Ricerche di Matematica
Vol. XXXVII, fasc. 1° (1988)

Optimal control for a nonlinear diffusion equation.

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ABSTRACT. – In this paper we study distributed control of problem for systems governed by a semilinear elliptique equation of the type

$$-\Delta y + f(y) = u \tag{0.1}$$

where f is monotone but not locally Lipschitz. We derive necessary optimality conditions. Then using these conditions and the method of symmetric rearrangements we obtain a bang-bang result in a special situation. At the end we deal with another concrete situation. We use a different symmetrization technique which has proved to be useful in many other problems such as the obstacle problem.

1. Introduction.

The aim of this paper is the study of the following control problem

(P) Min $\{g(y)+h(u)\}$ with (y, u) satisfying

$$-\Delta y + |y|^{\alpha-1} y = u, \qquad (0<\alpha<1) \text{ in } \Omega \tag{1.1}$$

$$y = \mu, \qquad \text{on } \Gamma = \partial\Omega \tag{1.2}$$

where μ is a real function on Γ and $\Omega \subset R^N$ is a bounded open domain with smooth boundary.

Such problem arise in chemistry (see e.g. [4]). Namely they model reaction-diffusion phenomenons when an irreversible reaction takes place in Ω. The reactant being consumed it is replaced through diffusion from the ambient region so that a steady state is possible. In (1.1)-(1.2) $y(x)$ denotes the density of reactant at a point $x \in \Omega$ while $u(x)$ denotes the density of catalyst.

The main feature of the problem under consideration is that the

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function $\beta(r)=|r|^{\alpha-1}r$ is not locally Lipschitz so that the results of [2] do not apply in this situation. This problem is an intermediate between the locally Lipschitz situation and the multivoque situation that arises e.g. in the obstacle problem.

In section **2** we shall review some known results for equations (1.1)-(1.2).

Section **3** is the core of the paper. Here we state and prove the main result: Theorem 3.1.

In section **4** we give some applications of Theorem 3.1. Some of the techniques developed there have proved to be useful in other problems such as the obstacle problem. We shall derive a bang-bang result for this problem.

2. Auxiliary results.

In this section we state some results concerning equations (1.1)-(1.2) in a form suitable for our considerations.

Existence: Let $f\colon \mathbb{R}\to\mathbb{R}$ be an increasing continuous function such that:

i) $f(0)=0$;

ii) there exists A, $B>0$ such that $f(r)\leq A|r|+B$, $\forall r\in\mathbb{R}$. Then for every $u\in L^p(\Omega)$ $(2\leq p<\infty)$ and for every $\mu\in C^{2,\delta}(\Gamma)$ $(0<\delta<1)$ there exists a unique $y\in W^{2,p}(\Omega)$ satisfying (0.1)-(1.2) and the estimate

$$\text{(E)} \qquad \|y\|_{W^{2,p}(\Omega)}\leq C(\|y\|_{L^2(\Omega)}+\|u\|_{L^p(\Omega)}+1)$$

where $C=C(\Omega,A,B,u)$.

The proof of this fact is a routine matter and relies on a result of Calderon-Zygmund (see e.g. [8]):

«If $F\in L^p(\Omega)$ than there exists $y\in W^{2,p}(\Omega)\cap W^{1,p}_\infty(\Omega)$ $(1<p<\infty)$ such that

$$-\Delta y=F, \qquad \text{a.e. in } \Omega.$$

Moreover the following estimate holds

$$\|y\|_{W^{2,p}}\leq C\|F\|_p, \qquad C=C(p,\Omega)».$$

Comparison result: Let $u_i\in L^p(\Omega)$, $\mu_i\in C^{2,\delta}(\Gamma)$, $i=1,2$ such that $u_1\leq u_2$ and $\mu_1\leq\mu_2$. If y_i $(i=1,2)$ denotes the solution of

$$-\Delta y_i+f(y_i)=u_i, \qquad y_i|_\Gamma=\mu_i$$

(where f is the same as above) then $y_1\leq y_2$.

3. Necessary optimality conditions for problem (P).

We shall consider a more general problem. We state the basic assumptions

(3.1) $g\colon (\Omega) \to \mathbb{R}$ is Frechet differentiable, locally Lipschitz and bounded from bellow: $g(y) \geq 0 \ \forall y \in L^2(\Omega)$,

(3.2) $h\colon L^s(\Omega) \to]-\infty, +\infty]$ is a proper lower semicontinuous convex function satisfying $h(u) \geq C_{1\|u\|_A} + C_2$, $C_1 > 0$, $C_2 \in \mathbb{R}$.

Here $s > \max\{2, N/2\}$ and $\|\cdot\|_s$ is the norm in $L^s(\Omega)$ (Ω is assumed to have a C^∞-boundary) $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous increasing function such that

(3.3.1) $$f \in C^1(\mathbb{R} \setminus \{0\}) \quad \text{and} \quad \lim_{x \to 0} f'(x) = +\infty$$

(3.3.2) $$\exists A, B > 0\colon |f(r)| \leq A|r| + B, \qquad \forall r \in \mathbb{R}$$

(3.3.3) $$\forall \varepsilon > 0 \ \exists C_\varepsilon > 0\colon |f'(r)| \leq C_\varepsilon, \qquad \forall |r| \geq \varepsilon$$

(3.3.4) $$f(0) = 0.$$

Such functions can be smoothened at the origin so that one can get a family $(f_\varepsilon(r))_{\varepsilon > 0}$ of functions with properties:

(3.4.1) $$f_\varepsilon \in C^1(\mathbb{R}), \quad f_\varepsilon(r) \xrightarrow[\varepsilon \to 0]{} f(x), \quad \forall r \in \mathbb{R},\ f_\varepsilon(0) = 0,\ \forall \varepsilon > 0$$

(3.4.2) $$\exists A, B > 0\colon |f_\varepsilon(r)| \leq A|r| + B, \qquad \forall r \in \mathbb{R},\ \varepsilon > 0$$

$$f'_\varepsilon(r_\varepsilon) \to +\infty \quad \textit{whenever} \quad \lim_{\varepsilon \to 0} r_\varepsilon = 0.$$

$(f_\varepsilon)_{\varepsilon > 0}$ can be obtained from f by C^1-interpolation with polynomials of degrees two. For example if $f(r) = |r|^{\alpha - 1} r (0 < \alpha < 1)$ then one can take

$$f(r) = \begin{cases} \dfrac{2\alpha - 1}{2} \varepsilon r^2 + (1 - \alpha)\, \varepsilon^{\alpha - 1} r, & 0 \leq r < \varepsilon \\ r^\alpha, & r \geq \varepsilon \\ -f_\varepsilon(-r), & r < 0. \end{cases}$$

We consider the problem

(P) Minimize $\{g(y) + h(u)\}$ over all (y, u) subject to

(3.5.1) $$-\Delta y + f(y) = u, \qquad \text{in } \Omega$$

(3.5.2) $$y = \mu, \qquad \text{on } \Gamma = \partial\Omega.$$

This problem has at least one optimal pair (cf. e.g. [2]). The main result of this paper if the following

THEOREM 3.1. – *Let (y^*, u^*) be an optimal pair for* (P). *Then there exists $p^* \in H_0^1(\Omega)$ satisfying along with y^* and u^* the equations*

$$-\Delta y^* + f(y^*) = u^*, \quad \text{in } \Omega \tag{3.5.1)'}$$

$$y^* = \mu, \quad \text{on } \Gamma \tag{3.5.2a}$$

$$\Delta p^* + f'(y^*)p^* = -\nabla g(y^*), \quad \text{a.e. in } [y^* \neq 0] \tag{3.6.1}$$

$$p^* = 0, \quad \text{a.e. in } [y^* = 0] \tag{3.6.2}$$

$$p^* \in \partial h(u^*). \tag{3.6.3}$$

Proof of Theorem 3.1. – We shall deduce the necessary optimality conditions by means of a method of penalization similar to that used in [2], [7].

Let us consider the following sequence of approximating problems.

(P) Minimize $g(y) + h(u) + \frac{1}{2}\|u^* - u\|_s^2$ over all (y, u) subject to

$$-\Delta y + f_\varepsilon(y) = u, \quad \text{in } \Omega \tag{3.5.1.ε}$$

$$y = \mu, \quad \text{on } \Gamma. \tag{3.5.2b}$$

Let y_ε^u denote the solution of (3.5.1.ε)-(3.5.2).

LEMMA 3.1. – *The mapping $u \in L^2(\Omega) \mapsto y_\varepsilon^u \in L^2(\Omega)$ is Gateaux differentiable.*

Proof. – Let $v \in L^2(\Omega)$. For every $\lambda \neq 0$ denote

$$y_\lambda = y_\varepsilon^{u+\lambda v} \quad \text{and} \quad y = y_\varepsilon^u, \qquad u \in L^2(\Omega).$$

Let $p \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of

$$-\Delta p + f'_\varepsilon(y)p = v. \tag{3.7}$$

We shall prove that

$$\lim_{\lambda \to 0} \left\| \frac{y_\lambda - y}{\lambda} - p \right\|_2 = 0. \tag{3.8}$$

Let $R_\lambda = (1/\lambda)(y_\lambda - y) - p$. Then

$$(3.9)\qquad \begin{cases} -\Delta R_\lambda + (1/\lambda)[f_\varepsilon(y_\lambda) - f_\varepsilon(y)] - f'_\varepsilon(y)p = 0\,, \\ -\Delta R_\lambda + f'_\varepsilon(y)R_\lambda = -(1/\lambda)[f_\varepsilon(y_\lambda) - f_\varepsilon(y) - f'_\varepsilon(y)(y_\lambda - y)]\,, \\ -\Delta R_\lambda + f'_\varepsilon(y)R_\lambda = -(1/\lambda)p_\lambda\,. \end{cases}$$

From the mean value theorem we infer

$$f_\varepsilon(y_\lambda) - f_\varepsilon(y) - f'_\varepsilon(y)(y_\lambda - y) = \int_0^1 [f'_\varepsilon(y + t(y_\lambda - y)) - f'_\varepsilon(y)](y_\lambda - y)]\,dt\,.$$

Hence

$$|P_\varepsilon(x)| \leq (y_\lambda(x) - y(x)| \cdot \left| \int_0^1 [f'_\varepsilon(y(x) + t(y_\lambda(x) - y(x)) - f'_\varepsilon(y(x))]\,dt \right|.$$

It is well-known that ([10])

$$\|y_\lambda - y\|_{H_0^1} \leq C\lambda \|v\|_2\,.$$

By Sobolev's inequality we get that

$$\|y_\lambda - y\|_p \leq C\lambda\|v\|_2\,, \qquad \text{for some } p > 2\,.$$

Let $1/q = 1/2 - 1/p$. By the extended Hölder inequality we get

$$\|P_\lambda\|_2 \leq \|y_\lambda - y\|_p \cdot \left\| \int_0^1 [f'_\varepsilon(y + t(y_\lambda - y)) - f'_\varepsilon(y)]\,dt \right\|_q \leq C\lambda \|v\|_2 \cdot \|D_\lambda\|_q\,.$$

By the dominated convergence theorem we deduce that $\|D_\lambda\|_q \to 0$ as $\lambda \to 0$. Hence

$$(1/\lambda)\|P_\lambda\|_2 \to 0\,, \qquad \text{as } \lambda \to 0\,.$$

(3.8) can now be obtained from (3.9) by using the well-known elliptique estimate

$$\|R_\lambda\|_{H^2} \leq \frac{C}{\lambda}\|P_\lambda\|_2 \quad (\text{see [8]})\,. \qquad \text{q.e.d.}$$

LEMMA 3.2. – *Let $(y_\varepsilon, u_\varepsilon)$ be an optimal pair for P_ε. Then there exists $p_\varepsilon \in H_0^1(\Omega)$ that satisfies along with $(y_\varepsilon, u_\varepsilon)$ the system (3.5.1.ε) + (3.5.2) and*

$$(3.10)\qquad -\Delta p_\varepsilon + f'_\varepsilon(y_\varepsilon)p_\varepsilon = -\nabla g(y_\varepsilon)\,,$$

$$(3.11)\qquad p_\varepsilon \in \partial h(u_\varepsilon) + F(u_\varepsilon - u^*)\,,$$

where $F: L^s(\Omega) \to (L^s(\Omega))^*$ is the duality mapping of the space $L^s(\Omega)$.

Proof. – We already know that $u \mapsto y_\varepsilon^u$ is Gâteaux differentiable from L^2 to L^2 and consequently also from L^s to L^2. $q_\varepsilon = (D_u y_\varepsilon^u) v$ satisfies (3.7) $[(D^u y_\varepsilon^u) v$ is the directional derivative of y_ε^u along $v]$. For each $v \in L^s(\Omega)$ and each λ_0

$$g(y_\varepsilon^{u_\varepsilon + \lambda v}) + h(u_\varepsilon + \lambda v) \geqslant g(y_\varepsilon) + h(u_\varepsilon) + \frac{1}{2}(\|u_\varepsilon - u^*\|_s^2 - \|u_\varepsilon + \lambda v - u^*\|_s^2).$$

Letting $\lambda \to 0$ this leads to

$$(\nabla g(y_\varepsilon), q_\varepsilon) + h'(u_\varepsilon, v) \geqslant \langle F(u^* - u_\varepsilon), v \rangle \qquad \forall v \in L^s(\Omega)$$

($h'(u_\varepsilon, v)$ is the directional derivative of h along $v \in L^s(\Omega)$).

Let p_ε be the solution of (3.10). We have

$$(-f_\varepsilon'(y_\varepsilon) p_\varepsilon + \Delta p_\varepsilon, q_\varepsilon)_{L^s} + h'(u_\varepsilon, v) \geqslant \langle F(u^* - u_\varepsilon), v \rangle_{L^s}.$$

Using Green's formula for $p_\varepsilon, q_\varepsilon \in H^2(\Omega) \cap H_0^1(\Omega)$ we get

$$\int_\Omega \Delta p_\varepsilon \cdot q_\varepsilon \, dx = \int_\Omega p_\varepsilon \cdot \Delta q_\varepsilon \, dx.$$

Hence

$$\langle p_\varepsilon, \Delta q_\varepsilon - f_\varepsilon'(y_\varepsilon) q_\varepsilon \rangle_{L^s} + h'(u_\varepsilon, v) \geqslant \langle F(u^* - u_\varepsilon), v \rangle_{L^s}$$
$$-\langle p_\varepsilon, v \rangle_{L^s} + h'(u_\varepsilon, v) \geqslant \langle F(u^* - u_\varepsilon), v \rangle_{L^s}$$
$$h'(u_\varepsilon, v) \geqslant \langle F(u^* - u_\varepsilon) + p_\varepsilon, v \rangle_{L^s}.$$

By a well-known results on subgradients (cf. e.g. [3]) follows (3.11).

q.e.d.

LEMMA 3.3. – *One can choose a subsequence $\varepsilon_n \to 0$ such that*

$$u_{\varepsilon_n} \to u^* \quad \textit{strongly in } L^s(\Omega)$$
$$y_{\varepsilon_n} \to y^* \quad \textit{weakly in } W^{2,s}(\Omega) \textit{ and uniformly in } \bar{\Omega}.$$

There exists also $p^* \in H_0^1(\Omega)$ such that

$$p_{\varepsilon_n} \to p \quad \textit{weakly in } H_0^1(\Omega) \textit{ and strongly in } L^2(\Omega).$$

(In fact $u_\varepsilon \to u^*$ and $y_\varepsilon \to y^*$ as $\varepsilon \to 0$).

Proof. – We see that (3.5.1.ε)-(5.5.2) can be restated as

(3.12.1) $$-\Delta z + \beta_\varepsilon(z) = u, \qquad \text{in } \Omega$$

(3.12.2) $$z = 0, \qquad \text{on } \Gamma$$

where $\beta_\varepsilon(z)=f_\varepsilon(q(x)+z(x))$. Here $q(x)$ denotes the function harmonic in Ω and taking value μ on Γ. (The existence of a such a function follows from a classical result of Kellog, see [8]).

Let (y, u) be an optimal pair. Hence

$$(3.13)\qquad g(y_\varepsilon)+h(u_\varepsilon)+(1/2)\|u_\varepsilon-u^*\|_s^2\leq g(y_\varepsilon^u)+h(u)+(1/2)\|u-u^*\|_s^2$$

when

$$-\Delta y_\varepsilon^u+f_\varepsilon(u_\varepsilon^u)=u_\varepsilon\,,\qquad \text{in } \Omega$$

$$y_\varepsilon^u=\mu\,,\qquad \text{in } \Gamma\,.$$

Denote $z_\varepsilon=y_\varepsilon-q$. Then z_ε satisfies (3.12.1)-(2.12.2).

By estimate (F) and (3.4.2) we obtain

$$\|y^u\|_{W^{2,s}}\leq C\quad \text{independent of } \varepsilon\,.$$

Hence on a subsequence (ε_n). $(y_{\varepsilon_n}^u)$ converges weakly in $W^{2,s}$ to the solution $y=y^u$ of equations (3.5.1)-(3.5.2). From $h(u)\geq C_1\|u\|_s+C_2$ we get

$$\|u\|_s\leq C$$

independent of ε. Hence we can choose a subsequence (u_{ε_n}) weakly convergent in $L^s(\Omega)$. We multiply (3.12.1) with $z_\varepsilon\in H_0^1(\Omega)$ we then integrate and use Green's formula. We get

$$\|z_\varepsilon\|_{H_0^1}^2\leq\|u_\varepsilon\|_2\cdot\|z_\varepsilon\|_2$$

i.e.

$$\|z_\varepsilon\|_{H_0^1}\leq C\quad \text{independent of } \varepsilon\,.$$

We can extract a subsequence denoted also with ε_n such that z_{ε_n} (and hence y_{ε_n} as well) is weakly convergent in $H_0^1(\Omega)$.

The estimate (E) combined with (3.4.2) leads us the conclusion that

$$\|y_{\varepsilon_n}\|_{W^{2,s}}\leq C(\|u_{\varepsilon_n}\|_s+\|y_{\varepsilon_n}\|_2+1)$$

with C independent of ε. Since $(\|u_{\varepsilon_n}\|_{L^s})$ and $(\|y_{\varepsilon_n}\|_2)$ are bounded sequence we get that

$$\|y_{\varepsilon_n}\|_{W^{2,s}}\leq C\quad \text{independent of } \varepsilon$$

and therefore we can suppose that (y_{ε_n}) is $W^{2,s}$-weakly convergent.

Owing to the fact that $s>\max\{N/2, 2\}$ we get from the classical imbedding theorems for Sobolev spaces that (y_{ε_n}) is also uniformly convergent in $\bar{\Omega}$. Let $\bar{y}$ the limit of the sequence (y_{ε_n}). Then $z_{\varepsilon_n} \to \bar{z} = \bar{y} - q$ weakly in $W^{2,s}$.

Let $\bar{u}$ be the L^s-weak limit of u_{ε_n}. We make $\varepsilon_n \to 0$ in (3.13). The L^s norm and h are weakly lower semicontinuous functions and hence

$$(3.13') \qquad g(\bar{y}) + h(\bar{u}) + \frac{1}{2}\|\bar{u} - u^*\|_s^2 \leq g(y^u) + h(u) + \frac{1}{2}\|u - u^*\|_s^2 .$$

If in (3.15′) we set $u = u^*$ we deduce from the optimality of u^* that in fact $\bar{u} = u^*$ and $u_{\varepsilon_n} \to u^*$ even strongly in $L^s(\Omega)$. Clearly $\bar{y} = y^*$ in this situation.

We multiply (3.10) with p_ε, then integrate and use Green's formula. The following inequality is obtained

$$\|p_\varepsilon\|_{H_0^1}^2 \leq \|p_\varepsilon\|_2 \cdot \|\nabla g(y_\varepsilon)\|_2$$

and consequently

$$\|p_\varepsilon\|_{H_0^1} \leq C \quad \text{independent of } \varepsilon .$$

since g locally Lipschitz and $y_{\varepsilon_n} \to y$ strongly in L^s. Hence on a subsequences $p_{\varepsilon_n} \to p^*$ weakly in $H_0^1(\Omega)$.

q.e.d.

We know that $p_{\varepsilon_n} \to p^*$ strongly in L^2 and hence strongly in $L^{s'}$ $(1/s = 1/s' = 1)$ $u_{\varepsilon_n} \to u^*$ strongly in L^s. The duality mapping F is uniformly continuous on bounded sets of L^s. If now $\varepsilon_n \to 0$ in we infer (according to [3]) that $p^* \in \partial h(u^*)$. Hence (3.6.3) is proved.

The proof of the remaining part of Theorem 3.1 relies on the following result

LEMMA 3.4.

$$(3.14) \qquad \int_\Omega f'_\varepsilon(y_\varepsilon)|p_\varepsilon|\, dx \leq C \quad \text{independent of } \varepsilon .$$

Proof. – Let $\theta_n \in C^1(R)$ be a sequence of increasing functions such that $\theta_n(0) = 0$ and $\theta_n(r) \to \operatorname{sgn} r \ \forall r \in \mathbb{R}$.

It is well-known that $\theta_n(p_\varepsilon) \in H_0^1(\Omega)$ (see e.g. [9]). We multiply (3.10) with $\theta_n(p_\varepsilon)$. Newt we integrate and use a Green formula to get:

$$\int_\Omega |\nabla p_\varepsilon|^2 \theta'_n(p)\, dx + \int_\Omega f'_\varepsilon(y_\varepsilon) p_\varepsilon \theta_n(p_\varepsilon)\, dx = - \int_\Omega \nabla g(y_\varepsilon)\, \theta_n(p_\varepsilon)\, dx .$$

Taking into account that $\theta'_n \geq 0$ we get

$$\int_\Omega f'_\varepsilon(y_\varepsilon) p_\varepsilon \theta_n(p_\varepsilon)\, dx \leq - \int_\Omega \nabla g(y_\varepsilon)\, \theta_n(p_\varepsilon)\, dx .$$

Let $n \to \infty$. Then $\int_\Omega f'_\varepsilon(y)|p_\varepsilon|\,dx \leq C$ independent of ε since g is locally Lipschitz. q.e.d.

The sequence $f'_\varepsilon(z_\varepsilon)p_\varepsilon$ is bounded in L^1. One can extract a subsequence weakly convergent in $(L^\infty(\Omega))^*$ i.e. there exist a measure $\lambda \in (L^\infty(\Omega))^*$ such that for every $\varphi \in L^\infty(\Omega)$

$$\lim_{\varepsilon_n \to 0} \int_\Omega f'_{\varepsilon_n}(y_{\varepsilon_n})p_{\varepsilon_n}\,dx = \int_\Omega \varphi\, d\lambda\,.$$

(3.10) may be rewritten as

$$\forall \varphi \in C_0^1(\Omega)\colon \int_\Omega \nabla p_\varepsilon \cdot \nabla\varphi\, dx + \int_\Omega f'_\varepsilon(y_\varepsilon)p_\varepsilon d\varphi\, dx = \int_\Omega \nabla g(y_\varepsilon)\,\varphi\, dx\,.$$

If in the above equality tends to zero we get

$$\int_\Omega \nabla p^* \cdot \nabla\varphi\, dx + \int_\Omega \varphi\, d\lambda = -\int_\Omega \nabla g(y)\varphi\, dx\,. \tag{3.15}$$

LEMMA 3.5:

$$\lambda = f'(y^*)p\,, \qquad \text{on } [y^* \neq 0]$$

in distribution sense i.e.

$$\int_\Omega \varphi\, d\lambda = \int_\Omega f'(y^*)p^*\varphi\, dx\,, \qquad \forall\varphi \in C_0^1(\Omega),\ \operatorname{supp}\varphi \subset [y^* \neq 0]\,.$$

Proof. – Let $\varphi \in C_0^1(\Omega)$, $\operatorname{supp}\varphi \subset [y^* \neq 0]$. Then there is $\delta_0 > 0$ such that $\operatorname{supp}\varphi \subset [|y^*| > \delta_0]$.

For every $\varepsilon > 0$ we denote

$$d_\varepsilon = \max_\Omega |y_\varepsilon(x) - y^*(x)|\,, \qquad \lambda_\varepsilon = \max\{\varepsilon, d_\varepsilon\}\,.$$

Owing to the fact that $y_\varepsilon \to y^*$ uniformly on $\bar\Omega$ we get $d_\varepsilon \to 0$ as $\varepsilon \to 0$. Therefore we can chose $n_0 > 0$ such that $\forall n \geq n_0$, $\lambda_{\varepsilon_n} < \delta_0/2$. We have

$$\int_\Omega f'_{\varepsilon_n}(y_{\varepsilon_n})p_{\varepsilon_n}\varphi\, dx = \int_{\operatorname{supp}\varphi} f'_{\varepsilon_n}(y_{\varepsilon_n})p_{\varepsilon_n}\varphi\, dx\,, \qquad \forall \varepsilon_n > 0\,. \tag{3.16}$$

Let $n \geq n_0$. Then $f'_{\varepsilon_n}(y_{\varepsilon_n}) = f'(y_{\varepsilon_n})$ on $\operatorname{supp}\varphi$. Indeed for $x \in \operatorname{supp}\varphi\, |y^*(x)| > \delta_0$.

Since $|y_{\varepsilon_n}(x) - y^*(x)| \leq \lambda_{\varepsilon_n} < \delta_0/2$ and therefore

$$|y_{\varepsilon_n}(x)| \geq |y^*(x)| - |y^*(x) - y_{\varepsilon_n}(x)| > \delta_0/2\,.$$

The equality $f_\varepsilon(r)=f(r)$, $\forall|r|\geqslant\varepsilon$ implies the desired equality (3.16) becomes

$$\int_\Omega f'_{\varepsilon_n}(y_{\varepsilon_n})p_{\varepsilon_n}\varphi\,dx=\int_{\operatorname{supp}\varphi} f'(y_{\varepsilon_n})p_{\varepsilon_n}\varphi\,dx\,,\qquad \forall n\geqslant n_0\,.$$

If we let $\varepsilon_n\to$ we get our lemma.

q.e.d.

(3.15) combined with the above lemma proves the assertion (3.6.1) of the theorem. The only thing left to be proved is (3.6.2) i.e. $p^*=0$ on $[y^*=0]$.

Let us suppose the contrary that is the measure of $[p^*\neq 0]\cap[y^*=0]$ is greater than zero. Then there is $\delta>0$ such that the set $[|p^*|>\delta]\cap[y^*=0]$ has non-zero measure. We denote this set with X

$$p_{\varepsilon_n}\to p^*\qquad \text{a.e. in } X\,.$$

From Egorov's theorem we infer the existence of a set $Y\subset X$ such that

$$m(X-Y)\subset m(X)/2\quad\text{ and }\quad p_{\varepsilon_n}\to p^*\text{ uniformly on } Y\,.$$

Hence there exists $\delta_0>0$ such that

$$\forall\varepsilon_n>\delta_0\,,\qquad |p_{\varepsilon_n}|>\delta/2\,.$$

Let us consider $\varphi\in C_0^\infty(\Omega)$ satisfying

$$\varphi=1\quad\text{on } Y\qquad\text{and}\qquad \varphi\geqslant\quad\text{on }\Omega$$

$f'_\varepsilon(y_\varepsilon)p_\varepsilon$ is bounded in L^1. Hence there is $C>0$:

$$\int_\Omega f'_\varepsilon(y_\varepsilon)p_\varepsilon\varphi\,dx\leqslant C\,.$$

On the other hand

$$\int_\Omega f'_\varepsilon(y_\varepsilon)|p_\varepsilon|\,\varphi\,dx\geqslant\delta/2\cdot\int_y f'_\varepsilon(y_\varepsilon)\,dx\,.$$

But

$$\int_y f'_\varepsilon(y_\varepsilon)\,dx\to\infty\qquad\text{as }\varepsilon\to 0\text{ owing to (5.4.3).}$$

We get to a contradiction that concludes the proof of Theorem 3.1.

q.e.d.

4. Examples.

We shall now show how the optimality conditions look like in some specific situation. Let us consider the problem

(P) Maximize $\int_\Omega y(x)\,dx$ over all (y, u) subject to

$$-\Delta y + |y|^{\alpha-1} y = u, \qquad \text{in } \Omega \tag{4.1.1}$$

$$y = \mu, \qquad \text{on } \Gamma \tag{4.1.2}$$

where $\mu \geqslant 0$ and

$$u \in U_{ad} = \left\{ v \in L^s(\Omega)/0 \leqslant v < L \cdot \int_\Omega v(x)\,dx \leqslant M \leqslant Lm(\Omega) \right\}. \tag{4.1.3}$$

We see that (P) lies in the general frame of Theorem 3.1 by setting

$$g(y) = -\int_\Omega y(x)\,dx$$

and

$$h(u) = I_{U_{ad}}(u) = \begin{cases} 0, & u \in U_{ad} \\ +\infty, & u \notin U_{ad}. \end{cases}$$

There is no difficulty in se that (P) is equivalent to the following problem.

(Q) Maximize $\int_\Omega y(x)\,dx$ over all (y, u)

subject to (4.1.1)-(4.1.2) and

$$u \in K = \left\{ v \in U_{ad}/\int_\Omega v(x)\,dx = M \right\}.$$

Indeed $\operatorname{Max} P \geqslant \operatorname{Max} Q$ because $K \subset U_{ad}$.

To prove the reverse inequality let (y^u, u) be an admisible pair for (P). Then there exists $v \in K$ such that $u(x) \leqslant v(x)$. E.g. set

$$v(x) = \begin{cases} u(x), & x \in X \\ L, & x \in \Omega \setminus X \end{cases}$$

such that $\int_\Omega v(x)\,dx = M$.

By the comparison result we infer

$$\int_\Omega y^v\,dx \geqslant \int_\Omega y^u\,dx \quad \text{i.e.} \quad \operatorname{Max}(Q) \geqslant \int_\Omega y^u\,dx\,, \qquad \forall u \in U_{ad}\,.$$

The above remark shows us that if (y^*, u^*) is an optimal pair for (P) then $u^* \in K$ and therefore (y^*, u^*) is an optimal pair for (Q). We shall henceforth focus on problem (Q).

We see that $K = K_1 \cap K_2$

$$K_1 = \{v \in L^s(\Omega)/0 \leqslant v \leqslant L\}$$

$$K_2 = \left\{v \in L^s(\Omega)/\int_\Omega v\,dx = M\right\}.$$

In this situation we have the relation

$$\partial I_K = \partial I_{K_1} + \partial I_{K_2}\ [3]\,.$$

An easy computation shows

$$\partial I_{K_1}(u) = \left\{ v \in L_{s'}(\Omega)/v(x) = \begin{cases} 0\,, & 0<u<L \\ \leqslant 0\,, & u=0 \\ \geqslant 0\,, & u=L \end{cases} \right\}$$

$$\partial I_{K_2}(u) = \{v \in L^{s'}(\Omega)/\exists\lambda \in \mathbb{R}\colon v(x) \equiv \lambda\} \left(\frac{1}{s} + \frac{1}{s'} = 1\right).$$

We draw the conclusion

$$\partial I_K(u) = \left\{ v \in L_{s'}(\Omega)/\exists\lambda\colon v(x) = \begin{cases} \lambda\,, & u<L \\ \leqslant \lambda\,, & u=0 \\ \geqslant \lambda\,, & u=L \end{cases} \right\}. \tag{4.2}$$

The optimality conditions can be written as:

$$\exists p^* \in H_1^0(\Omega) \qquad \text{such that}$$

$$-\Delta p^* + \frac{\alpha}{(y^*)^{1-\alpha}} p^* = 1\,, \qquad \text{on } [y^* \neq 0] \tag{4.3}$$

$$p^* = 0\,, \qquad \text{on } [y^* = 0]\,. \tag{4.4}$$

In view of (4.2) the relation (3.6.3) becomes

$$\exists\lambda\colon u^*(x) = \begin{cases} 0\,, & p^* < \lambda \\ L\,, & p^* > \lambda\,. \end{cases} \tag{4.5}$$

From the comparision result we see $y_\varepsilon > 0$ and $p_\varepsilon > 0$. Since $p_\varepsilon \to p^*$ a.e. on a subsequence we infer $p^* \geq 0$.

PROPOSITION 4.1. – *Let u^* an optimal control in problem* (Q).

a) If $M = Lm(\Omega)$ then $u^* \equiv L$.

b) If $M < Lm(\Omega)$ then in (4.5 $\lambda \geq 0$ and

$$u^*(x) = \begin{cases} 0, & p^* < \lambda \\ (\lambda\alpha)_{\alpha/(1-\alpha)}, & p^* = \lambda \\ L, & p^* > \lambda. \end{cases} \tag{4.6}$$

Proof. – *a*) It follows from the comparison principle that $y_{u^*} \geq y_u$ $\forall u \in K$.

b) We already know that $p^* \geq 0$. If $\lambda < 0$ then (4.5) would give us $u^* = L$ but such a function is not an admisible control. Hence $\lambda \geq 0$.

Let us observe that $p^* \neq 0$ a.e. on $[y^* \neq 0]$. Indeed from known elliptique regularity results we infer $p^* \in H^2_{\text{loc}}\, [y^* \neq 0]$ (cf. e.g. [6]). Let $S = [p^* = 0] \cap [y^* \neq 0]$. By Stampacchia's lemma (see [10]) it follows that $\Delta p^* = 0$ a.e. on S and if $m(S) > 0$ (4.3) wouldn't be statisfied. Therefore $m(S) = 0$.

We distinguish two situations

1) $\lambda = 0$.

Then $[p^* = 0] \subset [y^* = 0]$ by the previous remark. Since $u^* = 0$ a.e. on $[y^* = 0]$ Proposition 4.1 holds true in this situation

2) $\lambda > 0$.

Let $\Lambda = [p^* = \lambda]$. Then $\Delta p^* = 0$ a.e. on Λ by Stampacchia's lemma. Hence $\alpha/(y^*)_{(1-\alpha)} = 1$ a.e. on Λ from which we infer $y^* = (\lambda\alpha)_{1/(1-\alpha)}$ a.e. on Λ. Again using Stampacchia's lemma we deduce $\Delta y^* = 0$ a.e. on Λ and Proposition 4.1 follows upon inspecting (4.1.1). q.e.d.

Now we consider a further specialization: $\mu \equiv 0$ and $\Omega = B_R(0)$.

THEOREM 4.1. – *The function*

$$u(x) = \begin{cases} 0, & |x| > R_0 \\ L, & |x| \leq R_0, \ \omega_N R_N = M/L \end{cases}$$

(here ω_N is the volume of the unit ball in R_N) is the unique radially symmetric optimal control for problem (Q). *Moreover every optimal control is bang-bang.*

Proof. – We shall use the technique of symmetric rearrangements of Hardy and Littlewood. We first recall some basic facts concerning symmetric rearrangements. These are proved e.g. in [1], [9], [11],

Let u: $\Omega \to R$ be a measurable function ($\Omega \subset R_N$ is a bounded open set) $m(\Omega) = |\Omega|$. The distribution function of u is the function μ: $[0, +\infty[\to R_+$ is defined by

$$\mu(t) = m\{x \in \Omega / |u(x)| > t\}.$$

The decreasing rearrangement of u is defined by

$$\tilde{u}(s) = \inf\{t \geq 0 / \mu(t) < s\}, \qquad 0 \leq s \leq |\Omega|.$$

Finally, the symmetric rearrangement of u is given by

$$u^*(x) = \tilde{u}(\Omega_N |x|_N), \qquad x \in \Omega^*$$

where Ω^* is the ball centred at the origin of measure $|\Omega|$. We say that the concentration of $\phi \in L^1(\Omega)$ is less or equal to that of $\psi \in L^1(\Omega)$ written $\phi \lesssim \psi$ if

$$\int_0^t \tilde{\phi}(s)\, ds \leq \int_0^t \tilde{\psi}(s)\, ds, \qquad \forall t \in [0, \Omega].$$

The following inequality holds true

$$\int_\Omega u(x)\, dx \leq \int_{\Omega^*} u^*(x)\, dx. \tag{4.7}$$

Equality takes place for example when $u \geq 0$.

In the proof of theorem 4.1 we shall use a comparison results due to J. I. Diaz [4], [5] (we state only a special form of it)

THEOREM A. – *Let f be a continuous increasing function such that $f(0) = 0$. Let $g_i \in L^1(B_R(0))$, $i = 1, 2$ and $g_2 = g_2$. Let $u_i \in W_0^{1,1}(B_R(0))$. be such that*

$$-\Delta u_i + f(u_i) = g_i, \qquad i = 1, 2.$$

If $g_1^* \lesssim g_2$ then $f(u_1) \geq f(u_2)$.

In theorem A let $f(r) = |r|^{\alpha - 1} r$. Let $g_1 = u$ an admissible control and $g_2 = u^*$

$$\int_{B_R(0)} u\, dx = \int_{B_R(0)} u^*\, dx = M$$

hence u^* is an admissible control. In this case $g_1^* = g_2$ and therefore $g_{1\,9}{}^* \leq g_2$. Let us denote $y = y_u$ and $z = y_{u^*}$ the solutions to (4.1.1)-(4.1.2) (with $\mu = 0$) corresponding to u and respectively to u. $\tilde{y}(s)$ and $\tilde{z}(s)$ denote the decreasing rearrangements of these functions. Since f is increasing and continuous we have

$$(\widetilde{f \circ y})(s) = f(\tilde{y}(s)) \quad \text{and} \quad (\widetilde{f \circ z})(s) = f(\tilde{z}(s)).$$

By theorem A:

$$\int_0^t (\widetilde{f \circ y})(s)\, ds \leq \int_0^t (\widetilde{f \circ y})(s)\, ds, \qquad \forall s \in [0, |\Omega|].$$

Hence

$$\int_0^t f(\tilde{y}(s))\, ds \leq \int_0^t f(\tilde{z}(s))\, ds. \tag{4.8}$$

Now we recall an inequality of Hardy and Litlewood ([1]).

(I) Let $a, b \in L^1(0, M)$ two nonnegative and decreasing functions and suppose that

$$\int_0^t a(s)\, ds \leq \int_0^t b(s)\, ds, \qquad \forall t \in [0, M].$$

Then for every continuous convex function $\phi\colon \mathbb{R} \to \mathbb{R}$ we have

$$\int_0^t \phi(a(s))\, ds \leq \int_0^t \phi(b(s))\, ds, \qquad \forall t \in [0, M].$$

In (I) we set $a(s) = f(\tilde{y}(s))$ and $b(s) = f(\tilde{z}(s))$, $M = |\Omega|$ and $\phi(r) = f_{-1}(r) = r^{1/\alpha}$ (ϕ is convex since $\alpha \in \,]0, 1[$).

We get from (4.8)

$$\int_0^t \tilde{y}(s)\, ds \leq \int_0^t \tilde{z}(s)\, ds$$

or equivalently

$$\left(\int_{B_R} y(x)\, dx = \right) \int_{B_R} y^*(x)\, dx \leq \int_{B_R} z^*(x)\, dx = \int_{B_R} z(x)\, dx$$

i.e.

$$\int_{B_R} y^u\, dx \leq \int_{B_R} y^{u*}\, dx.$$

We have thus proved:

LEMMA 4.1. – *If u is an optimal control then so is its symmetric rearrangement u^*.*

Let u be an optimal control. By (4.6) we have

$$u(x)=\begin{cases}0, & p^*<\lambda\\ (\lambda\alpha)_{\alpha/(1-\alpha)}, & p^*=\lambda\\ L, & p^*>\lambda\end{cases}$$

u^*-its symmetric rearrangement is then equal to

$$u(x)=\begin{cases}L, & |x|\in[0,R_1]\\ (\lambda\alpha)_{\alpha/(1-\alpha)}, & |x|\in[R_1,R_2]\\ 0, & |x|\in[R_2,R].\end{cases}$$

Here $R_1\leqslant R_2$ satisfy $\omega_R R_1^N=m[p>\lambda]$, $\omega_N(R_2^N-R_1^N)=m[p=\lambda]$ where p is the adjoint state corresponding to u in virtue of Theorem 3.1.

We prove that either $R_1=R_2$, or $\lambda=0$, or $(\lambda\alpha)_{\alpha/(1-\alpha)}=L$ i.e. u has the form indicated in the statement of Theorem 4.1.

Let us suppose $R_1<R_2$ and $(\lambda\alpha)_{\alpha/(1-\alpha)}<L$. Let $z=y_{u^*}$ and q the adjoint state corresponding to the optimal control u^*.

According to Proposition 4.1 we get $q(x)=\lambda$ when $R_1\leqslant|x|\leqslant R_2$ and hence $z(x)=(\lambda\alpha)_{1/(1-\alpha)}$ when $R_1\leqslant|x|\leqslant R_2$. Hence $\nabla z(x)=0$ $R_1\leqslant|x|\leqslant R_2$. z satisfies the following equation with mixed boundary values

$$\begin{aligned}-\Delta z+z^\alpha&=0, & R_2<|x|<R\\ z&=0, & |x|=R\\ \partial z/\partial\nu&=0, & |x|=R_2.\end{aligned}$$

Hence $z\equiv0$ on $R_2\leqslant|x|\leqslant R$. Since $z\in C(\bar\Omega)$ we get $(\lambda\alpha)_{1/(1-\alpha)}=0$ that is $\lambda=0$.

We have thus proved that if u is an optimal control then so is u^* and moreover u^* has the form indicated in the statement.

To prove the remaining part of the theorem is suffices to observe that if the range of u consisted of more then three elements then so would be true for the range of its symmetric arrangement whish is necessarilly bang-bang.

q.e.d.

We give a nother application of general theory

(P) Minimize $\int_\Omega y(x)\,dx$ over all (y, u) subject to

$$-\Delta y + f(y) = u\,, \qquad \text{in } \Omega \tag{4.1.1}$$

$$y = \mu\,, \qquad \text{on } \Gamma. \tag{4.1.2}$$

Here $f(r) = |r|^{\alpha-1} r$, $\mu \geqslant 0$ and

$$u \in K = \left\{ v \in L^s(\Omega)/0 \leqslant v \leqslant L,\ \int_\Omega v\,dx = M \right\}. \tag{4.1.3}$$

The optimality conditions are

$$\begin{aligned} &\exists p^* \in H_0^1(\Omega):\ -\Delta p^* + \frac{\alpha}{(y^*)^{1-\alpha}} p^* = -1\,, && \text{a.e. in } [y^* > 0] \\ &p^* = 0\,, && \text{a.e. in } [y^* = 0] \\ &\exists \lambda:\ u^*(x) = \begin{cases} 0\,, & p^* < \lambda \\ L\,, & p^* > \lambda\,. \end{cases} \end{aligned} \tag{4.9}$$

Proceeding as in Proposition 4.1 we get $\lambda \leqslant 0$ and

$$u^*(x) = \begin{cases} 0\,, & p^* < \lambda \\ (-\lambda\alpha)_{\alpha/(1-\alpha)}\,, & p^* = \lambda \\ L\,, & p^* > \lambda\,. \end{cases}$$

When $\Omega = B_R(0)$ and $\mu \equiv$ const. then one can say more. It is convenient to consider a more general formulation

(P_0) Min $\int_\Omega H(y)\,dx$ over all (y, u) subject to

$$-\Delta y + f(y) = u\,, \qquad \text{in } \Omega = B_R(0)$$

$$y = \mu\,, \qquad \text{on } \Gamma\ u \in K\,.$$

Here $H: \mathbb{R} \to \mathbb{R}$ is a continuous convex increasing function while $f: \mathbb{R} \to \mathbb{R}$ denotes a concave increasing function satisfying the conditions in section **2.**

PROPOSIZION 4.2. – *The problem* (P_0) *has a least one radially symmetric optimal control.*

Proof. – We use another symmetrization method, yet very different of that used in Theorem 4.1.

Let $G=O(N)$ the N-th orthogonal group $(g \in O(N) \Leftrightarrow gg_t = 1)$ G is a compact Lie group. Our symmetrization method starts from the simple remark that a function u: $B_R(0) \to R$ is radially. symmetric if and only if

$$\forall g \in G \quad u \equiv u^g \quad \text{where} \quad u^g(x) = u(gx) \tag{4.10}$$

(gx denotes the action of G on $B_R(0)$). This remark is an easy consequence of the fact that G acts transitively on S^{N-1}.

One can define on G a Haar measure χ translation invariant such that $\chi(G)=1$ (see e.g. [12]). We denote $d\chi = dg$.

Let

$$u^G(x) = \int_G u^g(x)\, dg \tag{4.11}$$

u_G is radially symmetric since χ in translation invariant. For example when $N=1$, $\Omega =]-R, R[$

$$u^{O(1)}(x) = \frac{1}{2}[u(x) + u(-x)]$$

and when $N=2$, $\Omega = \{z/|z| < R\}$

$$u^{O(2)}(z) = \frac{1}{4\pi}\int_0^{2\pi} u(\exp[i\theta]\, z) + u(\exp[i\theta]\, \bar{z})\, d\theta\, .$$

By Fubini theorem we get

$$\int_G dg \int_{B_R} u^g(x)\, dx = \int_{B_R} dx \int_G u^g(x)\, dg\, .$$

Since

$$|\det g| = 1 \int_{B_R} u^g(x)\, dx = \int_{B_R} u(x)\, dx\, .$$

Therefore

$$\int_\Omega u(x)\, dx = \int_\Omega u^G(x)\, dx\, , \qquad (\chi(G)=1)\, . \tag{4.12}$$

If ϕ is any continuous convex (concave) function ϕ: $R \to R$ then

$$\phi\left(\int_G u^g(x)\, dx\right) \leqslant (\geqslant) \int_G \phi(u^g(x))\, dg \tag{4.13}$$

i.e.

(4.13′) $$\phi(u^G) \leqslant (\geqslant)(\phi \circ u)^G \quad \text{(see Appendix)}.$$

Because $gg' = 1$ then $\Delta y^g = (\Delta y^g)$ (Δ is rotation invariant) $\forall y \in C_2(\bar{\Omega})$ and passing to the limit

$$\Delta y^g = (\Delta y)^g, \qquad \forall y \in W^{2,1}(\Omega)$$

Let (y, u) be an admissible pair. y_g satisfies

$$-\Delta y^g + f(y^g) = u^g, \qquad \text{in } \Omega$$
$$y^g = \mu, \qquad \text{on } \Gamma.$$

We integrate with respect to $g \in G$. We get

$$\int_G -\Delta y^g \, dg + \int_G f(y^g)\, dg = u^G, \qquad \text{in } \Omega$$
$$y^G = \mu, \qquad \text{on } \Gamma.$$

Since

$$\int_G \Delta y^g \, dg = \Delta \int_G y^g \, dg, \qquad \forall y \in C^2(\bar{\Omega})$$

we obtain at limit

$$-\Delta y^G + \int_G f(y^g)\, dg = u^G, \qquad \text{in } \Omega$$
$$y^G = \mu, \qquad \text{on } \Gamma.$$

f is concave. Using the concave part of (4.13) we infer

$$-\Delta y^G + f(y^G) \geqslant u^G, \qquad \text{in } \Omega$$
$$y^G = \mu, \qquad \text{on } \Gamma$$

$u^G \in K$ owing to (4.12).

We infer from the comparison principle that $y^G \geqslant y^{u^G}$. H is increasing so

$$\int_\Omega H(y^G)\, dx \geqslant \int_\Omega H(y^{u^G})\, dx$$

H is also convex. From (4.13) we get

$$H(y^G) \leqslant [H(y)]^G.$$

Hence using (4.12)

$$\int_{B_R} H(y^{u^G})\,dx \leqslant \int_{B_R} H(y^u)\,dx\,.$$

We draw the conclusion:

«If u is an optimal control then so is u^G which is radially symmetric».
q.e.d.

Hence problem (P) has a radially optimal control.

Proposition 4.1 is useful in many other situations. Let us consider the following control problem

(OP) Minimize $\int_{B_R(0)} y(x)\,dx$ over all (y, u) subject to

$$\left.\begin{array}{lr} (4.14.1) & -\Delta y + u \geqslant 0 \\ & y \geqslant 0 \\ (4.14.2) & (-\Delta y + u)\,y = 0 \end{array}\right\} \quad \text{in } B_R(0)$$

(4.14.3) $\quad y|_\Gamma = 1$ and

(4.14.4) $$u \in K = \left\{ v \in L^2(B_R)/0 \leqslant v \leqslant L,\ \int_{B_R} v(x)\,dx = M \right\}.$$

(4.14.1)-(4.14.3) is the well-known obstable problem. For $N = 2$ it models the deformation of a circular membrane that has its boundary clamped at height 1 above a rigid obstacle in the horizontal plane.

The membrane is subjected to a distribution of forces of density u.

If we set $y = z + 1$ the state equations can be rewritten as

(4.15) $$-\Delta z = \beta(z+1) \ni -u\,, \qquad z|_\Gamma = 0$$

where

$$\beta(r) = \begin{cases} \phi\,, & r < 0\,, \\ ]-\infty, 0]\,, & r = 0\,, \\ 0\,, & r > 0\,. \end{cases}$$

Let us consider the following approximative problems

(OP) Minimize $\int_{B_R} y(x)\,dx$ over all (y, u) subject to

$$-\Delta z + \beta_\varepsilon(z+1) = -u\,, \qquad u \in K$$

$$z|_\Gamma = 0\,.$$

Here and $\beta_\varepsilon(r) = (1/\varepsilon)\,r_-$.

β_ε is a concave increasing function. Hence by Proposition 4.2 (OP$_\varepsilon$) has at least one radially symmetric optimal control. We denote it by u_ε^*. Let the corresponding state be $z_\varepsilon^* = y_\varepsilon{}^* \to 1$. z_ε^* is also radially symmetric.

It is well-known that

$$u_\varepsilon^* \to u^* \quad \text{weakly in } L^2(\Omega)$$

$$z_\varepsilon^* \to z^* \quad \text{weakly in } H_0^1(\Omega)\ (z^* = y^* - 1)$$

where (u^*, y^*) is an optimal pair for (OP) see for instance [2].

Since $y_\varepsilon^* \to y^*$ strongly in $L_2(\Omega)$ we get that $y_{\varepsilon 7}{}^* \to y^*$ a.e. on a subsequence. Therefore y^* is radially symmetric.

Thus we have proved that (OP) admits at least one optimal pair (y^*, u^*) such that state y^* is radially symmetric. Since the mapping $u \mapsto y^{u'}$ (= solution of (4.14.1)-(4.14.2) corresponding to u) is not necessarilly one-to-one it is to be expected that the same state can be obtained using different controls. However one can prove

PROPOSITION 4.4. – *Let (y^*, u^*) be an optimal pair for* (OP) *such that y^* is radially symmetric. Then u^* is bang-bang.*

Proof. – We use the necessary optimality conditions for (OP) (see [2]).

These conditions are:

«There exists $p^* \in H_0^1(B_R)$ such that

(4.16.1) $$-\Delta p^* + 1 = 0\,, \qquad \text{in } [y^* > 0]$$

(4.16.2) $$p^*(-\Delta y^* + u^*) = 0\,, \qquad \text{a.e. in } B_N$$

(4.16.3) $$p \in \partial I_K(u^*)\,, \qquad \text{a.e. in } B_R$$

(4.16.3) can still be defined to

(4.16.4) $$\exists\lambda:\ u^*(x) = \begin{cases} 0\,, & p^* > \lambda \\ L\,, & p^* > \lambda\,. \end{cases}$$

Inspecting the proof of these optimality conditions we find that p^* was obtained as a weak H_0^1 limit of negative functions and consequently

$$p^* \leqslant 0 .$$

Owing to this fact in (4.16.4) $\lambda \leqslant 0$.

u^* would be known if we knew its values on the set $[p^* = \lambda]$. This is what we want to find. We distinguish two situations

A. $\lambda < 0$

$$-\Delta y^* + u^* = 0 , \qquad \text{a.e. in } [p^* = \lambda] . \tag{4.17}$$

Furthermore $y^* = 0$ a.e. in $[p^* = \lambda]$. Indeed, if we supposed the contrary them $m([p^* = \lambda] \cap [y^* > 0]) > 0$.

$p^* \in H^2_{\text{loc}}[y^* > 0]$ (see [6]) and therefore $\Delta p^* = 0$ a.e. on $[p^* = \lambda] \cap$ $\cap [y^* > 0]$ by Stampacchia's lemma. This last conclusion does not agree with (4.16.1).

By standard regularity results ([10]) we infer $y \in W^{2,p}(B_R)$. Hence $\Delta y^* = 0$ a.e. in $[p^* = \lambda]$ since $y^* \equiv 0$ a.e. on $[p^* = \lambda]$. By (4.17) we deduce $u^* = 0$ a.e. in $[p^* = \lambda]$ i.e. u^* is bang-bang.

B. $\lambda = 0$. As in case A. we have $y^* = 0$ a.e. in $p^* = 0$. Recalling that $p^* \leqslant 0$ we rewrite (4.16.4) as

$$u^*(x) = \begin{cases} 0 , & p^* \neq 0 , \\ \text{undefined} , & p^* = 0 . \end{cases} \tag{4.18}$$

Since $\int_{B_R} u^* \, dx = M$ we get $m[p^* = 0] > 0$. Since $[p^* = 0] \subset [y^* = 0]$ we infer the coincidence set $C = [y^* = 0]$ has non-zero measure. This set has nice properties namely:

1) $[y^* = 0] \cap \partial B_R = \phi$ since $y^*|_{\partial B_R} = 1$.

2) Since $y^* \in W^{2,p}(B_R)$, $\forall p \in]1, \infty[$ and y attains its minimum on C we get $\nabla y^* = 0$ on C.

3) C is radially symmetric compact and $m(C) > 0$.

Let $r_0 = \min \{ r \in [0, R] / C \subset B_r(0) \}$. From the properties 1) and 3) above we get that $0 < r_0 < R$.

Moreover from the definition of r_0 we infer

$$y^*(x) = 0 , \quad \nabla y^*(x) = 0 , \qquad \forall |x| = r_0 , \tag{4.19}$$

$$y^*(x) > 0 , \qquad \forall |x| > r_0 . \tag{4.20}$$

We knaw that $p^* \neq 0$ on $[y^* > 0]$ and by (4.18) we get $u^* = 0$ on $[y^* > 0]$.

Therefore y^* satisfies the equation

$$(4.21)\qquad \begin{cases} -\Delta y^* = 0\,, & r_0 < |x| < R \\ y^*(x) = 0\,, & |x| = r_0 \\ y(x) = 1\,, & |x| = R \end{cases}$$

and

$$(4.22)\qquad \partial y^*/\partial \nu = 0\,, \qquad \text{on } |x| = r_0\,.$$

By Hopf's maximum principle we infer that y^* obtaines its minimum at a point x_0, $|x_0| = r_0$ and in this point $\partial y^*/\partial \nu \neq 0$ in contradiction to (4.22). Hence in (4.16.4) alway $\lambda < 0$ and therefore u^* is bang-bang. q.e.d.

As we have previously mentioned a given state can be eventually obtained using several controls. However Proposition 4.3 sets up severe restrictions on the optimal controls leading to radially symmetric states. Namely all such controls should be bang-bang.

We also see that this non-uniquenes of the controls derives from the fact that we don't know their behaviour on the coincidence set $[y^* = 0]$. Hence Proposition 4.3 restricts this behaviour on $[y^* = 0]$.

Let us observe that if $y = y^{u^1}$ $(u_1 \in K)$ then every $0 \leq u_2 \leq 1$ such that

(i) $u_1 = u_2$ on $[y > 0]$,

(ii) $\displaystyle\int_{[y=0]} u_1\, dx = \int_{[y=0]} u_2\, dx\,.$

We have $u \in K$ and $y = y^{u^2}$.

If now suppose $y = y^*$ a radially symmetric optimal state and u_1, u_2 the above two optimal controls leading to y then Proposition 4.3 says that all u_1, u_2 satisfuing (i)-(iii) should be bang-bang.

This additional condition leads us to the conclusion that every control corresponding to the optimal state y^* should be constant on the coinciden-ce set $[y^* = 0]$ (in a.e. sense).

Indeed if this thing did not happen then one can easily find a control v satisfying (i)-(ii) and which is not bang-bang.

Moreover the constant value of u on $y = 0$ can be uniquely determined from the value of

$$\int_{[y^*=0]} u\, dx$$

which is the same for all controls leading to y^*. Hence y^* can be obtained from a unique control u^* which is bang-bang and radially symmetric.

We have thus proved:

PROPOSITION 4.5. – *a) The obstacle problem* (OP) *admits at least one radially symmetric optimal state* y^*.

b) y^* *is obtained from a uniquely determined control* u^* *which is radially symmetric and bang-bang.*

c) u^* *has the additional property* $u^* \equiv \text{const}$ *on* $[y^* = 0]$.

APPENDIX. – Let (X, μ) be a measured space that $\mu(X) = 1$ and let $\phi: \mathbb{R} \to \mathbb{R}$ be a continuous convex function. Then for each $f: X \to R$ the following generalized Jensen inequality holds true:

$$\text{(J)} \qquad \phi\left(\int_X f(x)\, d\mu\right) \leq \int_X \phi(f(x))\, \partial\mu\,.$$

Proof. – Let $f_n: X \to R$ be a sequence of step mappings such that $f_n \to f$ a.e. and L^1. For step mappings (J) is another way of writing Jensen's inequality. Hence $\forall n \in \mathbb{N}$

$$\phi\left(\int_X f_n(x)\, d\mu\right) \leq \int_X \phi(f_n(x))\, du\,.$$

Letting $n \to \infty$ we get the general inequality. q.e.d.

Aknowledgements.

I wish to thank Professor V. BARBU whose valuable advice contributed to the improvement of this paper.

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Pervenuto alla Redazione il 29 aprile 1987