Morse theory on grassmanians

Liviu I. Nicolaescu
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556

0 Introduction

In the paper [N], while studying adiabatic deformations of Dirac operators on manifolds with boundary, we were led to the following finite dimensional dynamics problem.

Consider $\Lambda(n)$ the grassmannian of lagrangian subspaces in the canonical symplectic space $E = \mathbb{R}^{2n}$. If $A : E \to E$ is a selfadjoint operator anticommuting with the canonical complex structure J on E, then A belongs to the Lie algebra of the symplectic group $Sp(E)$ and thus e^{At} is a flow of symplectic matrices. It induces a flow on $\Lambda(n)$ via the transitive action of $Sp(E)$ on this grassmanian. This flow presents many similarities with a gradient-like flow. In particular it has a nice asymptotic behaviour. More precisely for $L \in \Lambda(n)$ $e^{At}L$ converges to some A-invariant lagrangian as t goes to infinity. In fact when n=1 so that $\Lambda(1) \cong S^1$ the phase portrait (see Fig.2 Sec.1) resembles the phase portrait of the gradient flow of a perfect Morse function on S^1. A natural question arises. Is this flow the gradient flow (in some appropiate metric) of a Morse function on $\Lambda(n)$?

This is one of the questions we address in these notes. The author was very pleased to find out that this question has the best answer one can hope for. Indeed this is the gradient flow (in a natural metric) of some Morse function. This function, which must depend on A, has a simple description

$$f_A(L) = -tr(AP_L) \tag{0.1}$$

where P_L is the orthogonal projection onto L. Its critical points are the A-invariant lagrangian subspaces and they admit a very nice combinatoric description. For a particular choice of A, this function is selfindexing and $\mathbb{Z}_2$-perfect. This allowed us to derive the $\mathbb{Z}_2$-Poincaré polynomial of $\Lambda(n)$.

The paper is divided in three Sections. In Section 1 we study the dynamics of this flow. In particular we describe its stationary points and their stable and unstable manifolds. They have an interesting combinatorial description and we spend some time analyzing it. This study is considerably facilitated by a very nice fact: the flow becomes **linear** in Arnold's coordinates.

In Section 2 we do some Morse theory on $\Lambda(n)$. Here we compute the Morse function . We show it is selfindexing and perfect and then we compute the Poincaré polynomial of $\Lambda(n)$

Section 3 is devoted to the study of a similar flow on complex grassmanians. Again we get a selfindexing $\mathbb{R}$-perfect Morse function and as a consequence we can compute the Poincaré polynomials of complex grassmanians.

Note: After this work was completed we found out t hat these types of Morse functions appeared in mathematics in various other contexts. The first references on this problem seem to be the classical by now [B1], [BS]. The authors construct perfect Morse functions on G/T where G is a compact Lie group and T is a maximal torus (see [A], p.65-90 for an excellent description of these results). The techniques of Bott were extended by Takeuchi in [T1, 2] to symmetric flag manifolds. More recently Duistermmat et al. ([DKV]) studied similar Morse functions in connections to stationary phase approximations for Harish-Chandra functions. These considerations extend to infinite dimensional situations, more precisely to loop groups (cf. [PS]). The perfect Morse function constructed is none other than the "energy" of a loop and the resulting Morse picture produces the famous Bott periodicity theorem. A splendid synthesis of these recent developments can be found in [B2].

1 Dynamics and combinatorics on lagrangian grassmanians

Consider the standard symplectic space $E = (\mathbb{R}^{2n}, \omega_0)$ and let $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denote the canonical complex structure

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

so that

$$\omega_0(x, y) = (x, Jy)$$

where $(\, , \,)$ denotes the Euclidean scalar product on $\mathbb{R}^{2n}$.

The symplectic group is defined as

$$Sp(n) = \{T \in GL(2n, \mathbb{R}) \ / \ \omega_0(Tx, Ty) = \omega_0(x, y) \quad \forall \, x \, , \, y \in \mathbb{R}^{2n}\}$$

i.e.

$$T \in Sp(n) \Longleftrightarrow T^* JT = J$$

The Lie algebra of $Sp(n)$ is

$$\underline{sp}(n) = \{A \in \underline{gl}(2n, \mathbb{R}) \ / \ A^* J + JA = 0\}$$

It has a Cartan decomposition (see [GS])

$$\underline{sp}(n) = \underline{k}(n) \oplus \underline{p}(n)$$

where $\underline{k}(n)$ consists of skewadjoint matrices in $\underline{sp}(n)$ and $\underline{p}(n)$ consists of symmetric ones. Alternatively

$$\underline{p}(n) = \{A \in \underline{gl}(2n, \mathbb{R}) \ / \ A = A^* \ \ , \ \ \{A, J\} = 0\}$$

where $\{ \, , \, \}$ denotes the anticommutator of two matrices.

If $V \subset E$ is a subspace then its annihilator , denoted by V^0 is defined as

$$V^0 = \{x \in E \ / \ \omega_0(x, v) = 0 \quad \forall v \in V\}$$

or equivalently

$$V^0 = JV^\perp$$

where $V^\perp$ is the orthogonal complement of V. V is called **lagrangian** if $V = V^0$. In particular if V is lagrangian $\dim V = 1/2 \dim E = n$. The condition that $L \subset E$ is lagrangian can be given an operator theoretic description as follows. Let R_L denote the (orthogonal) reflection through L defined by $R_L = 2P_L - 1$,where P_L denotes the orthogonal projection onto L. Then

$$L \text{ lagrangian} \Longleftrightarrow \{R_L, J\} = 0 \tag{1.1}$$

Denote by $\Lambda(n)$ the grassmanian of lagrangian subspaces of E. It is known ([GS]) that $\Lambda(n)$ is a homogeneous space for $Sp(n)$ of dimension $n(n+1)/2$. Alternatively, using (1.1) ,one can show that

$$\Lambda(n) = U(n)/O(n) \tag{1.2}$$

Now, given $A \in \underline{p}(n)$ we get a flow of symplectic matrices e^{At} and thus a flow on $\Lambda(n)$.

Any matrix $A \in \underline{p}(n)$ has real spectrum and since it anticommutes with J its spectrum is symmetric with respect to the origin. Such a matrix will be called **nondegenerate** if all its eigenvalues have multiplicity 1. In particular 0 is not in the spectrum of a nondegenerate matrix. Indeed if $Au = 0$ then $AJu = 0$ so that $\dim\ker A \geq 2$. The subset of nondegenerate matrices will be denoted by $\underline{p}(n)^*$.

Fix $A \in \underline{p}(n)^*$. Its spectrum is

$$\sigma(A) = \{\lambda_{-n} < \lambda_{-(n-1)} < \cdots \lambda_{-1} < 0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n \ / \ \lambda_j = -\lambda_{-j} \quad \forall j = 1, \cdots, n\}$$

with corresponding (normalized) eigenvectors

$$\{e_{-n}, \cdots, e_{-1}, e_1, \cdots e_n \ / \ Je_j = e_{-j} \quad \forall j = 1, \cdots, n\} \tag{1.3}$$

A stationary point for the flow e^{At} on $\Lambda(n)$ is an A-invariant lagrangian. Denote the set of A-invariant lagrangians by $\mathcal{L}_A$. If $L \in \mathcal{L}_A$ then it is spanned by a collection of n eigenvectors of A. Set

$$\mathcal{I} = \mathcal{I}_n = \{-n, \cdots, -1, 1, \cdots, n\} \ , \ \mathcal{I}^+ = \{j \in \mathcal{I} \ / \ j > 0\}$$

define the **indicator function** of L by

$$\iota_L : \mathcal{I} \to \{\pm 1\} \ , \ \iota_L(j) = \begin{cases} 1 & , \ e_j \in L \\ -1 & , \ e_j \notin L \end{cases}$$

and the **indicator set** of L

$$\Xi_L = \iota_L^{-1}(1)$$

The condition that L is lagrangian is then equivalent to the fact that its indicator is an odd function $\iota_L(j) = -\iota_L(-j)$, $\forall j \in \mathcal{I}$. Thus there is a bijective correspondence between $\mathcal{L}_A$ and the set of odd functions $\mathcal{I} \to \{\pm 1\}$. Such a function is uniquely determined by its restriction to $\mathcal{I}^+$ and in particular the set

$$\Xi_L^+ = \{j \in \mathcal{I}^+ \ / \ e_j \in L\} = \{j \in \mathcal{I}^+ \ / \ \iota_L(j) = 1\}$$

uniquely determines the lagrangian L. We deduce that $\mathcal{L}_A$ has cardinality 2^n.

The above discussion can be conveniently encoded in a black & white diagram as bellow. It is a sequence of n black / white circles : $\bullet/\circ$.

$$L \mapsto \delta(L) = | - \circ - \bullet - \bullet - \cdots - \circ$$

$\circ$ on the j-th spot $\Longleftrightarrow$ $j \in \mathcal{I}^+ \setminus \Xi_L^+$;
$\bullet$ on the k-th spot $\Longleftrightarrow$ $k \in \Xi_L^+$.

We will reserve the Greek letters α , β , γ , δ to denote such diagrams. The operation of attaching a $\circ/\bullet$ to a diagram δ of length n (thus producing a diagram of length $n+1$) will be denoted by $\delta + \circ/\bullet$.

To study the behaviour of the flow near the stationary points we will use the local coordinates introduced by Arnold in [Ar] in his description of the Maslov index . We will call these coordinates **Arnold coordinates**. To define them fix $L_0 \in \Lambda(n)$ and denote by $\mathcal{A}_{L_0}$ the family of lagrangians

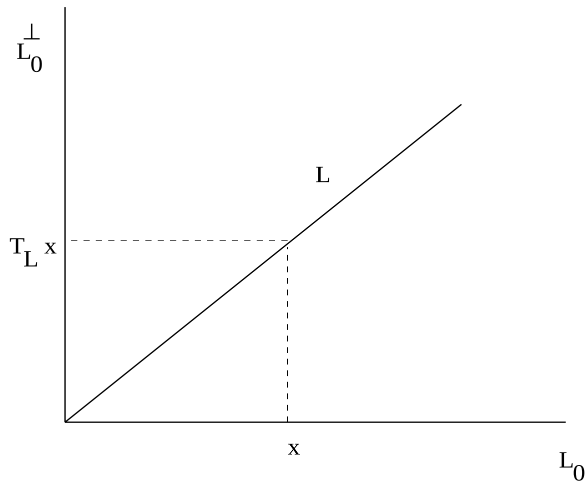

Figure 1: *L is the graph of $T_L : L_0 \to L_0^\perp$*

transversal to $L_0^\perp$. Any such lagrangian L can be represented as the graph of a linear operator $T = T_L : L_0 \to L_0^\perp$(Fig.1). Set $S = -JT : L_0 \to L_0$ so that

$$L = \{x + JSx \in L_0 \oplus L_0^\perp \;/\; x \in L_0\}$$

The fact that L is lagrangian imposes restriction on S. More precisely

$$L \text{ lagrangian} \Longleftrightarrow S \text{ symmetric}$$

If we denote by $Sym(L_0)$ the space of symmetric operators $L_0 \to L_0$ we see that the above construction defines a bijection

$$\Psi_{L_0} : \mathcal{A}_{L_0} \to Sym(L_0) \quad L \mapsto S_L$$

These are the Arnold coordinates .

In the paper [Ar] it is shown that the open sets $\{\mathcal{A}_L \;/\; L \in \mathcal{L}_A\}$ cover $\Lambda(n)$ so they form an atlas for the lagrangian grassmanian. We now analyze the flow e^{At} in a neighborhood of some A-invariant lagrangian L_0 . If L is a lagrangian close to L_0 then it is transversal to $L_0^\perp$ since L_0 is. This transversality is preserved for all the lagrangians $L_t = e^{At}L$ with t sufficiently small. Using the Arnold coordinates over $\mathcal{A}_{L_0}$ we get

$$L = \{x + JSx \;/\; x \in L_0\} \quad (S = \Psi_{L_0}(L))$$

Thus

$$\begin{aligned} L_t &= \{e^{tA}(x + JSx) \;/\; x \in L_0\} \\ &= \{e^{tA}x + Je^{-tA}Sx \;/\; x \in L_0\} \quad \text{since } \{A, J\} = 0 \\ &= \{u + Je^{-tA}Se^{-tA}u \;/\; u = e^{tA}x \in L_0\} \quad \text{since } AL_0 \subset L_0 \end{aligned}$$

Thus the Arnold coordinates of L_t are $e^{-tA}Se^{-tA}$. In particular we deduce

$$L \in \mathcal{A}_{L_0} \Rightarrow e^{tA}L \in \mathcal{A}_{L_0} \quad \forall t \in \mathbb{R} \tag{1.4}$$

A simple computation shows that

$$\frac{d}{dt}\,|_{t=0}\,\left(e^{-tA}Se^{-tA}\right) = -\{A, S\}$$

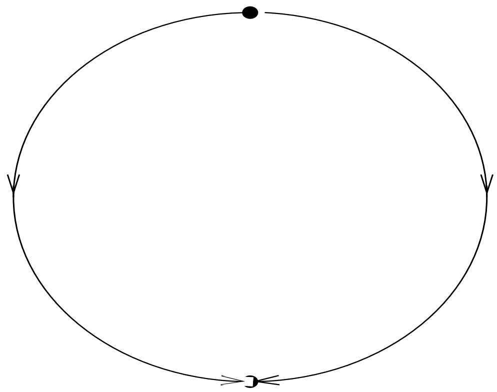

Figure 2: *The flow on* $\Lambda(1)$

so that in these coordinates the vector field X generating the flow is linear

$$X : S \mapsto -\{A, S\} \tag{1.5}$$

To compute the eigenvalues of X we use the basis $(e_i)_{i\in\Xi_{L_0}}$ of L_0.

$$S \;=\; (s^i_j)_{i,j\in\Xi_{L_0}} \;,\; s^i_j = s^j_i \;\;,\;\; A\,|_{L_0} = \mathrm{diag}(\lambda_i\,;\; i\in\Xi_{L_0})$$

Then

$$-\{A, S\} \;=\; -\left((\lambda_i+\lambda_j)s^i_j\right)_{i,j\in\Xi_{L_0}}$$

so that the eigenvalues of X are

$$\{-(\lambda_i+\lambda_j)\;/\;i,j\in\Xi_{L_0}\;,\;i\le j\}$$

An eigenvector for $\lambda_i+\lambda_j$ is the elementary symmetric matrix E_{ij} whose entries are equal to 1 on the (i,j) and (j,i) spots and 0 elsewhere. We now define the index $\mu(L_0)$ of L_0 as the number of positive eigenvalues of $X(L_0)$ (multiplicities included). Thus

$$\mu(L_0) \;=\; \#\{\lambda_i+\lambda_j<0\;/\;i,j\in\Xi_{L_0}\;,\;i\le j\} \tag{1.6}$$

We can write down a more explicit formula for this index.

Lemma 1.1 *Let $k = k(L_0)$ denote the number of $\bullet$'s in the diagram of L_0. For each $j\in\Xi^+_{L_0}$ denote by w_j the number of indices $i>j$ such that $\iota_{L_0}(i) = -1$. (In terms of the diagram of L_0 w_j is the number of $\circ$'s which follow the $\bullet$ sitting on the j-th spot). Then*

$$\mu(L_0) \;=\; (n-k)(n-k+1)/2 + \sum_{j\in\Xi^+_{L_0}} w_j \tag{1.7}$$

Proof Indeed $\lambda_i+\lambda_j$ with $i,j\in\Xi_{L_0}\;,\;i\le j$ can be negative in two instances :
(a) either both i and j are negative and ;
(b) or j is positive and $-i>j$ with $\iota_{L_0}(-i) = -1$
Correspondingly we get the two terms in (1.7) . ■

Example 1.2

$$\mu(| - \bullet - \bullet - \cdots - \bullet) \;=\; 0$$

so that $L_{min} \;=| - \bullet - \bullet - \cdots - \bullet \;=\; \mathrm{span}\{e_1, \cdots, e_n\}$ *is an* **attractor**.

$$\mu(| - \circ - \circ - \cdots - \circ) \;=\; n(n+1)/2$$

so that $L_{max} \;=\; | - \circ - \circ - \cdots - \circ \;=\; \mathrm{span}\{e_{-n}, \cdots, e_{-1}\}$ *is a* **repeller**.

When $n = 1$ *so that* $\Lambda(1) \;\cong S^1$ *these are the only stationary points and the phase portrait of the flow is depicted in Fig.2.*

For any A-invariant lagrangian L define its **energy**

$$\nu(L) \;=\; -\sum_{j \in \mathcal{I}} j\iota_L(j)$$

The energy is related to the index by the following nice formula (suggested by numerical experimentations).

Proposition 1.3 *For any* A*-invariant lagrangian*

$$\mu(L) \;=\; (\nu(L) + n(n+1))/4 \tag{1.8}$$

Proof We will give an inductive proof. Let $L \in \mathcal{L}_A$. An element $j \in \Xi_L^+$ will be called **mobile** if $j+1 \in \mathcal{I} \setminus \Xi_L^+$ and is called **immobile** if either j=n or $j+1 \in \Xi_L^+$. Using the $\circ/\bullet$ diagram associated to L we can rephrase the condition that j is immobile if there is a $\bullet$ on the j-th spot followed by a $\circ$ on the (j+1)-th spot

$$\text{mobile} \bullet \quad : \quad | - \cdots - \bullet - \circ - \cdots$$

A mobile element j defines an elementary transition which can be easily described in terms of the diagram

$$| - \cdots - \bullet - \circ - \cdots \;\mapsto\; | - \cdots - \circ - \bullet - \cdots \tag{1.9}$$

A lagrangian $L \in \mathcal{L}$ is called **immobile** if all $j \in \Xi_L^+$ are immobile. The diagram of an immobile lagrangian looks as below

$$| - \circ - \cdots - \circ - \bullet - \cdots - \bullet \quad (k \;\bullet' \, s \text{ and } (n-k) \;\; \circ' \, s) \tag{1.10}$$

The proof will be carried out in two steps.

Step 1 Equality (1.8) holds for immobile lagrangians. Let L as in (1.10). We have

$$\nu(L) \;=\; -2\left(\sum_{j=1}^{n-k}(-j) + \sum_{j=n-k+1}^{n} j\right) \;=\; (n-k+1)(n-k) \;-\; k(2n-k+1)$$

$$=\; (n-2k)(n-k+1) \;-\; kn$$

so that

$$(\nu(L) + n(n+1))/4 \;=\; \frac{1}{4}\left((n-2k)(n-k+1) \;-\; kn \;+\; n(n+1)\right)$$

$$=\; \frac{1}{4}\left((n-k+1)(n-2k) \;+\; n(n-k+1)\right) \;=\; (n-k)(n-k+1)/2 \;=\; \mu(L)$$

since the second term of the RHS of (1.7) is missing when L is immobile.

Step 2 Let L be an A-invariant lagrangian and $j_0 \in \Xi_L^+$ a mobile element. Denote by $\tilde{L}$ the A-invariant lagrangian obtained from L after an elementary transition on the j_0-th spot. Then

$$\mu(\tilde{L}) = \mu(L) - 1 \tag{1.11}$$

and

$$\nu(\tilde{L}) = \nu(L) - 4 \tag{1.12}$$

The proof of (1.12) is an elementary computation. (1.11) follows from the simple observation that in the sum $\sum_{j \in \Xi_L^+} w_j$ of (1.7) the only term that changes after this elementary transition is w_{j_0} which decreases by 1.

(1.11) and (1.12) show that if (1.8) holds for $\tilde{L}$ it holds also for L. Performing a finite number of elementary transitions we can transform any A-invariant lagrangian into an immobile one for which the equality (1.8) was proved at Step 1. Proposition 1.1 is proved. ∎

Define now the Morse polynomial

$$M_n(t) = \sum_{L \in \mathcal{L}_A} t^{\mu(L)} \tag{1.13}$$

Note that the above definition is purely combinatorial since it is independent of the choice of a nondegenerate A. We conclude this section with a more explicit description of $M_n(t)$.

Proposition 1.4

$$M_n(t) = \prod_{k=1}^{n} (1 + t^k)$$

Proof Set

$$M_n(t) = \sum_{k \geq 0} b_k(n) t^k$$

and artificially define $b_k(n) = 0$ for $k < 0$.

We will prove an induction formula for the coefficients $b_k(n)$ from which Proposition 1.2 will follow immediately.

The induction has its origin in some obvious addition formulae. For any diagram δ of length n we have

$$\nu(\delta + \circ) = \nu(\delta) - 2(n+1)$$

and

$$\nu(\delta + \bullet) = \nu(\delta) + 2(n+1)$$

Using the relation $(n+1)(n+2) - n(n+1) = 2(n+1)$ we deduce from (1.8) that

$$\mu(\delta + \bullet) = \mu(\delta)$$

and

$$\mu(\delta + \circ) = \mu(\delta) + (n+1)$$

Hence

$$b_k(n+1) = \#\{\delta \;/\; \mu(\delta) = k \;,\; \mathrm{length}(\delta) = n+1\}$$
$$= \#\{\alpha \;/\; \mu(\alpha) = k \;,\; \mathrm{length}(\alpha) = n\} \;+\; \#\{\beta \;/\; \mu(\beta) = k - (n+1) \;,\; \mathrm{length}(\beta) = n\}$$

$$= b_k(n) + b_{k-(n+1)}(n)$$

In terms of the Morse polynomial this can be rewritten as

$$M_{n+1}(t) = M_n(t)(1+t^{n+1})$$

Proposition 1.2 now follows from the obvious equality $M_1(t) = 1+t$. ■

2 Morse theory on lagrangian grassmanians

We will show (Prop. 2.1) that the flow studied in Sec.1 is in fact the gradient flow of a Morse function and for a special choice of A it is also selfindexing (Cor.2.1).Then we study a degenerate situation in which the Morse function turns out to be perfect. This allows one to compute the Poincaré polynomial of $\Lambda(n)$.

To start off consider a slightly more general problem. Denote by V the Euclidean space $\mathbb{R}^N$ and let $G_k(V)$ denote the grassmanian of k-dimensional subspaces in V. If A is a $N \times N$ symmetric matrix then e^{At} determines a flow in GL(V) and thus a flow in $G_k(V)$.

Any k-plane $L \subset V$ can be identified with the orthogonal projection P_L onto L. In this way $G_k(V)$ becomes a submanifold in the linear space $Sym(V)$ of symmetric linear operators $V \to V$. On $Sym(V)$ there is a natural scalar product

$$\langle S_1, S_2 \rangle = tr(S_1 S_2) \tag{2.1}$$

which induces a Riemann metric on $G_k(V)$.

Proposition 2.1 *The flow e^{At} on $G_k(V)$ is the negative gradient flow of the function*

$$f_A : G_k(V) \to \mathbb{R} \ , \ L \mapsto -tr(AP_L)$$

Proof We first describe the vector field determined by this flow. Let $L \in G_k(V)$ and set $L_t = e^{At}L$. A projection onto L_t (**not the orthonormal one !**) is given by

$$P_t = e^{At}Pe^{-At} \ , \ P = P_L = P^* \tag{2.2}$$

The orthogonal projection onto L_t can be obtained from (2.2) via the formula (cf. [BW], Chap.12)

$$P_{L_t} = P_t P_t^* (P_t P_t^* + Q_t^* Q_t)^{-1} \tag{2.3}$$

where $Q_t = 1 - P_t$. One computes easily

$$F(t) = P_t P_t^* = e^{At}Pe^{-2At}Pe^{At} \qquad (P = P^*)$$

$$G(t) = Q_t^* Q_t = e^{-At}Qe^{2At}Qe^{-At} \qquad (Q = Q^*)$$

so that

$$F(0) = P \ , \ \dot{F}(0) = AP - PAP + PA$$

and

$$G(0) = Q \ , \ \dot{G}(0) = -AQ + 2QAQ - QA$$

Using the above relations we deduce

$$\frac{d}{dt}P_{L_t} = \dot{F}(0)[F(0)+G(0)]^{-1} - F(0)[F(0)+G(0)]^{-1}(\dot{F}(0)+\dot{G}(0))[F(0)+G(0)]^{-1}$$

$$= AP - 2PAP + PA - P(AP - 2PAP + PA - AQ + 2QAQ - QA0) \quad (F(0) + G(0) = 1)$$
$$= AP - 2PAP + PA - (PAP - 2PAP + PA - PAQ) \qquad (PQ = 0)$$
$$= AP - 2PAP + PA + (PAP - PA + PAQ) = AP - 2PAP + PA \quad (Q = 1 - P)$$

i.e.

$$\frac{d}{dt} P_{L_t} = \{A, P\} - 2PAP \tag{2.4}$$

Now consider the functions α, $\beta : Sym(V) \to \mathbb{R}$ given by

$$\alpha : S \mapsto tr(AS^2) \ , \ \ \beta : S \mapsto tr(AS^3)$$

If we derivate α and β at a given point S along the direction $\dot{S}$ we get

$$d\alpha(S)(\dot{S}) = tr(A\{S, \dot{S}\}) = tr(\{A, S\}\dot{S}) = \langle \{A, S\}\dot{S} \rangle$$

so that the gradient of α is

$$\nabla \alpha(S) = \{A, S\} \tag{2.5}$$

A similar computation shows that

$$\nabla \beta(S) = \{A, S^2\} + SAS \tag{2.6}$$

Thus using (2.5) and (2.6) we can rewrite (2.4) as

$$\frac{d}{dt} P_{L_t} = \nabla \alpha(P) - 2\nabla \beta(P) + 2\{A, P^2\} = 3\nabla \alpha(P) - 2\nabla \beta(P) \ \ (P^2 = P)$$

Note that on $G_k(V)$ we have $\alpha = \beta$ so that

$$\frac{d}{dt} P_{L_t} = \nabla \alpha(P)$$

i.e. e^{At} is the negative gradient flow of $-\alpha \equiv f_A$. ■

Incidentally Prop.2.1 gives a different proof to Proposition 4.3 of [N]. We deduce in particular that the flow discussed in Sec.1 is a gradient flow. The critical points of the associated Morse functions are all nondegenerate iff A is nondegenerate.

Example 2.2 *Consider the case $n = 1$. Then $\Lambda(1) \cong S^1$ since any line in $\mathbb{R}^2$ is a lagrangian. Set $A = diag(1, -1)$. In particular for the line L of slope $m = \tan\theta$ the orthogonal projection is*

$$P_L = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

so that $f_A(L) = -2\cos 2\theta$. The negative gradient flow of this function is depicted in Fig.2

Going back to the general case consider $A_0 \in \underline{p}(n)^*$ given by

$$A_0 = diag(-n, \cdots, -1, 1, \cdots, n)$$

If $L \in \mathcal{L}_{A_0}$ is an A_0 invariant lagrangian with indicator function ι_L then the orthogonal reflection through L is given by

$$R_L = diag(\iota_L(-n), \cdots, \iota_L(-1), \iota_L(1), \cdots, \iota_L(n))$$

so that

$$-tr(AR_L) = -\sum_j j\iota_L(j) = \nu(L)$$

Since $R_L = 2P_L - 1$ and $tr(A_0) = 0$ we deduce $f_{A_0} = \frac{1}{2}\nu(L)$ so that Proposition 1.2 implies

Corollary 2.3 *For any $L \in \mathcal{L}_{A_0}$ we have*

$$2\mu(L) = f_{A_0}(L) + n(n+1)/2$$

i.e. f_{A_0} is selfindexing.

We now consider a degenerate situation. Let $e_0 \in E$ be a unit vector and denote by E' the orthogonal complement of *span* (e_0, Je_0) in E. Naturally associated to this decomposition is a degenerate matrix $A \in \underline{p}(n)$ defined by

$$Au = \begin{cases} 1 & , \quad u = e_0 \\ -1 & , \quad u = Je_0 \\ 0 & , \quad u \in E' \end{cases}$$

For each $u \in E'$ define for later use the skewsymmetric operator $J_k u : E \to E$ defined by

$$J_u v = \begin{cases} Jv & , \quad v \in \mathrm{span}(e_0, Je_0, u, Ju) \\ 0 & , \quad v \in \mathrm{span}(e_0, Je_0, u, Ju)^{\perp} \end{cases} \tag{2.7}$$

The 1-parameter group of symplectic matrices e^{At} defines as before a flow on $\Lambda(n)$ which is the negative gradient flow of the function $f(L) = -tr(AP_L)$. We will prove the following remarkable fact.

Proposition 2.4 *The function defined above is a $\mathbb{Z}_2$- perfect Morse-Bott function i.e its Morse polynomial equals the $\mathbb{Z}_2$ Poincaré polynomial of $\Lambda(n)$.*

Proof We will use the completion principle for degenerate situations in the form presented in [AB].

The critical points of f are the A-invariant lagrangians which in this case are grouped in two critical manifolds

$$C_- = \{L' \oplus (e_0) \;/\; L' \text{ lagrangian in } E'\}$$

and

$$C_+ = \{L' \oplus (Je_0) \;/\; L' \text{ lagrangian in } E'\}$$

Note in particular that both C_- and C_+ are diffeomorphic to $\Lambda(n-1)$ and

$$C_- = f^{-1}(-1) \quad , \quad C_+ = f^{-1}(1)$$

Lemma 2.5 *C_- is a nondegenerate critical submanifold and moreover its stable bundle is oriented.*

Proof of Lemma 2.1 Since C_- is diffeomorphic to $\Lambda(n-1)$ it has codimension n and thus the normal bundle has rank n.

For each $u \in E'$ the matrix J_u defined at (2.7) defines a 1-parameter group of unitary transformations e^{tJ_u} and thus a flow on $\Lambda(n)$. Denote by Y_u the vector field generating this flow. The proof of the lemma will be carried out in several steps.

Step 1 $\forall u \in E'$ the vector field Y_u defines a stable direction along C_-.

Pick $L \in C_-$, $L = span(e_0, e_2, \cdots, e_n)$ and set $L_t = e^{tJ_u} L$, $F(t) = f(L_t)$. We will prove that

$$\frac{d^2 F}{dt^2}(0) > 0 \tag{2.8}$$

Denote $P_t = P_{L_t}$ and for each $j = 0, \cdots n-1$ set $f_j = Je_j$. We have

$$F(t) = -\sum_{j=0}^{n-1}(AP_t e_j, e_j) - \sum_{j=0}^{n-1}(AP_t f_j, f_j)$$

$$= -\sum_{j=0}^{n-1}(P_t e_j, Ae_j) - \sum_{j=0}^{n-1}(P_t f_j, Af_j)$$

and since $Ae_j = Af_j = 0$ for $j > 0$ we get

$$F(t) = (P_t f_0, f_0) - (P_t e_0, e_0) \tag{2.9}$$

P_t can be equivalently described as $P_t = e^{tJ_u} P e^{-tJ_u}$, $(P = P_L)$ and using this in (2.9) we get

$$F(t) = (Pe^{-tJ_u} f_0, e^{-tJ_u} f_0) - (Pe^{-tJ_u} e_0, e^{-tJ_u} e_0) \tag{2.10}$$

A simple computation shows that $e^{-tJ_u} f_0 = \cos t f_0 + \sin t e_0$ and $e^{-tJ_u} e_0 = \cos t e_0 - \sin t f_0$. Since $Pf_0 = 0$ and $Pe_0 = e_0$ we deduce from (2.10) that

$$F(t) = \sin^2 t - \cos^2 t = -\cos 2t \tag{2.11}$$

so that

$$\frac{d^2 F}{dt^2}(0) = 4 > 0$$

and (2.8) is proved.

Let as before $L \in C_-$, $L = span(e_0, e_2, \cdots, e_n)$ and for $0 \leq k \leq n-1$ set $J_k = J_{e_k}$ and $Y_k = Y_{e_k}$ $(Y_{e_0} = Y_u$ with $u = 0)$.

Step 2 The vector fields Y_k are linearly independent at L.

Set $L_k(t) = e^{J_k t}L$. The orthonormal projection $P_k(t)$ onto $L_k(t)$ is as before $e^{J_k t} P e^{-J_k t}$. If we view $\Lambda(n)$ as an embedded submanifold of $Sym(E)$ via the map $L \to P_L$ then the tangent vectors to the curves $L_k(t)$ at $t = 0$ are the selfadjoint operators

$$\dot{P}_k(0) = [J_k, P] \tag{2.12}$$

where [,] denotes the commutator of two matrices. Now let $x_1, \cdots, x_n \in \mathbb{R}$ such that

$$\sum_{k=0}^{n-1} x_k [J_k, P] = 0 \tag{2.13}$$

If we set $B = \sum x_k J_k$ then (2.13) becomes

$$[B, P] = = \frac{1}{2}[B, R] = 0 \tag{2.14}$$

where R is the orthogonal reflection through L_0, $R = 2P - 1$. Note that each J_k anticommutes with R so that (2.14) becomes

$$BR = 0 \tag{2.15}$$

From the equality $BRe_m = 0$, $0 \leq m \leq n-1$ we deduce

$$x_0 + x_m = 0 \quad \forall 1 \leq m \leq n$$

so that $x_0 = \cdots = x_{n-1} = 0$.

Thus denote by ν_{min} the stable bundle of C_-.

Step 3 ν_{min} is orientable.

It suffices to show that ν_{min} can be trivialized along any loop in C_-. Since $C_- \cong \Lambda(n-1)$ we deduce that $\pi_1(C_-) \cong \mathbb{Z}$ and hence it suffices to show that ν_{min} can be trivialized along a generator of the fundamental group. It is easy to describe such a generator. Pick

$$L \in C_- \quad , \quad L = span(e_0, e_2, \cdots, e_n)$$

and let $B = J_{e_2} - J_{e_0}$. We get a loop of lagrangians in C_-

$$L(t) = e^{Bt}L \quad t \in [0, \pi] \tag{2.16}$$

This is a generator of$\pi_1(\Lambda(n-1))$ since its Maslov index is 1 (cf. [CLM]). Set $e_k(t) = e^{Bt}e_k$, $0 \leq k \leq n-1$ (note that only $e_2(t)$ is actually changing with t). Clearly $e_k(t)$ span $L(t)$ and from Step 2 we deduce that the collection of vectors $Y_{e_k(t)}$ trivializes ν_{min} along the path (2.16). Lemma 2.1 is proved. ■

Note that J induces a free $\mathbb{Z}_2$-action on $\Lambda(n)$ and moreover

$$f(JL) = -f(L) \tag{2.17}$$

Thus the gradient flow of f is $\mathbb{Z}_2$-equivariant and in particular

$$JC_+ = C_- \tag{2.18}$$

Using the $\mathbb{Z}_2$ symmetry we deduce that C_+ is also a nondegenerate critical submanifold and its unstable bundle is orientable.

Denote by ν_{max} the unstable bundle over C_+. The level set $f^{-1}(0)$ is naturally the unit sphere subbundle of both ν_{min} and ν_{max}. Denote the disk bundle of ν_{min} (resp. ν_{max}) by Δ_{min} (resp.Δ_{max}) and correspondingly the sphere bundles by Σ_{min} and Σ_{max}. We now show that our function satisfies the $\mathbb{Z}_2$ - completion principle as described in [AB] . We have to show that the composition bellow is trivial

$$H_{*-n}(C_+) \overset{\pi^{-1}}{\to} H_*(\Delta_{max}, \Sigma_{max}) \overset{\partial}{\to} H_{*-1}(\Sigma_{max} = \Sigma_{min}) \to H_{*-1}(\Delta_{min})$$

All the homology groups above have $\mathbb{Z}_2$- coefficients, the first arrow is the homological Thom isomorphism and ∂ is the connecting morphism in the long exact homological sequence of the pair $(\Delta_{max}, \Sigma_{max})$.

Let z' some cycle in $H_{*-n}(C_+)$ and denote by z its image in $H_*(\Sigma_{max})$ via the above composition. We have to show it bounds (mod 2) in Δ_{min}.

J defines an involution of $\Lambda(n)$ which invariates $\Sigma_{max} = f^{-1}(0)$ and switches Δ_{max} to Δ_{min} . Since z bounds in Δ_{max} the cycle J_*z will bound in Δ_{min}. On the other hand J_* defines an involutive automorphism of the $\mathbb{Z}_2$ homology of $\Sigma_{max} = \Sigma_{min}$. It is an elementary algebraic fact that the only involution of a vector space over $\mathbb{Z}_2$ is the identity. In particular, this shows that $J_*z = z$ in $H_*(\Sigma_{min}, \mathbb{Z}_2)$ so that z bounds mod 2 in Δ_{min}. Proposition 2.2 is proved. ■

We deduce immediately from Proposition 2.2 that

Corollary 2.6 *The $\mathbb{Z}_2$ Poincaré polynomial of $\Lambda(n)$ satisfies*

$$P_n(t) \;=\; (1+t^n)P_{n-1}(t)$$

i.e.

$$P_n(t) \;=\; \prod_{k=1}^{n}(1+t^k)$$

This computation agrees with the computation of the $\mathbb{Z}_2$ Poincaré polynomial in [MT] .

Coupling the above corollary with Proposition 1.2 we get

Corollary 2.7 *The Morse function f_A associated to a nondegenerate matrix $A \in \underline{p}(n)^*$ is $\mathbb{Z}_2$ perfect.*

3 Morse theory on complex grassmanians

Denote by $G_{k,n}$ the grassmanian of complex k-planes in $\mathbb{C}^n$. Consider the **complex** matrix

$$A \;=\; \mathrm{diag}(1,2,\cdots,n)$$

e^{At} defines a flow on $G_{k,n}$ and by Proposition 2.1 this is the negative gradient flow of the function

$$f : G_{k,n} \to \mathbb{R} \quad L \mapsto -tr(AP_L)$$

where P_L is the orthogonal projection onto L and when we compute the trace we think of A and P_L as **real** operators $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ commuting with the canonical complex structure on $\mathbb{R}^{2n}$.

The stationary points of this flow are the A-invariant complex k-planes. Denote the set of such k-planes by $\mathcal{L}_A$ and set $\mathcal{I} \;=\; \{1,\cdots,n\}$.

We can associate to any $L \in \mathcal{L}_A$ a cardinality k subset $\Xi_L^+ \subset \mathcal{I}$ such that

$$L \;=\; \mathrm{span}\{e_j \;/\; j \in \Xi_L^+\}$$

We will call Ξ_L^+ the **indicator set** of L. As in Section 1 we have an associated **indicator function**

$$\iota_L : \mathcal{I} \to \{\pm 1\} \quad \iota_L(j) \;=\; \begin{cases} 1 & , \quad j \in \Xi_L^+ \\ -1 & , \quad j \in \Xi_L^- = \mathcal{I} \setminus \Xi_L^+ \end{cases}$$

As in Section 1 it will be convenient to represent an A-invariant by a $\bullet/\circ$ diagram consisting of k $\bullet'$s and $(n-k)$ $\circ'$s

$$\bullet - \circ - \cdots - \circ - \bullet$$

$\bullet \in \Xi_L^+$ and $\circ \in \Xi_L^-$. Such a diagram will be said to have **type (k,n)**.

For any $L_0 \in \mathcal{L}_A$ consider the open subset

$$\mathcal{A}_{L_0} \;=\; \{L \in G_{k,n} \;/\; L \cap L_0^\perp \;=\; 0\}$$

Any $L \in \mathcal{A}_{L_0}$ is the graph of a unique (complex) linear operator

$$S : L_0 \to L_0^\perp$$

so that we can describe L as

$$L \;=\; \{x + Sx \;/\; x \in L_0\}$$

Set $S = \Psi_{L_0}(L)$. One can show immediately that the open sets $(\mathcal{A}_{L_0})_{L_0\in\mathcal{L}_A}$ cover $G_{k,n}$ so the above construction defines an atlas for the complex grassmanian (this is in fact the usual atlas).

We will analyze the structure of this flow e^{At} using these coordinates. Let $L \in \mathcal{A}_{L_0}$ and let $S = \Psi_{L_0}(L)$. Set $A_+ = A\,|_{L_0}$ and $A_- = A\,|_{L_0^\perp}$. We deduce

$$e^{At}L = \{e^{A_+t}x + e^{A_-t}Sx \;/\; x \in L_0\}$$

$$= \{u + e^{A_-t}Se^{-A_+t}u \;/\; u = e^{A_+t}x \in L_0\}$$

so that $S_t = \Psi_{L_0}(e^{At}L) = e^{A_-t}Se^{-A_+t}$ and

$$\dot{S}_0 = A_-S - SA_+ : L_0 \to L_0^\perp \tag{3.1}$$

Formula (3.1) shows that our choice of coordinates linearizes the flow. The eigenvalues of the linear operator

$$T : S \mapsto A_-S - SA_+$$

are easily computed using the canonical complex bases $(e_i)_{i\in\Xi_L^-}$ of $L_0^\perp$ and $(e_j)_{j\in\Xi_L^+}$ of L_0. These eigenvalues are

$$\{(i-j) \;/\; (i,j) \in \Xi_L^- \times \Xi_L^+\}$$

and the corresponding eigenvectors are the elementary complex matrices $E_{i,j}$ whose only nontrivial entry lies on the (i,j)-spot , $(i,j) \in \Xi_L^- \times \Xi_L^+$. Viewed over the reals each of these eigenvalues has multiplicity 2.

Define $\mu(L)$ the index of an $L \in \mathcal{L}_A$ as the **real** dimension of the space spanned by the negative eigenvalues of T. One sees immediately that

$$\mu(L) = 2\sum_{j\in\Xi_L^+} w_j \tag{3.2}$$

where w_j is the number of elements in Ξ_L^- greater than j. Using the $\bullet/\circ$ diagrams we can visualize w_j as the number of $\circ$'s which follow the $\bullet$ sitting on the j-th spot. In particular all critical points have even index so by Morse's lacunary principle we deduce

Proposition 3.1 *f is a $\mathbb{R}$-perfect Morse function.*

Define the **energy** of an A-invariant k-plane L by

$$\nu(L) = -\sum_{i\in\mathcal{I}} i\iota_L(i)$$

The reader can verify immediately that

$$\nu(L) = -tr(AR_L) = -tr(A(2P_L - 1) = 2f(L) + tr A \tag{3.3}$$

Proposition 3.2 *For any A-invariant k-plane L we have the equality*

$$\mu(L) = \nu(L) + n(n+1)/2 - (n-k)(n-k+1)/2$$

Proof The proof follows closely the proof of Proposition 1.1. We use the terminology defined there.

In this case there is a single immobile (k,n)-diagram $\circ - \cdots - \circ - \bullet - \cdots - \bullet$ and the reader can check immediately that

$$\mu(\circ - \cdots - \circ - \bullet - \cdots - \bullet) - \nu(\circ - \cdots - \circ - \bullet - \cdots - \bullet) = n(n+1)/2 - (n-k)(n-k+1)/2 \quad (3.4)$$

Then we study the effect of elementary transitions on the index and energy and we discover using (3.2) that

$$\mu(\cdots \bullet - \circ - \cdots) - \mu(\cdots \circ - \bullet \cdots) = \nu(\cdots \bullet - \circ - \cdots) - \nu(\cdots \circ - \bullet \cdots) = 2 \qquad (3.5)$$

Proposition 3.2 follows from (3.4) and (3.5). ■

Proposition 3.2 coupled with (3.3) yields

Corollary 3.3 *The function f is selfindexing.*

Let $M_{k,n}(t) = \sum_{L \in \mathcal{L}_A} t^{\mu(L)}$ denote the Morse polynomial of f. We rewrite it as

$$M_{k,n}(t) = \sum_{q \geq 0} m_{k,n}(q) t^q$$

We will produce a recurrence relation for these polynomials. We proceed as in Proposition 1.2 Using (3.2) we derive addition formulae

$$\mu(\delta + \bullet) = \mu(\delta)$$

and

$$\mu(\delta + \circ) = \mu(\delta) + 2k$$

for any (k,n) diagram δ. We put these together as

$$\begin{aligned} m_{k+1,n+1}(q) &= \#\{\alpha \,/\, \alpha - \text{(k,n) diagram}, \ \mu(\alpha) = q\} \\ &+ \#\{\beta \,/\, \beta - \text{(k+1,n) diagram}, \ \mu(\beta) = q - 2k - 2\} \end{aligned}$$

Hence

$$m_{k+1,n+1}(q) = m_{k,n}(q) + m_{k+1,n}(q - 2k - 2)$$

or equivalently

$$M_{k+1,n+1}(t) = M_{k,n}(t) + t^{2k+2} M_{k+1,n}(t) \qquad (3.6)$$

where we set $M_{0.n} = M_{n,n} = 1$. One immediate consequence of formula (3.6) is that the coefficients of $M_{k,n}$ stabilize as $n \to \infty$. In particular, if we let $n \to \infty$ in (3.6) we deduce

$$M_{k+1,\infty} = M_{k,\infty}(t) + t^{2k+2} M_{k+1,\infty}$$

so that

$$M_{k+1,\infty} = \frac{1}{1 - t^{2k+2}} M_{k,\infty} \qquad (3.7)$$

which yields the known result about the cohomology of the classifying space of $U(k)$ ([BT]):

$$P_t(BU(k)) = \frac{1}{(1-t^2)\cdots(1-t^{2k})} \qquad (3.8)$$

The formulae (3.6) can also be used to rederive the Poincaré polynomials of the complex grassmanians in the form presented e.g. in [BT]. We would also like to mention that the Schubert cells which describe the ring structure of the cohomology of $G_{k,n}$ can be given a Morse theoretic description. They are precisely the unstable manifolds of our Morse function f.

We conclude this section with a different description of our Morse function on $G_{1,n} \cong \mathbf{CP}^{n-1}$

Example 3.4 *Denote by* $(e_k)_{1\leq k\leq n}$ *the standard basis in* $\mathbf{C}^n$ *and set* $A = diag\{1,\cdots,n\}$. *Any complex line in* $\mathbf{C}^n$ *is determined by a nonzero vector* u. *The orthogonal projection onto* $L = span\,(u)$ *is*

$$P_u v \;=\; \frac{1}{|u|^2}\langle v,u\rangle u$$

For each $u\in\mathbf{C}^n\setminus\{0\}$ *let* $[u] = [u_1:\cdots:u_n]$ *denote its image in* $\mathbf{CP}^{n-1}$. *Our Morse function is*

$$f_A([u]) \;=\; -trAP_u \;=\; -\sum_k\langle AP_ue_k,e_k\rangle \;=\; -\sum_k k\langle P_ue_k,e_k\rangle$$

$$=\; -\frac{1}{|u|^2}\sum_k k\langle e_k,u\rangle\langle u,e_k\rangle \;=\; -\frac{1}{|u|^2}\sum_k k|u_k|^2$$

This is the favorite example of perfect Morse function (see[AB]).

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