chern-Lashof formula.
- The applicability of the formula is constrained by two headaches.

Remarks about the key formula

- The proof use the ubiquitous double-fibration trick in integral geometry applied to the double "fibration"

$$M \leftarrow \mathcal{I} \rightarrow S(\boldsymbol{U}),$$

  where $\mathcal{I}$ is the incidence variety

$$\mathcal{I} := \big\{ (p, \boldsymbol{u}) \in M \times S(\boldsymbol{U});\ \ d\boldsymbol{u}(p) = 0 \,\big\}.$$

- The key formula generalizes the Chern-Lashof formula.
- The applicability of the formula is constrained by two headaches.
  - The Jacobian $J_g(\boldsymbol{p})$ is difficult to compute.

Remarks about the key formula

- The proof use the ubiquitous double-fibration trick in integral geometry applied to the double "fibration"

$$M \leftarrow \mathcal{I} \rightarrow S(\boldsymbol{U}),$$

where $\mathcal{I}$ is the incidence variety

$$\mathcal{I} := \big\{ (p, \boldsymbol{u}) \in M \times S(\boldsymbol{U});\ \ d\boldsymbol{u}(p) = 0 \,\big\}.$$

- The key formula generalizes the Chern-Lashof formula.
- The applicability of the formula is constrained by two headaches.
  - The Jacobian $J_g(\boldsymbol{p})$ is difficult to compute.
  - The subspace $\boldsymbol{U}^0_{\boldsymbol{p}} \subset \boldsymbol{U}$ may be difficult to locate.

Remarks about the key formula

- The proof use the ubiquitous double-fibration trick in integral geometry applied to the double "fibration"

$$M \leftarrow \mathcal{I} \rightarrow S(\boldsymbol{U}),$$

  where $\mathcal{I}$ is the incidence variety

$$\mathcal{I} := \big\{ (p, \boldsymbol{u}) \in M \times S(\boldsymbol{U});\;\; d\boldsymbol{u}(p) = 0 \,\big\}.$$

- The key formula generalizes the Chern-Lashof formula.
- The applicability of the formula is constrained by two headaches.
  - The Jacobian $J_g(\boldsymbol{p})$ is difficult to compute.
  - The subspace $\boldsymbol{U}_{\boldsymbol{p}}^0 \subset \boldsymbol{U}$ may be difficult to locate.

We will aleviate these headaches using probabilistic methods.

Gaussian measures

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form

$$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form

$$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$

  When $\sigma = 0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m \in \mathbb{R}$.

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form

$$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$

  When $\sigma = 0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m \in \mathbb{R}$.

- A measure $d\gamma$ on a finite dimensional vector space $\boldsymbol{V}$ is called Gaussian if for any $\xi \in \boldsymbol{V}^{\vee} := \mathrm{Hom}(\boldsymbol{V}, \mathbb{R})$ the pushforward $\xi_* d\gamma$ is a Gaussian measure $d\gamma_{m(\xi),\sigma(\xi)}$ on $\mathbb{R}$.

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form
$$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$
When $\sigma = 0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m \in \mathbb{R}$.
- A measure $d\gamma$ on a finite dimensional vector space $\boldsymbol{V}$ is called Gaussian if for any $\xi \in \boldsymbol{V}^{\vee} := \operatorname{Hom}(\boldsymbol{V}, \mathbb{R})$ the pushforward $\xi_* d\gamma$ is a Gaussian measure $d\gamma_{m(\xi),\sigma(\xi)}$ on $\mathbb{R}$.
- The correspondence $\xi \mapsto m(\xi)$ is a linear map $\boldsymbol{V}^{\vee} \to \mathbb{R}$.

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form
$$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$
When $\sigma = 0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m \in \mathbb{R}$.
- A measure $d\gamma$ on a finite dimensional vector space $\boldsymbol{V}$ is called Gaussian if for any $\xi \in \boldsymbol{V}^\vee := \operatorname{Hom}(\boldsymbol{V}, \mathbb{R})$ the pushforward $\xi_* d\gamma$ is a Gaussian measure $d\gamma_{m(\xi),\sigma(\xi)}$ on $\mathbb{R}$.
- The correspondence $\xi \mapsto m(\xi)$ is a linear map $\boldsymbol{V}^\vee \to \mathbb{R}$. It can be identified with an element $\boldsymbol{m}$ of $\boldsymbol{V}$ called the barycenter or expectation of the Gaussian measure.

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form
$$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$
  When $\sigma = 0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m \in \mathbb{R}$.
- A measure $d\gamma$ on a finite dimensional vector space $\boldsymbol{V}$ is called Gaussian if for any $\xi \in \boldsymbol{V}^\vee := \mathrm{Hom}(\boldsymbol{V}, \mathbb{R})$ the pushforward $\xi_* d\gamma$ is a Gaussian measure $d\gamma_{m(\xi),\sigma(\xi)}$ on $\mathbb{R}$.
- The correspondence $\xi \mapsto m(\xi)$ is a linear map $\boldsymbol{V}^\vee \to \mathbb{R}$. It can be identified with an element $\boldsymbol{m}$ of $\boldsymbol{V}$ called the barycenter or expectation of the Gaussian measure.The measure is called centered if $\boldsymbol{m} = 0$.

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form

$$d\gamma_{m,\sigma}(x)=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{1}{2\sigma}(x-m)^2}dx,\quad m\in\mathbb{R},\quad \sigma\geq 0.$$

  When $\sigma=0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m\in\mathbb{R}$.
- A measure $d\gamma$ on a finite dimensional vector space $\boldsymbol{V}$ is called Gaussian if for any $\xi\in\boldsymbol{V}^\vee:=\operatorname{Hom}(\boldsymbol{V},\mathbb{R})$ the pushforward $\xi_*d\gamma$ is a Gaussian measure $d\gamma_{m(\xi),\sigma(\xi)}$ on $\mathbb{R}$.
- The correspondence $\xi\mapsto m(\xi)$ is a linear map $\boldsymbol{V}^\vee\to\mathbb{R}$. It can be identified with an element $\boldsymbol{m}$ of $\boldsymbol{V}$ called the barycenter or expectation of the Gaussian measure.The measure is called centered if $\boldsymbol{m}=0$.
- The correspondence $\xi\mapsto\sigma(\xi)$ is a nonnegative quadratic form on $\boldsymbol{V}^\vee$.

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form

$$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$

  When $\sigma = 0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m \in \mathbb{R}$.
- A measure $d\gamma$ on a finite dimensional vector space $\boldsymbol{V}$ is called Gaussian if for any $\xi \in \boldsymbol{V}^\vee := \mathrm{Hom}(\boldsymbol{V}, \mathbb{R})$ the pushforward $\xi_* d\gamma$ is a Gaussian measure $d\gamma_{m(\xi),\sigma(\xi)}$ on $\mathbb{R}$.
- The correspondence $\xi \mapsto m(\xi)$ is a linear map $\boldsymbol{V}^\vee \to \mathbb{R}$. It can be identified with an element $\boldsymbol{m}$ of $\boldsymbol{V}$ called the barycenter or expectation of the Gaussian measure.The measure is called centered if $\boldsymbol{m} = 0$.
- The correspondence $\xi \mapsto \sigma(\xi)$ is a nonnegative quadratic form on $\boldsymbol{V}^\vee$. The associated bilinear form on $\boldsymbol{V}^\vee$ is called the covariance form of the Gaussian measure. We will denote it by $\boldsymbol{\Sigma}$.

Gaussian measures

- A Gaussian measure on $\mathbb{R}$ is a measure of the form

  $$d\gamma_{m,\sigma}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma}(x-m)^2} dx, \quad m \in \mathbb{R}, \quad \sigma \geq 0.$$

  When $\sigma = 0$, $d\gamma_{m,\sigma}$ is the Dirac delta concentrated at $m \in \mathbb{R}$.
- A measure $d\gamma$ on a finite dimensional vector space $\boldsymbol{V}$ is called Gaussian if for any $\xi \in \boldsymbol{V}^\vee := \operatorname{Hom}(\boldsymbol{V}, \mathbb{R})$ the pushforward $\xi_* d\gamma$ is a Gaussian measure $d\gamma_{m(\xi),\sigma(\xi)}$ on $\mathbb{R}$.
- The correspondence $\xi \mapsto m(\xi)$ is a linear map $\boldsymbol{V}^\vee \to \mathbb{R}$. It can be identified with an element $\boldsymbol{m}$ of $\boldsymbol{V}$ called the barycenter or expectation of the Gaussian measure.The measure is called centered if $\boldsymbol{m} = 0$.
- The correspondence $\xi \mapsto \sigma(\xi)$ is a nonnegative quadratic form on $\boldsymbol{V}^\vee$. The associated bilinear form on $\boldsymbol{V}^\vee$ is called the covariance form of the Gaussian measure. We will denote it by $\boldsymbol{\Sigma}$. The measure $d\gamma$ is completely determined by $\boldsymbol{m}$ and $\boldsymbol{\Sigma}$.

Examples of Gaussian measures

Examples of Gaussian measures

- Suppose that S is a symmetric, positive definite $n \times n$ matrix.

Examples of Gaussian measures

- Suppose that S is a symmetric, positive definite $n \times n$ matrix. Then the measure $d\gamma_S(\boldsymbol{x})$ on $\mathbb{R}^n$ given by

  $$d\gamma_S(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det S}} e^{-\frac{1}{2}\left(S^{-1}\boldsymbol{x}, \boldsymbol{x} \right)} |d\boldsymbol{x}|,$$

  is a centered Gaussian measure on $\mathbb{R}^n$ with covariance form described by the matrix S.

Examples of Gaussian measures

- Suppose that S is a symmetric, positive definite $n \times n$ matrix. Then the measure $d\gamma_S(\boldsymbol{x})$ on $\mathbb{R}^n$ given by

$$d\gamma_S(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det S}} e^{-\frac{1}{2}\left(S^{-1}\boldsymbol{x},\boldsymbol{x} \right)} |d\boldsymbol{x}|,$$

  is a centered Gaussian measure on $\mathbb{R}^n$ with covariance form described by the matrix S.

- Suppose that $d\gamma$ is centered Gaussian measure on $\boldsymbol{V}$ and $T : \boldsymbol{V} \to \boldsymbol{W}$ is a linear map.

Examples of Gaussian measures

- Suppose that S is a symmetric, positive definite $n \times n$ matrix. Then the measure $d\gamma_S(\boldsymbol{x})$ on $\mathbb{R}^n$ given by

$$d\gamma_S(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det S}} e^{-\frac{1}{2}\left(S^{-1}\boldsymbol{x},\boldsymbol{x}\right)} |d\boldsymbol{x}|,$$

  is a centered Gaussian measure on $\mathbb{R}^n$ with covariance form described by the matrix S.
- Suppose that $d\gamma$ is centered Gaussian measure on $\boldsymbol{V}$ and $T : \boldsymbol{V} \to \boldsymbol{W}$ is a linear map. Then the pushforward $T_* d\gamma$ is a centered Gaussian measure on $\boldsymbol{U}$.

Gaussian vectors

Gaussian vectors

- Suppose that $(\mathcal{S}, d\mu)$ is a probability space and $\boldsymbol{V}$ is a finite dimensional real vector space.

Gaussian vectors

- Suppose that $(\mathcal{S}, d\mu)$ is a probability space and $\boldsymbol{V}$ is a finite dimensional real vector space.
- A Gaussian random vector is a measurable map $X : \mathcal{S} \to \boldsymbol{V}$ such that the pushforward $X_* d\mu$ is a Gaussian measure on $\boldsymbol{V}$.

Gaussian vectors

- Suppose that $(\mathcal{S}, d\mu)$ is a probability space and $\boldsymbol{V}$ is a finite dimensional real vector space.
- A Gaussian random vector is a measurable map $X : \mathcal{S} \to \boldsymbol{V}$ such that the pushforward $X_* d\mu$ is a Gaussian measure on $\boldsymbol{V}$.
- The random vector X is called centered if $X_* d\mu$ is centered.

Gaussian vectors

- Suppose that $(\mathcal{S}, d\mu)$ is a probability space and $\boldsymbol{V}$ is a finite dimensional real vector space.
- A Gaussian random vector is a measurable map $X : \mathcal{S} \to \boldsymbol{V}$ such that the pushforward $X_* d\mu$ is a Gaussian measure on $\boldsymbol{V}$.
- The random vector X is called centered if $X_* d\mu$ is centered.
- The covariance form of X is the covariance form of $X_* d\mu$. We denote it by $\boldsymbol{\Sigma}_X$.

Regression formula

Regression formula

- Suppose that $X_i : (\mathcal{S}, \mu) \to \boldsymbol{V}_i$, $i = 1, 2$, are two centered Gaussian random vectors such that the random vector $X_1 \oplus X_2$ is also Gaussian.

Regression formula

- Suppose that $X_i : (\mathcal{S}, \mu) \to \boldsymbol{V}_i$, $i = 1, 2$, are two centered Gaussian random vectors such that the random vector $X_1 \oplus X_2$ is also Gaussian.
- Assume that $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are equipped with inner products. The covariance of X_i is then a linear symmetric operator $\boldsymbol{\Sigma}_{X_i} : \boldsymbol{V}_i \to \boldsymbol{V}_i$.

Regression formula

- Suppose that $X_i : (\mathcal{S}, \mu) \to \boldsymbol{V}_i$, $i = 1, 2$, are two centered Gaussian random vectors such that the random vector $X_1 \oplus X_2$ is also Gaussian.
- Assume that $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are equipped with inner products. The covariance of X_i is then a linear symmetric operator $\boldsymbol{\Sigma}_{X_i} : \boldsymbol{V}_i \to \boldsymbol{V}_i$.
- Assume that X_2 is nondegenerate, i.e., $\boldsymbol{\Sigma}_2$ is invertible.

Regression formula

- Suppose that $X_i : (\mathcal{S}, \mu) \to \boldsymbol{V}_i$, $i = 1, 2$, are two centered Gaussian random vectors such that the random vector $X_1 \oplus X_2$ is also Gaussian.
- Assume that $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are equipped with inner products. The covariance of X_i is then a linear symmetric operator $\boldsymbol{\Sigma}_{X_i} : \boldsymbol{V}_i \to \boldsymbol{V}_i$.
- Assume that X_2 is nondegenerate, i.e., $\boldsymbol{\Sigma}_2$ is invertible.
- The covariance form of (X_1, X_2) is the bilinear map $\boldsymbol{\Sigma}_{X_1,X_2} : \boldsymbol{V}_1^{\vee} \times \boldsymbol{V}_2{}^{\vee} \to \mathbb{R}$ defined by

Regression formula

- Suppose that $X_i : (\mathcal{S}, \mu) \to \boldsymbol{V}_i$, $i = 1, 2$, are two centered Gaussian random vectors such that the random vector $X_1 \oplus X_2$ is also Gaussian.
- Assume that $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are equipped with inner products. The covariance of X_i is then a linear symmetric operator $\boldsymbol{\Sigma}_{X_i} : \boldsymbol{V}_i \to \boldsymbol{V}_i$.
- Assume that X_2 is nondegenerate, i.e., $\boldsymbol{\Sigma}_2$ is invertible.
- The covariance form of (X_1, X_2) is the bilinear map $\boldsymbol{\Sigma}_{X_1,X_2} : \boldsymbol{V}_1^\vee \times \boldsymbol{V}_2{}^\vee \to \mathbb{R}$ defined by

$$\boldsymbol{\Sigma}_{X_1,X_2}(\xi_1, \xi_2) = \boldsymbol{E}\big(\, \xi_1(X_1) \cdot \xi_2(X_2) \,\big), \;\; \forall \xi_i \in \boldsymbol{V}_i{}^\vee,$$

where $\boldsymbol{E}(\eta) =$ expectation of the random variable η.

Regression formula

- Suppose that $X_i : (\mathcal{S}, \mu) \to \boldsymbol{V}_i$, $i = 1, 2$, are two centered Gaussian random vectors such that the random vector $X_1 \oplus X_2$ is also Gaussian.
- Assume that $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are equipped with inner products. The covariance of X_i is then a linear symmetric operator $\boldsymbol{\Sigma}_{X_i} : \boldsymbol{V}_i \to \boldsymbol{V}_i$.
- Assume that X_2 is nondegenerate, i.e., $\boldsymbol{\Sigma}_2$ is invertible.
- The covariance form of (X_1, X_2) is the bilinear map $\boldsymbol{\Sigma}_{X_1,X_2} : \boldsymbol{V}_1^{\vee} \times \boldsymbol{V}_2{}^{\vee} \to \mathbb{R}$ defined by

$$\boldsymbol{\Sigma}_{X_1,X_2}(\xi_1, \xi_2) = \boldsymbol{E}\big(\, \xi_1(X_1) \cdot \xi_2(X_2)\,\big), \;\; \forall \xi_i \in \boldsymbol{V}_i{}^{\vee},$$

  where $\boldsymbol{E}(\eta) =$ expectation of the random variable η. Identify $\boldsymbol{\Sigma}_{X_1,X_2}$ with a linear map $\boldsymbol{V}_2 \to \boldsymbol{V}_1$. ($\Sigma_{X_1,X_2} = 0$ iff X_1 and X_2 are independent.)

The regression formula

The regression formula

Theorem (Regression formula)

Suppose that $f:\boldsymbol{V}_1\to\mathbb{R}$ *is is a measurable function. Then the conditional expectation*

$$\boldsymbol{E}\big(\,f(X_1)\,|\,X_2=0\,\big)$$

is equal to the unconditional expectation of a new random variable $f(Y)$*, where* $Y:\mathcal{S}\to\boldsymbol{V}_1$ *is a centered Gaussian random vector with covariance*

The regression formula

Theorem (Regression formula)

Suppose that $f:\boldsymbol{V}_1\to\mathbb{R}$ is is a measurable function. Then the conditional expectation

$$\boldsymbol{E}\big(\, f(X_1)\big|\, X_2=0\,\big)$$

is equal to the unconditional expectation of a new random variable $f(Y)$, where $Y:\mathcal{S}\to\boldsymbol{V}_1$ is a centered Gaussian random vector with covariance

$$\boldsymbol{\Sigma}_Y=\boldsymbol{\Sigma}_{X_1}-\boldsymbol{\Sigma}_{X_1,X_2}\boldsymbol{\Sigma}_{X_2}^{-1}\boldsymbol{\Sigma}_{X_1,X_2}^{\dagger}.$$

Above, $A^\dagger$ denotes the adjoint (transpose) of a linear operator A between Euclidean spaces.

Gaussian fields

Gaussian fields

- Suppose that T is a set.

Gaussian fields

- Suppose that T is a set. A random field on T is a family of random variables parametrized by T.

Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set. A random field on $\boldsymbol{T}$ is a family of random variables parametrized by $\boldsymbol{T}$.
- More precisely, a random field on $\boldsymbol{T}$ is a map

$$X : (\mathcal{S}, d\mu) \times \boldsymbol{T} \to \mathbb{R}, \quad (s, t) \mapsto F_t(s),$$

  where
  - $(\mathcal{S}, d\mu)$ is a probability space and,

Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set. A random field on $\boldsymbol{T}$ is a family of random variables parametrized by $\boldsymbol{T}$.
- More precisely, a random field on $\boldsymbol{T}$ is a map

$$X : (\mathcal{S}, d\mu) \times \boldsymbol{T} \to \mathbb{R}, \;\; (s, t) \mapsto F_t(s),$$

  where

  - $(\mathcal{S}, d\mu)$ is a probability space and,
  - for any $t \in \boldsymbol{T}$ the map $\mathcal{S} \ni s \mapsto F_t(s) \in \mathbb{R}$ is measurable.

Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set. A random field on $\boldsymbol{T}$ is a family of random variables parametrized by $\boldsymbol{T}$.
- More precisely, a random field on $\boldsymbol{T}$ is a map

$$X : (\mathcal{S}, d\mu) \times \boldsymbol{T} \to \mathbb{R}, \;\; (s, t) \mapsto F_t(s),$$

  where

  - $(\mathcal{S}, d\mu)$ is a probability space and,
  - for any $t \in \boldsymbol{T}$ the map $\mathcal{S} \ni s \mapsto F_t(s) \in \mathbb{R}$ is measurable.

- A random field $F : \mathcal{S} \times \boldsymbol{T} \to \mathbb{R}$ is called Gaussian if for any finite subset $A = \{a_1, \dots, a_n\} \subset \boldsymbol{T}$ the random vector

$$\mathcal{S} \ni s \mapsto F_A(s) := \bigl(F_{a_1}(s), \dots, F_{a_n}(s) \bigr) \in \mathbb{R}^n$$

  is a Gaussian random vector.

Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set. A random field on $\boldsymbol{T}$ is a family of random variables parametrized by $\boldsymbol{T}$.
- More precisely, a random field on $\boldsymbol{T}$ is a map

  $$X : (\mathcal{S}, d\mu) \times \boldsymbol{T} \to \mathbb{R}, \;\; (s, t) \mapsto F_t(s),$$

  where

  - $(\mathcal{S}, d\mu)$ is a probability space and,
  - for any $t \in \boldsymbol{T}$ the map $\mathcal{S} \ni s \mapsto F_t(s) \in \mathbb{R}$ is measurable.
- A random field $F : \mathcal{S} \times \boldsymbol{T} \to \mathbb{R}$ is called Gaussian if for any finite subset $A = \{a_1, \dots, a_n\} \subset \boldsymbol{T}$ the random vector

  $$\mathcal{S} \ni s \mapsto F_A(s) := \big(\, F_{a_1}(s), \dots, F_{a_n}(s) \,\big) \in \mathbb{R}^n$$

  is a Gaussian random vector. F is called centered if all the above Gaussian vectors F_A are centered.

Gaussian fields

Gaussian fields

- If $F : \mathcal{S} \times \boldsymbol{T} \to \mathbb{R}$ is a centered, Gaussian field on $\boldsymbol{T}$, then the covariance kernel of F is the function $\mathcal{E}_F : \boldsymbol{T} \times \boldsymbol{T} \to \mathbb{R}$ defined by

Gaussian fields

- If $F : \mathcal{S} \times \boldsymbol{T} \to \mathbb{R}$ is a centered, Gaussian field on $\boldsymbol{T}$, then the covariance kernel of F is the function $\mathcal{E}_F : \boldsymbol{T} \times \boldsymbol{T} \to \mathbb{R}$ defined by

$$\mathcal{E}_F(t_1, t_2) = \boldsymbol{E}\big(F_{t_1} \cdot F_{t_2} \big), \;\; \forall t_1, t_2 \in \boldsymbol{T}.$$

Gaussian fields

- If $F : \mathcal{S} \times \boldsymbol{T} \to \mathbb{R}$ is a centered, Gaussian field on $\boldsymbol{T}$, then the covariance kernel of F is the function $\mathcal{E}_F : \boldsymbol{T} \times \boldsymbol{T} \to \mathbb{R}$ defined by

$$\mathcal{E}_F(t_1, t_2) = \boldsymbol{E}\big(\, F_{t_1} \cdot F_{t_2} \,\big), \;\; \forall t_1, t_2 \in \boldsymbol{T}.$$

- The covariance kernel contains all the important information about the gaussian random field.

A fundamental example of Gaussian fields

A fundamental example of Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set and $\boldsymbol{U}$ is a finite vector space of functions $\boldsymbol{u} : \boldsymbol{T} \rightarrow \mathbb{R}$. Set $N := \dim \boldsymbol{U}$.

A fundamental example of Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set and $\boldsymbol{U}$ is a finite vector space of functions $\boldsymbol{u} : \boldsymbol{T} \to \mathbb{R}$. Set $N := \dim \boldsymbol{U}$.
- Fix an inner product on $\boldsymbol{U}$ and define the Gaussian measure

$$d\gamma(\boldsymbol{u}) = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{1}{2}|\boldsymbol{u}|^2} \, |d\boldsymbol{u}|.$$

A fundamental example of Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set and $\boldsymbol{U}$ is a finite vector space of functions $\boldsymbol{u} : \boldsymbol{T} \to \mathbb{R}$. Set $N := \dim \boldsymbol{U}$.
- Fix an inner product on $\boldsymbol{U}$ and define the Gaussian measure

$$d\gamma(\boldsymbol{u}) = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{1}{2}|\boldsymbol{u}|^2} \, |d\boldsymbol{u}|.$$

- We can view $(\boldsymbol{U}, d\gamma)$ as a probability space and we obtain a random field

$$F : \boldsymbol{U} \times \boldsymbol{T} \to \mathbb{R}, \;\; (\boldsymbol{u}, t) \mapsto F_t(\boldsymbol{u}) = \boldsymbol{u}(t).$$

A fundamental example of Gaussian fields

- Suppose that $\boldsymbol{T}$ is a set and $\boldsymbol{U}$ is a finite vector space of functions $\boldsymbol{u}: \boldsymbol{T} \to \mathbb{R}$. Set $N := \dim \boldsymbol{U}$.
- Fix an inner product on $\boldsymbol{U}$ and define the Gaussian measure

$$d\gamma(\boldsymbol{u}) = \frac{1}{(2\pi)^{\frac{N}{2}}} e^{-\frac{1}{2}|\boldsymbol{u}|^2} \, |d\boldsymbol{u}|.$$

- We can view $(\boldsymbol{U}, d\gamma)$ as a probability space and we obtain a random field

$$F : \boldsymbol{U} \times \boldsymbol{T} \to \mathbb{R}, \;\; (\boldsymbol{u}, t) \mapsto F_t(\boldsymbol{u}) = \boldsymbol{u}(t).$$

- This is a centered Gaussian field, with covariance kernel

$$\mathcal{E}(t_1, t_2) = \sum_{k=1}^{N} \Psi_k(t_1)\Psi_k(t_2),$$

where $(\Psi_k)_{1\leq k\leq N}$ is an orthonormal basis of $\boldsymbol{U}$.

The key formula through a probabilistic lens

The key formula through a probabilistic lens

- We return to our original geometric setup, (M, g) compact connected Riemann manifold of dimension $m > 1$, and

$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$

The key formula through a probabilistic lens

- We return to our original geometric setup, (M, g) compact connected Riemann manifold of dimension $m > 1$, and

$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$

- Assume L is large enough so that $\boldsymbol{U}_L$ is 2-ample.

The key formula through a probabilistic lens

- We return to our original geometric setup, (M, g) compact connected Riemann manifold of dimension $m > 1$, and

$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$

- Assume L is large enough so that $\boldsymbol{U}_L$ is 2-ample.
- Fix an orthonormal basis $(\Psi_n)_{n \geq 0}$ of $L^2(M, g)$ consisting of eigenfunctions

$$\Delta_g \Psi_n = \lambda_n \Psi_n, \;\; \lambda_0 \leq \lambda_2 \leq \cdots .$$

The key formula through a probabilistic lens

- We return to our original geometric setup, (M, g) compact connected Riemann manifold of dimension $m > 1$, and
$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$
- Assume L is large enough so that $\boldsymbol{U}_L$ is 2-ample.
- Fix an orthonormal basis $(\Psi_n)_{n \geq 0}$ of $L^2(M, g)$ consisting of eigenfunctions
$$\Delta_g \Psi_n = \lambda_n \Psi_n, \;\; \lambda_0 \leq \lambda_2 \leq \cdots .$$
- As explained above, $\boldsymbol{U}_L$ defines a random field on M with covariance kernel

The key formula through a probabilistic lens

- We return to our original geometric setup, (M, g) compact connected Riemann manifold of dimension $m > 1$, and

$$\boldsymbol{U}_L := \bigoplus_{\lambda \leq L} \ker(\lambda - \Delta_g).$$

- Assume L is large enough so that $\boldsymbol{U}_L$ is 2-ample.
- Fix an orthonormal basis $(\Psi_n)_{n \geq 0}$ of $L^2(M, g)$ consisting of eigenfunctions

$$\Delta_g \Psi_n = \lambda_n \Psi_n, \;\; \lambda_0 \leq \lambda_2 \leq \cdots .$$

- As explained above, $\boldsymbol{U}_L$ defines a random field on M with covariance kernel

$$\mathcal{E}_L(\boldsymbol{p}, \boldsymbol{q}) = \sum_{\lambda_n \leq L} \Psi_n(\boldsymbol{p}) \Psi_n(\boldsymbol{q}).$$

  This function is also known as the spectral function of the Laplacian $\Delta_g(M)$.

The key formula through a probabilistic lens

The key formula through a probabilistic lens

Fix $\boldsymbol{p} \in M$. We want to give probabilistic interpretations to the quantities

$$J_g(\boldsymbol{p}) \;\text{ and }\; h_L(\boldsymbol{p}) := \int_{\boldsymbol{U}^0_{\boldsymbol{p}}} |\det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{\dim \boldsymbol{U}^0_{\boldsymbol{p}}}{2}}} |d\boldsymbol{u}|.$$

The key formula through a probabilistic lens

Fix $\boldsymbol{p} \in M$. We want to give probabilistic interpretations to the quantities

$$J_g(\boldsymbol{p}) \;\text{ and }\; h_L(\boldsymbol{p}) := \int_{\boldsymbol{U}_{\boldsymbol{p}}^0} \left|\det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})\right| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{\dim \boldsymbol{U}_{\boldsymbol{p}}^0}{2}}} |d\boldsymbol{u}|.$$

- For any function $f \in C^\infty(M)$ we denote by $H_{\boldsymbol{p}}(f)$ the Hessian of f at $\boldsymbol{u}$ defined using the Levi-Civita connection on M.

The key formula through a probabilistic lens

Fix $\boldsymbol{p} \in M$. We want to give probabilistic interpretations to the quantities

$$J_g(\boldsymbol{p}) \text{ and } h_L(\boldsymbol{p}) := \int_{\boldsymbol{U}^0_{\boldsymbol{p}}} |\det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{\dim \boldsymbol{U}^0_{\boldsymbol{p}}}{2}}} |d\boldsymbol{u}|.$$

- For any function $f \in C^\infty(M)$ we denote by $H_{\boldsymbol{p}}(f)$ the Hessian of f at $\boldsymbol{u}$ defined using the Levi-Civita connection on M. We regard it as an element of $\mathcal{S}(T_{\boldsymbol{p}}M)$, the space of symmetric operators $T_{\boldsymbol{p}}M \to T_{\boldsymbol{p}}M$.

The key formula through a probabilistic lens

Fix $\boldsymbol{p} \in M$. We want to give probabilistic interpretations to the quantities

$$J_g(\boldsymbol{p}) \text{ and } h_L(\boldsymbol{p}) := \int_{\boldsymbol{U}_{\boldsymbol{p}}^0} |\det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{\dim \boldsymbol{U}_{\boldsymbol{p}}^0}{2}}} |d\boldsymbol{u}|.$$

- For any function $f \in C^\infty(M)$ we denote by $H_{\boldsymbol{p}}(f)$ the Hessian of f at $\boldsymbol{u}$ defined using the Levi-Civita connection on M. We regard it as an element of $\mathcal{S}(T_{\boldsymbol{p}}M)$, the space of symmetric operators $T_{\boldsymbol{p}}M \to T_{\boldsymbol{p}}M$.
- The pushforward of the Gaussian measure on $\boldsymbol{U}_L$ via the Hessian map $f \mapsto H_{\boldsymbol{p}}(f)$ is a centered Gaussian measure $d\boldsymbol{\Gamma}_{L,\boldsymbol{p}}$ on $\mathcal{S}(T_{\boldsymbol{p}}M)$.

The key formula through a probabilistic lens

Fix $\boldsymbol{p} \in M$. We want to give probabilistic interpretations to the quantities

$$J_g(\boldsymbol{p}) \;\text{ and }\; h_L(\boldsymbol{p}) := \int_{\boldsymbol{U}^0_{\boldsymbol{p}}} |\det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u})| \frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{\dim \boldsymbol{U}^0_{\boldsymbol{p}}}{2}}} |d\boldsymbol{u}|.$$

- For any function $f \in C^\infty(M)$ we denote by $H_{\boldsymbol{p}}(f)$ the Hessian of f at $\boldsymbol{u}$ defined using the Levi-Civita connection on M. We regard it as an element of $\mathcal{S}(T_{\boldsymbol{p}}M)$, the space of symmetric operators $T_{\boldsymbol{p}}M \to T_{\boldsymbol{p}}M$.
- The pushforward of the Gaussian measure on $\boldsymbol{U}_L$ via the Hessian map $f \mapsto H_{\boldsymbol{p}}(f)$ is a centered Gaussian measure $d\boldsymbol{\Gamma}_{L,\boldsymbol{p}}$ on $\mathcal{S}(T_{\boldsymbol{p}}M)$. We denote by $\boldsymbol{\Sigma}_{L,\boldsymbol{p}}$ its covariance form.

The key formula through a probabilistic lens

The key formula through a probabilistic lens

- The pushforward of the gaussian measure on $\boldsymbol{U}_L$ via the map $\boldsymbol{U}_L \ni \boldsymbol{u} \mapsto d\boldsymbol{u}(p) \in T^*_{\boldsymbol{p}}M$ is a centered Gaussian measure $d\gamma_{L,p}$ with covariance form $\boldsymbol{\sigma}_{L,\boldsymbol{p}}$. One can prove that

$$J_g(\boldsymbol{p}) = \sqrt{\det \boldsymbol{\sigma}_{L,p}}.$$

The key formula through a probabilistic lens

- The pushforward of the gaussian measure on $\boldsymbol{U}_L$ via the map $\boldsymbol{U}_L \ni \boldsymbol{u} \mapsto d\boldsymbol{u}(p) \in T^*_{\boldsymbol{p}}M$ is a centered Gaussian measure $d\gamma_{L,p}$ with covariance form $\boldsymbol{\sigma}_{L,\boldsymbol{p}}$. One can prove that

$$J_g(\boldsymbol{p}) = \sqrt{\det \boldsymbol{\sigma}_{L,p}}.$$

- The direct sum of the Gaussian random vectors $\boldsymbol{u} \mapsto H_{\boldsymbol{p}}(\boldsymbol{u})$ and $\boldsymbol{u} \mapsto d\boldsymbol{u}(\boldsymbol{p})$ is a Gaussian random vector. The co-area formula implies that the integral $h_L(\boldsymbol{p})$ is a conditional expectation

$$h_L(\boldsymbol{p}) = \boldsymbol{E}\big(\,|\det H_{\boldsymbol{p}}|\;\big|d\boldsymbol{u}(\boldsymbol{p}) = 0\big).$$

The key formula through a probabilistic lens

The key formula through a probabilistic lens

We now invoke the regression formula. We denote by $\boldsymbol{C}_{L,\boldsymbol{p}}$ the covariance form of the random vectors $H_{\boldsymbol{p}}$ and $\boldsymbol{u} \mapsto \boldsymbol{u}(\boldsymbol{p})$. We deduce

The key formula through a probabilistic lens

We now invoke the regression formula. We denote by $\boldsymbol{C}_{L,\boldsymbol{p}}$ the covariance form of the random vectors $H_{\boldsymbol{p}}$ and $\boldsymbol{u} \mapsto \boldsymbol{u}(\boldsymbol{p})$. We deduce

$$h_L(\boldsymbol{p}) = \boldsymbol{E}(|\det(Y)|),$$

The key formula through a probabilistic lens

We now invoke the regression formula. We denote by $\boldsymbol{C}_{L,\boldsymbol{p}}$ the covariance form of the random vectors $H_{\boldsymbol{p}}$ and $\boldsymbol{u} \mapsto \boldsymbol{u}(\boldsymbol{p})$. We deduce

$$h_L(\boldsymbol{p}) = \boldsymbol{E}(|\det(Y)|),$$

where $Y : \boldsymbol{U} \to \mathcal{S}(T_{\boldsymbol{p}}M)$ is a centered Gaussian random symmetric matrix with covariance

$$\boldsymbol{S}_{L,\boldsymbol{p}} = \boldsymbol{\Sigma}_{L,\boldsymbol{p}} - \boldsymbol{C}_{L,\boldsymbol{p}}\boldsymbol{\sigma}_{L,\boldsymbol{p}}^{-1}\boldsymbol{C}_{L,\boldsymbol{p}}^{\dagger}.$$

The key formula through a probabilistic lens

We now invoke the regression formula. We denote by $\boldsymbol{C}_{L,\boldsymbol{p}}$ the covariance form of the random vectors $H_{\boldsymbol{p}}$ and $\boldsymbol{u} \mapsto \boldsymbol{u}(\boldsymbol{p})$. We deduce

$$h_L(\boldsymbol{p}) = \boldsymbol{E}(|\det(Y)|),$$

where $Y : \boldsymbol{U} \to \mathcal{S}(T_{\boldsymbol{p}}M)$ is a centered Gaussian random symmetric matrix with covariance

$$\boldsymbol{S}_{L,\boldsymbol{p}} = \boldsymbol{\Sigma}_{L,\boldsymbol{p}} - \boldsymbol{C}_{L,\boldsymbol{p}}\boldsymbol{\sigma}_{L,\boldsymbol{p}}^{-1}\boldsymbol{C}_{L,\boldsymbol{p}}^{\dagger}.$$

The main point of the above equality is that the quantities $\boldsymbol{\sigma}_{L,\boldsymbol{p}}$, $\boldsymbol{\Sigma}_{L,\boldsymbol{p}}$ and $\boldsymbol{C}_{L,\boldsymbol{p}}$ can be explicitly described in terms of derivatives the spectral function along directions normal to the diagonal in $M \times M$.

Estimating $\boldsymbol{N}_L(M,g)$

Estimating $\boldsymbol{N}_L(M, g)$

The behavior of the spectral function along the diagonal is rather well understood.

Estimating $\boldsymbol{N}_L(M, g)$

The behavior of the spectral function along the diagonal is rather well understood.

The first major breakthrough is due to L. Hörmander (1968) who used the kernel of the wave operator to obtain estimates of the spectral functions more refined than the ones obtained via the heat equation.

Estimating $\boldsymbol{N}_L(M, g)$

The behavior of the spectral function along the diagonal is rather well understood.

The first major breakthrough is due to L. Hörmander (1968) who used the kernel of the wave operator to obtain estimates of the spectral functions more refined than the ones obtained via the heat equation.

X. Bin (2004) pushed Hörmander's technique further and obtained information about the partial derivatives of the spectral function along the diagonal.

Estimating $\boldsymbol{N}_L(M, g)$

Estimating $\boldsymbol{N}_L(M, g)$

Using these estimates of the spectral function one can prove that as $L \to \infty$ we have

$$\boldsymbol{S}_{L,\boldsymbol{p}} \sim a_m L^{\frac{m+4}{2}} \boldsymbol{S}_m, \;\; \boldsymbol{\sigma}_{L,\boldsymbol{p}} = b_m L^{\frac{m+2}{2}} \mathbb{1}_m,$$

where

Estimating $\boldsymbol{N}_L(M, g)$

Using these estimates of the spectral function one can prove that as $L \to \infty$ we have

$$\boldsymbol{S}_{L,\boldsymbol{p}} \sim a_m L^{\frac{m+4}{2}} \boldsymbol{S}_m, \;\; \boldsymbol{\sigma}_{L,\boldsymbol{p}} = b_m L^{\frac{m+2}{2}} \mathbb{1}_m,$$

where

- a_m, b_m are explicit positive constants,

Estimating $\boldsymbol{N}_L(M, g)$

Using these estimates of the spectral function one can prove that as $L \to \infty$ we have

$$\boldsymbol{S}_{L,\boldsymbol{p}} \sim a_m L^{\frac{m+4}{2}} \boldsymbol{S}_m, \;\; \boldsymbol{\sigma}_{L,\boldsymbol{p}} = b_m L^{\frac{m+2}{2}} \mathbb{1}_m,$$

where

- a_m, b_m are explicit positive constants,
- $\boldsymbol{S}_m$ is an explicit $O(m)$-invariant positive definite operator $\mathcal{S}_m \to \mathcal{S}_m$, $\mathcal{S}_m =$ the space of real, symmetric $m \times m$ matrices

Estimating $\boldsymbol{N}_L(M, g)$

Using these estimates of the spectral function one can prove that as $L \to \infty$ we have

$$\boldsymbol{S}_{L,\boldsymbol{p}} \sim a_m L^{\frac{m+4}{2}} \boldsymbol{S}_m, \;\; \boldsymbol{\sigma}_{L,\boldsymbol{p}} = b_m L^{\frac{m+2}{2}} \mathbb{1}_m,$$

where

- a_m, b_m are explicit positive constants,
- $\boldsymbol{S}_m$ is an explicit $O(m)$-invariant positive definite operator $\mathcal{S}_m \to \mathcal{S}_m$, $\mathcal{S}_m =$ the space of real, symmetric $m \times m$ matrices
- and we have identified $T_{\boldsymbol{p}}M$ with $\mathbb{R}^m$ via a g-orthonormal frame of $T_{\boldsymbol{p}}M$.

Estimating $\boldsymbol{N}_L(M,g)$

Using these estimates of the spectral function one can prove that as $L\to\infty$ we have

$$\boldsymbol{S}_{L,\boldsymbol{p}} \sim a_m L^{\frac{m+4}{2}} \boldsymbol{S}_m, \;\; \boldsymbol{\sigma}_{L,\boldsymbol{p}} = b_m L^{\frac{m+2}{2}} \mathbb{1}_m,$$

where

- a_m, b_m are explicit positive constants,
- $\boldsymbol{S}_m$ is an explicit $O(m)$-invariant positive definite operator $\mathcal{S}_m \to \mathcal{S}_m$, $\mathcal{S}_m =$ the space of real, symmetric $m\times m$ matrices
- and we have identified $T_{\boldsymbol{p}}M$ with $\mathbb{R}^m$ via a g-orthonormal frame of $T_{\boldsymbol{p}}M$.

Puting together these facts we obtained the claimed asymptotic estimate of $\boldsymbol{N}_L(M,g)$.

Estimating C_m

Estimating C_m

Using tricks from random matrix theory we can rewrite

$$C_{m-1} = const_m \int_{\mathbb{R}} \rho_m(s) \sqrt{\frac{2}{\pi}} e^{-2s^2} ds,$$

Estimating C_m

Using tricks from random matrix theory we can rewrite

$$C_{m-1} = const_m \int_{\mathbb{R}} \rho_m(s) \sqrt{\frac{2}{\pi}} e^{-2s^2} ds,$$

where $\rho_n(s)ds$ is the so called 1-point correlation function.

Estimating C_m

Using tricks from random matrix theory we can rewrite

$$C_{m-1} = const_m \int_{\mathbb{R}} \rho_m(s) \sqrt{\frac{2}{\pi}} e^{-2s^2} ds,$$

where $\rho_n(s)ds$ is the so called 1-point correlation function. Intuitively, $\rho_n(s)ds$ is the probability that a random symmetric $n \times n$ matrix has at least one eigenvalue in the interval $[s, s+ds]$.

Estimating C_m

Using tricks from random matrix theory we can rewrite

$$C_{m-1} = const_m \int_{\mathbb{R}} \rho_m(s) \sqrt{\frac{2}{\pi}} e^{-2s^2} ds,$$

where $\rho_n(s)ds$ is the so called 1-point correlation function. Intuitively, $\rho_n(s)ds$ is the probability that a random symmetric $n \times n$ matrix has at least one eigenvalue in the interval $[s, s + ds]$.

The function $\rho_n(s)$ can be expressed explicitly in terms of Hermite polynomials.

Estimating C_m

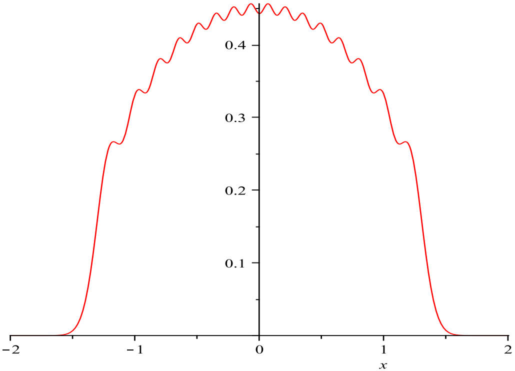

Figure: A depiction of $\rho_{16}(x)$, $|x| \leq 8$.

Estimating C_m

Estimating C_m

Theorem (Wigner's semi-circle law)

$$\frac{1}{\sqrt{n}}\rho_n(\sqrt{n}x) \to \bar{\rho}_\infty(x) := \frac{1}{\pi}\begin{cases}\sqrt{2-x^2}, & |x| \le \sqrt{2},\\ 0, & |x| > \sqrt{2},\end{cases} \quad \textit{as} \;\; n \to \infty.$$

Estimating C_m

Estimating C_m

- Set $\bar{\rho}_n(x) = \frac{1}{\sqrt{n}}\rho_n(\sqrt{n}x)$.

Estimating C_m

- Set $\bar{\rho}_n(x) = \frac{1}{\sqrt{n}}\rho_n(\sqrt{n}x)$.
- Then

$$C_{m-1} = const_m \int_{\mathbb{R}} \bar{\rho}_m(x) \underbrace{\frac{(3m)^{\frac{1}{2}} e^{-\frac{3mx^2}{2}}}{(2\pi)^{\frac{1}{2}}}}_{=:w_m(x)} dx.$$

Estimating C_m

- Set $\bar{\rho}_n(x) = \frac{1}{\sqrt{n}}\rho_n(\sqrt{n}x)$.
- Then

$$C_{m-1} = const_m \int_{\mathbb{R}} \bar{\rho}_m(x) \underbrace{\frac{(3m)^{\frac{1}{2}} e^{-\frac{3mx^2}{2}}}{(2\pi)^{\frac{1}{2}}}}_{=:w_m(x)} dx.$$

- $\lim_{m\to\infty} w_m(x)dx \to \delta_0 =$ Dirac delta at the origin.

Estimating C_m

- Set $\bar{\rho}_n(x) = \frac{1}{\sqrt{n}}\rho_n(\sqrt{n}x)$.
- Then

$$C_{m-1} = const_m \int_{\mathbb{R}} \bar{\rho}_m(x) \underbrace{\frac{(3m)^{\frac{1}{2}} e^{-\frac{3mx^2}{2}}}{(2\pi)^{\frac{1}{2}}}}_{=:w_m(x)} dx.$$

- $\lim_{m\to\infty} w_m(x)dx \to \delta_0 =$ Dirac delta at the origin.
- We deduce that the above integral is close to $\bar{\rho}_\infty(0) = \frac{\sqrt{2}}{\pi}$.

Estimating C_m

- Set $\bar{\rho}_n(x) = \frac{1}{\sqrt{n}}\rho_n(\sqrt{n}x)$.
- Then

$$C_{m-1} = const_m \int_{\mathbb{R}} \bar{\rho}_m(x) \underbrace{\frac{(3m)^{\frac{1}{2}} e^{-\frac{3mx^2}{2}}}{(2\pi)^{\frac{1}{2}}}}_{=:w_m(x)} dx.$$

- $\lim_{m\to\infty} w_m(x)dx \to \delta_0 =$ Dirac delta at the origin.
- We deduce that the above integral is close to $\bar{\rho}_\infty(0) = \frac{\sqrt{2}}{\pi}$.
- Finally, as $m \to \infty$,

$$C_m \sim \left(\frac{4\pi}{m+4}\right)^{\frac{m}{2}} \Gamma\left(1 + \frac{m}{2}\right) \times \frac{\Gamma\left(m + \frac{3}{2}\right)}{\pi 2^{m^2+m-2}(m+2)^{\frac{m-2}{2}}}.$$

Estimating C_m

- Set $\bar{\rho}_n(x) = \frac{1}{\sqrt{n}}\rho_n(\sqrt{n}x)$.
- Then

$$C_{m-1} = const_m \int_{\mathbb{R}} \bar{\rho}_m(x) \underbrace{\frac{(3m)^{\frac{1}{2}} e^{-\frac{3mx^2}{2}}}{(2\pi)^{\frac{1}{2}}}}_{=:w_m(x)} dx.$$

- $\lim_{m\to\infty} w_m(x)dx \to \delta_0 =$ Dirac delta at the origin.
- We deduce that the above integral is close to $\bar{\rho}_\infty(0) = \frac{\sqrt{2}}{\pi}$.
- Finally, as $m \to \infty$,

$$C_m \sim \left(\frac{4\pi}{m+4}\right)^{\frac{m}{2}} \Gamma\left(1+\frac{m}{2}\right) \times \frac{\Gamma\left(m+\frac{3}{2}\right)}{\pi 2^{m^2+m-2}(m+2)^{\frac{m-2}{2}}}.$$

- Now use Stirling's formula.

Random trigonometric polynomials

Random trigonometric polynomials

Denote by Z_ν the number of critical points of a trigonometric polynomial

$$P(\theta) = a_0 + \sum_{j=1}^{\nu}(a_j \cos j\theta + b_j \sin j\theta),$$

Random trigonometric polynomials

Denote by Z_ν the number of critical points of a trigonometric polynomial

$$P(\theta) = a_0 + \sum_{j=1}^{\nu} (a_j \cos j\theta + b_j \sin j\theta),$$

where a_i, b_j are normally distributed independent random variables with expectation 0 and variance 1.

Random trigonometric polynomials

Denote by Z_ν the number of critical points of a trigonometric polynomial

$$P(\theta) = a_0 + \sum_{j=1}^{\nu} (a_j \cos j\theta + b_j \sin j\theta),$$

where a_i, b_j are normally distributed independent random variables with expectation 0 and variance 1.

Theorem

As $\nu \to \infty$, we have

Random trigonometric polynomials

Denote by Z_ν the number of critical points of a trigonometric polynomial

$$P(\theta) = a_0 + \sum_{j=1}^{\nu} (a_j \cos j\theta + b_j \sin j\theta),$$

where a_i, b_j are normally distributed independent random variables with expectation 0 and variance 1.

Theorem

As $\nu \to \infty$, we have

$$\boldsymbol{E}(Z_\nu) \sim 2\sqrt{\frac{3}{5}}\nu, \quad \boldsymbol{var}(Z_\nu) \sim c\nu, \qquad \text{(Nic. 2010)}$$

Random trigonometric polynomials

Denote by Z_ν the number of critical points of a trigonometric polynomial

$$P(\theta) = a_0 + \sum_{j=1}^{\nu} (a_j \cos j\theta + b_j \sin j\theta),$$

where a_i, b_j are normally distributed independent random variables with expectation 0 and variance 1.

Theorem

As $\nu \to \infty$, we have

$$\boldsymbol{E}(Z_\nu) \sim 2\sqrt{\frac{3}{5}}\nu, \quad \boldsymbol{var}(Z_\nu) \sim c\nu, \qquad \text{(Nic. 2010)}$$

*where $c \approx 0.35$, **E**= expectation, **var**= variance.*

About the variance estimate

About the variance estimate

- The variance estimate shows a high concentration of Z_ν near its expectation.

About the variance estimate

- The variance estimate shows a high concentration of Z_ν near its expectation.
- The proof uses

About the variance estimate

- The variance estimate shows a high concentration of Z_ν near its expectation.
- The proof uses
  - Results in the theory of stationary gaussian processes to evaluate the second moment of Z_ν.

About the variance estimate

- The variance estimate shows a high concentration of Z_ν near its expectation.
- The proof uses
  - Results in the theory of stationary gaussian processes to evaluate the second moment of Z_ν.
  - Estimates of certain trigonometric sums.

About the variance estimate

- The variance estimate shows a high concentration of Z_ν near its expectation.
- The proof uses
  - Results in the theory of stationary gaussian processes to evaluate the second moment of Z_ν.
  - Estimates of certain trigonometric sums.
- Related recent results of A. Granville, I. Wigman suggest that Z_ν satisfies a central limit theorem.

The spectral function

The spectral function

To give an idea of the subtlety of the spectral function estimates we consider the special case when $M = S^2$.

The spectral function

To give an idea of the subtlety of the spectral function estimates we consider the special case when $M = S^2$.

- The spectrum of Δ_{S^2} is $n(n+1)$, $n = 0, 1, \dots$.

The spectral function

To give an idea of the subtlety of the spectral function estimates we consider the special case when $M = S^2$.

- The spectrum of Δ_{S^2} is $n(n+1)$, $n = 0, 1, \dots$.
- In this case, for any $\boldsymbol{x}, \boldsymbol{y} \in S^2$, we have

$$\mathcal{E}_{n(n+1)}(\boldsymbol{x}, \boldsymbol{y}) = E_n(\varphi) := \frac{(n+1)}{4\pi} \times \frac{P_{n+1}(\cos\varphi) - P_n(\cos\varphi)}{\cos\varphi - 1},$$

where

The spectral function

To give an idea of the subtlety of the spectral function estimates we consider the special case when $M = S^2$.

- The spectrum of Δ_{S^2} is $n(n+1)$, $n = 0, 1, \dots$.
- In this case, for any $\boldsymbol{x}, \boldsymbol{y} \in S^2$, we have

$$\mathcal{E}_{n(n+1)}(\boldsymbol{x}, \boldsymbol{y}) = E_n(\varphi) := \frac{(n+1)}{4\pi} \times \frac{P_{n+1}(\cos\varphi) - P_n(\cos\varphi)}{\cos\varphi - 1},$$

where

$$\cos\varphi = \boldsymbol{x} \cdot \boldsymbol{y}, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

The spectral function

To give an idea of the subtlety of the spectral function estimates we consider the special case when $M = S^2$.

- The spectrum of Δ_{S^2} is $n(n+1)$, $n = 0, 1, \dots$.
- In this case, for any $\boldsymbol{x}, \boldsymbol{y} \in S^2$, we have

  $$\mathcal{E}_{n(n+1)}(\boldsymbol{x}, \boldsymbol{y}) = E_n(\varphi) := \frac{(n+1)}{4\pi} \times \frac{P_{n+1}(\cos\varphi) - P_n(\cos\varphi)}{\cos\varphi - 1},$$

  where

  $$\cos\varphi = \boldsymbol{x} \cdot \boldsymbol{y}, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

- Along the diagonal in $S^2 \times S^2$ we have $\cos\varphi = 1$ and $\mathcal{E}_{n(n+1)}(\boldsymbol{x}, \boldsymbol{x}) = \frac{(n+1)^2}{4\pi}$.

The spectral function

To give an idea of the subtlety of the spectral function estimates we consider the special case when $M = S^2$.

- The spectrum of Δ_{S^2} is $n(n+1)$, $n = 0, 1, \dots$.
- In this case, for any $\boldsymbol{x}, \boldsymbol{y} \in S^2$, we have

$$\mathcal{E}_{n(n+1)}(\boldsymbol{x}, \boldsymbol{y}) = E_n(\varphi) := \frac{(n+1)}{4\pi} \times \frac{P_{n+1}(\cos\varphi) - P_n(\cos\varphi)}{\cos\varphi - 1},$$

  where

$$\cos\varphi = \boldsymbol{x} \cdot \boldsymbol{y}, \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n.$$

- Along the diagonal in $S^2 \times S^2$ we have $\cos\varphi = 1$ and $\mathcal{E}_{n(n+1)}(\boldsymbol{x}, \boldsymbol{x}) = \frac{(n+1)^2}{4\pi}$.
- The Legendre polynomials $P_n(x)$ have a highly oscillatory behavior.

The spectral function

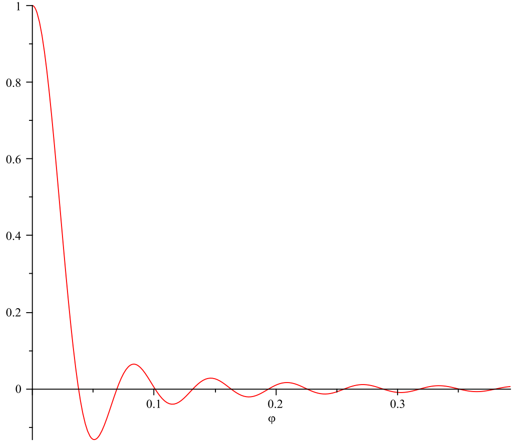

Figure: A depiction of $\frac{4\pi}{101^2}E_{100}(\varphi)$, $0 \leq \varphi \leq \frac{\pi}{8}$. Note that there are 11 zeros inside an interval of length $\frac{\pi}{8} < 0.4$

The spectral function

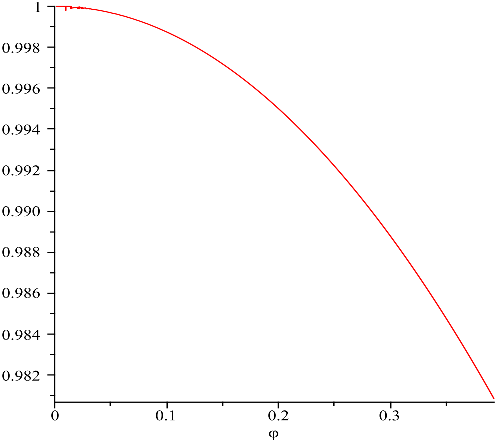

Figure: A depiction of $\frac{4\pi}{101^2}E_{100}(\frac{\varphi}{101})$, $0 \leq \varphi \leq \frac{\pi}{8}$. The rescaling $\varphi \mapsto \frac{\varphi}{101}$ moved the oscillations at ∞.

Universal rescaling of the spectral function

Universal rescaling of the spectral function

- Suppose (M, g) compact Riemann manifold of dimension m, $\boldsymbol{p}_0 \in M$, and $x = (x^1, \dots, x^m)$ normal coordinates at $\boldsymbol{p}_0$.

Universal rescaling of the spectral function

- Suppose (M, g) compact Riemann manifold of dimension m, $\boldsymbol{p}_0 \in M$, and $x = (x^1, \dots, x^m)$ normal coordinates at $\boldsymbol{p}_0$.
- In a neigborhood of $(\boldsymbol{p}_0, \boldsymbol{p}_0)$ we regard $\mathcal{E}_L$ as a function $\mathcal{E}_L(x, y)$ defined in a neighborhood of $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^m$.

Universal rescaling of the spectral function

- Suppose (M, g) compact Riemann manifold of dimension m, $\boldsymbol{p}_0 \in M$, and $x = (x^1, \dots, x^m)$ normal coordinates at $\boldsymbol{p}_0$.
- In a neigborhood of $(\boldsymbol{p}_0, \boldsymbol{p}_0)$ we regard $\mathcal{E}_L$ as a function $\mathcal{E}_L(x, y)$ defined in a neighborhood of $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^m$.
- Define

$$\bar{\mathcal{E}}_{L,\boldsymbol{p}_0}(x) := \frac{1}{L^{\frac{m}{2}}} \mathcal{E}_L\left(\frac{x}{L^{\frac{1}{2}}}, \frac{y}{L^{\frac{1}{2}}} \right) \Big|_{x+y=0}.$$

  We regard the space $V_{\boldsymbol{p}_0} := \{x + y = 0\}$ as the fiber over $(\boldsymbol{p}_0, \boldsymbol{p}_0)$ of the normal bundle of the diagonal.

Universal rescaling of the spectral function

Universal rescaling of the spectral function

Denote by r the distance-to-the diagonal function

$$r = \sqrt{2\sum_{i=1}^{m}(x^i - y^i)^2}.$$

Universal rescaling of the spectral function

Denote by r the distance-to-the diagonal function

$$r = \sqrt{2\sum_{i=1}^{m}(x^i - y^i)^2}.$$

Theorem (Lapointe-Polterovich-Safarov, 2009)

The functions $\bar{\mathcal{E}}_{L,\boldsymbol{p}_0}(x)$ converge as $L \to \infty$ to the $SO(m)$ invariant function on $V_{\boldsymbol{p}_0}$

$$E_m(r) := \frac{1}{(2\pi r)^{\frac{m}{2}}} J_{\frac{m}{2}}(r),$$

where J_ν denotes the Bessel function of the first kind and order ν.

The meaning of E_m

The meaning of E_m

- Denote by B_R^m the ball of radius R in $\mathbb{R}^m$ centered at the origin.

The meaning of E_m

- Denote by B^m_R the ball of radius R in $\mathbb{R}^m$ centered at the origin.
- The function $\boldsymbol{E}_L : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by

$$\boldsymbol{E}^L_m(x,y) = \frac{1}{(2\pi)^m} \int_{B^m_{\sqrt{L}}} e^{\boldsymbol{i}(\xi, x-y)} |d\xi|, \;\; \forall x, y \in \mathbb{R}^m$$

The meaning of E_m

- Denote by B_R^m the ball of radius R in $\mathbb{R}^m$ centered at the origin.
- The function $\boldsymbol{E}_L : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by

$$\boldsymbol{E}_m^L(x, y) = \frac{1}{(2\pi)^m} \int_{B_{\sqrt{L}}^m} e^{\boldsymbol{i}(\xi, x-y)} |d\xi|, \;\; \forall x, y \in \mathbb{R}^m$$

is the spectral function of the Laplacian on $\mathbb{R}^m$.

The meaning of E_m

- Denote by B_R^m the ball of radius R in $\mathbb{R}^m$ centered at the origin.
- The function $\boldsymbol{E}_L : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ given by

$$\boldsymbol{E}_m^L(x,y) = \frac{1}{(2\pi)^m} \int_{B^m_{\sqrt{L}}} e^{\boldsymbol{i}(\xi, x-y)} |d\xi|, \quad \forall x, y \in \mathbb{R}^m$$

  is the spectral function of the Laplacian on $\mathbb{R}^m$.
- For any $x \in \mathbb{R}^m$ we have

$$E_m(|x|) = \boldsymbol{E}_m^1(x,0) = \frac{1}{(2\pi)^m} \int_{B_1^m} e^{\boldsymbol{i}(\xi, x)} |d\xi|.$$