# THE COHOMOLOGY OF SHEAVES

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ABSTRACT. A fast introduction to the the construction of the cohomology of sheaves pioneered by A. Grothendieck and J.L. Verdier. The approach is from the point of view of derived categories, though this concept is never mentioned.

## 1. SHEAVES

Let $R$ commutative ring with 1. We denote by ${}_R\mathrm{Mod}$ the category of $R$-modules.

For any topological space $X$ we denote by $\mathrm{Open}(X)$ the collection of open subsets. It can be organized as a category in which the morphisms are given by inclusions. A pre-sheaf of $R$-modules is a contravariant functor

$$\mathcal{S} : \mathrm{Open}(X) \to {}_R\mathrm{Mod}, \ \ \mathrm{Open}(X) \ni U \mapsto \Gamma(U, \mathcal{S}) \in {}_R\mathrm{Mod}.$$

For any inclusion $U \subset V$, we have a *restriction* map $r = \mathcal{S}_{UV} : \Gamma(V, \mathcal{S}) \to \Gamma(U, \mathcal{S})$. The module $\Gamma(U, \mathcal{S})$ is called the module of *continuous sections of* $\mathcal{S}$ *over* $U$. If $s \in \Gamma(V, \mathcal{S})$, $V \subset U$, we set $s|_U := r_{UV}(s)$ when there is no danger of confusion.

A *morphism* of sheaves of $R$-modules $\mathcal{S}_0$ and $\mathcal{S}_1$ is a collection of morphisms of $R$-modules

$$\phi_U : \Gamma(U, \mathcal{S}_0) \to \Gamma(U, \mathcal{S}_1)$$

compatible in the obvious way with the restriction maps. We obtain a category $\boldsymbol{Psh}_R(X)$ of pre-sheaves of $R$-modules over $X$. For any morphism of presheaves $\phi : \mathcal{S}_0 \to \mathcal{S}_1$ we can define a kernel presheaf $\ker \phi$ and an image presheaf. For a sub presheaf $\mathcal{S}_0 \hookrightarrow \mathcal{S}_1$ we can define the quotient presheaf, $\mathcal{S}_1/\mathcal{S}_0$. We can also define the direct sum of two presheves. Rigourously, $\boldsymbol{Psh}_R(X)$ is an Abelian category in an obvious fashion.

A preseheaf $\mathcal{S} \in \boldsymbol{Psh}_R(X)$ is called a *sheaf* if for any open cover $(U_i)_{i \in I}$ of $X$, and for any collection of sections $s_i \in \Gamma(U_i, \mathcal{S})$ such that

$$s_i|_{U_i \cap U_j} = s_i|_{U_i \cap U_j}, \ \ \forall i, j$$

there exists a unique section $s \in \Gamma(X, \mathcal{S})$ such that $s|_{U_i} = s_i$, $\forall i$.

To any presheaf $\mathcal{S} \in \boldsymbol{Psh}_R(X)$ we can associate a canonical sheaf $\widetilde{\mathcal{S}} \in \boldsymbol{Sh}_R(X)$ as follows.

For $x \in X$ we define the *stalk* of $\mathcal{S}$ at $x$ to be the inductive limit

$$\mathcal{S}_x := \varinjlim_{U \ni x} \Gamma(U, \mathcal{S}).$$

Note that we have natural morphisms

$$\gamma_u : \Gamma(U, \mathcal{S}) \to \mathcal{S}_u, \ \ u \in U.$$

For $s \in \Gamma(U, \mathcal{S})$ and $u \in U$ the element $\gamma_u(s) \in \mathcal{S}_u$ is called the *germ of* $s$ *at* $u$.

We define $\Gamma(U, \widetilde{\mathcal{S}})$ to be the submodule of $\prod_{u \in U} \mathcal{S}_u$ consisting of collections $(s_u)_{u \in U}$ satisfying the conditions

$$s_u \in \mathcal{S}_u, \ \ \forall u \in U.$$

$$\forall u \in U, \ \ \exists V \in \mathrm{Open}(U), \ \ \exists s \in \Gamma(V, \mathcal{S}) \ \ \text{such that } u \in V \text{ and } \gamma_v(s) = s_v, \ \forall v \in V.$$

*Date*: Sometime during the spring of 2008.

The correspondence $U \mapsto \Gamma(U, \widetilde{\mathcal{S}})$ defines a sheaf called the *sheafification of* $\mathcal{S}$. If $\phi$ is a morphism of sheaves, its kernel is also a sheaf. However, its image is only a preshef, and we define the image sheaf to be the sheafification of the image presheaf. The quotient of two sheaves is a presheaf, and we define the quotient sheaf to be the sheafification of the quotient presheaf. We obtain in this fashion an (Abelian) category $\boldsymbol{Sh}_R(X)$ of sheaves of $R$-modules on $X$.

## 2. Complexes of sheaves

We denote by $C^+(X)$ the category of bonded from below complexes of sheaves, i.e., complexes of sheaves $(\mathcal{S}^\bullet = \oplus_{n\in\mathbb{Z}}\mathcal{S}^n, d)$ such that $\mathcal{S}^n = 0$ if $n \ll 0$. Let $K^+(X)$ be the category whose objects are bounded from below complex of sheaves on $X$, but whose morphisms are the *homotopy classes* of cochain maps. For a complex $(\mathcal{A}^\bullet, d_A) \in C^+(X)$ and $k \in \mathbb{Z}$ we denote by $(\mathcal{A}^\bullet[k], d_{A[k]})$ the complex defined by

$$\mathcal{A}^n[k] := \mathcal{A}^{n+k},\ \ d_{A[k]} = (-1)^k d_A.$$

The cohomology of a complex of sheaves $(\mathcal{A}^\bullet, d_A)$ is the direct sum of sheaves

$$H^\bullet((\mathcal{A}) = \bigoplus_n H^n(\mathcal{A}),\ \ H^n(\mathcal{A}) := \ker(d_A : \mathcal{A}^n \to \mathcal{A}^{n+1})/\,\mathbf{Im}(\phi : \mathcal{A}^{n-1} \to \mathcal{A}^n).$$

A complex is called *acyclic* if its cohomology is trivial. A morphism of complexes is called a *quasi-isomorphism* (qis for brevity) if it induces an isomorphism in cohomology. We will use the notation $\mathcal{A} \overset{\phi}{\rightsquigarrow} \mathcal{B}$ to denote a qis.

Observe that any sheaf $\mathcal{A} \in \boldsymbol{Sh}_R(X)$ can be tautologically identified with a complex $\mathcal{A}^\bullet$ where $\mathcal{A}^0 = \mathcal{A}$, $\mathcal{A}^n = 0$, $\forall n \neq 0$. We will denote by $[\mathcal{A}]$ this complex. A *resolution* of $\mathcal{A}$ is then a qis $[\mathcal{A}] \overset{\phi}{\rightsquigarrow} (\mathcal{S}^\bullet, d_S)$.

A morphism of complexes of sheaves $\phi : (\mathcal{A}^\bullet, d_A) \to (\mathcal{B}^\bullet, d_B)$ determines a new complex $\mathrm{Cone}(\phi)$ called the *cone* of $\phi$ defined by

$$\mathrm{Cone}(\phi)^n := \mathcal{B}^n \oplus \mathcal{A}[1]^n$$

and differential

$$d_\phi := \begin{bmatrix} d_B & \phi \\ 0 & d_{A[1]} \end{bmatrix}.$$

The cone fits in the middle of a short exact sequence of complexes

$$0 \to \mathcal{B} \xrightarrow{i} \mathrm{Cone}(\phi) \xrightarrow{\pi} \mathcal{A}[1] \to 0$$

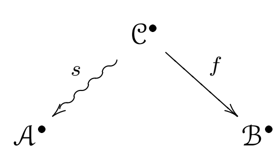
where $i$ and $\pi$ denote respectively the canonical inclusion and projection. From the above short exact sequence we obtain the following result.

**Proposition 2.1** (The cone trick). *Suppose $\phi : (\mathcal{A}^\bullet, d_A) \to (\mathcal{B}^\bullet, d_B)$ is a morphisms of complexes. We have a long exact sequence of sheaves*

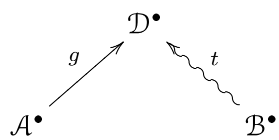
$$\cdots \xrightarrow{-\phi_*} H^n(B) \xrightarrow{i_*} H^n(\mathrm{Cone}(\phi)) \xrightarrow{\pi_*} H^{n+1}(\mathcal{A}) \xrightarrow{-\phi_*} H^{n+1}(\mathcal{B}) \xrightarrow{i_*} \cdots .$$

*In particular, $\phi$ is a qis if and only if* $\mathrm{Cone}(\phi)$ *is acyclic.* □

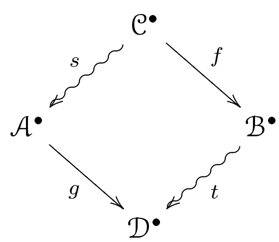
**Definition 2.2.** Let $\mathcal{A}^\bullet, \mathcal{B}^\bullet \in C^+(X)$.
(a) A *left* roof from $\mathcal{A}$ to $\mathcal{B}$ is a diagram of morphisms of complexes of sheaves of the form



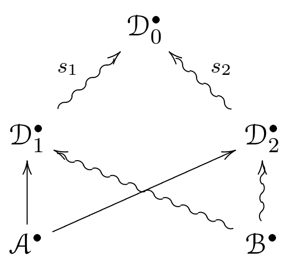
The complexes $\mathcal{A}^\bullet, \mathcal{B}^\bullet, \mathcal{C}^\bullet$ are called the *nodes* of the roof. We will use the notation $\mathcal{A}^\bullet \overset{s}{\leftsquigarrow} \mathcal{C}^\bullet \overset{f}{\to} \mathcal{B}^\bullet$ to denote the above left roof. We denote by $\mathcal{T}^\ell(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ the collection of left roofs from $\mathcal{A}^\bullet$ to $\mathcal{B}^\bullet$.
(b)A *right* roof from $\mathcal{A}$ to $\mathcal{B}$ is a diagram of morphisms of complexes of sheaves of the form




The nodes are defined in a similar way. We will use the notation $\mathcal{A}^\bullet \overset{g}{\to} \mathcal{D}^\bullet \overset{t}{\leftsquigarrow} \mathcal{B}^\bullet$ to denote the above right roof. We denote by $\mathcal{T}^r(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$ the collection of right roofs from $\mathcal{A}^\bullet$ to $\mathcal{B}^\bullet$.
(c) A left roof $\lambda = \mathcal{A}^\bullet \overset{s}{\leftsquigarrow} \mathcal{C}^\bullet \overset{f}{\to} \mathcal{B}^\bullet$ is said to be *equivalent* to the right roof $\rho = \mathcal{A}^\bullet \overset{g}{\to} \mathcal{D}^\bullet \overset{t}{\leftsquigarrow} \mathcal{B}^\bullet$, and we right this $\lambda \approx \rho$, if the diagram below




is commutative in $K^+(X)$, i.e., the morphism of complexes $t \circ f$ and $g \circ s$ are homotopic.
(d) Two right roofs $\rho_k = \mathcal{A}^\bullet \overset{g_k}{\to} \mathcal{D}_k^\bullet \overset{t_k}{\leftsquigarrow} \mathcal{B}^\bullet$, $k = 1, 2$ are called *equivalent*, and we write this $\rho_1 \sim_r \rho_2$ if there exists a third right roof $\mathcal{D}_1^\bullet \overset{s_1}{\rightsquigarrow} \mathcal{D}_0^\bullet \overset{s_2}{\leftsquigarrow} \mathcal{D}_2^\bullet$ such that the diagram below is homotopy commutative




We have the following important fact whose proof can be found in [1, §I.1,§I.5] and [3, Thm. III.4.4].

**Proposition 2.3** (Trading trick). *(a) Any left (respectively right) roof in $C^+(X)$ is equivalent to a right (respectively left) roof in $C^+(X)$.*
*(b) The binary relations "$\sim_\ell$" and "$\sim_r$" are equivalence relations.*
*(c) Suppose $\lambda_1, \lambda_2 \in \mathcal{T}^\ell(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$, $\rho_1, \rho_2 \in \mathcal{T}^\ell(\mathcal{A}^\bullet, \mathcal{B}^\bullet)$, and $\lambda_k \approx \rho_k$, $k = 1, 2$. Then*

$$\lambda_1 \sim_\ell \lambda_2 \Longleftrightarrow \rho_1 \sim_r \rho_2. \qquad \square$$

*Remark* 2.4. (a) Observe that any morphism of complexes $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ defines left and right triangle

$$\lambda(f) = \mathcal{A}^\bullet \overset{\mathbb{1}_{\mathcal{A}^\bullet}}{\leftsquigarrow} \mathcal{A}^\bullet \overset{f}{\to} \mathcal{B}^\bullet, \;\; \rho(f) = \mathcal{A}^\bullet \overset{f}{\to} \mathcal{B}^\bullet \overset{\mathbb{1}_{\mathcal{A}^\bullet}}{\leftsquigarrow} \mathcal{B}^\bullet$$

and $\lambda(f) \approx \rho(f)$.
(b) A left roof $\lambda = \mathcal{A}^\bullet \overset{s}{\leftsquigarrow} \mathcal{C}^\bullet \overset{f}{\to} \mathcal{B}^\bullet$ defines a morphism

$$\lambda_* : H^\bullet(\mathcal{A}^\bullet) \to H^\bullet(\mathcal{B}), \;\; \lambda_* = f_* \circ (s_*)^{-1},$$

while a right roof $\rho = \mathcal{A}^\bullet \xrightarrow{g} \mathcal{D}^\bullet \overset{t}{\leftsquigarrow} \mathcal{B}^\bullet$ defines a morphism

$$\rho_* = H^\bullet(\mathcal{A}^\bullet) \to H^\bullet(\mathcal{B}),\ \ \rho_* = (t_*)^{-1} \circ g_*.$$

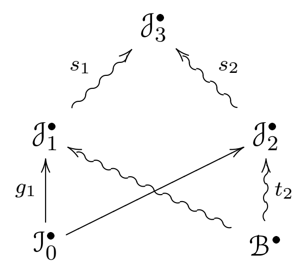
Let us observe that equivalent triangles induce identical morphisms in homology. □

## 3. Generating subcategories

**Definition 3.1** (Generating subcategory)**.** A *generating subcategory* of $\boldsymbol{Sh}_R(X)$ is a collection $\boldsymbol{I}(X)$ of sheaves of $R$-modules on $X$ satisfying the following conditions.

(1) $0 \in \boldsymbol{I}(X)$.
(2) If $\mathcal{I} \in \boldsymbol{I}(X)$, and $\mathcal{J} \in \boldsymbol{Sh}(X)$ is isomorphic to $\mathcal{I}$, then $\mathcal{J} \in \boldsymbol{I}(X)$.
(3) If $\mathcal{I}, \mathcal{J} \in \boldsymbol{I}(X)$, then $\mathcal{I} \oplus \mathcal{J} \in \boldsymbol{I}(X)$.
(4) If $0 \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to$ is a short exact sequence in $\boldsymbol{Sh}_R(X)$ and $\mathcal{I}_0, \mathcal{I}_1 \in \boldsymbol{I}(X)$, then $\mathcal{I}_2 \in \boldsymbol{I}(X)$.
(5) For any $\mathcal{A} \in \boldsymbol{Sh}_R(X)$ there exists short exact sequence $0 \to \mathcal{A} \to \mathcal{I}$, with $\mathcal{I} \in \boldsymbol{I}(X)$.

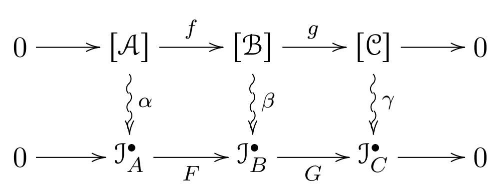
We denote by $C^+_{\boldsymbol{I}}(X)$ the bounded from below complex of sheaves $(\mathcal{I}^\bullet, d_I)$ such that $\mathcal{I}^n \in \boldsymbol{I}(X)$. We will refer to the complexes in $C^+_{\boldsymbol{I}}(X)$ as $\boldsymbol{I}(X)$-complexes. A left/right $\boldsymbol{I}(X)$-roof is a left/right roof such that all its nodes are $\boldsymbol{I}(X)$-complexes. Two right $\boldsymbol{I}(X)$-triangles $\mathcal{I}_0^\bullet \xrightarrow{g_k} \mathcal{J}_k \overset{t_k}{\leftsquigarrow} \mathcal{I}_1^\bullet$, $k = 1, 2$, are called $\boldsymbol{I}(X)$*-equivalent* if there exists a $\boldsymbol{I}(X)$-triangle $\mathcal{J}_1 \overset{s_1}{\rightsquigarrow} \mathcal{J}_3^\bullet \overset{s_2}{\leftsquigarrow} \mathcal{J}_2^\bullet$ such that the diagram below is commutative




We have the following important result whose proof can be found in [1, §I.7].

**Theorem 3.2** (Approximation principle)**.** *(a) For any complex $(\mathcal{A}^\bullet, d_A) \in C^+(X)$ there exists a qis $\mathcal{A}^\bullet \overset{\phi}{\rightsquigarrow} \mathcal{I}^\bullet$, where $\mathcal{I}^n \in \boldsymbol{I}(X)$, $\forall n \in \mathbb{Z}$. We say that $\mathcal{A}^\bullet \overset{\phi}{\rightsquigarrow} \mathcal{I}^\bullet$ is a* $\boldsymbol{I}(X)$-resolution *of $\mathcal{A}^\bullet$.*

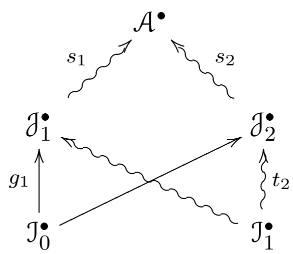
*(b) Any short exact sequence of sheaves $0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{B} \to 0$ can be completed to a diagram commutative in $K^+(X)$ (that is homotopy commutative) of the form*



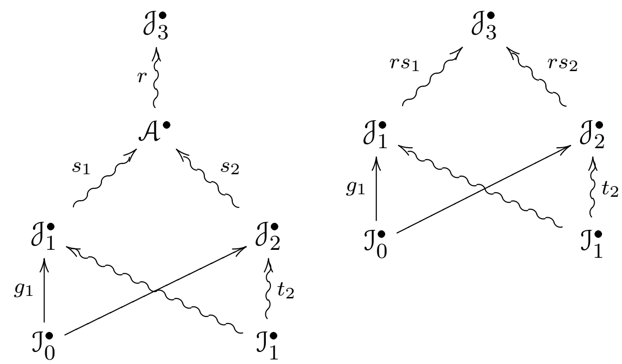

*where $\mathcal{I}_A^\bullet, \mathcal{I}_B^\bullet, \mathcal{I}_C^\bullet \in C^+_{\boldsymbol{I}}(X)$.* □

**Proposition 3.3.** *Consider two right $\boldsymbol{I}(X)$-triangles $\rho_k = \mathcal{I}_0^\bullet \xrightarrow{g_k} \mathcal{J}_k \overset{t_k}{\leftsquigarrow} \mathcal{I}_1^\bullet$, $k = 1, 2$. The following statements are equivalent.*
*(a) The triangles $\rho_1$ and $\rho_2$ are $\boldsymbol{I}(X)$-equivalent.*
*(b) The triangles $\rho_1$ and $\rho_2$ are equivalent.*

*Proof.* Clearly (a) $\Rightarrow$ (b). Assume (b). We then have a homotopy commutative diagram



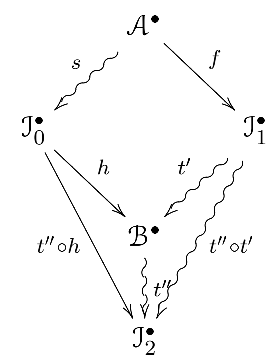
where the $\mathcal{I}$'s and the $\mathcal{J}$'s are $\boldsymbol{I}(X)$-complexes. Choose a $\boldsymbol{I}(X)$-resolution $\mathcal{A}^\bullet \overset{r}{\rightsquigarrow} \mathcal{J}_3^\bullet$. We obtain the homotopy commutative diagrams



which show that $\rho_1$ and $\rho_2$ are $\boldsymbol{I}(X)$-equivalent. □

**Proposition 3.4** (Refined trading trick)**.** *Any left roof $\lambda = \mathcal{I}_0^\bullet \overset{s}{\leftsquigarrow} \mathcal{A}^\bullet \overset{f}{\to} \mathcal{I}_1^\bullet$, where $\mathcal{I}_0^\bullet, \mathcal{I}_1^\bullet \in C^+_{\boldsymbol{I}}(X)$, $\mathcal{A}^\bullet \in C^+(X)$ is equivalent to a right $\boldsymbol{I}(X)$-roof $\mathcal{I}_0^\bullet \overset{g}{\to} \mathcal{I}_2^\bullet \overset{t}{\leftsquigarrow} \mathcal{I}_1^\bullet$. Moreover, any two $\boldsymbol{I}(X)$-roofs $\rho$, $\rho'$ equivalent to $\lambda$ are $\boldsymbol{I}(X)$-equivalent.*

*Proof.* Using the trading trick we can find a right triangle $\mathcal{I}_0 \overset{h}{\to} \mathcal{B}^\bullet \overset{t'}{\leftsquigarrow} \mathcal{I}_1^\bullet$. From the approximation theorem (a) we can find a $\boldsymbol{I}(X)$-resolution $\mathcal{B}^\bullet \overset{t''}{\rightsquigarrow} \mathcal{I}_2^\bullet$. Now look at the homotopy commutative diagram of complexes of sheaves



If we set $g = t'' \circ h$ and $t = t'' \circ t'$ we see that the triangle $\mathcal{I}_0 \overset{g}{\to} \mathcal{I}_2^\bullet \overset{t}{\leftsquigarrow} \mathcal{I}_1^\bullet$ has all the desired properties. The second part of the proposition follows from Proposition 3.3. □

## 4. Derived functors

Suppose $X, Y$ are topological spaces and we have a left exact Abelian covariant functor

$$\boldsymbol{F} : \boldsymbol{Sh}_R(X) \to \boldsymbol{Sh}_R(Y),$$

i.e., a functor satisfying the following conditions.

$$\boldsymbol{F}(0)=0,\;\; \boldsymbol{F}(\mathcal{A}\oplus\mathcal{B})=\boldsymbol{F}(\mathcal{A})\oplus\mathcal{B}),\;\; \forall\mathcal{A},\mathcal{B}\in\boldsymbol{Sh}_R(X)$$

• The induced map

$$\boldsymbol{F}_*:\operatorname{Hom}_{\boldsymbol{Sh}_R(X)}\big(\mathcal{A},\mathcal{B}\,\big)\to\operatorname{Hom}_{\boldsymbol{Sh}_R(Y)}\big(\,\boldsymbol{F}(\mathcal{A}),\boldsymbol{F}(\mathcal{B})\,\big)$$

is a morphism of Abelian groups.

• If $0\to\mathcal{A}\stackrel{f}{\to}\mathcal{B}\stackrel{g}{\to}\mathcal{C}$ is an exact sequence in $\boldsymbol{Sh}_R(X)$, then the sequence

$$0\to\boldsymbol{F}(\mathcal{A})\stackrel{\boldsymbol{F}_*(f)}{\longrightarrow}\boldsymbol{F}(\mathcal{B})\stackrel{\boldsymbol{F}_*(g)}{\longrightarrow}\boldsymbol{F}(\mathcal{C})$$

is exact in $\boldsymbol{Sh}_R(Y)$.

Suppose further that $\boldsymbol{I}(X)$ is an $\boldsymbol{F}$-acyclic generating subcategory of $\boldsymbol{Sh}_R(X)$, i.e., $\boldsymbol{I}(X)$ is a generating subcategory of $\boldsymbol{Sh}_R(X)$ satisfying the additional condition

If $0\to\mathcal{I}_0\stackrel{f_0}{\to}\mathcal{I}_1\stackrel{f_1}{\to}\mathcal{I}_2\to 0$ is a short exact sequence of sheaves in $\boldsymbol{I}(X)$, then the sequence of sheaves on $Y$

$$0\to\boldsymbol{F}(\mathcal{I}_0)\stackrel{\boldsymbol{F}_*(f_0)}{\longrightarrow}\boldsymbol{F}(\mathcal{I}_1)\stackrel{\boldsymbol{F}_*(f_1)}{\longrightarrow}\boldsymbol{F}(\mathcal{I}_2)\to 0$$

is also exact.

We list a few elementary properties of the the pair $(\boldsymbol{F},\boldsymbol{I}(X))$.

**Proposition 4.1.** *(a) If $(\mathcal{I}^\bullet,d)\in C^+_{\boldsymbol{I}}(X)$ is acyclic, then so is $(\boldsymbol{F}(\mathcal{I}^\bullet),\boldsymbol{F}_*(d))$.*
*(b) If $\phi:(\mathcal{A}^\bullet,d_A)\to(\mathcal{B}^\bullet,d_B)$ is a morphism of complexes of sheaves on $X$ then*

$$\boldsymbol{F}\big(\operatorname{Cone}(\phi)\,\big)=\operatorname{Cone}\big(\,\boldsymbol{F}_*(\phi)\,\big)$$

*(c) For any qis $\mathcal{I}^\bullet\stackrel{\phi}{\rightsquigarrow}\mathcal{J}^\bullet$, $\mathcal{I}^\bullet,\mathcal{J}^\bullet\in C^+_{\boldsymbol{I}}(X)$ the induced morpism $\boldsymbol{F}_*(\phi)$ is also a qis $\boldsymbol{F}(\mathcal{I}^\bullet)\stackrel{\boldsymbol{F}_*(\phi)}{\rightsquigarrow}\boldsymbol{F}(\mathcal{J}^\bullet)$.*

*Proof.* (a) We can assume without any loss of generality that $\mathcal{I}^n=0$, $\forall n<0$ so that the acyclic complex $\mathcal{I}^\bullet$ leads to a long exact sequence.

$$0\to\mathcal{J}^0\stackrel{d_0}{\to}\mathcal{I}^1\stackrel{d_1}{\to}\mathcal{I}^2\stackrel{d_2}{\to}\cdots$$

We obtain short exact sequences

$$0\to\mathcal{I}_0\to\mathcal{I}_1\to\ker d_2\to 0,\;\; 0\to\ker d_1\to\mathcal{I}^1\to\mathcal{I}^2\to\ker d_3\to 0,\;\; 0\to\ker d_n\to\mathcal{I}^n\to\ker d_{n+1}\to 0.$$

Observe that $\ker d_1\cong\mathcal{I}_0\in\boldsymbol{I}_X$. From Definition 3.1(4) we deduce inductively that $\ker d_n\in\boldsymbol{I}(X)$, $\forall n$. Because $\boldsymbol{I}(X)$ is $\boldsymbol{F}$-acyclic we deduce that we have short exact sequences

$$0\to\ker\boldsymbol{F}_*(d_n)\to\mathcal{I}^n\stackrel{\boldsymbol{F}_*(d_n)}{\to}\ker\boldsymbol{F}_*(d_{n+1})\to 0$$

which shows that the complex $(\boldsymbol{F}(\mathcal{I}^\bullet),\boldsymbol{F}_*(d))$ is acyclic.

Part (b) is obvious. For part (c) let us observe that Definition 3.1(3) implies that $\operatorname{Cone}(\phi)\in C^+_{\boldsymbol{I}}(X)$. From the cone trick we deduce that $\operatorname{Cone}(\phi)$ is acyclic. Using (a) and (b) we deduce that $\operatorname{Cone}(\boldsymbol{F}_*(\phi)$ is also acyclic. Invoking the cone trick again we deduce that $\boldsymbol{F}_*(\phi)$ is a qis. □

**Proposition 4.2.** *Suppose $\mathcal{A}^\bullet\in C^+(X)$, and $\mathcal{A}^\bullet\stackrel{s_k}{\rightsquigarrow}\mathcal{I}^\bullet_k$, $k=0,1$ are two $\boldsymbol{I}(X)$-resolutions. Then the complexes $\boldsymbol{F}(\mathcal{I}^\bullet_k)\in C^+(Y)$ , $k=1,2$, have isomorphic cohomology.*

*Proof.* Using the refined trading trick we deduce that the left roof $\mathcal{I}^\bullet_0\stackrel{s_0}{\leftsquigarrow}\mathcal{A}^\bullet\stackrel{s_1}{\rightsquigarrow}\mathcal{I}^\bullet_1$, is equivalent to a right roof $\mathcal{I}^\bullet_0\stackrel{t_0}{\to}\mathcal{I}^\bullet_2\stackrel{t_1}{\leftsquigarrow}\mathcal{I}^\bullet_1$. Clearly $t_0$ must be a qis as well because $t_0\circ s_0=t_1\circ s_1$, and $s_0,s_1,t_1$ are qis. From Proposition 4.1(c) we deduce that the maps

$$\boldsymbol{F}_*(t_k):\boldsymbol{F}(\mathcal{I}^\bullet_2)\to\boldsymbol{F}(\mathcal{I}^\bullet_k),\;\; k=0,1,$$

are qis. □

**Definition 4.3.** For every $\mathcal{A}^\bullet \in C^+(X)$ we set $R^\bullet \boldsymbol{F}(\mathcal{A}^\bullet) := H^\bullet\big(\boldsymbol{F}(\mathcal{I}^\bullet)\big)$ where $\mathcal{I}^\bullet$ is a $\boldsymbol{I}(X)$-resolution of $\mathcal{A}^\bullet$. □

**Proposition 4.4.** *Suppose $\mathcal{A}^\bullet, \mathcal{B}^\bullet \in C^+(X)$ and $\mathcal{A}^\bullet \overset{\alpha}{\rightsquigarrow} \mathcal{I}^\bullet_A$, $\mathcal{B}^\bullet \overset{\beta}{\rightsquigarrow} \mathcal{I}^\bullet_B$ are $\boldsymbol{I}(X)$-resolutions. Then any morphism of complexes $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ defines a unique equivalence class of right roofs from $\boldsymbol{F}(\mathcal{I}^\bullet_A)$ to $\boldsymbol{F}(\mathcal{I}^\bullet_B)$ called the* equivalence class *of $(f, \alpha, \beta)$*

*Proof.* Consider the left roof

$$\begin{array}{ccccc} & & \mathcal{A}^\bullet & & \\ & \overset{\alpha}{\swarrow} & & \overset{\beta\circ f}{\searrow} & \\ \mathcal{I}^\bullet_A & & & & \mathcal{I}^\bullet_B \end{array} \tag{4.1}$$

Choose any equivalent right roof

$$\begin{array}{ccccc} \mathcal{I}^\bullet_A & & & & \mathcal{I}^\bullet_B\ , \quad \mathcal{I}^\bullet \in C^+_{\boldsymbol{I}}(X). \\ & \overset{\phi}{\searrow} & & \overset{s}{\swarrow} & \\ & & \mathcal{I}^\bullet & & \end{array}$$

Using Proposition 4.1(c) we get a roof

$$\begin{array}{ccccc} \boldsymbol{F}(\mathcal{I}^\bullet_A) & & & & \boldsymbol{F}(\mathcal{I}^\bullet_B)\ . \\ & \searrow & & \swarrow & \\ & & \boldsymbol{F}(\mathcal{I}^\bullet) & & \end{array}$$

If

$$\begin{array}{ccccc} \mathcal{I}^\bullet_A & & & & \mathcal{I}^\bullet_B\ , \quad \mathcal{I}^\bullet \in C^+_{\boldsymbol{I}}(X). \\ & \overset{\psi}{\searrow} & & \overset{t}{\swarrow} & \\ & & \mathcal{J}^\bullet & & \end{array}$$

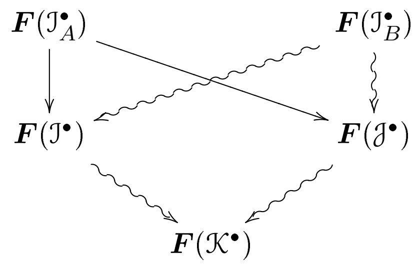
is another right roof equivalent to the left roof (4.1), then the roofs

$$\mathcal{I}^\bullet_A \overset{\phi}{\to} \mathcal{I}^\bullet \overset{s}{\leftsquigarrow} \mathcal{I}^\bullet_B \quad \text{and} \quad \mathcal{I}^\bullet_A \overset{\psi}{\to} \mathcal{I}^\bullet \overset{t}{\leftsquigarrow} \mathcal{I}^\bullet_B$$

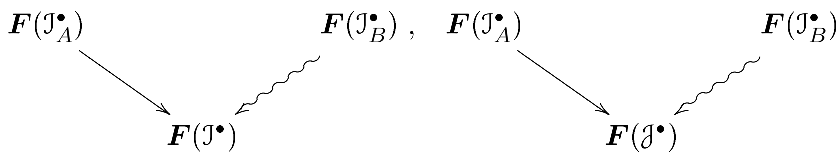
are equivalent and thus, according to Proposition 3.3, they are also $\boldsymbol{I}(X)$-equivalent. Hence there exists a $\boldsymbol{I}(X)$-roof $\mathcal{I}^\bullet \overset{p}{\rightsquigarrow} \mathcal{K}^\bullet \overset{q}{\leftsquigarrow} \mathcal{J}^\bullet$ such that the diagram below is homotopy commutative

$$\begin{array}{ccccc} \mathcal{I}^\bullet_A & & & & \mathcal{I}^\bullet_B \\ \downarrow & & \times & & \downarrow \\ \mathcal{I}^\bullet & & & & \mathcal{J}^\bullet \\ & \searrow & & \swarrow & \\ & & \mathcal{K}^\bullet & & \end{array}$$

We get a homotopy commutative diagram of complexes of sheaves on $Y$



Above the qis's are transformed into qis's by $\boldsymbol{F}$ due to Proposition 4.1(c). This proves that the two right roofs



are equivalent. □

Let $\mathcal{A}^\bullet \overset{\alpha}{\rightsquigarrow} \mathcal{I}^\bullet_A$, $\mathcal{B}^\bullet \overset{\beta}{\rightsquigarrow} \mathcal{I}^\bullet_B$, and $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ as in the above proposition. For any right roof

$$\boldsymbol{F}(\mathcal{I}^\bullet_A) \overset{\phi}{\to} \mathcal{C}^\bullet \overset{s}{\leftsquigarrow} \boldsymbol{F}(\mathcal{I}^\bullet_B) \tag{4.2}$$

in the equivalence class of $(f, \alpha, \beta)$ we get a morphism in cohomology

$$s_*^{-1} \circ \phi_* : H^\bullet\big(\boldsymbol{F}(\mathcal{I}^\bullet_A)\big) \to H^\bullet\big(\boldsymbol{F}(\mathcal{I}^\bullet_B)\big)$$

This morphism depends only on the equivalence class of the roof (4.2), and thus depends only on $(f, \alpha, \beta)$. We will denote it by $\boldsymbol{F}_{\beta\alpha}(f)$

Observe that if $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ and $g : \mathcal{B}^\bullet \to \mathcal{C}^\bullet$ are morphisms of complexes and $\mathcal{A}^\bullet \overset{\alpha}{\rightsquigarrow} \mathcal{I}^\bullet_A$, $\mathcal{B}^\bullet \overset{\beta}{\rightsquigarrow} \mathcal{I}^\bullet_B$, $\mathcal{C}^\bullet \overset{\gamma}{\rightsquigarrow} \mathcal{I}^\bullet_C$ are $\boldsymbol{I}(X)$-resolutions, then

$$\boldsymbol{F}_{\gamma\alpha}(g \circ f) = \boldsymbol{F}_{\gamma\beta}(g) \circ \boldsymbol{F}_{\beta\alpha}(f).$$

## 5. Examples

**Example 5.1.** Note that when $Y$ consists of a single point $*$ then the category $\boldsymbol{Sh}_R(*)$ can be identified with the category ${}_R\mathrm{Mod}$ of left $R$-modules. The sections functor

$$\Gamma : \boldsymbol{Sh}_R(X) \to {}_R\mathrm{Mod},\ \ \mathcal{S} \mapsto \Gamma(X, \mathcal{S})$$

can be viewed as a left-exact Abelian functor $\boldsymbol{Sh}_R(X) \to \boldsymbol{Sh}_R(*)$.

The functor

$$\Gamma_c : \boldsymbol{Sh}_R(X) \to {}_R\mathrm{Mod},\ \ \mathcal{S} \mapsto \Gamma_c(X, \mathcal{S}) = \text{section of } \mathcal{S} \text{ with compact support}$$

is also a left exact Abelian functor $\boldsymbol{Sh}_R(X) \to \boldsymbol{Sh}_R(*)$. □

**Example 5.2.** Any continuous map $f : X \to Y$ defines a functor $f_* : \boldsymbol{Sh}_R(X) \to \boldsymbol{Sh}_R(Y)$

$$\boldsymbol{Sh}_R(X) \ni \mathcal{A} \mapsto f_*\mathcal{A} := \text{the sheaf associated to the presheaf } f_\#\mathcal{A},$$

where

$$\Gamma(U, f_\#\mathcal{A}) := \Gamma\big(f^{-1}(U), \mathcal{A}\big),\ \ \forall U \subset X,\ \ U \text{ open}.$$

The functor $f_* : \boldsymbol{Sh}_R(X) \to \boldsymbol{Sh}_R(Y)$ is left exact and Abelian. Note that when $Y = \{*\}$ and $f : X \to Y$ is the collapse map $f = c$ then we can identify the functor $c_*$ with the functor $\Gamma$ in the previous example. □

**Example 5.3.** For any continuous map $g : Y \to X$ we get a functor $g^{-1} : \boldsymbol{Sh}_R(X) \to \boldsymbol{Sh}_R(Y)$, where for any $\mathcal{S} \in \boldsymbol{Sh}_R(X)$, $g^{-1}\mathcal{S}$ denotes the sheaf associated to the presheaf given by

$$Y \supset U \mapsto = \varinjlim_{V \supset g(U)} \Gamma(V, \mathcal{S}), \;\; U \text{ open subset of } Y,$$

where the above inductive limit is taken over all open sets $V$ containing $g(U)$. The functor $g^{-1}$ is exact in the sense that if $0 \to \mathcal{S}_0 \to \mathcal{S}_1 \to \mathcal{S}_1 \to 0$ is a short exact sequence of sheaves on $X$ then $0 \to g^{-1}\mathcal{S}_0 \to g^{-1}\mathcal{S}_1 \to g^{-1}\mathcal{S}_1 \to 0$ is a short exact sequence of sheaves on $Y$.

Observe that if $S \subset X$ is a closed subset of $X$, $i : S \to X$ denotes the natural inclusion, and $\mathcal{F} \in \boldsymbol{Sh}_R(X)$ then we define the space of sections of $\mathcal{F}$ on $S$ to be

$$\Gamma(S, \mathcal{F}) := \Gamma(S, i^{-1}\mathcal{F}).$$

Note that we have a natural map

$$\varinjlim_{V \supset S} \Gamma(V, \mathcal{F}) \to \Gamma(S, \mathcal{F}).$$

If $X$ is paracompact then the above map is an isomorphism. □

**Example 5.4.** (a) For any closed subset $S \subset X$ we denote by $i : S \to X$ the canonical inclusion map and we define the *adjunction functor* $\boldsymbol{a}_S : \boldsymbol{Sh}_R(X) \to \boldsymbol{Sh}_R(X)$ by the equality $\boldsymbol{a}_S = i_* i^{-1}$. The adjunction functor is left exact. □

**Definition 5.5.** (a) A sheaf $\mathcal{F} \in \boldsymbol{Sh}_R(X)$ is called *flabby* if for any open subset $U \subset X$ the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is surjective.
(b) A sheaf $\mathcal{S} \in \boldsymbol{Sh}_R(X)$ is called *soft* if for every closed subset $C \subset X$ the natural map $\Gamma(X, \mathcal{S}) \to \Gamma(C, \mathcal{S})$ is surjective.
(c)A sheaf $\mathcal{S} \in \boldsymbol{Sh}_R(X)$ is called *c-soft* if for every compact subset $K \subset X$ the natural map $\Gamma(X, \mathcal{S}) \to \Gamma(K, \mathcal{S})$ is surjective. □

*Remark* 5.6. If $X$ is a paracompact space then a sheaf $\mathcal{F} \in \boldsymbol{Sh}_R(X)$ is soft if for any closed set $C$, any open set $U \supset C$ and any section $u \in \Gamma(U, \mathcal{F})$ there exists an open subset $C \subset V \subset U$ and a section $v \in \Gamma(V, \mathcal{S})$ such that the restriction of $u$ to $V$ coincides with $v$. For example if $X$ is a smooth (paracompact) manifold and $E$ is a smooth vector bundle on $X$ then the sheaf $C^\infty_E$ of smooth sections of $E$ is soft. In particular, the sheaf $\Omega^p_X$ of smooth forms of degree $p$ is a soft sheaf. Let us observe that on a paracompact space any flabby sheaf is automatically soft and $c$-soft. □

**Proposition 5.7.** *(a) On any topological space $X$ the subcategory of flabby sheaves of $R$-modules is a generating subcategory of $\boldsymbol{Sh}_R(X)$.*
*(b) On any paracompact space $X$ the category of soft sheaves of $R$-modules is a generating subcategory of $\boldsymbol{Sh}_R(X)$.*

*Proof.* (a) The conditions (1),(2),(3) in Definition 3.1 are obvious. To deal with the condition (4) we need the following result. See [2, Chap. IV] for a proof.

**Lemma 5.8.** *Consider a short exact sequence of sheaves*

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

*such that $\mathcal{F}_0$ is flabby. Then for any open subset $U \subset X$ the sequence*

$$0 \to \Gamma(U, \mathcal{F}_0) \to \Gamma(U, \mathcal{F}_1) \to \Gamma(U, \mathcal{F}_2) \to 0$$

*is exact.*

If now we have a short exact sequence $0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$ such that $\mathcal{F}_0$ and $\mathcal{F}_1$ are flabby, then for every open subset $U \subset X$ we obtain a commutative diagram in which the rows are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X,\mathcal{F}_0) & \longrightarrow & \Gamma(X,\mathcal{F}_1) & \longrightarrow & \Gamma(X,\mathcal{F}_2) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle r_0} & & \Big\downarrow{\scriptstyle r_1} & & \Big\downarrow{\scriptstyle r_2} & & \\ 0 & \longrightarrow & \Gamma(U,\mathcal{F}_0) & \longrightarrow & \Gamma(U,\mathcal{F}_1) & \longrightarrow & \Gamma(U,\mathcal{F}_2) & \longrightarrow & 0 \end{array}$$

Since $\mathcal{F}_0$ and $\mathcal{F}_2$ are flabby we deduce that the morphisms $r_0$, $r_1$ are surjective. This forces $r_2$ to be surjective so that $\mathcal{F}_2$ is also

To conclude the proof of part (a) it suffices to show that any sheaf $\mathcal{S}$ injects in some flabby sheaf $\widetilde{\mathcal{S}}$. We define sheaf $\widetilde{\mathcal{S}}$ to consist of the discontinuous sections of $\mathcal{S}$, i.e.,

$$\Gamma(U,\widetilde{\mathcal{S}}) = \Big\{\, f : U \coprod_{u\in U} \mathcal{S}_0;\quad f(u) \in \mathcal{S}_u \,\Big\}.$$

It is not difficult to see that $\widetilde{\mathcal{S}}$ is a flabby sheaf and $\mathcal{S}$ is a subsheaf of $\widetilde{\mathcal{S}}$.

(b) The conditions (1),(2),(3) in Definition 3.1 are obvious. The condition (5) is satisfies since any sheaf is a subsheaf of a flabby sheaf, and thus a subsheaf of a soft sheaf. To deal with the condition (4) we need the following result. See [2, Chap. IV] for a proof.

**Lemma 5.9.** *Consider a short exact sequence of sheaves*

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

*such that $\mathcal{F}_0$ is soft. Then for any closed subset $C \subset X$ the sequence*

$$0 \to \Gamma(C,\mathcal{F}_0) \to \Gamma(C,\mathcal{F}_1) \to \Gamma(C,\mathcal{F}_2) \to 0$$

*is exact.*

We now conclude as in part (a). □

**Proposition 5.10.** *Suppose $f : X \to Y$ is a continuous map, $X$ paracompact, $S$ is a closed subset of $X$, and $\boldsymbol{I}(X)$ denotes either the category of flabby sheaves or the category of soft sheaves. Then $\boldsymbol{I}(X)$ is both an $\boldsymbol{a}_S$ and $f_*$-acyclic generating subcategory of $\boldsymbol{Sh}_R(X)$.*

**Definition 5.11.** For any continuous map $f : X \to Y$, and any sheaf $\mathcal{S} \in \boldsymbol{Sh}_R(X)$ we will refer to the sheaves $R^\bullet f_*(\mathcal{S}) \in \boldsymbol{Sh}_R(Y)$ as the higher direct images of $\mathcal{S}$ via $f$. When $f$ is the collapse map $f = c : X \to \{*\}$ we set

$$H^\bullet(\mathcal{S}) := R^\bullet c_*(\mathcal{S}),$$

and we will refer to these modules as the *cohomology of* $\mathcal{S}$. □

The cohomology of a sheaf $\mathcal{S} \in \boldsymbol{Sh}_R(X)$ on a paracompact space $X$ can be computed a follows. Choose a soft (or flabby) resolution of $\mathcal{S}$, i.e., a complex of soft sheaves

$$\mathcal{S}_0 \xrightarrow{d} \mathcal{S}_1 \xrightarrow{d} \cdots$$

and a morphism of sheaves $\mathcal{S} \to \mathcal{S}_0$ such that the resulting long sequence of sheaves

$$0 \to \mathcal{S} \to \mathcal{S}_0 \xrightarrow{d} \mathcal{S}_1 \xrightarrow{d} \cdots$$

is exact. The the cohomology of $\mathcal{S}$ is the cohomology of the complex of $R$-modules $(\Gamma(X,\mathcal{S}^\bullet), d)$.

**Example 5.12.** Suppose $M$ is a (paracompact) smooth manifold. Consider the constant sheaf $\underline{\mathbb{R}}_M$,

$$\Gamma(U,\underline{\mathbb{R}}_M) = \text{locally constant functions } U \to \mathbb{R}.$$

The Poincaré lemma shows that the DeRham complex of sheaves

$$\Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \cdots$$

is a resolution of $\underline{\mathbb{R}}$, i.e., the sequence of sheaves and morphisms of sheaves

$$0 \to \underline{\mathbb{R}}_M \to \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \cdots$$

is exact. We deduce that the cohomology of the sheaf $\underline{\mathbb{R}}_M$ is isomorphic to the cohomology of the complex $(\Gamma(M, \Omega^\bullet_M), d)$ which is precisely the DeRham cohomology of $M$. □

## References

[1] A. Borel et al: *Algebraic D-modules*, Academic Press, 1987.
[2] J.P. Demailly: *Complex Analytic and Algebraic Geometry* http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf
[3] S.I. Gelfand, Yu.I. Manin: *Methods of Homological Algebra*, 2nd Edition,Springer-Verlag, 2003.