

Let X be a closed manifold having integral homology of $S^1 \times S^3$. Define $\rho(X)$ to be the Rohlin invariant of any spin 3-manifold in X carrying $H_3(X)$. A well known question is whether $\pi_1(X) = \mathbb{Z}$ implies that $\rho(X) = 0$. Motivated by Casson's approach to proving that the Rohlin invariant of a homotopy 3-sphere vanishes, we study an analogue of Casson's invariant in dimension four, obtained by "creatively" counting irreducible flat connections over X . This invariant was introduced by Furuta and Ohta, who also conjectured that its modulo 2 reduction equals $\rho(X)$.

We derive explicit formulas for the Furuta-Ohta invariant for mapping tori of finite order diffeomorphisms and for certain circle bundles. In both instances, it is expressed in terms of classical invariants, and reduces modulo 2 to the Rohlin invariant. As a part of our broader surgery program aimed at proving the Furuta-Ohta conjecture, we investigate an analogue of the Furuta-Ohta invariant for homology 4-tori T and prove that its mod 2 reduction equals the quadruple cup product of vectors forming a basis in $H^1(T)$.

Finally, we relate the Furuta-Ohta invariant to the Floer Lefschetz number and show how a positive solution of the Furuta-Ohta conjecture would imply existence of non-triangulable topological manifolds of dimension five and higher.

This is a joint work with Daniel Ruberman.