KNOTS AND THEIR CURVATURES

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ABSTRACT. I discuss an old result of John Milnor stating roughly that if a closed curve in space is not too curved then it cannot be knotted.

CONTENTS

1. THE TOTAL CURVATURE OF A POLYGONAL CURVE

An (oriented) *polygonal knot* (or curve) is a closed curve C in $\mathbb{R}^3$, *without selfintersections*, obtained by successively joining n *distinct* points

$$\boldsymbol{p}_1, \dots, \boldsymbol{p}_n,\ \boldsymbol{p}_{n+1} = \boldsymbol{p}_1 \in \mathbb{R}^3$$

via straight line segments

$$[\boldsymbol{p}_1\boldsymbol{p}_2], \dots, [\boldsymbol{p}_{n-1}\boldsymbol{p}_n],\ [\boldsymbol{p}_n, \boldsymbol{p}_1].$$

The points $\boldsymbol{p}_i$ are called the *vertices* of the polygonal knot C. We denote by $\mathcal{V}_C$ the set of vertices. To each oriented edge $[\boldsymbol{p}_i, \boldsymbol{p}_{i+1}]$, $1 \leq i \leq n$, we associate the unit vector

$$\boldsymbol{\gamma}_i := \frac{1}{|\overrightarrow{\boldsymbol{p}_i\boldsymbol{p}_{i+1}}|} \cdot \overrightarrow{\boldsymbol{p}_i\boldsymbol{p}_{i+1}}.$$

Denote by $\boldsymbol{S}^2$ the *unit* sphere in $\mathbb{R}^3$ centered at the origin. We obtain in this fashion a map

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}_C : \mathcal{V}_C \to \boldsymbol{S}^2,\ \ \boldsymbol{\gamma}(\boldsymbol{p}_i) = \boldsymbol{\gamma}_i.$$

This is known as the *Gauss map* of the polygonal knot C.

Let $\alpha_i \in [0, \pi)$ be the angle between $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{i+1}$; see Figure 1. We obtain in this fashion a map

$$\alpha = \alpha_C : \mathcal{V}_C \to [0, \pi),\ \ \alpha(\boldsymbol{p}_i) = \alpha_i.$$

We define the *total curvature* of C to be the *positive* real number

$$K(C) = \frac{1}{2\pi} \sum_{\boldsymbol{p} \in \mathcal{V}_C} \alpha_C(\boldsymbol{p}) = \frac{1}{2\pi} \sum_{i=1}^{n} \alpha_i. \tag{1.1}$$

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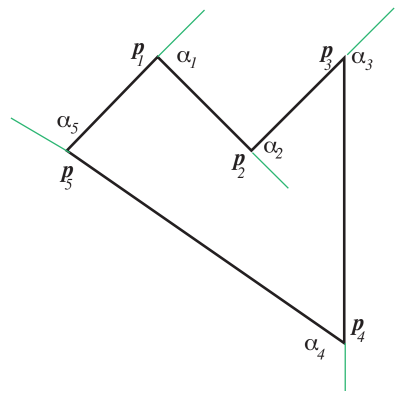

FIGURE 1. *A planar polygonal knot.*

Observe that if C is a convex, planar polygonal curve then $K(C) = 1$.

We can give a simple geometric interpretation to the total curvature. The points $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{i+1}$ on $\boldsymbol{S}^2$ determine a great circle (think Equator) on the sphere obtained by intersecting the sphere with the plane Π_i through the origin and containing these two points. This great circle is divided into two arcs by the points $\boldsymbol{\gamma}_i$ and $\boldsymbol{\gamma}_{i+1}$. We let σ_i denote the shorter of the two arcs. Note that

$$\alpha_i = \mathrm{length}(\sigma_i).$$

The collection of curves σ_i trace a closed curve σ_C on $\boldsymbol{S}^2$ called the *gaussian image* of C. We deduce

$$K(C) = \frac{1}{2\pi}\,\mathrm{length}(\sigma_C).$$

2. A PROBABILISTIC INTERPRETATION OF THE TOTAL CURVATURE

Every unit vector $\boldsymbol{u} \in \boldsymbol{S}^2$ determines a linear map

$$L_{\boldsymbol{u}} : \mathbb{R}^3 \to \mathbb{R},\ \ L_{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}\cdot\boldsymbol{x},$$

where "$\cdot$" denotes the dot product in $\mathbb{R}^3$. This induces by restriction a continuous map

$$\ell_{\boldsymbol{u}} = L_{\boldsymbol{u}}|_C : C \to \mathbb{R}.$$

A vertex $\boldsymbol{p}$ of C is a local minimum of $\ell_{\boldsymbol{u}}$ if

$$\ell_{\boldsymbol{u}}(\boldsymbol{p}) \leq \ell_{\boldsymbol{u}}(\boldsymbol{x}),\ \ \text{for all } \boldsymbol{x} \in C \text{ situated in a neighborhood of } \boldsymbol{p}.$$

We now define

$$\mu_C : \boldsymbol{S}^2 \times \mathcal{V}_C \to \mathbb{R},\ \ \boldsymbol{S}^2 \times \mathcal{V}_C \ni (\boldsymbol{u},\boldsymbol{p}) \mapsto \mu_C(\boldsymbol{u},\boldsymbol{p}) = \begin{cases} 1, & \text{if } \boldsymbol{p} \text{ is a local minimum of } \ell_{\boldsymbol{u}},\\ 0, & \text{otherwise.}\end{cases}$$

We set

$$\mu_C : \boldsymbol{S}^2 \to \mathbb{R},\ \ \mu_C(\boldsymbol{u}) = \text{the number of vertices of } C \text{ that are local minima of } \ell_{\boldsymbol{u}}.$$

Let us point out that that $\mu_C(\boldsymbol{u}) = \infty)$ for some $\boldsymbol{u}$'s. Observe that

$$\mu_C(\boldsymbol{u}) = \sum_{\boldsymbol{p}\in\mathcal{V}_C} \mu_C(\boldsymbol{u},\boldsymbol{p}). \tag{2.1}$$

Let us have a look at the function μ_C. First let us call a unit vector $\boldsymbol{u} \in \boldsymbol{S}^2$ *nondegenerate* (with respect to C) if the restriction $\ell_{\boldsymbol{u}} : \mathcal{V}_C \to \mathbb{R}$, i.e., the function ℓ_u takes different values on different vertices of C. Otherwise, we say that $\boldsymbol{u}$ is *degenerate* (with respect to C). We denote by $\Delta_C \subset \boldsymbol{S}^2$ the collection of degenerate vectors.

Note that $\boldsymbol{u}$ is degenerate if and only if there exist $\boldsymbol{p}_i, \boldsymbol{p}_j \in \mathcal{V}_C$ such that $\boldsymbol{u} \cdot (\boldsymbol{p}_i - \boldsymbol{p}_j)$, i.e., $\boldsymbol{u}$ is perpendicular to the line ℓ_{ij} determined by $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$. In other words, $\boldsymbol{u}$ belongs to the great circle $E_{ij} \subset \boldsymbol{S}^2$ obtained by intersecting $\mathbb{S}^2$ with the plane through origin perpendicular to ℓ_{ij}. Thus

$$\Delta_C = \bigcup_{1 \leq i < j \leq n} E_{ij}.$$

In particular, the set Δ_C has zero area, i.e., most vectors $\boldsymbol{u} \in \boldsymbol{S}^2$ are nondegenerate. Set

$$\boldsymbol{S}^2_C := \boldsymbol{S}^2 \setminus \Delta_C.$$

Let us point out that

$$\boldsymbol{u} \in \boldsymbol{S}^2_C \Rightarrow \mu_C(\boldsymbol{u}) < \infty.$$

The set $\boldsymbol{S}^2_C$ is the complement of finitely many great circles, and thus consists of the interiors of finitely many spherical polygons,

$$\boldsymbol{S}^2_C = P_1 \cup \cdots \cup P_\nu.$$

Let us observe that if $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \boldsymbol{S}^2_C$ belong to the interior of the same polygon P_k then

$$\mu_C(\boldsymbol{u}_0, \boldsymbol{p}) = \mu_C(\boldsymbol{u}_1, \boldsymbol{p}), \;\; \forall \boldsymbol{p} \in \mathcal{V}_C.$$

To see this we choose a continuous path $\boldsymbol{u} : [0,1] \to P_k$ such that

$$\boldsymbol{u}(0) = u_0, \;\; \boldsymbol{u}(1) = \boldsymbol{u}_1.$$

We set $\ell_t := \ell_{\boldsymbol{u}(t)}$, we consider a vertex $\boldsymbol{p}$ of C and we denote by $\boldsymbol{p}'$ and $\boldsymbol{p}''$ its neighbors. Since the vector $\boldsymbol{u}(t)$ is nongenerate the quantities

$$d'_t = \ell_t(\boldsymbol{p}') - \ell(\boldsymbol{p}) \;\; \text{and} \;\; d''_t = \ell_t(\boldsymbol{p}'') - \ell_t(\boldsymbol{p})$$

are nonzero for any $t \in [0,1]$. In particular, the signs of these quantities are independent of t. Observe that $\boldsymbol{p}$ is a local minimum for ℓ_0 if and only if both d'_0 and d''_0 are positive, that is, if and only if d'_1 and d''_1 are positive. Thus $\boldsymbol{p}$ is a local minimum for ℓ_0 if and only if it is a local minimum for ℓ_1, i.e.,

$$\mu_C(\boldsymbol{u}_0, \boldsymbol{p}) = \mu_C(\boldsymbol{u}_1, \boldsymbol{p}).$$

This shows that the function μ_C is constant and finite on each of the regions P_k and in particular, it is integrable.

Now define

$$\boldsymbol{\kappa}(C) := \frac{1}{\operatorname{area}(\boldsymbol{S}^2)} \int_{\boldsymbol{S}^2} \mu_C(\boldsymbol{u})\, dA_{\boldsymbol{u}} = \frac{1}{4\pi} \int_{\boldsymbol{S}^2} \mu_C(\boldsymbol{u})\, dA_{\boldsymbol{u}},$$

where dA denotes the area element on $\boldsymbol{S}^2$. In other words $\boldsymbol{\kappa}(C)$ is the average number of local minima of the collection of function

$$\{ \, \ell_{\boldsymbol{u}} : C \to \mathbb{R}; \;\; \boldsymbol{u} \in \boldsymbol{S}^2 \, \}.$$

We have the following beautiful result due to Milnor [2]

Theorem 2.1. *For any polygonal curve $C \subset \mathbb{R}^3$ we have $K(C) = \boldsymbol{\kappa}(C)$.*

Proof. The proof is based on one of the oldest tricks in the book, namely, changing the order of summation (or integration) in a double sum (or integral). We have

$$\boldsymbol{\kappa}(C) = \frac{1}{4\pi} \int_{\boldsymbol{S}^2} \mu_C(\boldsymbol{u})\, dA_{\boldsymbol{u}} = \frac{1}{4\pi} \int_{\boldsymbol{S}^2} \left(\sum_{\boldsymbol{p} \in \mathcal{V}_C} \mu_C(\boldsymbol{u}, \boldsymbol{p}) \right) dA_{\boldsymbol{u}} = \frac{1}{4\pi} \sum_{\boldsymbol{p} \in \mathcal{V}_C} \int_{\boldsymbol{S}^2} \mu_C(\boldsymbol{u}, \boldsymbol{p})\, dA_{\boldsymbol{u}}.$$

Let $\mathcal{V}_C = \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\}$. We want to compute the integral

$$\int_{\boldsymbol{S}^2} \mu_C(\boldsymbol{u}, \boldsymbol{p}_i)\, dA_{\boldsymbol{u}}.$$

Above, for almost all $\boldsymbol{u}$ we have $\mu_C(\boldsymbol{u}, \boldsymbol{p}_i) = 0, 1$. Note that $\boldsymbol{p}_i$ is a local minimum of $\ell_{\boldsymbol{u}}$ if and only if $\boldsymbol{u}$ belongs to the lune $L_i \subset \boldsymbol{S}^2$ defined as follows.

FIGURE 2. *A planar section of a dihedral angle and the associated lune with opening* $\beta_i = \alpha_i$*.*

Consider the planes π_i and π_{i-1} perpendicular to the lines $\boldsymbol{p}_i\boldsymbol{p}_{i+1}$ and respectively $\boldsymbol{p}_{i-1}\boldsymbol{p}_i$; see Figure 2. The planes π_i and π_{i-1} determine four dihedral angles. Let let D_i denote the dihedral angle characterized by the inequalities

$$\boldsymbol{u} \in D_i \Longleftrightarrow \boldsymbol{u} \cdot \boldsymbol{p}_i \leq \boldsymbol{u} \cdot \boldsymbol{p}_{i-1}, \;\; \boldsymbol{u} \cdot \boldsymbol{p}_{i+1}.$$

Then $L_i = D_i \cap \boldsymbol{S}^2$. The area of the lune L_i is twice the measure β_i of the dihedral angle D_i (can you argue why?) and upon inspecting Figure 2 we see that $\beta_i = \alpha_i$ Hence

$$\frac{1}{4\pi} \int_{\boldsymbol{S}^2} \mu_C(\boldsymbol{u}, \boldsymbol{p}_i)\, dA_{\boldsymbol{u}} = \frac{1}{4\pi} \text{area}\,(L_i) = \frac{\alpha_i}{2\pi}.$$

Hence

$$\boldsymbol{\kappa}(C) = \frac{1}{2\pi} \sum_{i=1}^{n} \alpha_i = K(C).$$

□

Remark 2.2. For a different probabilistic interpretation of $K(C)$ we refer to the paper of Istvan Fáry [1]. □

3. THE TOTAL CURVATURE OF A SMOOTH CLOSED CURVE

Suppose now that C is a C^2 closed curve in $\mathbb{R}^3$ without self-intersections. In other words we can find a twice continuously differentiable map $\boldsymbol{r} : \mathbb{R} \to \mathbb{R}^3$, $t \mapsto \boldsymbol{r}(t)$ that is 1-periodic,

$$\boldsymbol{r}(t + n) = \boldsymbol{r}(t), \;\; \forall t \in \mathbb{R}, \;\; n \in \mathbb{Z},$$

its restriction to $[0, 1)$ is injective, and

$$\dot{\boldsymbol{r}}(t) \neq 0, \;\; \forall t \in \mathbb{R},$$

where the dot indicates a t-derivative, such that C coincides with the image of $\boldsymbol{r}$. The parametrization $\boldsymbol{r}$ induces an orientation on C. We set $\boldsymbol{p}_0 = \boldsymbol{r}(0)$.

For every $\boldsymbol{p}=\boldsymbol{r}(t)\in C$ we denote by $\boldsymbol{\gamma}_C(p)$ the unit vector tangent to C at $\boldsymbol{p}$ and pointing in the same direction as the velocity vector $\dot{\boldsymbol{r}}(t)$ at $\boldsymbol{p}$. More formally,

$$\boldsymbol{\gamma}_C(\boldsymbol{p})=\frac{1}{|\dot{\boldsymbol{r}}(t)|}\dot{\boldsymbol{r}}(t).$$

We the resulting C^1-map

$$\boldsymbol{\gamma}_C : C\to \boldsymbol{S}^2$$

is called the Gauss map of the oriented closed curve C. Its image σ_C is a C^1 curve on $\boldsymbol{S}^2$ called the gaussian image of C.

We denote by ds the arclength element along C, $ds=|\dot{\boldsymbol{r}}(t)|dt$ so that

$$L_C:=\operatorname{length}(C)=\int_C ds=\int_0^1|\dot{\boldsymbol{r}}(t)|dt.$$

For every $\boldsymbol{p}\in C\setminus\{\boldsymbol{p}_0\}$ we denote by $s(\boldsymbol{p})$ the length of the arc of C connecting $\boldsymbol{p}_0$ to $\boldsymbol{p}$ following the orientation given by $\boldsymbol{r}$. Set $s(\boldsymbol{p}_0)=0$. We can use the quantity s to indicate the position of a point on C. Thus we can view $\boldsymbol{r}$ as a function of s, $\boldsymbol{r}=\boldsymbol{r}(s)$. Note that

$$\left|\frac{d\boldsymbol{r}}{ds}\right|=1,\;\;\frac{d\boldsymbol{r}}{ds}=\boldsymbol{\gamma}(s).$$

We approximate C by a sequence of inscribed polygonal curves C_n, obtained inductively as follows.

- The polygonal curve C_1 has 2^k vertices $\boldsymbol{p}_0,\boldsymbol{p}_1,\dots,\boldsymbol{p}_{2^k-1},\boldsymbol{p}_{2^k}=\boldsymbol{p}_0$ oriented following the orientation of C, and $s(\boldsymbol{p}_i)-s(\boldsymbol{p}_{i-1})=\frac{L_C}{2^k}$.
- $\mathcal{V}_{C_n}\subset\mathcal{V}_{C_{n+1}}$ and new vertices of C_{n+1} are the midpoints of the arcs of C formed by the consecutive vertices of C_n.

Observe that the set

$$\mathcal{V}_\infty=\bigcup_{n\geq 1}\mathcal{V}_{C_n}$$

can be identified with the dense subset of $[0,L_C]$

$$V_\infty=\Big\{\,s\in[0,L],\;\;s=\frac{m}{2^n}L_C;\;\;m,n\in\mathbb{Z}_{\geq 0},\;\;n\geq k,\;\;m\leq 2^n\,\Big\}.$$

Note that if $\boldsymbol{p}\in\mathcal{V}_\infty$, then $\boldsymbol{p}\in\mathcal{V}_{C_n}$ for all $n\gg 1$. Denote by $\boldsymbol{p}_{i,n}$ the vertex i of C_n that coincides with p, and by $\boldsymbol{p}_{i+1,n}$ its succesor. We set

$$s_{i,n}:=s(\boldsymbol{p}_{i,n}),\;\;s_{i+1,n}:=s(\boldsymbol{p}_{i+1,n}).$$

Note that

$$\gamma_{C_n}(\boldsymbol{p})=\frac{1}{|\boldsymbol{r}(s_{i+1,n})-\boldsymbol{r}(s_{i,n})|}\big(\,\boldsymbol{r}(s_{i+1,n})-\boldsymbol{r}(s_{i,n})\,\big),$$

so that

$$\lim_{n\to\infty}\boldsymbol{\gamma}_{C_n}(\boldsymbol{p})=\lim_{n\to\infty}\frac{1}{|\boldsymbol{r}(s_{i+1,n})-\boldsymbol{r}(s_{i,n})|}\big(\,\boldsymbol{r}(s_{i+1,n})-\boldsymbol{r}(s_{i,n})\,\big)\gamma_C(\boldsymbol{p}).$$

$$=\lim_{n\to\infty}\frac{1}{s_{i+1,n}-s_{i,n}}\big(\,\boldsymbol{r}(s_{i+1,n})-\boldsymbol{r}(s_{i,n})\,\big)=\gamma_C(\boldsymbol{p}).$$

Thus the gaussian images of C_n are curves converging to the gaussian image of C, so we could expect that

$$\lim_{n\to\infty}\operatorname{length}(\sigma_{C_n})=\operatorname{length}(\sigma_C).$$

In fact something more precise is true. We set

$$K(C) = \frac{1}{2\pi}\,\mathrm{length}\,\sigma_C = \frac{1}{2\pi}\int_C \left|\frac{d\boldsymbol{\gamma}}{ds}\right| ds. \tag{3.1}$$

The quantitity $K(C)$ is called the total curvature of C and it is a measure of the total "bending" of C.

Theorem 3.1. *(a)* $K(C_n) \leq K(C_{n+1})$, $\forall n \geq 1$ *and*

$$\lim_{n\to\infty} K(C_n) = K(C).$$

(b) There exists $n_0 > 0$ *such that for any* $n \geq n_0$ *and any* $\boldsymbol{u} \in \boldsymbol{S}^2$ *we have*

$$\mu_{C_n}(\boldsymbol{u}) \leq \mu_{C_{n+1}}(\boldsymbol{u}) = \mu_C(\boldsymbol{u}) := \text{ the number of local minimal of } L_{\boldsymbol{u}}|_C.$$

Moreover

$$\lim_{n\to\infty}\int_{\boldsymbol{S}^2}\mu_{C_n}(\boldsymbol{u})\,dA_{\boldsymbol{u}} = \int_{\boldsymbol{S}^2}\mu_C(\boldsymbol{u})\,dA_{\boldsymbol{u}}.$$

The proof is not very hard, but it is rather technical and we refer for details to [2]. In particular we deduce that for any closed C^2 curve we have

$$K(C) = \boldsymbol{\kappa}(C), \tag{3.2}$$

where the left-hand side is the bending measure (3.1) and it is a purely geometric quantity, while $\boldsymbol{\kappa}(C)$ is a probabilistic quantity

$$\boldsymbol{\kappa}(C) = \frac{1}{4\pi}\int_{\boldsymbol{S}^2}\mu_C(\boldsymbol{u})\,dA_{\boldsymbol{u}}. \tag{3.3}$$

4. Total curvature and knotting

The topologists refer to closed C^2 curves in $\mathbb{R}^3$ as knots. In the 40s K Borsuk ask the following question

Is it true that if a knot C is "not too bent", then it is not really knotted? More precisely, he sked to prove that if $K(C) \leq 2$ then C is not knotted.

In 1949, while an undergraduate at Princeton, J. Milnor gave a proof to this conjecture in the beautiful paper [2] that served as inspiration for this talk. At about the same time, in Europe, I. Fáry gave a different but related proof of this fact. We want to prove a slightly weaker result.

Theorem 4.1 (Milnor-Fáry)**.** *If C is a knot and $K(C) < 2$, then C is not knotted.*

Proof. Here is briefly Milnor's strategy. He introduced an invariant $m(C)$ of a knot C, called *crookedness* and he showed that if $m(C) = 1$ then C is not knotted. A simple argument based on (3.2) then shows that $K(C) \leq 2$ implies that $m(C) = 1$.

The crookedness $m(C)$ is the integer

$$m(C) := \min_{\boldsymbol{u}\in\boldsymbol{S}^2}\mu_C(\boldsymbol{u}).$$

Lemma 4.2. *If $m(C) = 1$ then C is not knotted.*

Proof of the lemma. Since $m(C) = 1$ there exists $\boldsymbol{u} \in \boldsymbol{S}^2$ such that the function $L_{\boldsymbol{u}}|_C$ has a unique local minimum, which has to be a global minimum. In particular this function must have a unique local maximum, because between two local maxima there must be a local minimum.

By a suitable choice of coordinates we can assume that $\boldsymbol{u}$ is the basic vector $\boldsymbol{k}$, so that

$$L_{\boldsymbol{u}}(x\boldsymbol{i} + y\boldsymbol{i} + z\boldsymbol{k}) = z$$

i.e., $L_{\boldsymbol{u}}$ is the altitude function.

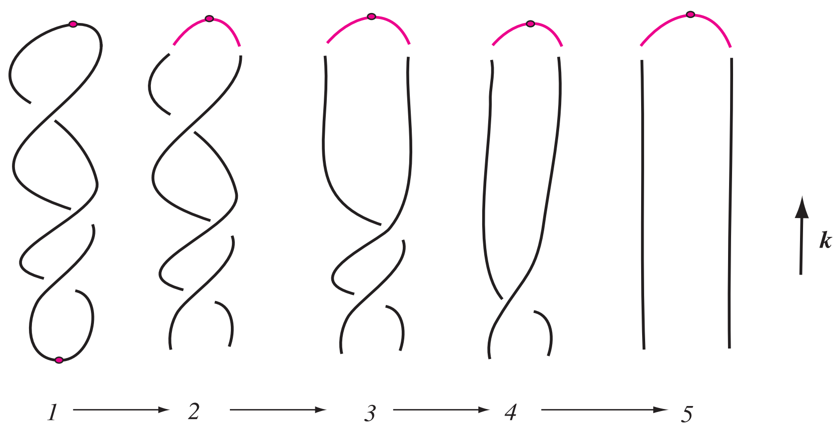

FIGURE 3. *Unknotting a curve with small crookedness.*

By removing two small caps, i.e., small connected neighborhoods of the minimum and the maximum points we obtain two disjoint arcs in $\mathbb{R}^3$ as depicted in Figure 3-2. The restriction of the altitude along each of these arcs is a continuous injective function. These two arcs start at the same altitude z_0 and end at the same altitude $z_1 > z_0$. For $t \in [z_0, z_1]$ these two arcs intersect the horizontal plane $\{z = t\}$ in two points p_t and q_t. Denote by S_t line segment connecting p_t to q_t. The union of these segments spans a ribbon between the two arcs which shows that they can be untwisted, as in Figure 3-3,4,5. To unknot C we let the boundary of the caps follow the boundaries of the two arcs as they are untwisted. □

We can now complete the proof of Theorem 4.1. We observe that

$$K(C) = \frac{1}{4\pi}\int_{\boldsymbol{S}^2} \mu_C(\boldsymbol{u})\, dA_{\boldsymbol{u}} \geq \frac{1}{4\pi}\int_{\boldsymbol{S}^2} m(C)\, dA_{\boldsymbol{u}} = m(C).$$

Thus if $K(C) < 2$ then the positive integer $m(C)$ is strictly less than 2 so that $m(C) = 1$. From Lemma 4.2 we deduce that C is not knotted. □

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