Residues and Hodge theory

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Abstract

We discuss some basic applications of higher dimensional residues as presented in [7] and [8, Chap. V].

Contents

1 Gysin maps and Leray residues: the topological picture

We begin by briefly describing the topological facts underpinning the residue construction. For more details we refer to the original source [9].

Suppose X is a complex manifold, $D \hookrightarrow X$ is a smooth divisor. Denote by $N(D)$ a tubular neighborhood of D in X, by $\sigma : D \to N(D)$ the inclusion as a zero section, and by $\pi : N \to D$ the natural projection. Denote by

$$\langle \bullet, \bullet \rangle : H^q_{DR} \times H_q \to \mathbb{C}$$

the Kronecker pairing (integration). We then have (see [4, §VIII. 12]) the Gysin long exact sequence (we use complex coefficients)

$$\cdots \to H_k(N - D) \stackrel{i_*}{\to} H_k(N) \stackrel{\sigma^!}{\to} H_{k-2}(D) \stackrel{\pi^!}{\to} H_{k-1}(N - D) \to$$

Intuitively, the map $H_k(N) \ni c \mapsto \sigma^!(c) \in H_{k-2}(D)$ is given by the intersection of the k-cycle c with D, $c \mapsto c \cap D$. More rigorously, if we denote by τ_D the Thom class of the normal bundle of $D \hookrightarrow X$, then we can view τ_D as an element of $H^2(N, \partial N)$ and then $\varpi(c)$ is defined by the equality

$$\langle \tau_D \wedge \pi^* \varphi, c \rangle = \langle \varphi, \sigma^!(c) \rangle, \;\; \forall \varphi \in H^{k-2}(D).$$

*Notes for Karen Chandler's *Absolutely fabulous seminar in algebraic geometry.*

The morphism $\pi^!$ is called the *tube map* or the *Leray coboundary*. If $c \in H_m(D)$ is represented by a compact smooth submanifold $M \hookrightarrow D$, we then get by restriction a disk bundle $\pi^{-1}(M) \to M$. Its boundary is a S^1-bundle over M, $S^1 \hookrightarrow \partial\pi^{-1}(M) \to M$ whose total space caries the homology class of $\pi^!(c)$. We will refer to the composition

$$H_k(D) \stackrel{\pi^!}{\to} H_{k+1}(\partial N) \to H_{k+1}(X - D)$$

as the tube maps as well.

Dualizing we get a morphism

$$\pi_! : \mathrm{Hom}(H_{k+1}(X - D), \mathbb{C}) = H^{k+1}_{DR}(X - D) \to \mathrm{Hom}(H_k(D), \mathbb{C}) = H^k_{DR}(D)$$

We will refer to it as the *topological Leray residue* and we will denote it by Res_D.

Suppose more generally that we have smooth divisors $D_1, \cdots, D_m$, $1 \leq m \leq \dim_{\mathbb{C}} X$, intersecting transversally. We set $D = \cup D_i$, $X'_0 = X \setminus D$,

$$X'_1 = D_1 \setminus \bigcup_{1<k\leq m} D_k, \;\; X'_2 = (D_1 \cap D_2) \setminus \bigcup_{2<k\leq m} D_k, \cdots, X'_j = \bigcap_{1\leq i\leq j} D_i \setminus \bigcup_{j<k\leq m} D_k, \cdots .$$

Iterating the above construction we get a sequence

$$H^k_{DR}(X'_0) \stackrel{\mathrm{Res}_{D_1}}{\longrightarrow} H^{k-1}_{DR}(X'_1) \stackrel{\mathrm{Res}_{D_2}}{\longrightarrow} H^{k-2}_{DR}(X'_2) \longrightarrow \cdots \stackrel{\mathrm{Res}_{D_m}}{\longrightarrow} H^{k-m}_{DR}(D_1 \cap \cdots \cap D_m)$$

We would like to have a better understanding of these maps at the level of differential forms, and in particular we would like to understand how these maps interact with the Hodge structures on the various DeRham cohomology groups. We will study two extreme cases: $m = \dim_{\mathbb{C}} X$ and $m = 1$. The first case leads to the *Grothendieck residue* map and an explicit description of the Grothendieck-Serre duality for zero dimensional schemes, while the second case leads to a classical construction going back to Poincaré usually referred to as the *Poincaré residue*.

2 Residues: local aspects

§2.1 The Grothendieck residue We denote by $\mathcal{O} = \mathcal{O}_{\mathbb{C}^n}$ the sheaf of holomorphic functions in n variables $z^1, \cdots, z^n$ on $\mathbb{C}^n$. We denote by $\mathcal{O}_0$ the stalk at $0 \in \mathbb{C}^n$. Fix an open polydisk U centered at $0 \in \mathbb{C}^n$. For every $\vec{F} = (f_1, \cdots, f_n) \in \mathcal{O}(U)^n$ we se

$$D_i = D_i(\vec{F}) = \{\vec{z} \in U;\;\; f_i\vec{z}) = 0\}, D = D_1 \cup \cdots \cup D_n \;\; \vec{F}^{-1}(0) := \bigcap D_i = \{\vec{z} \in U;\;\; f_i(\vec{z}) = 0,\;\; \forall i\}.$$

$\vec{F}$ defines a holomorphic map $\vec{F} : U \to \mathbb{C}^n$ and we denote its Jacobian determinant by $J_{\vec{F}}$ or $\frac{\partial \vec{F}}{\partial \vec{z}}$.

Let $g \in \mathcal{O}(U)$, and $\vec{F} \in \mathcal{O}^n$ such that $\vec{F}^{-1}(0) = \{0\}$. We define

$$\omega = \omega_{\vec{z}}(g, \vec{F}) = \frac{g}{f_1 \cdots f_n} dz^1 \wedge \cdots \wedge dz^n.$$

Observe that ω is a meromorphic n-form on U such that $\omega\,|_{U\setminus D}$ is holomorphic. For dimension reasons we have

$$\partial\omega = 0 \Longrightarrow d\omega = (\bar{\partial} + \partial)\omega = 0$$

so that ω is closed. We denote by $[\omega]$ its cohomology class in $H^n_{DR}(U \setminus D)$. For very small positive real numbers δ_i define

$$\Gamma = \Gamma_{\vec{\delta}}(g, \vec{F}) := \{\vec{z};\ \ |f_i(\vec{z})| = \delta_i,\ \ 1 \le i \le n\},$$

Γ is a compact, real n-dimensional submanifold of $U \setminus D$ oriented by the n-form

$$d \arg f_1 \wedge \cdots \wedge d \arg f_n.$$

It determines a homology class $[\Gamma] \in H_n(U \setminus D)$ which is independent of the radii δ_i.

The *Grothendieck residue* of ω at 0 is by definition the complex number

$$\operatorname{Res}_0 \omega := \left(\frac{1}{2\pi \mathbf{i}}\right)^n \int_\Gamma \omega = \left(\frac{1}{2\pi \mathbf{i}}\right)^n \langle [\omega], [\Gamma] \rangle.$$

Example 2.1. (a) Suppose we holomorphically change the coordinates near 0, $\vec{z} = \vec{z}(\vec{u})$. Then ω changes to

$$\omega_{\vec{z}} = \frac{\partial \vec{z}}{\partial \vec{u}} \cdot \omega_{\vec{u}}.$$

In particular, if $\vec{F}$ is biholomorphic, then we use F as new coordinates then

$$\omega_{\vec{z}} = \frac{\partial \vec{z}}{\partial \vec{F}} \cdot g(\vec{z}) \cdot \frac{\partial f_1}{f_1} \wedge \cdots \wedge \frac{\partial f_n}{f_n}$$

and we deduce

$$\operatorname{Res}_0 = \frac{g(0)}{J_{\vec{F}}(0)}.$$

In this case the cycle Γ is an n-torus.

(b) Suppose $f_i(z) = (z^i)^{a_i}$, and $g(z) = J_{\vec{F}}(z) = \prod_{j=1}^n a_j (z^j)^{a_j - 1}$. Then

$$\operatorname{Res}_0 \omega(g, \vec{F}) = \operatorname{Res}_0 \Big(\frac{J_{\vec{F}}(z)}{f_1 \cdots f_n} dz^1 \wedge \cdots \wedge dz^n \Big) \prod_{j=1}^n \frac{1}{2\pi \mathbf{i}} \int_{|z_j| = \varepsilon} \frac{a_j dz^j}{z^j} = \prod_{k=1}^n a_k.$$

□

The residue depends only on the germs of g and $\vec{F}$ at 0. It depends linearly on g. Denote by $I_{\vec{F}}$ the ideal in $\mathcal{O}_0$ generated by the germs $f_1, \cdots, f_n$. Observe that

$$g \in I_{\vec{F}} \Longrightarrow \operatorname{Res}_0 \omega = 0.$$

This is clear when $f = h \cdot f_i$ for some i, and the general case can be reduced to this one by linearity.

Set $U_i := U \setminus D_i$. Then the collection (U_i) is an open cover of $U^* := U \setminus 0$.

Proposition 2.2. *There exists a closed form $\eta_\omega \in \Omega^{2n-1}(U^*)$ depending smoothly on g and $\vec{F}$ such that*

$$\operatorname{Res}_0 \omega = \int_{S^{2n-1}(r)} \eta_\omega$$

where $S^{2n-1}(r) \subset U^*$ is a sphere of small radius r centered at 0.

Proof We prove this only in the case $n = 2$. We have an open cover

$$U^* = U_1 \cup U_2.$$

Set $U_{12} = U_1 \cap U_2 = U \setminus D$. Denote by j_α the inclusion $U_\alpha \hookrightarrow U^*$, and by i_α the inclusion $U_{12} \hookrightarrow U_\alpha$. Denote by $\mathcal{A}^{p,q}(V)$ the space of smooth (p, q)-forms on an open subset $D \hookrightarrow \mathbb{C}^n$.

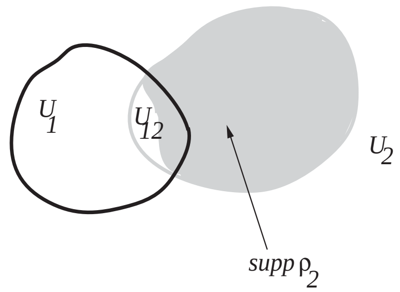

Figure 1: $\rho_2\omega$ *extends by zero to the rest of* U_1.

Lemma 2.3. *We then have a short exact sequence of cochain complexes*

$$0 \to \Big(\mathcal{A}^{n,*}(U^*), \bar\partial\Big) \stackrel{f=j_1^*\oplus j_2^*}{\longrightarrow} \Big(\mathcal{A}^{n,*}(U_1), \bar\partial\Big) \oplus \Big(\mathcal{A}^{n,*}(U_2), \bar\partial\Big) \stackrel{g=i_1^*-i_2^*}{\longrightarrow} \Big(\mathcal{A}^{n,*}(U_{12}), \bar\partial\Big) \to 0 \qquad (2.1)$$

called the Mayer-Vietoris sequence of the cover $\{U_1, U_2\}$.

Proof of the lemma The only non-obvious part is the surjectivity of g. Let $\eta \in \mathcal{A}^{n,k}(U_{12})$. We want to show that there exist $\eta_\alpha \in \mathcal{A}^{n,k}(U_\alpha)$ are such that $g(\eta_1 \oplus \eta_2) = \eta$, i.e.

$$\eta_1\,|_{U_{12}} - \eta_2\,|_{U_{12}} = \eta$$

Choose a smooth partition of unity $\{\rho_1, \rho_2\}$ subordinated to $\{U_1, U_2\}$, i.e. $0 \le \rho_\alpha \le 1$, $\operatorname{supp}\rho_\alpha \subset U_\alpha$, and $\rho_1 + \rho_2 = 1$.

Note that we can extend $\rho_2\eta$ to a smooth form on U_1 by setting $\rho_2\eta = 0$ on $U_1 \setminus U_{12}$ (see Figure 1. Similarly, we can extend $\rho_1\eta$ to a form on U_2. Clearly

$$g(\rho_2\eta, -\rho_1\eta) = (\rho_2\eta)\,|_{U_{12}} - (-\rho_1\eta)\,|_{U_{12}} = \eta.$$

□

From the above short exact sequence we obtain the long exact sequence

$$0 \to H^{n,0}(U^*) \to H^{n,0}(U_1) \oplus H^{n,0}(U_2) \to H^{n,0}(U_{12}) \stackrel{\delta}{\to} H^{n,1}(U^*) \to \cdots$$

The meromorphic form ω defines an element $\omega \in H^{n,0}(U_{12})$ and we set

$$\eta_\omega := \delta\omega \in H^{n,1}(U^*).$$

In our case $n = 2$ and $H^{2,1} \cong H^3(U^*)$. Using the partition of unity in the proof of Lemma 2.3 we can be much more explicit about the form of η_ω. More precisely

$$\eta_\omega = C_n \begin{cases} \bar\partial(\rho_2\omega) & \text{on} \quad U_1 \\ -\bar\partial(\rho_1\omega) & \text{on} \quad U_2 \end{cases} = C_n \begin{cases} d(\rho_2\omega) & \text{on} \quad U_1 \\ -d(\rho_1\omega) & \text{on} \quad U_2 \end{cases},$$

where C_n is a constant to be determined a bit latter. Observe that the above definition correctly defines a form on U^* since on the overlap U_{12} we have

$$d(\rho_2\omega) - \big(-d(\rho_1\omega)\big) = d(\rho_2\omega) + d(\rho_1\omega) = d\Big((\rho_1+\rho_2)\omega\Big) = d\omega = 0.$$

In our very concrete situation we can take

$$\rho_1 = \frac{|f_1|^2}{\|\vec{F}\|^2}, \ \ \rho_2 = \frac{|f_2|^2}{\|\vec{F}\|^2}, \ \ \|\vec{F}\|^2 := |f_1|^2 + |f_2|^2$$

and we have

$$\rho_1\omega = \frac{\bar{f}_1 g}{\|\vec{F}\|^2 f_2} dz_1 \wedge dz_2, \ \ \rho_2\omega = \frac{\bar{f}_2 g}{\|\vec{F}\|^2 f_1} dz_1 \wedge dz_2.$$

The cycle $\Gamma = \{|f_i|^2 = \varepsilon^2, \ \ i = 1, 2\}$ lies on the hypersurface

$$\Sigma := \{\|\vec{F}\|^2 = 2\varepsilon^2\}, \ \ \dim_{\mathbb{R}} \Sigma = 3.$$

Moreover, Γ divides Σ into two parts

$$\Sigma_1 = \{\vec{z} \in \Sigma; \ |f_2| \leq \varepsilon\}, \ \ \Sigma_2 = \{\vec{z} \in \Sigma; \ |f_1| \leq \varepsilon\}, \ \ \partial\Sigma_1 = -\partial\Sigma_2 = \Gamma$$

Note that

$$\eta_\omega |_{\Sigma_1} = C_n d(\rho_2\omega), \ \ \eta_\omega |_{\Sigma_2} = -C_n d(\rho_1\omega)$$

Hence

$$\int_\Sigma \eta_\omega = C_n \int_{\Sigma_1} d(\rho_2\omega) - C_n \int_{\Sigma_2} d(\rho_1\omega)$$

(use Stokes' formula)

$$C_n \int_\Gamma (\rho_2\omega + \rho_1\omega) = \int_\Gamma \omega.$$

Hence if we take $C_n = (2\pi\mathbf{i})^n$ we have

$$\int_\Sigma \eta_\omega = \mathrm{Res}_0\, \omega.$$

Finally observe that Σ is homologous in U^* to a small sphere centered at 0.

We can give an even more explicit description of η_ω. As we have noticed on the overlap U_{12} we have

$$\bar{\partial}(\rho_2\omega) = -\bar{\partial}(\rho_1\omega)$$

In fact both forms above are *globally defined*. Thus (recalling that $n = 2$)

$$\eta_\omega = \frac{C_2}{2}\Big(\bar{\partial}(\rho_2\omega) - \bar{\partial}(\rho_1\omega)\Big).$$

A simple computation shows that

$$\eta_\omega = \frac{C_2 g}{2 f_1 f_2} \sum_\alpha (-1)^\alpha \Big(\frac{\bar{\partial}|f_\alpha|^2 \cdot \|\vec{F}\|^2 - |f_\alpha|^2 \bar{\partial}\|\vec{F}\|^2}{\|\vec{F}\|^4}\Big) dz_1 \wedge dz_2$$

$$= \frac{C_2 g}{2 f_1 f_2 \|\vec{F}\|^4}\Big(\|\vec{F}\|^2 f_2 d\bar{f}_2 - |f_2|^2(f_1 d\bar{f}_1 + f_2 d\bar{f}_2) - \|\vec{F}\|^2 f_1 d\bar{f}_1 + |f_1|^2(f_1 d\bar{f}_1 + f_2 d\bar{f}_2)\Big)$$

$$= \frac{C_2 g}{2 f_1 f_2 \|\vec{F}\|^4}\Big((f_2 d\bar{f}_2 - f_1 d\bar{f}_1)(|f_1|^2 + |f_2|^2) + (|f_1|^2 - |f_2|^2)(f_1 d\bar{f}_1 + f_2 d\bar{f}_2)\Big)$$

$$= \frac{C_2 g}{\|\vec{F}\|^4 f_1 f_2}\Big(|f_1|^2 f_2 d\bar{f}_2 - |f_2|^2 f_1 d\bar{f}_1\Big) = \frac{C_2 g}{\|\vec{F}\|^4}\Big(\bar{f}_1 d\bar{f}_2 - \bar{f}_2 d\bar{f}_1\Big).$$

The continuous dependence of η_ω on g and $\vec{F}$ is now clear.

□

Remark 2.4. Let us say a few things about the proof of Proposition 2.2 in the case $n > 2$. This requires a detour in the world of sheaf cohomology and spectral sequences.

Suppose X is a locally compact space. We will be interested mostly in the case when X is an n-dimensional complex manifold. A sheaf of Abelian groups $\mathcal{F}$ on X is called *soft* if for every closed set $C \subset X$ the restriction map $\mathcal{F}(X) \to \mathcal{F}(C)$ is onto. $\mathcal{F}$ is called *fine* if the sheaf $Hom_{\mathbb{Z}}(\mathcal{F}, \mathcal{F})$ is soft. Intuitively, the fine sheaves are the sheaves for which we can use the partition of unity trick. More precisely, given an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of the locally compact space X, a partition of unity subordinated to this cover is a collection of endomorphisms $\rho_\alpha \in \operatorname{Hom}(\mathcal{F}(X), \mathcal{F}(X)$, $\alpha \in A$ such that

$$\operatorname{supp}(\rho_\alpha) \subset U_\alpha, \ \ \forall \alpha, \ \ \sum_\alpha \rho_\alpha = \mathbf{1}.$$

We have the following implications

$$\mathcal{F} \ \text{fine} \Longrightarrow \mathcal{F} \ \text{soft} \Longrightarrow \mathcal{F} \ \text{acyclic}.$$

For example the sheaves of *smooth* sections of a smooth vector bundle over a smooth manifold are all fine.

A *resolution* of a sheaf $\mathcal{F}$ is an exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{S}^0 \stackrel{d_0}{\to} \mathcal{S}^1 \to \cdots$$

We will denote this by $\mathcal{F} \hookrightarrow (\mathcal{S}^*, d)$. The resolution is called *fine* if all the sheaves $\mathcal{S}_i$ are fine. For example, if X is a complex n-dimensional manifold, $\mathcal{F} = \Omega^{p,0}$ is the sheaf of *holomorphic* p-forms, $\mathcal{A}^{p,q}$ is the sheaf of *smooth* (p, q)-forms, then $\mathcal{A}^{p,q}$ is a fine sheaf, and Dolbeault lemma shows that

$$0 \to \Omega^{p,0} \hookrightarrow \mathcal{A}^{p,0} \stackrel{\bar{\partial}}{\to} \mathcal{A}^{p,1} \stackrel{\bar{\partial}}{\to} \mathcal{A}^{p,2} \to \cdots$$

is a fine resolution of $\Omega^{p,0}$. We have the following fundamental result, [6, Thm. 4.7.1].

Theorem 2.5 (Generalized DeRham theorem). *If $\mathcal{F} \hookrightarrow (\mathcal{S}^*, d)$ is a fine resolution of $\mathcal{F}$ then $H^*(X, \mathcal{F})$, the cohomology $\mathcal{F}$, is isomorphic to the cohomology of the co-chain complex*

$$\mathcal{S}^0(X) \stackrel{d_0}{\to} \mathcal{S}^1(X) \stackrel{d_1}{\to} \mathcal{S}^2(X) \stackrel{d_1}{\to} \cdots$$

Suppose $\mathcal{F} \hookrightarrow (\mathcal{S}^*, d)$ is a fine resolution of $\mathcal{F}$ and $\mathcal{U} := (U_\alpha)_{\alpha \in A}$ is an open cover of X, where A is a *linearly ordered set*. The *nerve* of the cover $\mathcal{U}$ consists of the finite subsets $F \subset A$ such that

$$U_F := \bigcap_{\alpha \in F} U_\alpha \neq \emptyset.$$

We denote by $\mathcal{N} = \mathcal{N}(\mathcal{U})$ the nerve of the cover. We can associate a simplicial complex to the nerve, with one d-dimensional simplex for each $F \in \mathcal{N}$ such that $|F| = d + 1$. We will denote it by $\Delta(\mathcal{U})$. For example, the nerve of the cover in Figure 1 consists of $\{1\}, \{2\}$ and $\{1, 2\}$. The associated simplicial complex is depicted in Figure 2. For each sheaf $\mathcal{S}$ on X we set

$$C^p(\mathcal{U}, \mathcal{S}) = \prod_{|F|=p+1} \mathcal{S}(U_F)$$

We identify the elements of $C^p(\mathcal{U}, \mathcal{S})$ with families of section $(s_F)_{F \subset A}$, $|F| = p + 1$, $s_F \in \mathcal{S}(U_F)$. Since $\mathcal{F}$ is a *sheaf* we have a natural inclusion

$$i : \mathcal{S}(X) \to C^0(\mathcal{U}, \mathcal{S}), \ \ \mathcal{S}(X) \ni u \mapsto \prod_{\alpha \in A} u|_{U_\alpha}.$$

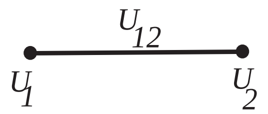

Figure 2: *The simplicial complex associated to the cover in Figure 1.*

For the cover in Figure 1 and the sheaf $\mathcal{S} = \Omega^{n,0}$ we have

$$C^0(\mathcal{U}, \mathcal{S}) = \Omega^{n,0}(U_1) \oplus \Omega^{n,0}(U_2), \;\; C^1(\mathcal{U}, \mathcal{S}) = \Omega^{n,0}(U_{12}).$$

For each $p \geq 0$ we have a *Čech coboundary map*

$$\delta : C^p(\mathcal{U}, \mathcal{S}) \to C^{p+1}(\mathcal{U}, \mathcal{S}), \;\; \delta s = \prod_{|T|=p+2} (\delta s)_T \in \prod_{|T|=p+2} \mathcal{S}(U_T) = C^{p+1}(\mathcal{U}, \mathcal{S})$$

defined as follows. Let $T = \{t_0, t_1, \cdots, t_{p+1}\} \in \mathcal{N}$, $t_0 < t_1 < \cdots < t_{p+1}$. Then for every $f \in C^p(\mathcal{U}, \mathcal{S})$ we define

$$(\delta s)_T = \sum_{i=0}^{p+1} (-1)^i \big(s_{T-\{t_i\}}\big) \,|_{U_T} \;.$$

One can verify that the resulting sequence

$$0 \to \mathcal{S}(X) \to C^0(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{S}) \xrightarrow{\delta} \cdots \tag{2.2}$$

is a cochain complex. It is called the *augmented Čech complex* of the cover $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$. When $\mathcal{S}$ is the constant sheaf $\underline{\mathbb{Z}}$ then the resulting Čech complex coincides with the augmented simplicial cochain complex determined by the simplicial complex $\Delta(\mathcal{U})$.

For example, $\delta : \Omega^{n,0}(U_1) \oplus \Omega^{n,0}(U_2) \to \Omega^{n,0}(U_{12})$ is given by

$$\delta(\omega_1, \omega_2) = \omega_2 \,|_{U_{12}} - \omega_1 \,|_{U_{12}} \;.$$

We see that the sequence

$$0 \to \mathcal{A}^{n,q}(U^*) \to \Omega^{n,0}(U_1) \oplus \mathcal{A}^{n,q}(U_2) \xrightarrow{\delta} \mathcal{A}^{n,q}(U_{12})$$

is precisely the Mayer-Vietoris sequence (2.1). The next result generalizes Lemma 2.3

Lemma 2.6. *If $\mathcal{S}$ is a fine sheaf on a locally compact space X and $\mathcal{U}$ is an open cover then the Čech complex is acyclic. Moreover, any partition of unity subordinated to this cover canonically determines a cochain contraction,*

$$\boldsymbol{k} : C^p(\mathcal{U}, \mathcal{S}) \to C^{p-1}(\mathcal{U}, \mathcal{S}), \;\; \boldsymbol{k}\delta + \delta\boldsymbol{k} = \mathbf{1}.$$

For a proof of this result we refer to [1, §8] and [6, §5.2].

Suppose now that $\mathcal{F}$ is a sheaf on a locally compact space and $\mathcal{F} \hookrightarrow (\mathcal{S}^*, d)$ is a fine resolution. Set

$$K^{p,q} := C^q(\mathcal{U}, \mathcal{S}^p),$$

The sheaf morphisms $d_p : \mathcal{S}^p \to \mathcal{S}^{p+1}$ induce morphisms

$$d_I = (-1)^p d_p : K^{p,q} = C^q(\mathcal{U}, \mathcal{S}^p) \to C^q(\mathcal{U}, \mathcal{S}^{p+1}) = K^{p+1,q},$$

while the Čech coboundary operator induces morphisms

$$d_{II} = \delta : K^{p,q} = C^q(\mathcal{U}, \mathcal{S}^p) \to C^{q+1}(\mathcal{U}, \mathcal{S}^p) = K^{p,q+1}$$

such that the diagram below is anti-commutative for any p, q, $d_I D_{II} = -d_{II} d_I$,

$$\begin{array}{ccc} C^{p,q+1} & \xrightarrow{d_I} & K^{p+1,q+1} \\ {\scriptstyle d_{II}}\uparrow & & \uparrow{\scriptstyle d_{II}} \\ C^{p,q} & \xrightarrow{d_I} & K^{p+1,q} \end{array}$$

Now form the total complex (K^*, D)

$$K^m := \bigoplus_{p+q=m} K^{p,q}, \;\; D = d_I + d_{II} : K^m \to K^{m+1}.$$

The anti-commutativity of the above diagram implies that $D^2 = 0$. We have a natural chain morphism induced by restriction

$$r : \mathcal{S}^m(X) \to K^m, \;\; \mathcal{S}^m(X) \to C^0(\mathcal{U}, \mathcal{S}^m) = C^{m,0} \hookrightarrow K^m.$$

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ K^{0,2} & \xrightarrow{d_I} & K^{1,2} & \xrightarrow{d_I} & K^{2,2} & \cdots & \\ \uparrow{\scriptstyle d_{II}} & & \uparrow{\scriptstyle d_{II}} & & \uparrow{\scriptstyle d_{II}} & & \\ K^{0,1} & \xrightarrow{d_I} & K^{1,1} & \xrightarrow{d_I} & K^{2,1} & \cdots & \\ \uparrow{\scriptstyle d_{II}} & & \uparrow{\scriptstyle d_{II}} & & \uparrow{\scriptstyle d_{II}} & & \\ K^{0,0} & \xrightarrow{d_I} & K^{1,0} & \xrightarrow{d_I} & K^{2,0} & \cdots & \\ \updownarrow{\scriptstyle r} & & \updownarrow{\scriptstyle r} & & \updownarrow{\scriptstyle r} & & \\ \mathcal{S}^0(X) & \xrightarrow{d} & \mathcal{S}^1(X) & \xrightarrow{d} & \mathcal{S}^2(X) & \cdots & \end{array} \tag{2.3}$$

Observe that since $(\mathcal{S}^*, d)$ is a fine resolution of $\mathcal{F}$ we deduce from the generalized DeRham theorem that the cohomology of the bottom row is isomorphic to the cohomology $H^*(X, \mathcal{F})$.

Theorem 2.7 (The generalized Mayer-Vietoris principle). *The morphism*

$$r : (\mathcal{S}^*(X), d) \to (K^*, D)$$

is a quasi-isomorphism, i.e. it induces isomorphisms in cohomology

$$H^*(X, \mathcal{F}) \xrightarrow{\cong} H^*(K^*, D).$$

For a proof of this result we refer to [1, §8] and [6, §5.2].

We analyze how this works in the special case when $X = U^*$, the cover $\mathcal{U}$ consists of the open sets $U_i = U \setminus D_i$, $\mathcal{F} = \Omega^{n,0}$, and the resolution is $(\mathcal{S}^q, d) = (\mathcal{A}^{n,q}, \bar{\partial})$. Fix a partition of unity subordinated to this cover and denote by $\boldsymbol{k}$ the corresponding co-chain contractions

$$\boldsymbol{k} = \boldsymbol{k}_p : K^{p,q} = C^q(\mathcal{U}, \mathcal{A}^{n,p}) \to C^{q-1}(\mathcal{U}, \mathcal{A}^{n,p}) = K^{p,q-1}, \;\; \boldsymbol{k} d_{II} + d_{II} \boldsymbol{k} = \mathbf{1}$$

In this case $C^q(\mathfrak{U}, \mathcal{A}^{n,p}) = 0$ for $q \geq n$, and we have

$$K^{p,n-1} = C^{n-1}(\mathfrak{U}, \mathcal{A}^{n,p}) = \mathcal{A}^{n,p}(U \setminus D).$$

The meromorphic form ω defines an element $\omega \in C^{n-1}(\mathfrak{U}, \mathcal{A}^{n,0}) = K^{0,n-1}$ satisfying $d_{II}\omega = 0$ (since $K^{0,n} = 0$ and $d_I\omega = \bar{\partial}\omega = 0$. Hence $D\omega = 0$ so that ω is a D-cocycle and thus defines an element

$$[\omega] = H^{n-1}(K^*, D) \cong H^{n-1}(U^*, \Omega^{n,0}) = H^{n,n-1}(U^*) \cong H^{2n-1}_{DR}(U^*).$$

Thus we can represent $[\omega]$ by a closed $(2n-1)$-form η_ω. To find an explicit representative we need to use the diagram (2.3) and the contractions $\boldsymbol{k}$.

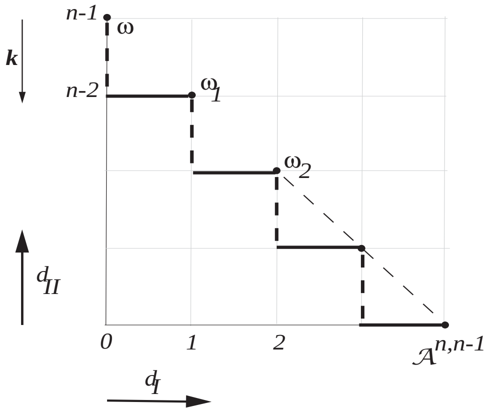

Figure 3: *Tracing a zig-zag.*

Observe that since $\boldsymbol{k}d_{II} + d_{II}\boldsymbol{k} = \mathbf{1}$ we deduce that for any Čech cocycle c we have

$$d_{II}\boldsymbol{k}c = c$$

so that on these cocycles $\boldsymbol{k}$ behaves like d_{II}^{-1}. Thus $d_{II}^{-1}\omega$ is well defined and we set

$$\omega_1 = -d_I d_{II}^{-1}\omega = d_I\boldsymbol{k}\omega \in K^{1,n-2}.$$

Now observe that $d_I\omega_1 = 0$,

$$d_{II}\omega_1 = d_{II}d_I\boldsymbol{k}\omega = -d_I d_{II}\boldsymbol{k}\omega = -d_I\omega = 0$$

so that ω_1 is a D- cocycle as well. It is D-cohomologous to ω since

$$\omega - \omega_1 = D\boldsymbol{k}\omega.$$

We set again

$$\omega_2 = -d_I\boldsymbol{k}\omega_I \in K^{2,n-3}$$

and we get a new D-cocycle cohomologous to ω_1. We iterate this procedure until we get

$$\omega_{n-1} \in K^{n-1,0} = C^0(\mathfrak{U}, \mathcal{A}^{n,n-1}).$$

This is a collection of holomorphic forms $\eta_i \in \Omega^{n,n-1}(U_i)$ which agree on the overlaps. They patch-up to a global holomorphic $(n,n-1)$-form $\eta_\omega \in \Omega^{n,n-1}(U^*)$. Again we can form

$$\eta_\omega := \frac{1}{n}\sum_{i=1}^{n} \eta_i$$

and a simple computation gives

$$\eta_\omega = C_n g(z) dz^1 \wedge \cdots \wedge dz^n \frac{\left(\sum_{j=1}^{n} (-1)^{j-1} \bar{f}_j d\bar{f}_1 \wedge \cdots \wedge \widehat{d\bar{f}_j} \wedge \cdots \wedge d\bar{f}_n\right)}{\|\vec{F}\|^{2n}}. \tag{2.4}$$

□

§2.2 The global residue theorem Suppose now that M is a compact Kähler manifold of complex dimension n and $D_1, \cdots, D_n$ are effective divisors such that

$$P := \bigcap_{j=1}^{n} \operatorname{supp}(D_j)$$

is a finite set. Set $D := D_1 + \cdots + D_n$. Suppose $\tilde{\omega}$ is a holomorphic section of $K_M \otimes \mathcal{O}(D)$. For each $1 \le j \le n$ fix a holomorphic section u_j of $\mathcal{O}(D_j)$ such that $(s_j) = D_j$. Then

$$\omega = \frac{1}{u_1 \cdots u_n} \tilde{\omega} \in \Omega^{n,0}(M \setminus D)$$

is a meromorphic n-form on M with polar set contained in D. For each $p \in P$ *fix* local coordinates (z^i) and holomorphic trivializations of $\boldsymbol{e}_j$ of $[D_j]$ near p. We deduce that near $p \in P$ the form ω has the description.

$$\omega = \frac{g(z)}{f_1 \cdots f_n} dz^1 \wedge \cdots \wedge dz^n, \;\; u_j := f_j \cdot \boldsymbol{e}_j.$$

For each $p \in P$ define $\operatorname{Res}_p \omega$ as in the previous section. Set $U_i = M \setminus D_i$ so that $M \setminus D = \cap U_i$. Denote by $\mathcal{U}$ the open cover $U_1 \cup \cdots \cup U_n$ of $M \setminus P$. Then $\omega \in \Omega^n(M \setminus D)$ is an $(n-1)$-Čech cocycle

$$\omega \in C^{n-1}(\mathcal{U}, \Omega^n_{M\setminus P})$$

so that it defines a homology class

$$[\omega] \in H^{n-1}(M - P, \Omega^n) \cong H^{n,n-1}(M\setminus) \cong H^{2n-1}_{DR}(M \setminus P)$$

This is represented as a $(n,n-1)$-form $\eta_\omega \in \Omega^{2n-1}(M \setminus P)$ and as above we deduce that

$$\operatorname{Res}_p \omega = \int_{\partial B_\varepsilon(p)} \eta_\omega.$$

Theorem 2.8 (Global residue theorem).

$$\sum_{p \in P} \operatorname{Res}_p \omega = 0.$$

Proof Set $M_\varepsilon = M \setminus \bigcup_{p\in P} B_\varepsilon(p)$. Then

$$\sum_{p\in P} \operatorname{Res}_p \omega = \int_{-\partial M_\varepsilon} \eta_\omega = -\int_{M_\varepsilon} d\eta_\omega = 0.$$

□

§2.3 Applications of the global residue theorem We will concentrate exclusively on the case $M = \mathbb{P}^n$. Assume the divisor D_i is described as the zero locus of a homogeneous polynomial f_i of degree d_i in the variables $[z^0, z^1, \cdots, z^n]$. We identify $\mathbb{C}^n$ with the finite part in $\mathbb{P}^n$ described by the condition $z^0 \neq 0$. We introduce affine coordinates

$$x^j = \frac{z^j}{z^0}, \quad 1 \leq j \leq n.$$

We assume none of the divisors D_i contains the divisor at infinity $z^0 = 0$. A meromorphic form on $\mathbb{P}^n$ with poles along D_i is a linear combination of terms

$$\omega = \frac{g(x)}{f_1(x)\cdots f_n(x)} dx^1 \wedge \cdots \wedge dx^n, \quad f_i(x) = \frac{f_i(z)}{(z^0)^{d_i}}, \quad (z^0)^d g(x) = g(z),$$

where g is a homogeneous polynomial of degree d in $[z^0, \cdots, z^n]$. The degree d of g is constrained by requirement that ω has no pole at ∞. To find this constraint we switch to a coordinate system near ∞

$$x^1 = \frac{z^1}{z^0} = \frac{1}{y_1}, \cdots, x^k = \frac{z^k/z^1}{z^0/z^1} = \frac{y^k}{y^1}, \quad k \geq 2.$$

In these coordinates ω has the local description

$$\omega = (-1)^n \frac{(y^1)^{\sigma - d}\tilde{g}(y)}{\tilde{f}_1(y)\cdots \tilde{f}_n(y)} dy^1 \wedge \cdots \wedge dy^n,$$

where $\sigma = d_1 + \cdots + d_n - (n+1)$, $\tilde{f}_i(y) = (y^1)^{d_i} f_i(x)$, $\tilde{g} = (y^1)^d g(x)$. Since $\{y^1 = 0\} \not\subseteq \tilde{f}_1 \cdots \tilde{f}_n$ we deduce that ω has no poles at ∞ if $d \leq \sigma$.

In case $D_1, \cdots, D_\nu$ meet transversally at $d_1 \cdots d_n$ distinct points (P_ν) away from the divisor at ∞ the global residue formula implies the following classical result of Jacobi

$$\sum_\nu \frac{g(P_\nu)}{\frac{\partial(f_1, \cdots, f_n)}{\partial(x^1, \cdots, x^n)}(P_\nu)} = 0. \tag{2.5}$$

for every polynomial of degree $d \leq d_1 + d_2 + \cdots + d_n - (n+1)$. When $n = 1$ we obtain Lagrange interpolation formula

$$\sum_{f(P)=0} \frac{g(P)}{f'(P)} = 0, \quad \forall g \in \mathbb{C}[x], \quad \deg g \leq \deg f - 2. \tag{2.6}$$

Corollary 2.9 (Cayley-Bacharach). *Suppose two smooth plane curves $C_1, C_2 \subset \mathbb{P}^2$ intersect in $\deg C_1 \cdot \deg C_2$ distinct points. Then any curve D of degree $\deg C_1 + \deg C_2 - 3$ which passes through all but one point of $C_1 \cap C_2$ necessarily contains the remaining point as well. In particular if two smooth plane cubics C_1, C_2 intersect in 9 distinct points, then any other cubic containing 8 of the intersection points must contain the ninth point as well.*

Proof Suppose C_i is described by the equation $f_i = 0$ where f_i is a homogeneous polynomial of degree $d_i = \deg C_i$ in the variables $[z^0, z_1, z^2]$. Assume that E is described by the equation $g = 0$, where g is a homogeneous polynomial of degree $d_1 + d_2 - 3 = d_1 + d_2 - (2+1)$. Via a linear change of coordinates we can assume that $C_1 \cap C_2$ does not intersect the divisor at ∞, $\{z^0 = 0\}$. The result now follows from Jacobi's identity (2.5).

□

3 The Poincaré residue

§3.1 Residues of top degree meromorphic forms with simple poles Suppose X is a connected complex manifold of dimension n and $D \hookrightarrow X$ is a smooth hypersurface. D defines a holomorphic line bundle $[D]$ on X, and a holomorphic section $\mathcal{S}_D \in \mathcal{O}(D)$ of $[D]$ determined by the condition $(s_D) = D$ uniquely up to multiplication by a nonzero holomorphic function.

The normal bundle of $D \hookrightarrow X$ along D is isomorphic to $[D]\,|_D$ so that we obtain an isomorphism

$$T^{1,0}X\,|_D \cong T^{1,0}D \oplus [D]\,|_D\ .$$

By dualizing and then passing to determinants we deduce the *adjunction formula*

$$K_X\,|_D \cong K_D \otimes [-D]\,|_D \Longleftrightarrow K_X\,|_D \otimes [D] \cong K_D. \tag{3.1}$$

The *Poincaré residue* Res_D is a global incarnation of the above local construction. Observe first that the sections of $K_X \otimes D$ consists of meromorphic n-forms on X with at most a simple pole along D. Then Res_D is a map

$$\mathrm{Res}_D : H^0(X, K_X \otimes D) \to H^0(D, K_D),$$

defined as follows. Suppose $\omega \in \Omega^{n,0}(D)$ is a meromorphic form with at most a simple pole along D, and $f = 0$ is a local equation of D in a coordinate patch U. Then $f \cdot \omega$ is a holomorphic n form. We claim that for every $p \in D$ there exists a neighborhood V_p of p in U and a holomorphic $(n-1)$-form $\eta_p \in \Omega^{n-1,0}(V_p)$ such that

$$f\omega = df \wedge \eta_p \quad \text{on} \quad V_p.$$

Indeed, since $df(p) \neq 0$ we can find another coordinate system $(y^1, \cdots, y^n)$ near p such that $f = y^1$ so that

$$f\omega = g dy^1 \wedge \cdots \wedge dy^n \Longrightarrow \eta_p = g dy^2 \wedge \cdots \wedge dy^n.$$

Note that formally

$$\eta_p = \frac{\omega}{df/f}.$$

Observe that if $\eta, \eta' \in \Omega^{n-1,0}(V_p)$ are two holomorphic $(n-1)$-forms such that

$$df \wedge \eta = df \wedge \eta' = f\omega$$

then

$$df \wedge (\eta - \eta') = 0.$$

At this pint we want to use the following elementary result.

Lemma 3.1 (DeRham). *The Koszul sequence at p*

$$0 \to \mathcal{O}_{X,p} \stackrel{df}{\longrightarrow} \Omega^{1,0}_{X,p} \stackrel{df\wedge}{\longrightarrow} \cdots \stackrel{df\wedge}{\longrightarrow} \Omega^{n-1,0}_{X,p} \stackrel{df\wedge}{\longrightarrow} \Omega^{n,0}_{X,p} \to 0$$

is exact.

We deduce that there exists $u \in \Omega^{n-2,0}_{X,p}$ such that

$$(\eta - \eta') = df \wedge u.$$

Since $df\,|_D \equiv 0$ it follows that $\eta\,|_D\; D \equiv \eta'\,|_D$ in a possible smaller neighborhood $p \in V'_p \subset V_p$.

The above argument shows that forms $\eta_p\,|_D$ agree on the overlaps of their domains and thus define a global holomorphic form $\eta_\omega \in \Omega^{n-1,0}(D)$. We set

$$\operatorname{Res}_D \omega := \eta_\omega.$$

We can write formally

$$\operatorname{Res}_D \omega = \frac{\omega}{\frac{df}{f}}\,|_D\;.$$

Recall the Leray coboundary $\gamma : H_{n-1}(D) \to H_n(X \setminus D)$. The residue map satisfies the identity

$$\langle \operatorname{Res}_D \omega, c\rangle = \frac{1}{2\pi\mathbf{i}}\langle \omega, \lambda c\rangle,\;\; \forall \omega \in \Omega^{n,0}(D),\;\; c \in H_{n-1}(D).$$

In local coordinates, if $\omega = \frac{g}{f} dz^1 \wedge \cdots \wedge dz^n$, and along $f = 0$ we have $\frac{\partial f}{\partial z^k} \neq 0$, then near $f = 0$ we have

$$\frac{g}{f} dz^1 \wedge \cdots \wedge dz^n = (-1)^{k-1}\frac{g}{f} dz^k \wedge dz^1 \wedge \cdots \wedge \widehat{dz^k} \wedge \cdots \wedge dz^n$$

$$= (-1)^{k-1}\frac{g}{\frac{\partial f}{\partial z^k}}\frac{1}{f}\Big(\frac{\partial f}{\partial z^1}dz^1 + \cdots + \frac{\partial f}{\partial z^k}dz^k + \cdots + \frac{\partial f}{\partial z^n}dz^n\Big) dz^1 \wedge \cdots \wedge \widehat{dz^k} \wedge \cdots \wedge dz^n$$

$$= (-1)^{k-1}\frac{df}{f}\frac{g}{\frac{\partial f}{\partial z^k}} dz^1 \wedge \cdots \wedge \widehat{dz^k} \wedge \cdots \wedge dz^n.$$

so that locally

$$\operatorname{Res}_D \omega = (-1)^{k-1}\frac{g}{\frac{\partial f}{\partial z^k}} dz^1 \wedge \cdots \wedge \widehat{dz^k} \wedge \cdots \wedge dz^n.$$

This shows that the residue map induces an injection

$$\operatorname{Res}_D : H^0(X, K_X \otimes D) \to H^0(D, K_D).$$

§3.2 Smooth plane curves Suppose $C \hookrightarrow \mathbb{P}^2$ is a smooth degree d pane curve described by the equation $P = 0$, where P is a homogeneous polynomial of degree d in the variables $[z^0, z^1, z^2]$. We then have a Leray coboundary map

$$\lambda : H_1(C) \to H_2(\mathbb{P}^2 - C).$$

Proposition 3.2. *The Leray coboundary is an isomorphism.*

Proof We follow the presentation in [2].

Surjectivity. Suppose $\sigma \in H_2(\mathbb{P}^2 - C)$. Then $\sigma \cdot [C] = 0 \in H_2(\mathbb{P}^2)$. Hence $[\sigma] = 0$ in $H_2(\mathbb{P})^2$ so that σ bounds a 3-chain Σ in $\mathbb{P}^2$. If we choose Σ carefully the intersection $c = \Sigma \cot C$ is a 1-cycle on C such that $\tau(c) = \sigma$.

Injectivity. Consider a cycle $c \in H_1(C)$ such that $\lambda(c) = 0 \in H_2(\mathbb{P}^2 - C)$. Denote by N a thin tubular neighborhood of $C \hookrightarrow \mathbb{P}^2$, and denote by $\pi : N \to C$ the natural projection. Then

$$\lambda(c) = \partial\alpha,\;\; \alpha := \pi^{-1}(c).$$

Since $\lambda(c) = 0$ in $H_2(\mathbb{P}^2 - C)$ there exists a 3-chain β in $\mathbb{P}^2 - C$ such that $\partial\beta = \lambda(c)$. We deduce that $\alpha - \beta$ is a 3-cycle in $\mathbb{P}^2$ since $H_3(\mathbb{P}^2) = 0$ we conclude that $\alpha - \beta = \partial U$, where U is a 4-chain in n$\mathbb{P}^2$. Then $U \cap C$ is a 2-chain on C such that

$$\partial(U \cap C) = c$$

i.e. c is homologous to zero.

□

Using the Poincaré residue map we can considerably refine the above result to capture rather subtle interactions between the geometry of C and its complement $\mathbb{P}^2 - C$. Arguing as in §2.3 we deduce the following.

Proposition 3.3. *The holomorphic 2-forms on $\mathbb{P}^2$ with a simple pole along C have the form*[1]

$$\omega = \frac{g(x^1, x^2)}{P(x^1, x^2)} dx^1 \wedge dx^2, \;\; x^j = z^j / z^0,$$

where $(z^0)^{d-3} g(z^1/z^0, z^2/z^0)$ is homogeneous polynomial of degree $d-3$ in the variables z^0, z^1, z^2. In other words

$$K_{\mathbb{P}}^2 \otimes C = (d-3)H.$$

In particular

$$K_{\mathbb{P}^2} = -3H. \tag{3.2}$$

Using (3.2) in the adjunction formula (3.2) we deduce

$$-3[H]\,|_C = K_C - [C]\,|_C$$

so that

$$-3\langle c_1([H]), C\rangle = \langle c_1(K_C), C\rangle - \langle c_1(N_C), C\rangle.$$

Note that $\langle c_1(N_C), C\rangle = C \cdot C = d^2$ while Gauss-Bonnet theorem implies

$$\langle c_1(K_C), C\rangle = 2g(C) - 2.$$

The last equality becomes

$$-3d = 2g(C) - 2 - d^2 \Longleftrightarrow g(C) = \frac{(d-1)(d-3)}{2}.$$

Hence

$$\dim_{\mathbb{C}} H^1(C, K_C) = \dim H^{1,0}(C) = g(C) = \frac{(d-1)(d-3)}{2}.$$

On the other hand, Proposition 3.3 shows that

$$\dim_{\mathbb{C}} H^0(\mathbb{P}^2, K_{\mathbb{P}^2} \otimes C) = \dim_{\mathbb{C}} H^0(\mathbb{P}^2, (d-3)H)$$

$$= \binom{d-3+2}{3-1} = \binom{d-1}{2} = \frac{(d-1)(d-2)}{2} = \dim_{\mathbb{C}} H^1(C, K_C).$$

We have thus proved the following dual to Proposition 3.2.

Proposition 3.4. *If C is a smooth plane curve then the Poincaré residue map defines an isomorphism*

$$\mathrm{Res}_C : H^0(\mathbb{P}^2, K_{\mathbb{P}^2} \otimes C) \to H^0(C, K_C).$$

[1] The polynomials g in Proposition 3.3 are classically known as polynomials *adjoint to C.*

§3.3 Abel's theorem for smooth plane curves Suppose C is a smooth plane curve of degree d described as above by an equation $P=0$. Fix a point $p_0 \in C$, and consider a family of homogeneous polynomials $Q_t = Q_t(z^0, z^1, z^2)$ of degree n depending holomorphically on the parameter $t \in \mathbb{C}$. Denote by D_t the divisor $\{Q_t = 0\} \cap C$ on C. For generic t it consists of $\deg C \cdot \deg Q_t$ distinct points. We would like to construct an invariant of this family of linearly equivalent divisors.

Fix a holomorphic form $\omega \in H^{1,0}(C) = H^0(C, K_C)$. The *periods* of ω are the complex numbers $\langle \omega, c\rangle \in \mathbb{C}$, $c \in H_1(C,\mathbb{Z})$. More invariantly, every cycle $c \in H_1(C, \mathbb{Z}$ defines a linear map

$$\mathcal{P}_c : H^{1,0}(C) \to \mathbb{C},\ \ \omega \mapsto \int_c \omega.$$

Hodge theory implies that the correspondence $c \mapsto \mathcal{P}_c$ defines an injection

$$H_1(C,\mathbb{Z}) \hookrightarrow H^{1,0}(C)^*.$$

Note that

$$\operatorname{rank} H_1(C,\mathbb{Z}) = 2g = \dim_{\mathbb{R}} H^0(C, K_C)^*.$$

We can invoke Hodge theory again to conclude that $H_1(C,\mathbb{Z})$ sits as a *lattice* in $H^0(C,K_C)^*$, called the *lattice of periods*. We denote it by Λ_C. Thus the quotient

$$J(C) := H^0(C, K_C)^*/\Lambda_C$$

is a $2g$-dimensional torus. It called the *Jacobian of* C.

For every point $p \in C$ and every path γ in C from p_0 to p we obtain a linear map

$$\int_\gamma : H^{1,0}(C) \to \mathbb{C},\ \ \omega \mapsto \int_\gamma \omega.$$

Observe that if we choose a different such path γ' then

$$\int_\gamma - \int_{\gamma'} = \int_{\gamma-\gamma'} \in \Lambda_C.$$

Thus the image of $\int_\gamma$ in $H^{1,0}(C)^*/\Lambda_C$ is independent of the path γ connecting p_0 to p. We will denote this image by $\int_{p_0}^p$. Every $\omega \in H^{1,0}(C)$ defines an Abelian group

$$\Lambda_\omega := \{\int_c \omega \in \mathbb{C};\ \ c \in H_1(C,\mathbb{Z})\}$$

and a projection

$$H^{1,0}(C)^*/\Lambda_C \to \mathbb{C}/\Lambda_\omega,\ \ L \mod \Lambda_C \mapsto L(\omega) \mod \Lambda_\omega.$$

In particular, for every $\omega \in H^{1,0}(C)$ and every $p \in C$ we have an element

$$\int_{p_0}^p \omega \in \mathbb{C}/\Lambda_\omega.$$

For every divisor $D = \sum_i n_i p_i$ on C and every $\omega \in H^{1,0}(C)$ we set

$$\mathcal{P}_D = \int_{p_0}^D := \sum_i n_i \int_{p_0}^{p_i} \in J(C),\ \ \omega(D) = (\mathcal{P}_D)(\omega) \bmod \Lambda_\omega = \int_{p_0}^D \omega \bmod \Lambda_\omega \in \mathbb{C}/\Lambda_\omega.$$

Observe that $\mathcal{P}_{D+D'} = \mathcal{P}_D + \mathcal{P}_{D'}$ and

$$\mathcal{P}_D = \mathcal{P}_{D'} \Longleftrightarrow \omega(D) = \omega(D') \in \mathbb{C}/\Lambda_\omega,\ \ \forall \omega \in H^{1,0}(C).$$

Theorem 3.5 (Abel). *The element $\mathcal{P}_{D_t} \in J(C)$ is independent of t, that is for every $\omega \in H^0(C, K_C)$ the element $\omega(D_t) \in \mathbb{C}/\Lambda_\omega$ is independent of t.*

Before we present the proof of this theorem we discuss one important consequence. Denote by "$\sim$" the linear equivalence relation between divisors.

Corollary 3.6.

$$D \sim D' \Longrightarrow \omega(D) = \omega(D'),\ \ \forall \omega \in H^{1,0}(C) \Longleftrightarrow \mathcal{P}_D = \mathcal{P}_{D'}.$$

In particular if D is a principal divisor then $\mathcal{P}_D = 0$.[2]

Proof Choose a meromorphic function R such that

$$D - D' = (R).$$

Then by Chow's theorem (see [8]) there exist two homogeneous polynomials of identical degrees in the variables $[z^0, z^1, z^2]$ such that

$$(R) = \frac{Q_0}{Q_1}|_C\ .$$

Set $Q_t := (1-t)Q_0 + tQ_1$, $D_t := Q_t \cap C$. Then

$$\omega(D) - \omega(D') = \omega((R)) = \omega(D_0) - \omega(D_1) \stackrel{Abel}{=} 0.$$

□

Proof of Theorem 3.5. Denote by Δ the discriminant locus of the family D_t, i.e. the set of points $t \in \mathbb{C}$ such that D_t contains multiple points. Δ is a finite set. Let $t_0 \in \mathbb{C} \setminus \Delta$ and set $\nu := \deg C \cdot \deg Q_{t_0}$. Assume for simplicity that $t_0 = 0$. We can find $\varepsilon > 0$ and ν disjoint holomorphic paths

$$\gamma_j : \{|s| < \varepsilon\} \to C$$

such that

$$D_s = \big\{\gamma_j(s);\ \ 1 \le j \le \nu\big\}.$$

Assume that for $|s| < \varepsilon$ the divisor D_s does not intersect the divisor $z^0 = 0$. Thus we can work in the "finite" part of C where we can use the local coordinates

$$x = z^1/z^0,\ \ y = z^2/z^0.$$

We can then find a homogeneous degree $(d-3)$ polynomial g in the variables z^0, z^1, z^2 such that

$$\omega = \mathrm{Res}_C\Big(\frac{g}{P}dx \wedge dy).$$

For each $|s| < \varepsilon$ and $1 \le j \le \nu$ we consider the paths

$$u_j(t) = \gamma_j(ts),\ \ t \in [0,1].$$

In local coordinates they are given by

$$u_j(t) = (x_j(t), y_j(t)),\ \ t \in [0,1].$$

[2]The converse is also true. Theorem 3.7 establishes this converse in the special case of plane cubics.

Then

$$\omega(D_s) - \omega(D_0) = \sum_j \int_{u_j} \omega \mod \Lambda_\omega.$$

We will prove that

$$\sum_j \int_{u_j} \omega = 0.$$

We study each of the integrals in the above sum separately. Fix j. Near $u_j(0)$ we have $dP \neq 0$ so we can assume $\frac{\partial P}{\partial x} \neq 0$ near $p_j(0)$. Then near $p_j(0)$

$$\omega = \frac{g}{P'_x} dy$$

so that

$$\int_{u_j} \omega = \int_0^1 \frac{g}{P'_x} \dot{y}_j dt.$$

Differentiating the equalities

$$P(x_j(t), y_j(t)) = 0, \;\; Q(x_j(t), y_j(t); t) = 0$$

we deduce

$$P'_x \dot{x}_j + P'_y \dot{y}_j = 0, \quad Q'_x \dot{x}_j + Q'_y \dot{y}_j + Q'_t = 0.$$

Solving for $\dot{x}_j$ and $\dot{y}_j$ we obtain

$$\dot{y}_j = -\frac{Q'_t P'_x}{\partial(P,Q)/\partial(x,y)} \Longrightarrow \frac{\dot{y}_j}{P'_x} = -\frac{Q'_t}{\partial(P,Q)/\partial(x,y)}.$$

Set $H_t := -g \cdot Q'_t$. Note that $\deg H_t < \deg P + \deg Q - 3$ We deduce that

$$\sum_j \int_{u_j} \omega = \int_0^1 \Big(\sum_j \frac{H_t}{\partial(P,Q)/\partial(x,y)} (p_j(t)) \Big) dt$$

The integrand is zero by Jacobi's formula (2.5). Thus the *continuous* function

$$\mathbb{C} \to J(C), \;\; t \mapsto \mathcal{P}_{D_t}$$

is constant outside the finite set Δ. We deduce that it must be constant.

□

§3.4 The addition law for smooth plane cubics Consider a smooth plane cubic, i.e. smooth plane curve C described by an equation $P = 0$ where P is a homogeneous polynomial of degree 3 in the variables $[z^0, z^1, z^2]$. We can linearly change the coordinates $[z^0, z^1, z^2]$ such that in the finite part $z^0 \neq 0$ the polynomial P has the Legendre form (see [2, §7.3.11] or [3, §2.2])

$$P(1, x, y) = y^2 - x(x-1)(x-\lambda), \;\; x = z^1/z^0, \;\; y = z^2/z_0$$

that is

$$P(z^0, z^1, z^2) = z^0(z^2)^2 - z^1(z^1 - z^0)(z^1 - \lambda z^0).$$

The point $p_\infty := [0, 0, 1]$ lies on this curve. If we use the coordinates

$$u = z^0/z^2, \;\; v = z^1/z^2$$

we deduce that near p_∞ the curve C is described by the equation

$$u = v(v-u)(v-\lambda u),\ \ p_\infty = (0,0)$$

which shows that the line $u = 0$ has a third order contact with C at p_∞.

The space of holomorphic differentials on C is one dimensional and is generated by

$$\omega = \frac{dx}{y}.$$

Fix a basis a, b of $H_1(C,\mathbb{Z})$ such that $a \cdot b = 1$. The Jacobian of C is the torus

$$J(C) \cong \mathbb{C}/\mathrm{span}_{\mathbb{Z}}(\langle\omega, a\rangle, \langle\omega, b\rangle).$$

For every line L in $\mathbb{P}^2$ we get a divisor $L \cap C$ and Abel's theorem implies that

$$\omega(L \cap C) \in J(C)$$

is independent of L. In particular, if L is the line $L_\infty = \{u = 0\}$ we deduce

$$L_\infty \cap C = 3 \cdot p_\infty$$

so that $\omega(L_\infty \cap C) = 0 \in J(C)$. We deduce that

$$\omega(L \cap C) = 0 \in J(C),\ \ \text{for any line } L.$$

Thus if we set

$$u(p) := \int_{p_\infty}^{p} \omega \in J(C)$$

we deduce that

$$u(p_1) + u(p_2) + u(p_3) = 0$$

for any three collinear points $p_1, p_2, p_3 \in C$.

We regard the correspondence $p \to u(p)$ as a holomorphic map

Theorem 3.7 (Abel). *The map*

$$C \ni p \mapsto u(p) \in J(C).$$

is one-to-one.

Proof We argue by contradiction. Here is briefly the strategy. Suppose $u(p_1) = u(p_2)$. If we denote by D the divisor $p_2 - p_1$ this condition implies $\mathcal{P}_D = 0$. We know that this would happen if D were a principal divisor. We will show that this is indeed the case. A simple counting argument will then show that D cannot be a principal divisor, thus yielding a contradiction.

Step 1. There *exists* a meromorphic form η on C with simple poles at p_1 and p_2 such that

$$\mathrm{Res}_{p_k}\,\eta = (-1)^k,\ \ k = 1,2. \tag{3.3}$$

Indeed consider the short exact sequence of sheaves

$$0 \to K_C \to K_C \otimes \mathcal{O}(p_1 + p_2) \xrightarrow{\mathrm{Res}} \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2} \to 0.$$

We get a long exact sequence

$$0 \to H^0(C, K_C) \to H^0(C, K_C(p_1 + p_2)) \xrightarrow{r} \mathbb{C}_{p_1} \oplus \mathbb{C}_{p_2} \to H^1(C, K_C) \to$$

Since the sum of residues of a meromorphic form is 0 we deduce that $\dim \operatorname{coker} r \geq 1$. On the other hand

$$H^1(C, K_C) \cong H^0(C, \mathcal{O}) \cong \mathbb{C}$$

so that we deduce $\dim \operatorname{coker} r = 1$. Thus for any two complex numbers a_1, a_2 such that $a_1 + a_2 = 0$ we can find a meromorphic form on C with simple poles at p_1, p_2 such that $\operatorname{Res}_{p_k} = a_k$.

Denote by $\mathcal{X}$ the space of meromorphic forms with simple poles at p_1, p_2 satisfying (3.3). Fix $\eta_0 \in \mathcal{X}$. Then

$$\mathcal{X} = \eta_0 + H^0(C, K_C) = \eta_0 + \mathbb{C}\langle\omega\rangle.$$

Step 2.[3] There *exists* a meromorphic form $\eta \in \mathcal{X}$ on C

$$\int_c \eta \in 2\pi\mathbf{i}\mathbb{Z}, \quad \forall c \in H_1(C, \mathbb{Z}). \tag{3.4}$$

We will present two proofs of this fact. The first proof is inspired by [5, Lecture 2] and uses in a more visible fashion the Hodge structure on $H^1(C, \mathbb{C})$. The second is the classical proof based on Riemann's bilinear relations.

1st Proof. Fix a smooth path γ from p_1 to p_2. We assume that γ has no self-intersections and that for every $\varepsilon \ll 1$ the closed set

$$T_\varepsilon := \{p \in C;\ \operatorname{dist}(p, \gamma) \leq \varepsilon\}$$

is diffeomorphic to a disk (see Figure 4). Set $C_\varepsilon := C \setminus T_\varepsilon$.

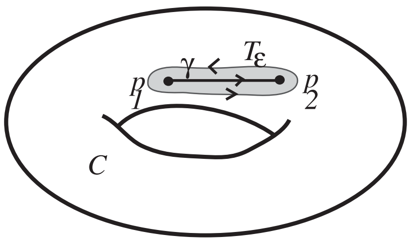

Figure 4: *Constructing meromorphic forms on a cubic*

For each $c \in H_1(C, \mathbb{Z})$ denote by φ_c the *harmonic* 1-form representing the Poincaré dual of c. It is uniquely determined by the equality (see [1, Chap.I, §5])

$$\int_C \beta \wedge \varphi_c = \int_c \beta, \quad \forall \beta \in H^1(C, \mathbb{C}).$$

Since $u(p_1) = u(p_2)$ there exists $c_0 \in H_1(C, \mathbb{Z})$ such that

$$\int_\gamma \omega = \int_{c_0} \omega = \int_C \omega \wedge \varphi_{c_0}.$$

Denote by α the *harmonic* 1-form on C such that

$$\int_c \alpha = \frac{1}{2\pi\mathbf{i}} \int_c \eta_0, \quad \forall c \in H_1(C, \mathbb{Z}).$$

[3] This is the key moment in the proof.

We have a *Hodge decomposition* (see [8])

$$\alpha = \alpha^{1,0} + \alpha^{0,1},\ \ \alpha^{1,0} \in H^0(C, K_C) \cong H^{1,0}(C),\ \ \alpha^{0,1} \in H^0(C, K_C^{-1}) \cong H^{0,1}(C).$$

Since

$$\frac{1}{2\pi\mathbf{i}} \int_{\partial T_\varepsilon} \eta_0 = \int_{\partial T_\varepsilon} \alpha = 0$$

we deduce that α and $\frac{1}{2\pi\mathbf{i}}\eta_0$ are cohomologous on $C \setminus \gamma$. Hence there exists

$$f : C \setminus \gamma \to \mathbb{C} :\ \ \frac{1}{2\pi\mathbf{i}}\eta_0 - \alpha = df.$$

Since $\operatorname{Res}_{p_2} \eta_0 = 1$ we deduce that f increases by 1 when we cross γ. Then

$$\int_C \omega \wedge \varphi_{c_0} = \int_\gamma \omega = \lim_{\varepsilon \searrow 0} \int_{\partial C_\varepsilon} \omega f = \lim_{\varepsilon \searrow 0} \int_{C_\varepsilon} \omega \wedge df = \lim_{\varepsilon \searrow 0} \int_{C_\varepsilon} \omega \wedge (\eta_0 - \alpha).$$

Now observe that since ω and η_0 are $(1,0)$ forms we have $\omega \wedge \eta_0 = 0$. Hence

$$\int_C \omega \wedge \varphi_{c_0}^{0,1} = -\int_C \omega \wedge \alpha^{0,1}.$$

Since ω spans $H^{1,0}(C)$ we deduce from Hodge theory that $\alpha^{0,1} = -\varphi_{c_0}^{0,1}$. Now observe that

$$\int_c \Big(\frac{1}{2\pi\mathbf{i}}\eta_0 - \alpha^{1,0}\Big) = \int_c (\alpha - \alpha^{1,0}) = -\int_C \varphi_c \wedge (\alpha - \alpha^{1,0}) = -\int_C \varphi_c \wedge \alpha^{0,1}$$

$$= \int_C \varphi_c \wedge \varphi_{c_0}^{0,1} = \int_C \varphi_c \wedge \varphi_{c_0} = c \cdot c_0$$

The above identity shows that the periods of $\eta_0 - 2\pi\mathbf{i}\alpha^{1,0} \in \mathfrak{X}$ are in $2\pi\mathbf{i}\mathbb{Z}$.

2nd Proof. Fix an integral basis $\boldsymbol{a}, \boldsymbol{b}$ of $H_1(C, \mathbb{Z})$ such that $\langle \omega, \boldsymbol{a} \rangle \neq 0$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 1$. We assume $\boldsymbol{a}, \boldsymbol{b}$ are represented by simple closed curves which do note pass through p_1, p_2 and p_∞. Fix a path γ connecting p_1 to p_2 and denote by C_k a small circle centered at p_k oriented as boundary component of $C \setminus \{p_1, p_2\}$. Now cut C along $\boldsymbol{a}$ and $\boldsymbol{b}$ to obtain the standard cut-and-paste description of the torus C (see Figure 5).

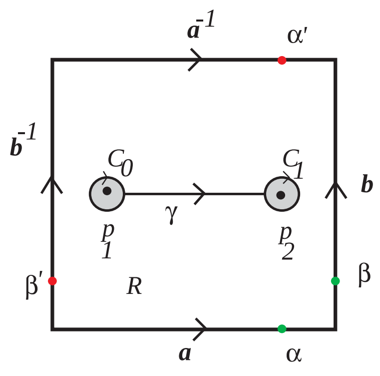

Figure 5: *Cutting a torus*

Since the rectangle in Figure 5 is simply connected the function u lifts to a holomorphic function $u : R \to \mathbb{C}$. More precisely we define $u(p)$ by integrating ω along a path *inside* R from p_∞ to p To every point α on $\boldsymbol{a}$ there corresponds a point α' on $\boldsymbol{a}^{-1}$ which glue to the same point on C. Now that

$$u(\alpha') - u(\alpha) = \langle \omega, \boldsymbol{b}\rangle.$$

Similarly, for every point $\beta \in \boldsymbol{b}$ there exists a point $\beta' \in \boldsymbol{b}^{-1}$ which is identified with β in C. In this case

$$u(\beta') - u(\beta) = -\langle \omega, \boldsymbol{a}\rangle.$$

Observe now that for every $\eta \in \mathcal{X}$ we have

$$2\pi\mathbf{i}\big(u(p_2) - u(p_1)\big) = 2\pi\mathbf{i}\big(\operatorname{Res}_{p_1}(u\eta) + \operatorname{Res}_{p_2}(u, \eta)\big) = \int_{\partial R} u\eta$$

$$= \Big(\int_{\boldsymbol{a}} + \int_{\boldsymbol{a}^{-1}}\Big)(u\eta) + \Big(\int_{\boldsymbol{b}} + \int_{\boldsymbol{b}^{-1}}\Big)(u\eta) = \int_{\boldsymbol{a}}\big(u(\alpha) - u(\alpha')\big)\eta + \int_{\boldsymbol{b}}\big(u(\beta) - u(\beta')\big)\eta$$

$$= -\langle \omega, \boldsymbol{b}\rangle \int_{\boldsymbol{a}} \eta + \langle \omega, \boldsymbol{a}\rangle \int_{\boldsymbol{b}} \eta.$$

Hence[4]

$$u(p_2) - u(p_1) = \frac{1}{2\pi\mathbf{i}}\Big(-\langle \omega, \boldsymbol{b}\rangle \cdot \langle \eta, \boldsymbol{a}\rangle + \langle \omega, \boldsymbol{a}\rangle \cdot \langle \eta, \boldsymbol{b}\rangle\Big) = \frac{1}{2\pi\mathbf{i}}\begin{vmatrix} \langle \omega, \boldsymbol{a}\rangle & \langle \eta, \boldsymbol{a}\rangle \\ \langle \omega, \boldsymbol{b}\rangle & \langle \eta, \boldsymbol{b}\rangle \end{vmatrix}, \quad \forall \eta \in \mathcal{X}. \tag{3.5}$$

The condition $u(p_1) = u(p_2) \in J(C)$ implies that there exist two integers m, n such that

$$u(p_2) - u(p_1) = \int_{\gamma} \omega = m\int_{\boldsymbol{a}} \omega + n\int_{\boldsymbol{b}} \omega. \tag{3.6}$$

Using (3.5) we deduce

$$\int_{m\boldsymbol{a}+n\boldsymbol{b}} \omega = -\frac{1}{2\pi\mathbf{i}}\begin{vmatrix} \langle \omega, \boldsymbol{a}\rangle & \langle \eta, \boldsymbol{a}\rangle \\ \langle \omega, \boldsymbol{b}\rangle & \langle \eta, \boldsymbol{b}\rangle \end{vmatrix}, \quad \forall \eta \in \mathcal{X} \tag{3.7}$$

Define

$$\eta_1 = \eta_0 - \frac{\langle \eta_0, \boldsymbol{a}\rangle}{\langle \omega, \boldsymbol{a}\rangle}\omega \in \mathcal{X}.$$

By design

$$\langle \eta_1, \boldsymbol{a}\rangle = 0.$$

Using (3.7) we deduce

$$\langle \eta_1, \boldsymbol{b}\rangle = \frac{2\pi\mathbf{i}}{\langle \omega, \boldsymbol{a}\rangle}\langle \omega, m\boldsymbol{a} + n\boldsymbol{b}\rangle = 2m\pi\mathbf{i} + n\frac{2\pi\mathbf{i}}{\langle \omega, \boldsymbol{a}\rangle}\langle \omega, \boldsymbol{b}\rangle$$

Now define

$$\eta_2 = \eta_1 - \frac{2n\pi\mathbf{i}}{\langle \omega, \boldsymbol{a}\rangle}\omega \in \mathcal{X}.$$

Then

$$\langle \eta_2, \boldsymbol{a}\rangle = -2n\pi\mathbf{i}, \ \ \langle \omega, \boldsymbol{b}\rangle = 2m\pi\mathbf{i}$$

[4]The identity (3.5) is a special case of Riemann's bilinear relations.

so η_2 satisfies (3.4).

Step 3. There exists a meromorphic function h on C with a simple zero at p_1 and a simple pole at p_2. Consider the meromorphic form η_2 constructed at step 2. Set

$$h(p) = \exp(\int_{p_\infty}^{p} \eta_2).$$

The condition (3.4) implies that h is well defined. Since $\frac{dh}{h} = \eta_2$, the condition (3.3) implies that h has a simple zero at p_2 and a simple pole at p_1.

Step 4. There exists no meromorphic function R on C with a simple pole and a simple zero. To see this we write R as a quotient f/g where P and Q are homogeneous polynomials of identical degrees $n > 0$. Then the zero set consists of $3n > 1$ points counting multiplicities.

This concludes the proof of Theorem 3.7.

□

We now have a one-to-one map $C \to J(C)$, $p \mapsto u(p)$. Observe that $du = \omega$. This implies that this map is *biholomorphic*. The biholomorphic map $C \ni p \mapsto u \in J(C)$ has thus introduced a group law on C by the rule

$$p_3 = -(p_1 + p_2) \Longleftrightarrow p_1, p_2, p_3 \text{ are colinear.}$$

We can invert this function, and regard p as a function of u. We deduce that

$$u_1 + u_2 + u_3 = 0 \Longleftrightarrow p(u_1), p(u_2), p(u_3) \text{ are colinear} \tag{3.8}$$

Observe that $u(p_\infty) = 0 \in J(C)$. Denote by $x(u)$ and $y(u)$ the coordinates of $p(u)$ $u \neq 0$.

Given two points $p_1, p_2 \in C$ then the coordinates of the third intersection point of the line $[p_1 p_2]$ with C are rational functions of the coordinates of p_1 and p_2. Hence

$$x(-(u_1 + u_2)) = R(x(u_1), x(u_2), y(u_1), y(u_2)), \quad y(-(u_1 + u_2)) = S(x(u_1), x(u_2), y(u_1), y(u_2)).$$

The above identities are classically known as the addition laws for the (inverse of the) elliptic integrals. The function $x(u)$ is none other than Weierstrass $\wp$-function. From the equality

$$du = \omega = \frac{dx}{y} = \frac{x'(u)}{y(u)} du$$

we deduce $y(u) = x'(u)$ and the equality $y^2 = P_3(x) := x(x-1)(x-\lambda)$ becomes the known differential equation satisfied by the Weierstrass function

$$(\wp')^2 = P_3(\wp).$$

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