Notes on the Atiyah-Singer Index Theorem

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Math 658, Spring 2004: *The Atiyah-Singer Index Theorem*
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This is arguably one of the deepest and most beautiful results in modern geometry, and in my view is a must know for any geometer/topologist. It has to do with elliptic partial differential operators on a compact manifold, namely those operators P with the property that $\dim\ker P, \dim\operatorname{coker} P < \infty$. In general these integers are very difficult to compute without some very precise information about P. Remarkably, their difference, called the *index* of P, is a "soft" quantity in the sense that its determination can be carried out relying only on topological tools. You should compare this with the following elementary situation.

Suppose we are given a linear operator $A : \mathbb{C}^m \to \mathbb{C}^n$. From this information alone we *cannot* compute the dimension of its kernel or of its cokernel. We *can* however compute their difference which, according to the rank-nullity theorem for $n \times m$ matrices must be $\dim\ker A - \dim\operatorname{coker} A = m - n$.

Michael Atiyah and *Isadore Singer* have shown in the 60's that the index of an elliptic operator is determined by certain cohomology classes on the background manifold. These cohomology classes are in turn *topological invariants* of the vector bundles on which the differential operator acts and the homotopy class of the principal symbol of the operator. Moreover, they proved that in order to understand the index problem for an arbitrary elliptic operator it suffices to understand the index problem for a very special class of first order elliptic operators, namely the *Dirac type* elliptic operators. Amazingly, most elliptic operators which are relevant in geometry are of Dirac type. The index theorem for these operators contains as special cases a few celebrated results: the Gauss-Bonnet theorem, the Hirzebruch signature theorem, the Riemann-Roch-Hirzebruch theorem.

In this course we will be concerned only with the index problem for the Dirac type elliptic operators. We will adopt an analytic approach to the index problem based on the heat equation on a manifold and Ezra Getzler's rescaling trick.

☞ **Prerequisites:** Working knowledge of smooth manifolds, and algebraic topology (especially cohomology). Some familiarity with basic notions of functional analysis: Hilbert spaces, bounded linear operators, L^2-spaces.

☞ **Syllabus:** *Part I.* Foundations: connections on vector bundles and the Chern-Weil construction, calculus on Riemann manifolds, partial differential operators on manifolds , Dirac operators, [16].

Part II. The statement and some basic applications of the index theorem, [19].

Part III. The proof of the index theorem, [19].

☞ **About the class** There will be 3-4 homeworks containing routine exercises which involve the basic notions introduced during the course. We will introduce a fairly large number of new objects and ideas and solving these exercises is the only way to gain something form this class and appreciate the rich flavor hidden inside this theorem.

Notations and conventions

• $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

• For every finite dimensional $\mathbb{K}$-vector space V we denote by $\mathrm{Aut}_{\mathbb{K}}(V)$ the Lie group of $\mathbb{K}$-linear automorphisms of V.

• We will orient the manifolds with boundary using the outer normal first convention.

Contents

Chapter 1

Geometric Preliminaries

1.1. Vector bundles and connections

1.1.1. Smooth vector bundles. The notion of smooth $\mathbb{K}$-vector bundle of rank r formalizes the intuitive idea of a smooth family of r-dimensional $\mathbb{K}$-vector spaces.

Definition 1.1.1. A *smooth* $\mathbb{K}$*-vector bundle* of rank r over a smooth manifold B is a quadruple (E, B, π, V) with the following properties.

(a) E, B are smooth manifolds and V is a r-dimensional $\mathbb{K}$-vector space.

(b) $\pi : E \to B$ is a surjective submersion. We set $E_b := \pi^{-1}(b)$ and we will call it the *fiber (of the bundle) over* b.

(c) There exists a *trivializing cover*, i.e. an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of B and diffeomorphisms

$$\Psi_\alpha : E \,|\, U_\alpha = \pi^{-1}(U_\alpha) \to V \times U_\alpha$$

with the following properties.

(c1) For every $\alpha \in A$ the diagram below is commutative.

$$\begin{array}{ccc} E\,|_{U_\alpha} & \xrightarrow{\Psi_\alpha} & V \times U_\alpha \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{proj}} & \\ & U_\alpha & \end{array} .$$

(c2) For every $\alpha, \beta \in A$ there exists a *smooth* map

$$g_{\beta\alpha} : U_{\beta\alpha} := U_\alpha \cap U_\beta \to \mathrm{Aut}(V), \;\; u \mapsto g_{\beta\alpha}(u)$$

such that for every $u \in U_{\alpha\beta}$ we have the commutative diagram

$$\begin{array}{ccc} & & V \times \{u\} \\ & {\scriptstyle \Psi_\alpha|_{E_u}} \nearrow & \\ E_u & & \downarrow {\scriptstyle g_{\beta\alpha}(u)} \\ & {\scriptstyle \Psi_\beta|_{E_u}} \searrow & \\ & & V \times \{u\} \end{array}$$

B is called the *base*, E is called *total space*, V is called the *model (standard) fiber* and π is called the *canonical (or natural) projection*. A $\mathbb{K}$-line bundle is a rank 1 $\mathbb{K}$-vector bundle.

Remark 1.1.2. The condition (c) in the above definition implies that each fiber E_b has a natural structure of $\mathbb{K}$-vector space. Moreover, each map Ψ_α induces an isomorphism of vector spaces

$$\Psi_\alpha\,|_{E_b} \to V \times \{b\}.$$

□

Here is some terminology we will use frequently. Often instead of (E, π, B, V) we will write $E \stackrel{\pi}{\to} B$ or simply E. The inverses of Ψ_α^{-1} are called *local trivializations* of the bundle (over U_α). The map $g_{\beta\alpha}$ is called the *gluing map* from the α-trivialization to the β-trivialization. The collection

$$\Big\{ g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{Aut}(V);\ \ U_{\alpha\beta} \neq \emptyset \Big\}$$

is called a $(\mathrm{Aut}(V))$-*gluing cocycle* (subordinated to $\mathfrak{U}$) since it satisfies the *cocycle condition*

$$g_{\gamma\alpha}(u) = g_{\gamma\beta}(u) \cdot g_{\beta\alpha}(u),\ \ \forall u \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma, \tag{1.1.1}$$

where ”·” denotes the multiplication in the Lie group $\mathrm{Aut}(V)$. Note that (1.1.1) implies that

$$g_{\alpha\alpha}(u) \equiv \mathbb{1}_V,\ \ g_{\beta\alpha}(u) = g_{\alpha\beta}(u)^{-1},\ \ \forall u \in U_{\alpha\beta}. \tag{1.1.2}$$

Example 1.1.3. (a) A vector space can be regarded as a vector bundle over a point.

(b) For every smooth manifold M and every finite dimensional $\mathbb{K}$-vector space we denote by $\underline{V}_M$ the *trivial* vector bundle

$$V \times M \to M,\ \ (v, m) \mapsto m.$$

(c) The tangent bundle TM of a smooth manifold is a smooth tangent bundle.

(d) If $E \stackrel{\pi}{\to} B$ is a smooth vector bundle and $U \hookrightarrow B$ is an open set then $E\,|_U \stackrel{\pi}{\to} U$ is the vector bundle

$$\pi^{-1}(U) \stackrel{\pi}{\to} U.$$

(e) Recall that $\mathbb{CP}^1$ is the space of all one-dimensional subspaces of $\mathbb{C}^2$. Equivalently, $\mathbb{CP}^1$ is the quotient of $\mathbb{C}^2 \setminus \{0\}$ modulo the equivalence relation

$$p \sim p' \Longleftrightarrow \exists \lambda \in \mathbb{C}^* :\ \ p' = \lambda p.$$

For every $p = (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$ we denote by $[p] = [z_0, z_1]$ its $\sim$-equivalence class which we vie as the line containing the origin and the point (z_0, z_1). We have a nice open cover $\{U_0, U_1\}$ of $\mathbb{CP}^1$ defined by

$$U_i := \{[z_0, z_1];\ \ z_i \neq 0\}.$$

U_0 consists of the lines transversal to the vertical axis, while U_1 consists of the lines transversal to the horizontal axis. The slope $m_0 = z_1/z_0$ of the line through (z_0, z_1) is a local coordinated over U_0 and the slope $m_1 = z_0/z_1$ is a local coordinate over U_1. On the overlap we have

$$m_1 = 1/m_0.$$

Let

$$E = \{(x, y; [z_0, z_1]) \in \mathbb{C}^2 \times \mathbb{CP}^1;\ \ \frac{y}{x} = \frac{z_1}{z_0},\ \ \text{i.e. } yz_0 - xz_1 = 0\}$$

The natural projection $\mathbb{C}^2 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ induces a surjection $\pi : E \to \mathbb{CP}^1$. Observe that for every $[p] \in \mathbb{CP}^1$ the fiber $\pi^{-1}(p)$ can be naturally identified with the line through p. We can thus regard

E as a family of 1-dimensional vector spaces. We want to show that π actually defines a structure of smooth complex line bundle over $\mathbb{CP}^1$. Set

$$E_i := \pi^{-1}(U_i) = \Big\{(x, y; [z_0, z_1]) \in E;\ \ z_i \neq 0\Big\}.$$

We construct a map

$$\Psi_0 : E_0 \to \mathbb{C} \times U_0,\ \ E_0 \ni (x, y; [z_0, z_1]) \mapsto (x, [z_0, z_1])$$

and

$$\Psi_1 : E_1 \to \mathbb{C} \times U_1,\ \ E_1 \ni (x, y; [z_0, z_1]y) \mapsto (y, [z_0, z_1])$$

Observe that Ψ_0 is bijective with inverse $\Psi_0^{-1} : \mathbb{C} \times U_0 \to E_0$ is given by

$$\mathbb{C} \times U_0 \ni (t; [z_0, z_1]) \mapsto (t, \frac{z_1}{z_0}t; [z_0, z_1]) = (t, m_0 t; [z_0, z_1]).$$

The composition

$$\Psi_1 \circ \Psi_0^{-1} : \mathbb{C} \times U_{01} \to \mathbb{C} \times U_{01}$$

is given by

$$\mathbb{C} \times U_{01} \ni (s; [p]) \mapsto (g_{10}([p])s, [p]),$$

where

$$U_{01} \ni [p] = [z_0, z_1] \mapsto g_{10}([p]) = z_1/z_0 = m_0([p]) \in \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C}).$$

The complex line bundle constructed above is called the *tautological line bundle*. □

Given a smooth manifold B, a vector space V, an open cover $\mathfrak{U} = (U_\alpha)_{\alpha \in A}$ of B, and a gluing cocycle subordinated to $\mathfrak{U}$

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{Aut}(V)$$

we can construct a smooth vector bundle as follows. Consider the disjoint union

$$X = \coprod_{\alpha \in A} \underline{V}_{U_\alpha}.$$

Denote by E the quotient space of X modulo the equivalence relation

$$\underline{V}_{U_\alpha} \ni (v_\alpha, u_\alpha) \sim (v_\beta, u_\beta) \in \underline{V}_{U_\beta} \Longleftrightarrow u_\alpha = u_\beta = u \in U_{\alpha\beta},\ \ v_\beta = g_{\beta\alpha}(u)v_\alpha.$$

Since we glue open sets of smooth manifolds via diffeomorphisms we deduce that E is naturally a smooth manifold. Moreover, the natural projections $\pi_\alpha : \underline{V}_{U_\alpha} \to U_\alpha$ are compatible with the above equivalence relation and define a smooth map

$$\pi : E \to B.$$

The natural maps $\Phi_\alpha : \underline{V}_{U_\alpha} \to E\,|_{U_\alpha}$ are diffeomorphisms and their inverses $\Psi_\alpha = \Phi_\alpha^{-1}$ satisfy all the conditions in Definition 1.1.1. We will denote the vector bundle obtained in this fashion by $(\mathfrak{U}, g_{\bullet\bullet}, V)$ or by $(B, \mathfrak{U}, g_{\bullet\bullet}, V)$.

Definition 1.1.4. Suppose (E, π_E, B, V) and (F, π_F, B, W) are smooth $\mathbb{K}$-vector bundles over B of ranks p and respectively q. Assume $\{U_\alpha, \Psi_\alpha\}_\alpha$ is a trivializing cover for π_E and $\{V_\beta, \Phi_\beta\}_{\beta \in B}$ is a trivializing cover for π_F. A *vector bundle morphism* from $E \xrightarrow{\pi_E} B$ to $F \xrightarrow{\pi_F} B$ is a smooth map $T : E \to F$ satisfying the following conditions.

(a) The diagram bellow is commutative.

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ & {\scriptstyle\pi_E}\searrow \quad \swarrow{\scriptstyle\pi_F} & \\ & B & \end{array}.$$

(b) T is linear along the fibers, i.e. for every $b \in B$ and every $\alpha \in A$, $b \in B$ such that $b \in U_\alpha \cap V_\beta$ the composition $\Phi_\beta T|_{F_b}\, \Psi_\alpha|_{E_b}\colon V \to W$ is linear,

$$\begin{array}{ccc} E_b & \xrightarrow[\Psi_\alpha|_{E_b}]{} & V \times \{b\} \\ {\scriptstyle T|E_b}\downarrow & & \vdots\ {\scriptstyle \text{linear}} \\ F_b & \xrightarrow[\Phi_\beta|_{F_b}]{} & W \times \{b\} \end{array}.$$

T is called an *isomorphism* if it is a diffeomorphism. We denote by $\underline{Hom}(E,F)$ the space of bundle morphisms $E \to F$. When $E = F$ we set $\underline{End}(E) := \underline{Hom}(E,E)$. A *gauge transformation* of E is a bundle automorphism $E \to E$. We will denote the space of gauge transformations of E by $\underline{Aut}(E)$ or $\mathcal{G}_E$.

We will denote by $\mathcal{VB}_{\mathbb{K}}(M)$ the set of isomorphism classes of smooth $\mathbb{K}$-vector bundles over M.

Definition 1.1.5. A subbundle of $E \stackrel{\pi}{\to} B$ is a smooth submanifold $F \hookrightarrow E$ with the property that $F \stackrel{\pi}{\to} B$ is a vector bundle and the inclusion $F \hookrightarrow E$ is a bundle morphism.

Definition 1.1.6. Suppose $E \to M$ is a rank r $\mathbb{K}$-vector bundle over M. A *trivialization of* E is a bundle isomorphism

$$\underline{\mathbb{K}}^r_M \to E.$$

The bundle E is called *trivializable* if it admits trivializations. A *trivialized* vector bundle is a pair (vector bundle, trivialization).

Example 1.1.7. (a) A bundle morphism between two trivial vector bundles

$$T : \underline{V}_B \to \underline{W}_B$$

is a smooth map

$$T : B \to \mathrm{Hom}(V, W).$$

(b) If we are given two vector bundles over B described by gluing cocycles subordinated to the same open cover

$$(\mathcal{U}, g_{\bullet\bullet}, V), \quad (\mathcal{U}, h_{\bullet\bullet}, W)$$

then a bundle morphism can be described as a collection of smooth maps

$$T_\alpha : U_\alpha \to \mathrm{Hom}(V, W)$$

such that for any α, β and any $u \in U_{\alpha\beta}$ the diagram below is commutative.

$$\begin{array}{ccc} V & \xrightarrow{T_\alpha(u)} & W \\ {\scriptstyle g_{\beta\alpha}(u)}\downarrow & & \downarrow{\scriptstyle h_{\beta\alpha}(u)} \\ V & \xrightarrow{T_\beta(u)} & W \end{array}$$

□

There are a few basic methods of producing new vector bundles from given ones. The first methods reproduce some fundamental operations for vector spaces, i.e. vector bundles over a point. We list below a few of them.

$$V \rightsquigarrow V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}) \quad - \quad \text{the dual of } V,$$

$$V, W \rightsquigarrow V \oplus W \quad - \quad \text{the direct sum of } V \text{ and } W,$$

$$V, W \rightsquigarrow V \otimes W \quad - \quad \text{the tensor product of } V \text{ and } W,$$

$$V \rightsquigarrow \mathrm{Sym}^m V \quad - \quad \text{the } m\text{-th symmetric product of } V,$$

$$V \rightsquigarrow \Lambda^k V \quad - \quad \text{the } k\text{-th exterior power of } V,$$

$$V \rightsquigarrow \det V := \Lambda^{\dim V} V \quad - \quad \text{the determinat line of } V.$$

These constructions are natural in the following sense. Given linear maps $V_i \stackrel{T_i}{\to} W_i$, $i = 0, 1$ we have induced maps

$$^tT_0 : W_0^* \longrightarrow V_0^*,$$

$$T_0 \oplus T_1 : V_0 \oplus V_1 \to W_0 \oplus W_1, \quad T_0 \otimes T_1 : V_0 \otimes V_1 \longrightarrow W_0 \otimes W_1,$$

$$\mathrm{Sym}^k T_0 : \mathrm{Sym}^k V_0 \longrightarrow \mathrm{Sym}^k W_0, \quad \Lambda^k T_0 : \Lambda^k V_0 \longrightarrow \Lambda^k W_0.$$

If $\dim V_0 = \dim W_0 = n$ then the map $\Lambda^n T_0$ will be denoted by $\det T_0$.

These operations for vector spaces can also be performed for smooth families of vector spaces, i.e. bundles over arbitrary smooth manifolds.

Given two bundles E, F over *the same manifold* M described by the gluing cocycles

$$E = (\mathcal{U}, g_{\bullet\bullet}, V), \quad F = (\mathcal{U}, h_{\bullet\bullet}, W)$$

we can form

$$E^* = (\mathcal{U}, (^tg_{\bullet\bullet})^{-1}, V^*$$

$$E \oplus F = (\mathcal{U}, g_{\bullet\bullet} \oplus h_{\bullet\bullet}, V \oplus W), \quad E \otimes F = (\mathcal{U}, g_{\bullet\bullet} \otimes h_{\bullet\bullet}, V \otimes W,$$

$$\mathrm{Sym}^m E = (\mathcal{U}, \mathrm{Sym}^m g_{\bullet\bullet}, \mathrm{Sym}^m V), \quad \Lambda^k E = (\mathcal{U}, \Lambda^k g_{\bullet\bullet}, \Lambda^k V),$$

$$\det_{\mathbb{K}} E = (\mathcal{U}, \det g_{\bullet\bullet}, \det V).$$

$\det_{\mathbb{K}} E$ is called the *determinant line bundle* of E

Definition 1.1.8. (a) Suppose $E \to M$ is a $\mathbb{K}$-vector bundle. A $\mathbb{K}$-*orientation* of E is an equivalence class of trivializations of $\tau : \underline{\mathbb{K}}_M \to \det_{\mathbb{K}} E$, where two trivializations $\tau_i : \underline{\mathbb{K}}_M \to \det_{\mathbb{K}} E$, $i = 0, 1$ are considered equivalent if there exists a smooth function $\mu : M \to \mathbb{R}$ such that

$$\tau_1(s) = \tau_0(e^{\mu} s), \quad \forall s \in C^\infty(\underline{\mathbb{K}}_M).$$

A bundle is called $\mathbb{K}$-*orientable* if it admits $\mathbb{K}$-orientations. An *oriented* $\mathbb{K}$-vector bundle is a pair (vector bundle, $\mathbb{K}$-orientation).

Example 1.1.9. (a) A smooth manifold M is orientable if its tangent bundle TM is $\mathbb{R}$-orientable. □

When $\mathbb{K} = \mathbb{R}$ and when no confusion is possible we will use the simpler terminology of orientation rather than $\mathbb{R}$-orientation.

Another important method of producing new vector bundles is the *pullback construction*. More precisely given a vector bundle $E \stackrel{\pi}{\to} M$ described by the gluing cocycle

$$(M, \mathcal{U}, g_{\bullet\bullet}, V)$$

and a smooth map $f : N \to M$ then we can construct a bundle $f^*E \to N$ described by the gluing cocycle

$$(N, f^{-1}(\mathcal{U}), g_{\bullet\bullet} \circ f, V).$$

There is a natural smooth map $f_* : f^*E \to E$ such that the diagram below is commutative

$$\begin{array}{ccc} f^*E & \stackrel{f_*}{\dashrightarrow} & E \\ \Big\downarrow & & \Big\downarrow \\ M & \stackrel{f}{\longrightarrow} & N \end{array}$$

and for every $m \in M$ the induced map $(f^*E)_m \to E_{f(m)}$ is linear.

Remark 1.1.10. The above construction is a special case of the *fibered product* construction,

$$f^*(E) \to N \leftrightsquigarrow E \times_M N \stackrel{\pi \times_M f}{\longrightarrow} N,$$

$$E \times_M N := \Big\{ (e, n) \in E \times N;\ \ \pi(e) = f(n) \Big\},\ \ (\pi \times_M f)(e, n) = n.$$

Equivalently $E \times_M N$ is the preimage of the diagonal $\Delta \subset M \times M$ via the map

$$\pi \times f : E \times N \to M \times M.$$

This is a smooth manifold since π is a submersion. □

Example 1.1.11. If V is a vector space, M is a smooth manifold and $c : M \to \{pt\}$ is the collapse to a point, then the trivial bundle $\underline{V}_M$ is the pullback via c of the vector bundle over pt which is the vector space V itself

$$\underline{V}_M = c^*V.$$

□

Definition 1.1.12. A (smooth) *section* of a vector bundle $E \stackrel{\pi}{\to} B$ is a (smooth) map $s : B \to E$ such that

$$s(b) \in E_b,\ \ \forall B$$

If $U \subset B$ is an open subset then a smooth section of E over U is a (smooth) section of $E\,|_U$. We denote by $C^\infty(U, E)$ the set of smooth sections of U over E. When $U = B$ we will write simply $C^\infty(E)$.

Observe that $C^\infty(E)$ is a vector space where the sum of two sections $s, s' : B \to E$ is the section $s + s'$ defines by

$$(s + s')(b) := s(b) + s'(b) \in E_b.\ \ \forall b \in B.$$

If the vector bundle $E \to B$ is given by the local gluing data $(\mathcal{U}, g_{\bullet\bullet}, V)$ then a section of E can be described as a collection $s_\bullet$ of smooth functions

$$s_\bullet : U_\bullet \to V$$

with the property that $\forall \alpha, \beta$ and $\forall u \in U_{\alpha\beta}$ we have

$$s_\beta(u) = g_{\beta\alpha}(u) s_\alpha(u).$$

This shows that there exists at least one section 0 defined by the collection $s_\bullet \equiv 0$. It is called the *zero section of* E.

Given two sections $s = (s_\bullet)$, $s' = (s'_\bullet)$ their sum is the section described locally by the collection $(s_\bullet + s'_\bullet)$.

Example 1.1.13. (a) If M is a smooth manifold then a smooth section of the trivial line bundle $\underline{\mathbb{C}}_M$ is a smooth function $M \to \mathbb{C}$.

(b) A smooth section of the tangent bundle of M is a *vector field* over M. We will denote by $\mathrm{Vect}(M)$ the set of smooth vector fields on M.

(c) A smooth section of the cotangent bundle T^*M is called a differential 1-form. A smooth section of the k-th exterior power of T^*M is called a *differential form of degree* k. We will denote by $\Omega^k(M)$ the space of such differential forms.

(d) Suppose $E \to M$ is a smooth vector bundle. Then an E-valued differential form of degree k is a section of $\Lambda^k T^*M \otimes E$. The space of such sections will be denoted by $\Omega^k(E)$. Observe that

$$\Omega^k(M) = \Omega^k(\underline{\mathbb{R}}_M).$$

(e) Suppose that $E, F \to M$ are smooth $\mathbb{K}$-vector bundles over M. Then

$$C^\infty(E^* \otimes F) \cong \underline{Hom}(E, F).$$

For this reason we set

$$\mathrm{Hom}(E, F) := E^* \otimes F.$$

When $E = F$ we set

$$\mathrm{End}(E) := \mathrm{Hom}(E, E).$$

If E is a line bundle then

$$\mathrm{End}(E) \cong \underline{\mathbb{K}}_M.$$

We want to emphasize that $\underline{Hom}(E, F)$ is an infinite dimensional vector space while $\mathrm{Hom}(E, F)$ is a finite dimensional vector bundle and

$$C^\infty(\mathrm{Hom}(E, F)) = \underline{Hom}(E, F).$$

Let us also point out that a $\mathbb{K}$-linear map $T : C^\infty(E) \to C^\infty(F)$ is induced by a bundle morphism $E \to F$ iff and only if T is a morphism of $C^\infty(M)$-modules, i.e. for any smooth function $f : M \to \mathbb{K}$ we have

$$T(fu) = fTu, \quad u \in C^\infty(E).$$

(e) Suppose that $E \to M$ is a real vector bundle. A *metric* on E is then a section h of $\mathrm{Sym}^2 E^*$ with the property that for every $m \in M$ the symmetric bilinear form $h_m \in \mathrm{Sym}^2 E^*$ is an Euclidean metric on the fiber E_m. A *Riemann metric* on a manifold M is a metric on the tangent bundle TM. A metric on E induces metrics on all the bundles E^*, $E^{\otimes k}$, $\mathrm{Sym}^k E$, $\Lambda^k E$.

Observe that if h is a metric on E and F is a sub-bundle of E then h induces a metric on F. In particular, the tautological line bundle $L \to \mathbb{CP}^1$ is by definition a subbundle of the trivial vector bundle $\underline{\mathbb{C}}^2_{\mathbb{CP}^1}$ and as such it is equipped with a natural metric.

(f) Suppose that $E \to M$ is a complex vector of rank r described by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{C}^r)$. Then the *conjugate* of E is the complex vector bundle $\bar{E}$ described by the gluing cocycle $(\mathcal{U}, \bar{g}_{\bullet\bullet}, \mathbb{C}^r)$ where for any matrix $g \in \mathrm{GL}_r(\mathbb{C})$ we have denoted by $\bar{g}$ its complex conjugate. Note that there exists a *canonical* isomorphism of *real* vector bundles

$$C : E \to \bar{E}$$

called the *conjugation*.

A section u of $\bar{E}^*$ defines for every $m \in M$ a $\mathbb{R}$-linear map $u_m : E_m \to \mathbb{C}$ which is *complex conjugate linear* i.e.

$$u_m(\lambda e) = \bar{\lambda} u_m(e), \ \ \forall e \in E_m, \quad \lambda \in \mathbb{C}.$$

A *hermitian metric* on H is a section h of $E^* \otimes_{\mathbb{C}} \bar{E}^*$ satisfying for every $m \in M$ the following properties.

h_m defines a $\mathbb{R}$-bilinear map $E \times E \to \mathbb{C}$ which is complex linear in the first variable and conjugate linear in the second variable.

$$h_m(e_1, e_2) = \overline{h(e_2, e_1)}, \ \ \forall e_1, e_2 \in E_m.$$

$$h_m(e, e) > 0, \ \ \forall e \in E_m \setminus \{0\}.$$

If E is a vector bundle equipped with a metric h (riemannian or hermitian) then we denote by $\mathrm{End}_h^- E$ the real subbundle of $\mathrm{End}(E)$ whose sections are the endomorphisms $T : E \to E$ satisfying

$$h(Tu, v) = -h(u, Tv), \ \ \forall u, v \in C^\infty(E).$$

(g) A $\mathbb{K}$-vector bundle is $\mathbb{K}$-orientable iff $\det_{\mathbb{K}} E$ admits a nowhere vanishing section. Indeed since $\det_{\mathbb{K}} E \cong (\underline{\mathbb{K}}_M)^* \otimes \det_{\mathbb{K}} E \cong \mathrm{Hom}(\underline{\mathbb{K}}_M, E)$ a section of E can be identified with a bundle morphism $\underline{\mathbb{K}}_M \to E$. This is an isomorphism since the section is nowhere vanishing.

(h) Every complex vector bundle $E \to M$ is $\mathbb{R}$-*orientable*. To construct it we need to produce a nowhere vanishing section of $\det_{\mathbb{R}} E$. Suppose E is described by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{C}^r)$. Using the inclusion

$$i : \mathrm{GL}_r(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2r}(\mathbb{R})$$

we get maps

$$\hat{g}_{\bullet\bullet} = i \circ g_{\bullet\bullet} : U_{\bullet\bullet} \to GL_{2r}(\mathbb{R})$$

satisfying

$$w_{\bullet\bullet} := \det \hat{g}_{\bullet\bullet} = |\det g_{\bullet\bullet}|^2 > 0.$$

Let

$$f_{\bullet\bullet} := \log w_{\bullet\bullet} \Longleftrightarrow w_{\bullet\bullet} = \exp(f_{\bullet\bullet}).$$

Since $w_{\bullet\bullet}$ defines a gluing cocycle for $\det_{\mathbb{R}} E$ and in particular

$$w_{\gamma\alpha}(u) = w_{\gamma\beta}(u) w_{\beta\alpha}(u).$$

We deduce

$$f_{\gamma\alpha}(u) = f_{\gamma\beta}(u) + f_{\beta\alpha}(u), \ \ \forall \alpha, \beta, \gamma, \ \ \forall u \in U_{\alpha\beta\gamma}.$$

Consider now a partition of unity (θ_α) subordinated to $\mathcal{U}$, $\mathrm{supp}\, \theta_\alpha \subset U_\alpha$. Define

$$f_\alpha : U_\alpha \to \mathbb{R}, \ \ f_\alpha(u) = \sum_{U_\beta \ni u} \theta_\beta(u) f_{\beta\alpha}(u) = \sum_\beta \theta_\beta(u) f_{\beta\alpha}(u)$$

Observe first that f_α is smooth. Using the equalities

$$f_{\gamma\alpha} - f_{\gamma\beta} = f_{\gamma\alpha} + f_{\beta\gamma} = f_{\beta\alpha}$$

we deduce[1]

$$f_\alpha - f_\beta = \sum_\gamma \theta_\gamma (f_{\gamma\alpha} - f_{\gamma\beta}) = \sum_\gamma \theta_\gamma f_{\beta\alpha} = \Big(\sum_\gamma \theta_\gamma\Big) f_{\beta\alpha} = f_{\beta\alpha}.$$

Equivalently

$$-f_\beta = f_{\beta\alpha} - f_\alpha \Longrightarrow e^{-f_\beta} = w_{\beta\alpha} e^{-f_\alpha} = (\det \hat{g}_{\beta\alpha}) e^{-f_\alpha}.$$

This shows that the collection $s_\alpha = e^{-f_\alpha}$ is a nowhere vanishing section of $\det_{\mathbb{R}} E$.

(i) Suppose $E \to N$ is a smooth bundle and $f : M \to N$ is a smooth map. Then f induces a linear map

$$f^* : C^\infty(E) \to C^\infty(f^* E)$$

which associates to each section s of $E \to N$ a section $f^* s$ of $f^* E \to M$ called the *pullback* of s by f. If s is described by a collection of smooth maps $s_\bullet : U_\bullet \to \mathbb{K}^r$, then $f^* s$ is described by the collection

$$s_\bullet \circ f : f^{-1}(U_\bullet) \to \mathbb{K}^r.$$

Moreover we have a commutative diagram

$$\begin{array}{ccc} f^*E & \xrightarrow{f_*} & E \\ {\scriptstyle f^*s}\uparrow & & \uparrow{\scriptstyle s} \\ M & \xrightarrow{f} & N \end{array}$$

□

Definition 1.1.14. Suppose $E \to B$ is a smooth $\mathbb{K}$-vector bundle. A *local frame* over the open set $U \to B$ is an ordered collection of smooth sections $e_1, \cdots, e_r$ of $E\,|_U$ such that for every $u \in U$ the vectors $\vec{e} = (e_1(u), \cdots, e_r(u))$ form a basis of the fiber E_u.

Given a local frame $\vec{e} = (e_1, \cdots, e_r)$ of $E \to B$ over U we can represent a section s of E over U as a linear combination

$$s = s^1 e_1 + \cdots + s^r e_r$$

where $s_i : U \to \mathbb{K}$ are smooth functions.

1.1.2. Principal bundles. Fix a Lie group G. For simplicity, we will assume that it is a matrix Lie group[2], i.e. it is a closed subgroup of some $\mathrm{GL}_n(\mathbb{K})$. A *principal G-bundle over a smooth manifold* B is a triple (P, π, B) satisfying the following conditions.

$$P \xrightarrow{\pi} B$$

is a surjective submersion. We set $P_b := \pi^{-1} b$

There is a right free action

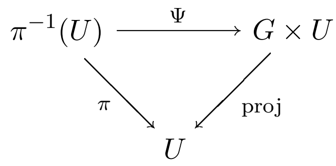
$$P \times G \to P, \quad (p, g) \mapsto pg$$

[1] *For the cognoscienti.* The collection of smooth functions$(f_{\alpha\beta})$ is a Čech 1-cocycle of the fine sheaf of smooth functions. Since the cohomology of a fine sheaf is trivial in positive dimensions this collection must be a Čech coboundary, i.e there exists a collection of smooth functions (f_α) such that $f_\alpha - f_\beta = f_{\beta\alpha}$; see [9]

[2] Any compact Lie group is a matrix Lie group

such that for every $p \in P$ the G-orbit containing p coincides with the fiber of π containing p.

π is locally trivial, i.e. every point $b \in B$ has an open neighborhood U and a diffeomorphism $\Psi_U : \pi^{-1}(U) \to G \times U$ such that the diagram below is commutative



and

$$\Psi(pg) = \Psi(p)g, \ \ \forall p \in \pi^{-1}(U), \ g \in G,$$

where the right action of G on $G \times U$.

Any principal bundle can be obtained by gluing trivial ones. Suppose we are given an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of M and for every $\alpha, \beta \in A$ smooth maps

$$g_{\alpha\beta} : U_{\alpha\beta} \to G$$

satisfying the cocycle condition

$$g_{\gamma\alpha}(u) = g_{\gamma\beta}(u) \cdot g_{\beta\alpha}(u), \ \ \forall u \in U_{\alpha\beta\gamma}$$

Then, exactly as in the case of vector bundles we can obtain a principal bundle by gluing the trivial bundles $P_\alpha = G \times U_\alpha$. More precisely we consider the disjoint union

$$X = \bigcup_\alpha P_\alpha \times \{\alpha\}$$

and the equivalence relation

$$G \times U_\alpha \times \{\alpha\} \ni (g, u, \alpha) \sim (h, v, \beta) \in G \times U_\beta \times \{\beta)\} \Longleftrightarrow u = v \in U_{\alpha\beta}, \ \ h = g_{\beta\alpha}(u)g.$$

Then $P = X/\sim$ is the total space of a principal G-bundle. We will denote this bundle by $(B, \mathcal{U}, g_{\bullet\bullet}, G)$.

Example 1.1.15 (Fundamental example)**.** Suppose $E \to M$ is a $\mathbb{K}$-vector bundle over M of rank r, described by the gluing data $(\mathcal{U}, g_{\bullet\bullet}, V)$, where V is a r-dimensional $\mathbb{K}$-vector space. A *frame* of V is by definition an ordered basis $\vec{e} = (e_1, \cdots, e_r)$ of V. We denote by $\mathbf{Fr}(V)$ the set of frames of V. We have a free and transitive *right* action

$$\mathbf{Fr}(V) \times \mathrm{GL}_r(\mathbb{K}) \to \mathbf{Fr}(V), \ \ (e_1, \cdots, e_r) \cdot g = (\sum_i g_1^i e_i, \cdots, \sum_i g_r^i e_i),$$

$$\forall g = [g_j^i]_{1 \le i,j \le r} \in \mathrm{GL}_r(\mathbb{K}), \ \ (e_1, \cdots, e_r) \in \mathbf{Fr}(V).$$

In particular, the set of frames is naturally a smooth manifold diffeomorphic to $\mathrm{GL}_r(\mathbb{K})$. Note that a frame $\vec{e}$ of V associates to every vector $v \in V$ a vector $v(\vec{e}) \in \mathbb{K}^r$, the coordinates of v with respect to the frame $\vec{e}$. For every $g \in \mathrm{GL}_r(\mathbb{K})$ we have

$$v(\vec{e} \cdot g) = g^{-1} v(\vec{e}).$$

If we let $\mathrm{GL}_r(\mathbb{K})$ act on the right on $\mathbb{K}^r$,

$$\mathbb{K}^r \times \mathrm{GL}_r(\mathbb{K}) \ni (u, g) \mapsto u \cdot g = g^{-1} u \in \mathbb{K}^r$$

then we see that the coordinate map induced by $v \in V$,

$$v : \mathbf{Fr}(V) \to \mathbb{K}^r, \ \ \vec{e} \to v(\vec{e})$$

is G-equivariant.

An isomorphism $\Psi : V \to \mathbb{K}^r$ induces a diffeomorphism

$$\vec{\Phi} : \mathrm{GL}_r(\mathbb{K}) \to \mathbf{Fr}\,(V), \ \ g \mapsto \vec{\Phi}(g) = \Psi^{-1}(\vec{\delta}) \cdot g,$$

where $\vec{\delta}$ denotes the canonical frame of $\mathbb{K}^r$. Observe that

$$\vec{\Phi}(g \cdot h) = \vec{\Phi}(g) \cdot h.$$

To the bundle E we associate the principal bundle $\mathbf{Fr}\,(E)$ given by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathrm{GL}_r(\mathbb{K}))$. The fiber of this bundle over $m \in U_\alpha$ can be identified with the space $\mathbf{Fr}(E_m)$ of frames in the fiber E_m via the map $\vec{\Phi}$ and the local trivialization

$$\Psi_\alpha : E_m \to \mathbb{K}^r.$$

□

To any principal bundle $P = (B, \mathcal{U}, g_{\bullet\bullet}, G)$ and representation $\rho : G \to \mathrm{Aut}_{\mathbb{K}}(V)$ of G on a finite dimensional $\mathbb{K}$-vector space V we can associate a vector bundle $E = (B, \mathcal{U}, \rho(g_{\bullet\bullet}), V)$. We will denote it by $P \times_\rho V$. Equivalently, $P \times_\rho V$ is the quotient of $P \times V$ via the *left* G-action

$$g(p, v) = (pg^{-1}, \rho(g)v).$$

A vector bundle E on a smooth manifold M is said to have *(G, ρ)-structure* if $E \cong P \times_\rho V$ for some principal G-bundle P.

We denote by $\mathfrak{g} = T_1 G$ the Lie algebra of G. We have an *adjoint representation*

$$\mathrm{Ad} : G \to \mathrm{End}\,\mathfrak{g}, \ \ \mathrm{Ad}(g)X = gXg^{-1} = \frac{d}{dt}\,|_{t=0}\, g\exp(tX)g^{-1}, \ \ forall g \in G.$$

The associated vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ is denoted by $\mathrm{Ad}(P)$.

For any representation $\rho : G \to \mathrm{Aut}(V)$ we denote by ρ_* the differential of ρ at 1

$$\rho_* : \mathfrak{g} \to \mathrm{End}\,V.$$

Observe that for every $X \in \mathfrak{g}$ we have

$$\rho_*(\mathrm{Ad}(g)X) = \rho_*(gXg^{-1}) = \rho(g)(\rho_* X)\rho(g)^{-1}. \tag{1.1.3}$$

If we set $\mathrm{End}_\rho\,V := \rho_*(\mathfrak{g}) \subset \mathrm{End}\,V$ we have an induced action

$$\mathrm{Ad}_\rho : G \to \mathrm{End}_\rho(V), \ \ \mathrm{Ad}_\rho(g)T := \rho(g)T\rho(g)^{-1}, \ \ \forall T \in \mathrm{End}\,V, \ g \in G.$$

If $E = P \times_\rho V$ then we set

$$\mathrm{End}_\rho(V) := P \times_{\mathrm{Ad}_\rho} \mathrm{End}_\rho(V).$$

This bundle can be viewed as the bundle of infinitesimal symmetries of E.

Example 1.1.16. (a) Suppose G is a Lie subgroup of $\mathrm{GL}_m(\mathbb{K})$. It has a tautological representation

$$\tau : G \hookrightarrow \mathrm{GL}_m(\mathbb{K}) = \mathrm{Aut}(\mathbb{K}^m).$$

A rank m $\mathbb{K}$-vector bundle $E \to M$ is said to have *G-structure* if it has a (G, τ)-structure. This means that E can be described by a gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^m)$ with the property that the matrices $g_{\bullet\bullet}$ belong to the subgroup G.

For example, $\mathrm{SO}(m), \mathrm{O}(m) \subset \mathrm{GL}_m(\mathbb{R})$ and we can speak of $\mathrm{SO}(m)$ and $\mathrm{O}(m)$ structures on a real vector bundle of rank m. Similarly we can speak of $\mathrm{U}(m)$ and $\mathrm{SU}(m)$ structures on a complex vector bundle of rank m.

A hermitian metric on a rank r complex vector bundle defines a $U(r)$-structure on E and in this case

$$\operatorname{Ad} P = \operatorname{End}_\rho(E) = \operatorname{End}_h^-(E).$$

□

1.1.3. Connections on vector bundles. Roughly speaking, a connection on a smooth vector bundle is a "coherent procedure" of differentiating the smooth sections.

Definition 1.1.17. Suppose $E \to M$ is a $\mathbb{K}$-vector bundle. A *smooth connection* on E is a $\mathbb{K}$-linear operator

$$\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$$

satisfying the product rule

$$\nabla(fs) = s \otimes df + f\nabla s, \ \ \forall f \in C^\infty(M), \ \ s \in C^\infty(E).$$

We say that ∇s is the covariant derivative of s with respect to ∇. We will denote by $\mathcal{A}_E$ the space of smooth connections on E.

Remark 1.1.18. (a) For every section s of E the covariant derivative ∇s is a section of $T^*M \otimes E \cong \operatorname{Hom}(TM, E)$. i.e.

$$\nabla s \in \underline{Hom}(TM, E).$$

As such, ∇s associates to each vector field X on M a section of E which we denote by $\nabla_X s$. We say that $\nabla_X s$ is the derivative of s in along the vector field X with respect to the connection ∇. The product rule can be rewritten

$$\nabla_X(fs) = (L_X f)s + f\nabla s, \ \ \forall X \in \operatorname{Vect}(M), \ \ f \in C^\infty(M), \ \ s \in C^\infty(M),$$

where $L_X f$ denotes the Lie derivative of f along the vector field X.

(b) Suppose $E, F \to M$ are vector bundles and $\Psi : E \to F$ is a bundle isomorphism. If ∇ is a connection of E then $\Psi\nabla\Psi^{-1}$ is a connection on F.

(c) Suppose ∇^0 and ∇^1 are two connections on E. Set

$$A := \nabla^1 - \nabla^0 : C^\infty(E) \to C^\infty(T^*M \times E).$$

Observe that for every $f \in C^\infty(M)$ and every $s \in C^\infty(E)$ we have

$$A(fs) = fA(s)$$

so that

$$A \in \underline{Hom}(E, T^*M \otimes E) \cong C^\infty(E^* \otimes T^*M \otimes E) \cong C^\infty(T^*M \otimes E^* \otimes E)$$
$$\cong C^\infty(T^*M, \operatorname{End}(E) = \Omega^1(\operatorname{End}(E)).$$

In other words, the difference between two connections is a $\operatorname{End} E$-valued 1-form. Conversely, if

$$A \in \Omega^1(\operatorname{End} E) \cong \underline{Hom}(TM \otimes E, E)$$

then for every connection ∇ on E the sum $\nabla + A$ is a gain a connection on E. This shows that the space $\mathcal{A}_E$, if nonempty, is an affine space modelled by the vector space $\Omega^1(\operatorname{End} E)$. □

Example 1.1.19. (a) Consider the trivial bundle $\underline{\mathbb{R}}_M$. The sections of $\underline{\mathbb{R}}_M$ are smooth functions $M \to \mathbb{R}$. The differential

$$d : C^\infty(M) \to \Omega^1(M), \quad f \mapsto df$$

is a connection on $\underline{\mathbb{R}}_M$ called the *trivial connection*.

Observe that $\operatorname{End}(\underline{\mathbb{R}}_M) \cong \underline{\mathbb{R}}_M$ so that any other connection on M has the form

$$\nabla = d + a, \quad a \in \Omega^1(\underline{\mathbb{R}}_M) = \Omega^1(M).$$

(b) Consider similarly the trivial bundle $\underline{\mathbb{K}}^r_M$. Its smooth sections are r-uples of smooth functions

$$s = \begin{bmatrix} s^1 \\ \vdots \\ s^r \end{bmatrix} : M \to \mathbb{K}^r.$$

$\underline{\mathbb{K}}^r$ is equipped with a *trivial connection* ∇^0 defined by

$$\nabla^0 \begin{bmatrix} s^1 \\ \vdots \\ s^r \end{bmatrix} = \begin{bmatrix} ds^1 \\ \vdots \\ ds^r \end{bmatrix}.$$

Any other connection on $\underline{\mathbb{K}}^r$ has the form

$$\nabla = \nabla^0 + A, \quad A \in \Omega^1(\operatorname{End} \underline{\mathbb{K}}^r).$$

More concretely, A is an $r \times r$ matrix $[A^a_b]_{1 \leq a,b \leq r}$, where each entry A^a_b is a $\mathbb{K}$-valued 1-form. If we choose local coordinates $(x^1, \cdots, x^n)$ on M then we can describe A^i_j locally as

$$A^a_b = \sum_k A^a_{kb} dx^k.$$

We have

$$\nabla s = \begin{bmatrix} ds^1 \\ \vdots \\ ds^r \end{bmatrix} + \begin{bmatrix} \sum_b A^1_b s^b \\ \vdots \\ \vdots \\ \sum_b A^r_b s^b \end{bmatrix}.$$

(c) Suppose $E \to B$ is a $\mathbb{K}$-vector bundle of rank r and $\vec{e} = (e_1, \cdots, e_r)$ is a local frame of E over the open set U. Suppose ∇ is a connection on E. Then for every $1 \leq b \leq r$ we get section ∇e_b of $T^*M \otimes E$ over U and thus decompositions

$$\nabla e_b = \sum_a A^a_b e_a, \quad A^a_b \in \Omega^1(U), \quad \forall 1 \leq a, b \leq r. \tag{1.1.4}$$

Given a section $s = \sum^b s^b e_b$ of E over U we have

$$\nabla s = \sum_b ds^b e_b + \sum_b s_b \sum_a A^a_b e_a = \sum_a \Big(ds^a + \sum_b A^a_b s^b \Big) e_a.$$

This shows that the action of ∇ on any section over U is completely determined by the action of ∇ on the local frame, i.e by the matrix (A^a_b). We can regard this as a 1-form whose entries are $r \times r$ matrices. This is known as the *connection* 1*-form* associated to ∇ by the local frame $\vec{e}$. We will denote it by $A(\vec{e})$. We can rewrite (1.1.4) as

$$\nabla(\vec{e}) = \vec{e} \cdot A(\vec{e}).$$

Suppose $\vec{f} = (f_1, \cdots, f_r)$ is another local frames of E over U related to $\vec{e}$ by the equalities

$$f_a = \sum_b e_b g_a^b, \tag{1.1.5}$$

where $U \ni u \mapsto g(u) = (g_a^b(u))_{1 \leq a,b \leq r} \in \mathrm{GL}_r(\mathbb{K})$ is a smooth map. We can rewrite (1.1.5) as

$$\vec{f} = \vec{e} \cdot g.$$

Then $A(\vec{f})$ is related to $A(\vec{e})$ by the equality

$$A(\vec{f}) = g^{-1}A(\vec{e})g + g^{-1}dg. \tag{1.1.6}$$

Indeed

$$\vec{f} \cdot A(\vec{f}) = \nabla(\vec{f}) = \nabla(\vec{e}g) = (\nabla(\vec{e}))g + \vec{e}dg = (\vec{e}A(\vec{e}))g + \vec{f}g^{-1}dg = \vec{f}(g^{-1}A(\vec{e})g + g^{-1}dg).$$

Suppose now that E is given by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, , \mathbb{K}^r)$. Then the canonical basis of $\mathbb{K}^r$ induces via the natural isomorphism $\underline{\mathbb{K}}^r_{U_\alpha} \to E\,|_{U_\alpha}$ a local frame $\vec{e}(\alpha)$ of $E\,|_{U_\alpha}$. We set

$$A_\alpha = A(\vec{e}(\alpha)).$$

On the overlap $U_{\alpha\beta}$ we have the equality $\vec{e}(\alpha) = \vec{e}(\beta)g_{\beta\alpha}$ so that on these overlaps the $\underline{gl}_r(\mathbb{K})$-valued 1-forms A_α satisfy the transition formulæ

$$A_\alpha = g_{\beta\alpha}^{-1}A_\beta g_{\beta\alpha} + g_{\beta\alpha}^{-1}dg_{\beta\alpha} \Longleftrightarrow A_\beta = g_{\beta\alpha}A_\alpha g_{\beta\alpha}^{-1} - (dg_{\beta\alpha})g_{\beta\alpha}^{-1}. \tag{1.1.7}$$

□

Proposition 1.1.20. *Suppose E is a rank r vector bundle over M described by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$. Then a collection of 1-forms*

$$A_\alpha = \Omega^1(U_\alpha) \otimes \underline{gl}_r(\mathbb{K}).$$

satisfying the gluing conditions (1.1.7) determine a connection on E.

Proposition 1.1.21. *Suppose $E \to M$ is a smooth vector bundle. Then there exist connections on E, i.e. $\mathcal{A}_E \neq \emptyset$.*

Proof Suppose that E is described by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$, $r = \mathrm{rank}(E)$.

Denote by $\Psi_\alpha : \underline{\mathbb{K}}^r_{U_\alpha} \to E\,|_{U_\alpha}$ the local trivialization over U_α and by ∇^α the trivial connection on $\underline{\mathbb{K}}^r_{U_\alpha}$ Set

$$\hat{\nabla}^\alpha := \Psi_\alpha \nabla^\alpha \Psi_\alpha^{-1}.$$

Then (see Remark 1.1.18(b)) $\hat{\nabla}^\alpha$ is a connection on $E\,|_{U_\alpha}$. Fix a partition of unity (θ_α) subordinated to (U_α). Observe that for every α and every $s \in C^\infty(E)$ $\theta_\alpha s$ is a section of E with support in U_α. In particular $\hat{\nabla}^\alpha(\theta_\alpha s)$ is a section of $T^*M \otimes E$ with support in U_α. Set

$$\nabla s = \sum_{\alpha,\beta} \theta_\beta \hat{\nabla}^\alpha(\theta_\alpha s)$$

If $f \in C^\infty(M)$ then

$$\nabla(fs) = \sum_{\alpha,\beta} \theta_\beta \hat{\nabla}^\alpha(\theta_\alpha fs) = \sum_\beta \theta_\beta \Big(\sum_\alpha df \otimes (\theta_\alpha s) + f\hat{\nabla}^\alpha(\theta_\alpha s) \Big)$$

$$= df \otimes s \sum_{\alpha,\beta} \theta_\alpha \theta\beta + f\nabla s = df \otimes s \underbrace{\Big(\sum\nolimits_\alpha \theta_\alpha\Big)}_{=1} \underbrace{\Big(\sum\nolimits_\beta \theta_\beta\Big)}_{=1} + f\nabla s = df \otimes s + f\nabla s.$$

Hence ∇ is a connection on E.

□

Definition 1.1.22. Suppose $E_i \to M$, $i = 0, 1$ are two smooth vector bundles over M. Suppose also ∇^i is a connection on E_i, $i = 0, 1$. A morphism $(E_0, \nabla^0) \to (E_1, \nabla^1)$ is a bundle morphism $T : E_0 \to E_1$ such that for every $X \in \mathrm{Vect}(M)$ the diagram below is commutative.

$$\begin{array}{ccc} C^\infty(E_0) & \xrightarrow{T} & C^\infty(E_1) \\ {\scriptstyle \nabla^0_X}\downarrow & & \downarrow{\scriptstyle \nabla^1_X} \\ C^\infty(E_0) & \xrightarrow{T} & C^\infty(E_1) \end{array}$$

An isomorphism of vector bundles with connections is defined in the obvious way. We denote by $\mathcal{VB}^c_{\mathbb{K}}(M)$ the collection of isomorphism classes of $\mathbb{K}$-vector bundles with connections over M.

Observe that we have a forgetful map

$$\mathcal{VB}^c(M) \to \mathcal{VB}(M), \;\; (E, \nabla) \mapsto E.$$

The tensorial operations $\oplus, {}^*, \otimes, \mathfrak{S}$ and Λ^* on $\mathcal{VB}(M)$ have lifts to the richer category of vector bundles with connections. We explain this construction in detail. Suppose $(E_i, \nabla^i) \in \mathcal{VB}^c(M)$, $i = 0, 1$.

• We obtain a connection $\nabla = \nabla^0 \oplus \nabla^1$ on $E_0 \oplus E_1$ via the equality

$$\nabla(s_0 \oplus s_1) = (\nabla^0 s_0 \oplus \nabla^1 s_1), \;\; \forall s_0 \in C^\infty(E_0), \;\; s_1 \in C^\infty(E_1).$$

• The connection ∇^0 induces a connection $\check{\nabla}^0$ on E_0^* defined by the equality

$$L_X\langle U, v\rangle = \langle \check{\nabla}^0_X u, v\rangle + \langle u, \nabla_X v\rangle, \;\; \forall X \in \mathrm{Vect}(M), u \in C^\infty(E_0^*), \;\; v \in C^\infty(E_0),$$

where $\langle \bullet, \bullet \rangle \in \underline{Hom}(E_0^* \otimes E_0, \underline{\mathbb{K}}_M)$ denotes the natural bilinear pairing between a bundle and its dual.

Suppose $\vec{e} = (e_1, \cdots, e_r)$ is a local frame of E and $A(\vec{e})$ is the connection 1-form associated to ∇,

$$\nabla \vec{e} = \vec{e} \cdot A(\vec{e})$$

Denote by ${}^t\vec{e} = (e^1, \cdots, e^r)$ the dual local frame of E_0^* defined by

$$\langle e^a, e_b\rangle = \delta^a_b.$$

We deduce that $\langle \check{\nabla}^0 e^a, e_b\rangle = -\langle e^a, \nabla^0 e_b\rangle = -A^a_b$ so that

$$\check{\nabla}^0 e^a = -\sum_b A^a_b e^b.$$

We can rewrite this

$$\check{\nabla}^{0\,t}\vec{e} = {}^t\vec{e} \cdot \left(-{}^tA(\vec{e})\right)$$

that is

$$A({}^t\vec{e}) = -{}^tA(\vec{e}).$$

• We get a connection $\nabla^0 \otimes \nabla^1$ on $E_0 \otimes E_1$ via the equality

$$(\nabla^0 \otimes \nabla^1)(s_0 \otimes s_1) = (\nabla s_0) \otimes s_1 + s_1 \otimes (\nabla^1 s_1).$$

• We get a connection on $\Lambda^k E_0$ via the equality

$\nabla^0_X(s_1\wedge\cdots\wedge s_k) = (\nabla_X s_1)\wedge s_2\wedge\cdots\wedge s_k + s_1\wedge(\nabla^0_X s_2)\wedge\cdots\wedge s_k + \cdots s_1\wedge s_2\wedge\cdots\wedge(\nabla^0_X s_k)$

$\forall s_1,\cdots,s_k\in C^\infty(M)$, $X\in \mathrm{Vect}(M)$.

• If E is a complex vector bundle, then any connection ∇ on E induces a connection $\bar{\nabla}$ on the conjugate bundle $\bar{E}$ defined via the conjugation operator $C: E\to\bar{E}$

$$\bar{\nabla} = C\nabla C^{-1}.$$

Suppose $E\to N$ is a vector bundle over the smooth manifold N, $f: M\to N$ is a smooth map, and ∇ is a connection on E. Then ∇ induces a connection $f^*\nabla$ on f^* defined as follows. If E is defined by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$ and ∇ is defined by the collection $A_\bullet\in\Omega^1(\bullet)\otimes \underline{gl}_r(\mathbb{K})$, then f^∇ is defined by the collection $f^*A_\bullet\in\Omega^1(f^{-1}(U_\bullet))\otimes\underline{gl}_r(\mathbb{K})$. It is the unique connection on f^*E which makes commutative the following diagram.

$$\begin{array}{ccc} C^\infty(E) & \xrightarrow{\ f^*\ } & C^\infty(f^*E) \\ \downarrow{\scriptstyle\nabla} & & \downarrow{\scriptstyle f^*\nabla} \\ C^\infty(T^*N\otimes E) & \xrightarrow{\ f^*\ } & C^\infty(T^*M\otimes f^*E) \end{array}$$

Definition 1.1.23. Suppose ∇ is a connection on the vector bundle $E\to M$.

(a) A section $s\in C^\infty(E)$ is called (∇)-*covariant constant* or *parallel* if

$$\nabla s = 0.$$

(b) A section $s\in C^\infty(E)$ is said to be *parallel* along the smooth path $\gamma:[0,1]\to M$ if the pullback section γ^*s of $\gamma^*E\to[0,1]$ is parallel with respect to the connection $f^*\nabla$.

Example 1.1.24. Suppose $\gamma:[0,1]\to M$ is a smooth path whose image lies entirely in a single coordinate chart U of M. Denote the local coordinates by $(x^1,\cdots,x^n)$ so we can represent γ as a n-uple of functions $(x^1(t),\cdots,x^n(t))$. Suppose $E\to M$ is a rank r vector bundle over M which can be trivialized over U. If ∇ is a connection on E then with respect to some trivialization of $E|_U$ can be described as

$$\nabla = d + A = d + \sum_i dx^i\otimes A_i,\quad A_i: U\to\underline{gl}_r(\mathbb{K}).$$

The tangent vector $\dot\gamma$ along γ can be described in the local coordinates as

$$\dot\gamma = \sum_i \dot{x}^i\partial_i.$$

A section s is the parallel along γ if $\nabla_{\dot\gamma}s = 0$. More precisely, if we regard s as a smooth function $s: U\to\mathbb{K}^r$ then we can rewrite this condition as

$$0 = \frac{d}{d\dot\gamma}s + \sum_i dx^i(\dot\gamma)A_i s = (\sum_i \dot{x}^i\partial_i)s + \sum_i \dot{x}^i A_i s$$

$$\frac{ds}{dt} + \sum_i \dot{x}^i A_i s = 0. \tag{1.1.8}$$

Thus a section which is parallel over a path $\gamma(0)$ satisfies a first order linear differential equation. The existence theory for such equations shows that given any initial condition $s_0 \in E_{\gamma(0)}$ there exists a unique parallel section $[0,1] \ni t \mapsto S(t;s_0) \in E_{\gamma(t)}$. We get a linear map

$$E_{\gamma(0)} \ni s_0 \to S(t;s_0)\,|_{t=1} \in E_{\gamma(1)}.$$

This is called the *parallel transport* along γ (with respect to the connection ∇). □

Suppose E is a real vector bundle, g is a metric on E. A connection ∇ on E is called *compatible with the metric* g (or a *metric connection*) if g is a section of $E^* \otimes E^*$ covariant constant with respect to the connection on $E^* \otimes E^*$ induced by ∇. More explicitly, this means that for every sections u, v of E and every vector field X on M we have

$$L_X g(u,v) = g(\nabla_X u, v) + g(u, \nabla_X v).$$

One can define in a similar fashion the connections on a complex vector bundle compatible with a hermitian metric h.

Proposition 1.1.25. *Suppose h is a metric (riemannian or hermitian) on the vector bundle E. Then there exists connections compatible with h. Moreover the space $\mathcal{A}_{E,h}$ of connections compatible with h is an affine space modelled on the vector space $\Omega^1(\mathrm{End}_h^-(E))$.*

Suppose that ∇ is a connection on a vector bundle $E \to M$. For any vector fields X, Y over M we get three linear operators

$$\nabla_X, \nabla_Y, \nabla_{[X,Y]} : C^\infty(E) \to C^\infty(E),$$

where $[X,Y] \in \mathrm{Vect}(M)$ is the Lie bracket of X and Y. Form the linear operator

$$F_\nabla(X,Y) : C^\infty(E) \to C^\infty(E), \;\; F_\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Observe two things. First,

$$F_\nabla(X,Y) = -F_\nabla(Y,X).$$

Second, if $f \in C^\infty(M)$ and $s \in C^\infty(E)$ then

$$F_\nabla(X,Y)(fs) = fF_\nabla(X,Y)s = F_\nabla(fX,Y)s = F_\nabla(X,fY)s$$

so that for every $X, Y \in \mathrm{Vect}(M)$ the operator $F_\nabla(X,Y)$ is an endomorphism of E and the correspondence

$$\mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \underline{End}(E), \;\; (X,Y) \mapsto F_\nabla(X,Y)$$

is $C^\infty(M)$-bilinear and skew-symmetric. In other words $F_\nabla(\bullet,\bullet)$ is a 2-form with coefficients in $\mathrm{End}\, E$, i.e. a section of $\Omega^2(\mathrm{End}\, E)$.

Definition 1.1.26. The $\mathrm{End}\, E$-valued 2-form $F_\nabla(\bullet,\bullet)$ constructed above is called the *curvature* of ∇.

Example 1.1.27. (a) Consider the trivial vector bundle $E = \underline{\mathbb{K}}_U^r$, where U is an open subset in $\mathbb{R}^n$. Denote by $(x^1, \cdots, x^n)$ the Euclidean coordinates on U. Denote by d the trivial connection on E. Any connection ∇ on E has the form

$$\nabla = d + A = d + \sum_i dx^i A_i, \;\; A_i : U \to \underline{gl}_r(\mathbb{K}).$$

Set $\partial_i := \frac{\partial}{\partial x^i}$, $\nabla_i = \nabla_{\partial_i}$. Then for every $s : U \to \mathbb{K}^r$ we have

$$F_\nabla(\partial_i, \partial_j)s = [\nabla_i, \nabla_j]s = \nabla_i(\nabla_j s) - \nabla_j(\nabla_i s)$$
$$= \nabla_i(\partial_j s + A_j s) - \nabla_j(\partial_i s + A_i s) = (\partial_i + A_i)(\partial_j s + A_j s) - (\partial_j + A_j)(\partial_i s + A_i s)$$
$$= \Big(\partial_i A_j - \partial_j A_i + [A_i, A_j]\Big)s.$$

Hence

$$\sum_{i<j} F(\partial_i, \partial_j)dx^i \wedge dx^j = \Big(\partial_i A_j - \partial_j A_i + [A_i, A_j]\Big)dx^i \wedge dx^j.$$

We can write this formally as

$$F_\nabla = dA + A \wedge A = -\sum_i dx^i d(A_i) + \big(\sum_i dx^i A_i\big) \wedge \big(\sum_j dx^j A_j\big).$$

Observe that if $r = 1$, so that E is the trivial line bundle $\underline{\mathbb{K}}_U$ then we can identify $\underline{gl}_1(\mathbb{K}) \cong \mathbb{K}$ so the components A_i are scalars. In particular $[A_i, A_j] = 0$ so that in this case

$$F_\nabla = dA.$$

(b) If E is a vector bundle described by a gluing cocycle $(\mathcal{U}, g_{\bullet\bullet}, \mathbb{K}^r)$ and ∇ is a connection described by the collection of 1-forms $A_\alpha \in \Omega^1(U_\alpha) \otimes \underline{gl}_r(\mathbb{K})$ satisfying (1.1.7) then the curvature of ∇ is represented by the collection of 2-forms

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$$

satisfying the compatibility conditions

$$F_\beta = g_{\beta\alpha} F_\alpha g_{\beta\alpha}^{-1} \;\text{ on }\; U_{\alpha\beta}. \tag{1.1.9}$$

(c) If ∇ is a connection on a complex line bundle $L \to M$ then its curvature F_∇ can be identified with a complex valued 2-form. If moreover, ∇ is compatible with a hermitian metric then $\mathbf{i}F_\nabla$ is a *real* valued 2-form. □

We define an operation

$$\wedge : \Omega^k(\operatorname{End} E) \times \Omega^\ell(\operatorname{End} E) \to \Omega^{k+\ell}(\operatorname{End} E),$$

by setting

$$(\omega^k \otimes S) \wedge (\eta^\ell \otimes T) = (\omega^k \wedge \eta^\ell) \otimes (ST)$$

for any $\Omega^k \in \Omega^k(M)$, $\eta^\ell \in \Omega^\ell(M)$, $S, T \in \underline{End}(E)$.

Using a connection ∇ on E we can produce an exterior derivative

$$d^\nabla : \Omega^k(\operatorname{End} E) \to \Omega^{k+1}(\operatorname{End} E)$$

defined by

$$d^\nabla(\omega^k \otimes S) = (d\omega^k) \otimes S + (-1)^k(\omega \otimes \mathbb{1}_E) \wedge \nabla^{\operatorname{End} E} S,$$

We have the following result.

Proposition 1.1.28. *Suppose ∇', ∇ are two connections on the vector bundle $E \to M$. Their difference $B = \nabla^1 - \nabla^0$ is an* End E*-valued* 1*-form. Then*

$$F_{\nabla'} = F_\nabla + d^\nabla B + B \wedge B.$$

Proof The result is local so we can assume E is the trivial bundle over an open subset $M \hookrightarrow \mathbb{R}^n$. Let $r = \operatorname{rank} E$. We can write

$$\nabla = d + A, \quad \nabla' = d + A', \quad A, A' \in \Omega^1(M) \otimes \underline{gl}_r(\mathbb{K}).$$

Then $B = A' - A$,

$$F' = F_{\nabla'} = dA' + A' \wedge A', \quad F = F_{\nabla} = dA + A \wedge A$$

and thus

$$\begin{aligned} F' - F = d(A' - A) + (A' \wedge A') - (A \wedge A) &= d(A' - A) + (A + B) \wedge (A + \Gamma) - B \wedge B \\ &= dB + B \wedge A + A \wedge B + B \wedge B. \end{aligned}$$

In local coordinates d^{∇} we have (see Exercise 1.4.6)

$$\begin{aligned} d^{\nabla}(\sum_i dx^i \otimes B_i) &= -\sum_i dx^i \wedge \left(\sum_j dx^j \otimes \nabla_j B_i \right) \\ &= -\sum_i dx^i \wedge \left(\sum_j dx^j \otimes (\partial_j B_i + [A_j, B_i]) \right) \\ &= \sum_{i<j} dx^i \wedge dx^j \otimes (\partial_i B_j - \partial_j B_i) - \sum_{i,j} dx^i \wedge dx^j \otimes (A_j B_i - B_i A_j) \\ &= dB + \left(\sum_j dx^j \otimes A_j \right) \wedge \left(\sum_i dx^i \otimes B_i \right) + \left(\sum_i dx^i \otimes B_i \right) \wedge \left(\sum_j dx^j \otimes A_j \right) \\ &= dB + A \wedge B + B \wedge A. \end{aligned}$$

□

1.2. Chern-Weil theory

1.2.1. Connections on principal G-bundles. In the sequel we will work exclusively with *matrix Lie groups*, i.e. closed subgroups of some $\mathrm{GL}_r(\mathbb{K})$.

Fix a (matrix) Lie group G and a principal G-bundle $P = (M, \mathfrak{U}, g_{\bullet\bullet})$ over the smooth manifold M. Denote by $\mathfrak{g} = T_1 G$ the Lie algebra of G. A *connection* on P is a collection

$$A = \{A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}\}$$

satisfying the following conditions

$$A_\beta(u) = g_{\beta\alpha}(u) A_\alpha(u) g_{\beta\alpha}^{-1}(u) - d(g_{\beta\alpha}) g_{\beta\alpha}(u)^{-1}, \ \ \forall u \in U_{\alpha\beta}. \tag{1.2.1}$$

We denote by $\mathcal{A}_P$ the space of connections on P.

Proposition 1.2.1. *$\mathcal{A}_P$ is an affine space modelled on $\Omega^1(\mathrm{Ad}\, P)$.*

Proof We will show that given two connections (A_α^1), (A_α^0) their difference $C_\alpha = A_\alpha^1 - A_0^\alpha$ defines a global section of $\Lambda^1 T^*M \otimes \mathrm{Ad}\, P$, i.e. on the overlaps $U_{\beta\alpha}$ we have the equality

$$C_\beta = \mathrm{Ad}(g_{\beta\alpha}) C_\alpha = g_{\beta\alpha} C_\alpha g_{\beta\alpha}^{-1}.$$

This follows immediately by taking the difference of the transition equalities (1.2.1) for $A_\bullet^1$ and $A_\bullet^0$. □

To formulate our next result let us introduce an operation

$$[-,-] : \Omega^k(U_\alpha) \otimes \mathfrak{g} \times \Omega^\ell(U_\alpha) \otimes \mathfrak{g} \to \Omega^{k+\ell}(U_\alpha) \otimes \mathfrak{g},$$

$$[\omega^k \otimes X, \eta^\ell \otimes Y] := (\omega^k \wedge \eta^\ell) \otimes [X, Y],$$

where $[X, Y]$-denotes the Lie bracket in $\mathfrak{g}$, or in the case of a matrix Lie group, $[X, Y] = XY - YX$ is the commutator of the matrices X, Y. Let us point out that if $A, B \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ we have

$$[A, B] = A \wedge B + B \wedge A.$$

We define

$$F_\alpha := dA_\alpha + \frac{1}{2}[A_\alpha, A_\alpha] = dA_\alpha + A_\alpha \wedge A_\alpha \in \Omega^2(U_\alpha) \otimes \mathfrak{g}.$$

For a proof of the following result we refer to [**16**, Chap.8].

Proposition 1.2.2. *(a) The collection F_α defines a global section $F(A)$ of $\Lambda^2 T^*M \otimes \mathrm{Ad}\, P$, i.e. on the overlaps $U_{\alpha\beta}$ it satisfies the compatibility conditions,*

$$F_\beta = g_{\beta\alpha} F_\alpha g_{\beta\alpha}^{-1} = Ad(g_{\beta\alpha}) F_\alpha.$$

(b) **(The Bianchi Identity)**

$$dF_\alpha + [A_\alpha, F_\alpha] = 0, \ \ \forall \alpha.$$

The 2-form $F(A) \in \Omega^2(\mathrm{Ad}\, P)$ is called the *curvature* of A.

Consider now a representation $\rho : G \to \mathrm{Aut}(V)$ and the vector bundle $E = P \times_\rho V$. Denote by ρ_* the differential of ρ at $1 \in G$

$$\rho_* : \mathfrak{g} \to \mathrm{End}\, V.$$

We recall that $\mathrm{End}_\rho(V) = \rho_*\mathfrak{g}$ and $\mathrm{End}_\rho E = P \times_{\mathrm{Ad}_\rho} \mathrm{End}_\rho(V)$. The identity (1.1.3) shows that any connection (A_α) on P defines a connection $\nabla = (\rho_* A_\alpha)$ on E. We say that this connection is *compatible with the* (G,ρ)*-structure*. Observe that

$$F_\nabla\,|_{U_\alpha} = \rho_* F_\alpha.$$

In particular $F_\nabla \in \Omega^2(\mathrm{End}_\rho E)$.

Example 1.2.3. Suppose $E \to M$ is a complex vector bundle of rank r. A hermitian metric h on E defines a $U(r)$-structure. A connection ∇ is compatible with this structure if and only if it is compatible with the metric. In this case $\mathrm{End}_\rho E$ is the subbundle $\mathrm{End}_h^- E$ of $\mathrm{End}\, E$ and we have

$$F(\nabla) \in \Omega^2(\mathrm{End}_h^- E).$$

□

1.2.2. The Chern-Weil construction. Suppose $P \to M$ is a principal G-bundle over M defined by the gluing cocycle $(\mathcal{U}, g_{\bullet\bullet})$. To formulate the Chern-Weil construction we need to introduce first the concept of Ad-invariant polynomials on $\mathfrak{g}$. .

The adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ induces an adjoint representation

$$\mathrm{Ad}^k : G \to \mathrm{GL}(\mathrm{Sym}^k\, \mathfrak{g}_{\mathbb{C}}^*),\ \ \mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}.$$

We denote by $I_k(\mathfrak{g})$ the Ad^k-invariant elements of $\mathrm{Sym}^k\, \mathfrak{g}^*$. Equivalently, they are k-multilinear maps

$$P : \underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{k} \to \mathbb{C},$$

such that

$$P(X_{\varphi(1)}, \cdots, X_{\varphi(k)}) = P(gX_1g^{-1}, \cdots, gX_kg^{-1}) = P(X_1, \cdots, X_k)$$

for any $X_1, \cdots, X_k \in \mathfrak{g}$, $g \in G$ and any permutation φ of $\{1, \cdots, k\}$. If in the above equality we take $g = \exp(tY)$, $Y \in \mathfrak{g}$ and then we differentiate with respect to t at $t = 0$ we obtain

$$P([Y, X_1], X_2, \cdots, X_k) + \cdots + P(X_1, \cdots, X_{k-1}, [Y, X_k]) = 0,\ \ \forall Y, X_1, \cdots, X_k \in \mathfrak{g}. \tag{1.2.2}$$

For $P \in I_k(\mathfrak{g})$ and $X \in \mathfrak{g}$ we set

$$P(X) := P(\underbrace{X, \cdots, X}_{k}).$$

We have the *polarization formula*

$$P(X_1, \cdots, X_k) = \frac{1}{k!}\frac{\partial^k}{\partial t_1 \cdots \partial t_k} P(t_1X_1 + \cdots + t_kX_k).$$

More generally, given $P \in I_k(\mathfrak{g})$ and (not necessarily commutative) $\mathbb{C}$-algebra $\mathcal{R}$ we define $\mathcal{R}$-multilinear map

$$P : \underbrace{\mathcal{R} \otimes \mathfrak{g} \times \cdots \times \mathcal{R} \otimes \mathfrak{g}}_{k} \to \mathcal{R}$$

by

$$P(r_1 \otimes X_1, \cdots, r_k \otimes X_k) = r_1 \cdots r_k P(X_1, \cdots, X_k).$$

Let us emphasize that when $\mathcal{R}$ is not commutative the above function *is not* symmetric in its variables. For example if $r_1r_2 = -r_2r_1$ then

$$P(r_1X_1, r_2X_2, \cdots) = -P(r_2X_2, r_1X_1, \cdots).$$

It will be so if $\mathcal{R}$ is commutative. For applications to geometry $\mathcal{R}$ will be the algebra $\Omega^\bullet(M)$ of complex valued differential forms on a smooth manifold M. When restricted to the commutative subalgebra

$$\Omega^{even}(M) = \bigoplus_{k\geq 0} \Omega^{2k}(M) \otimes \mathbb{C}.$$

we do get a symmetric function.

Let us point out a useful identity. If $P \in I_k(g)$, U is an open subset of $\mathbb{R}^n$,

$$F_i = \omega_i \otimes X_i \in \Omega^{d_i}(U) \otimes \mathfrak{g}, \quad A = \omega \otimes X \in \Omega^d(U) \otimes \mathfrak{g}$$

then

$$P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1} \cdots, F_k) = (-1)^{d(d_1+\cdots d_{i-1})}\omega\omega_1 \cdots \omega_k P(X_1, \cdots, [X, X_i], \cdots X_k).$$

In particular, if $F_1, \cdots, F_{k-1}$ have even degree we deduce that for every $i = 1, \cdots, k$ we have

$$P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1}, \cdots, F_k) = \omega\omega_1 \cdots \omega_k P(X_1, \cdots, [X, X_i], \cdots X_k)$$

Summing over i and using the Ad-invariance of P we deduce

$$\sum_{i=1}^k P(F_1, \cdots, F_{i-1}, [A, F_i], F_{i+1}, \cdots, F_k) = 0, \tag{1.2.3}$$

$\forall F_1 \cdots, F_{k-1} \in \Omega^{even}(U) \otimes \mathfrak{g}, \quad F_k, A \in \Omega^*(U) \otimes \mathfrak{g}$.

Theorem 1.2.4 (Chern-Weil)**.** *Suppose $A = (A_\bullet)$ is a connection on the principal G-bundle $(M, \mathcal{U}, g_{\bullet\bullet})$, with curvature $F(A) = (F_\bullet)$, and $P \in I_k(\mathfrak{g})$. Then the following hold.*

(a) The collection of $2k$-forms $P(F_\alpha) \in \Omega^{2k}(U_\alpha)$ define a global $2k$-form $P(F(A))$ on M, i.e.

$$P(F_\alpha) = P(F_\beta) \ \text{ on } U_{\alpha\beta}.$$

(b) The form $P(F(A))$ is closed

$$dP(F(A)) = 0.$$

(c) For any two connections $A^0, A^1 \in \mathcal{A}_P$ the closed forms $P(F(A^0))$ and $P(F(A^1)$ are cohomologous, i.e their difference is an exact form.

Proof (a) On the overlap $U_{\alpha\beta}$ we have

$$P(F_\beta) = P(\mathrm{Ad}(g_{\beta\alpha})F_\alpha) = P(F_\alpha)$$

due to the Ad-invariance of P.

(b) Observe first that the Bianchi indentity implies that $dF_\alpha = -[A_\alpha, F_\alpha]$. From the product formula we deduce

$$dP(F_\alpha) = dP(\underbrace{F_\alpha, \cdots, F_\alpha}_{k}) = P(dF_\alpha, F_\alpha, \cdots, F_\alpha) + \cdots + P(F_\alpha, \cdots, F_\alpha, dF_\alpha)$$

$$= -P([A_\alpha, F_\alpha], F_\alpha, \cdots, F_\alpha) - \cdots - P(F_\alpha, \cdots, F_\alpha, [A_\alpha, F_\alpha]) \stackrel{(1.2.3)}{=} 0.$$

(c) Consider two connections $A^1, A^0 \in \mathcal{A}_P$. We need to find a $(2k-1)$ form η such tha

$$P(F(A^1)) - P(F(A^0) = d\eta.$$

Let $C := A^1 - A^0 \in \Omega^1(\operatorname{Ad} P)$. We get a path of connections $t \mapsto A^t = A^0 + tC$ which starts at A^0 and ends at A^1. Set $F^t := F(A^t)$ and

$$P(t) = P(F_{A_t}).$$

We want to show that $P(1) - P(0)$ is exact. We will prove a more precise result. Define *the local transgression forms*

$$T_\alpha P(A^1, A^0) := k \int_0^1 P(F^t_\alpha, \cdots, F^t_\alpha, C_\alpha) dt$$

The Ad-invariance of P implies that

$$T_\alpha P(A^1, A^0) = T_\beta P(A^1, A^0), \ \text{ on } U_{\alpha\beta}$$

so that these forms define a global form $T(A^1, A^0) \in \Omega^{2k-1}(M)$ called the *transgression form* from A^0 to A^1. We will prove that

$$P(1) - P(0) = dTP(A^1, A^0).$$

We work locally on U_α we have

$$P(1) - P(0) = \int_0^1 \frac{d}{dt} P(F^t_\alpha, \cdots, F^t_\alpha) dt$$

$(\dot{F}^t_\alpha = \frac{d}{dt} F^t_\alpha)$

$$= \int_0^1 \Big(P(\dot{F}^t_\alpha, F^t_\alpha, \cdots, F^t_\alpha) + \cdots + P(F^t_\alpha, \cdots, F^t_\alpha, \dot{F}^t_\alpha) \Big) dt$$

$$= k \int_0^1 P(F^t_\alpha, \cdots, F^t_\alpha, \dot{F}^t_\alpha) dt.$$

We have

$$F^t_\alpha = dA^t_\alpha + \frac{1}{2}[A^t_\alpha, A^t_\alpha] = F^0_\alpha + t(dC_\alpha + [A^0_\alpha, C_\alpha]) + \frac{t^2}{2}[C_\alpha, C_\alpha]$$

so that

$$\dot{F}^t_\alpha = dC_\alpha + [A^0_\alpha, C_\alpha] + t[C_\alpha, C_\alpha] = dC_\alpha + [A^t_\alpha, C_\alpha].$$

Hence

$$P(F^t_\alpha, \cdots, F^t_\alpha, \dot{F}^t_\alpha) = P(F^t_\alpha, \cdots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha]).$$

To finish the proof of the theorem it suffices to show that

$$dP(F^t_\alpha, \cdots, F^t_\alpha, C_\alpha) = P(F^t_\alpha, \cdots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha]).$$

Indeed we have

$$dP(F^t_\alpha, \cdots, F^t_\alpha, C_\alpha) = P(dF^t_\alpha, \cdots, F^t_\alpha, C_\alpha) + \cdots + P(F^t_\alpha, \cdots, dF^t_\alpha, C_\alpha) + P(F^t_\alpha, \cdots, F^t_\alpha, dC_\alpha)$$

$(dF^t_\alpha = -[A^t_\alpha, F^t_\alpha])$

$$= -P([A^t_\alpha, F^t_\alpha], \cdots, F^t_\alpha, C_\alpha,) - \cdots - P(F^t_\alpha, \cdots, [A^t_\alpha, F^t_\alpha], C_\alpha) + P(F^t_\alpha, \cdots, F^t_\alpha, dC_\alpha)$$

$$= P(F^t_\alpha, \cdots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha])$$

$$-\Big(P(F^t_\alpha, \cdots, F^t_\alpha, [A^t_\alpha, C_\alpha]) + P([A^t_\alpha, F^t_\alpha], \cdots, F^t_\alpha, C_\alpha) + \cdots + P(F^t_\alpha, \cdots, [A^t_\alpha, F^t_\alpha], C_\alpha) \Big)$$

$$= P(F^t_\alpha, \cdots, F^t_\alpha, dC_\alpha + [A^t_\alpha, C_\alpha])$$

since the term in parentheses vanishes[3] due to (1.2.3).

□

We set

$$\mathbb{C}[\mathfrak{g}^*]^G = \bigoplus_{k\geq 0} I_k(\mathfrak{g}), \;\; \mathbb{C}[[\mathfrak{g}^*]]^G = \prod_{k\geq 0} I_k(\mathfrak{g}).$$

$\mathbb{C}[\mathfrak{g}^*]^G$ is the ring of Ad-invariant polynomials and $\mathbb{C}[[\mathfrak{g}^*]]^G$ is the ring of Ad-invariant formal power series. We have

$$\mathbb{C}[\mathfrak{g}^*]^G \subset \mathbb{C}[[\mathfrak{g}^*]]^G$$

Suppose A is a connection on the principal G-bundle $P \to M$. Then for every $f = \sum_{k\geq 0} f_k \in \mathbb{C}[[\mathfrak{g}^*]]^G$ we get an element

$$f(F(A)) = \sum_{k\geq 0} f_k(F(A))$$

Observe that $f_k(F(A)) \in \Omega^{2k}(M)$. In particular $f_{2k}(A) = 0$ for $2k > \dim M$ so that in the above sum only finitely many terms are non-zero. We obtain a well defined correspondence

$$\mathbb{C}[[\mathfrak{g}^*]]^G \times \mathcal{A}_P \to \Omega^{even}(M), \;\; (f, A) \mapsto f(F(A)).$$

This is known as the *Chern-Weil correspondence*. The image of the Chern-Weil correspondence is a subspace of $\mathcal{Z}^*(M)$, the vector space of closed forms on M. We have also constructed a canonical map

$$T : \mathbb{C}[[\mathfrak{g}^*]]^G \times \mathcal{A}_P \times \mathcal{A}_P \to \Omega^{odd}(M), \;\; (f, A_0, A_1) \mapsto Tf(A_1, A_0)$$

such that

$$f(F(A_1) - f(F(A_0)) = dTf(A_1, A_0).$$

We will refer to it as the *Chern-Weil transgression*.

The Chern construction is *natural* in the following sense. Suppose $P = (M, \mathcal{U}, g_{\bullet\bullet}, G)$ is a principal G-bundle over M and $f : N \to M$ is a smooth map. Then we get a pullback bundle f^*P over N described by the gluing data $(N, f^{-1}(\mathcal{U}), f^*(g_{\bullet\bullet}), G$. For any connection $A = (A_\bullet)$ on P we get a connection $f^*A = (f^*A_\bullet)$ on f^*P such that

$$F(f^*A) = f^*F(A).$$

Then for every element $h \in \mathbb{C}[[\mathfrak{g}^*]]^G$ we have

$$h(f^*F(A)) = f^*h(F(A)).$$

1.2.3. Chern classes. We consider now the special case $G = U(n)$. The Lie algebra of $U(n)$, denoted by $\underline{u}(n)$ is the space of skew-hermitian matrices. Observe that we have a natural identification

$$\underline{u}(1) \cong \mathbf{i}\mathbb{R}.$$

The group $U(n)$ acts on $\underline{u}(n)$ by conjugation

$$U(n) \times \underline{u}(n) \ni (g, X) \mapsto gXg^{-1} \in \underline{u}(n).$$

It is a basic fact of linear algebra that for every skew-hermitian endomorphism of $\mathbb{C}^n$ can be diagonalized, or in other words, every skew-hermitian matrix is conjugate to a diagonal one. The space

[3] The order in which we wrote the terms, $F^t, \cdots, F^t, C$ instead of $C, F^t, \cdots, F^t$ is **very important** in view of the asymmetric definition of

$$P : \mathcal{R}\otimes\mathfrak{g} \times \cdots \times \mathcal{R}\otimes\mathfrak{g} \to \mathcal{R}.$$

of diagonal skew-hermitian matrices forms a commutative Lie subalgebra of $\underline{u}(n)$ known as the *Cartan subalgebra* of $\underline{u}(n)$. We will denote it by $\mathbf{Cartan}(\underline{u}(n))$.

$$\mathbf{Cartan}(\underline{u}(n)) = \Big\{ \mathrm{Diag}(\mathbf{i}\lambda_1, \cdots, \mathbf{i}\lambda_n);\ \ (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \Big\}.$$

The group $W_{U(n)}$[4] of permutations of n objects acts on $\mathbf{Cartan}(\underline{u}(n)$ is the obvious way and two diagonal matrices are conjugate if and only if we can obtain one from the other by a permutation of its entries. Thus an Ad-invariant polynomial on $\underline{u}(n)$ is determined by its restriction to the Cartan algebra. Thus we can regard every Ad-invariant polynomial as a polynomial function $P = P(\lambda_1, \cdots, \lambda_n)$. This polynomial is also invariant under the permutation of its variables and thus can de described as a polynomial in the elementary symmetric quantities

$$c_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k},\ \ x_j = \frac{\mathbf{i}}{2\pi}(\mathbf{i}\lambda_j) = -\frac{\lambda_j}{2\pi}.$$

The factor $\frac{\mathbf{i}}{2\pi}$ appears due to historical and geometric reasons. The variables x_j are also known as the *Chern roots*. More elegantly, if we set

$$D = D(\vec{\lambda}) = \mathrm{Diag}(\mathbf{i}\lambda_1, \cdots, \mathbf{i}\lambda_n) \in \underline{u}(n)$$

then

$$\det(1 + \frac{\mathbf{i}t}{2\pi} D) = 1 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

Instead of the elementary sums we can consider the momenta

$$s_r = \sum_i x_i^r.$$

The elementary sums can be expressed in terms of the momenta via the *Newton relation*

$$s_1 = c_1,\ \ s_2 = c_1^2 - 2c_2,\ \ s_3 = c_1^2 - 3c_1c_2 + 3c_3,\ \ \sum_{j=1}^{r} (-1)^j s_{r-j} c_j = 0. \tag{1.2.4}$$

Using again the matrix D we have

$$\sum_{r \geq 0} \frac{s_r}{r!} t^r = \mathrm{tr}\exp(\frac{\mathbf{i}t}{2\pi} D).$$

Motivated by these examples we introduce the *Chern polynomial*

$$c \in \mathbb{C}[\underline{u}(n)^*]^{U(n)},\ \ c(X) = \det(\mathbb{1}_{\mathbb{C}^n} + \frac{\mathbf{i}}{2\pi} X),\ \ \forall X \in \underline{u}(n).$$

Now define the *Chern character*

$$\mathbf{ch} \in \mathbb{C}[[\underline{u}(n)]]^{U(n)},\ \ \mathbf{ch}(X) = \mathrm{tr}\exp(\frac{\mathbf{i}}{2\pi} X).$$

Using (1.2.4)

$$\mathbf{ch} = n + c_1 + \frac{1}{2!}\big(c_1^2 - 2c_2\big) + \frac{1}{3!}\big(c_1^2 - 3c_1c_2 + 3c_3\big) + \cdots. \tag{1.2.5}$$

[4]We use the notation $W_{U(n)}$ because this group is in this case the symmetric group is isomorphic to the Weyl group of $U(n)$.

Example 1.2.5. Suppose

$$F = \begin{bmatrix} \mathbf{i}F_1^1 & F_2^1 \\ F_1^2 & \mathbf{i}F_2^2 \end{bmatrix} \in \underline{u}(2) \Longleftrightarrow F_1^2 = -\bar{F}_2^1.$$

Then

$$c_1(F) = -\frac{1}{2}(F_1^1 + F_2^2), \;\; c_2(F) = -\frac{1}{4\pi^2}(F_2^1 \wedge \bar{F}^{1,2} - F_1^1 \wedge F_2^2).$$

□

Our construction of the Chern polynomial is a special case of the following general procedure of constructing symmetric elements in $\mathbb{C}[[\lambda_1, \cdots, \lambda_n]]$. Consider a formal power series

$$f = a_0 + a_1 x + a_2 x^2 + \cdots \in \mathbb{C}[[x]], \;\; a_0 = 1.$$

Then if we set $\vec{x} = (x_1, \cdots, x_n)$ the function

$$\mathbf{G}_f(\vec{x}) = f(x_1) \cdots f(x_n) \in \mathbb{C}[[x_1, \cdots, x_n]]$$

is a symmetric power series in $\vec{x}$ with leading coefficient 1. Observe that if $D = \mathrm{Diag}(\mathbf{i}\vec{\lambda})$ then

$$f(\frac{\mathbf{i}}{2\pi}D) = \mathrm{Diag}\big(f(x_1), \cdots f(x_n)\big) \Longrightarrow f(\vec{x}) = \det f\big(\frac{\mathbf{i}}{2\pi}D\big).$$

We thus get an element $\mathbf{G}_f \in \mathbb{C}[[\underline{u}(n)]]^{U(n)}$ defined by

$$\mathbf{G}_f(X) = \det f(\frac{\mathbf{i}}{2\pi}X).$$

It is called the f-*genus* or the genus associated to f. When $f(x) = 1 + x$ we obtain the Chern polynomial.

Of particular relevance in geometry is the *Todd genus*, i.e. the genus associated to the function[5]

$$\mathrm{td}\,(x) := \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \cdots = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!}x^{2k}.$$

The coefficients B_k are known as the *Bernoulli numbers*. Here are a few of them

$$B_2 = \frac{1}{6}, \;\; B_4 = -\frac{1}{30}, \;\; B_6 = \frac{1}{42},$$
$$B_8 = -\frac{1}{30}, \;\; B_{10} = \frac{5}{66}, \;\; B_{12} = -\frac{691}{2730}.$$

We set

$$\mathbf{td} := \mathbf{G}_{\mathrm{td}}.$$

Consider now a rank n complex vector bundle $E \to M$ equipped with a hermitian metric h. We denote by $\mathcal{A}_{E,h}$ the affine space of connections on E compatible with the metric h and by $P_h(E)$ the principal bundle of h-orthonormal frames. Then the space of connections $\mathcal{A}_{E,h}$ can be naturally identified with the space of connections on $P_h(E)$. For every $A \in \mathcal{A}_{E,h}$ we can regard the curvature $F(A)$ as a $n \times n$ matrix with entries even degree forms on M. We get a non-homogeneous even degree form

$$c(A) = c(F(A)) = \det\big(\mathbb{1}_E + \frac{\mathbf{i}}{2\pi}F(A)\big) \in \Omega^{even}(M).$$

---

[5] **Warning.** The literature is not consistent on the definition of the Todd function. We chose to work with Hirzebruch's definition in [11]. This agrees with the definition in [2, 14], but it differs from the definitions in [4, 19] where $\mathrm{td}\,(x)$ is defined as $\frac{x}{e^x - 1}$.

According to the Chern-Weil theorem this form is closed and its cohomology class is independent of the metric[6] h and the connection A. It is thus a *topological invariant* of E. We denote it by $c(E)$ and we will call it the *total Chern class of* E . It has a decomposition into homogeneous components

$$c(E) = 1 + c_1(E) + \cdots + c_n(E), \quad c_k(E) \in H^{2k}(M, \mathbb{R}).$$

We will refer to $c_k(E)$ as the *k-th Chern class*. More generally for any $f = 1 + a_1 x + \cdots \in \mathbb{C}[[x]]$ we define $\mathbf{G}_f(E)$ to be the cohomology class carried by the form

$$\mathbf{G}_f(A) = \det f(F(A)).$$

In particular, $\mathbf{td}\,(E)$ is the cohomology class carried by the closed form

$$\mathbf{td}\,(A) := \det\Big(\frac{\frac{\mathbf{i}}{2\pi}F}{\exp(\frac{\mathbf{i}}{2\pi}F) - \mathbb{1}_E}\Big)$$

(see [11, I.§1])

$$= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \cdots .$$

Similarly we define the *Chern character of* E as the cohomology class $\mathbf{ch}(E)$ carried by the form

$$\mathbf{ch}(A) = \operatorname{tr}\exp\big(\frac{\mathbf{i}}{2\pi}F(A)\big)$$

$$= \operatorname{rank} E + c_1(E) + \frac{1}{2}\big(c_1(E)^2 - 2c_2(E)\big) + \frac{1}{3!}\big(c_1(E)^2 - 3c_1(E)c_2(E) + 3c_3(E)\big) + \cdots$$

Due to the naturality of the Chern-Weil construction we deduce that for every smooth map $f : M \to N$ and every complex vector bundle $E \to N$ we have

$$c(f^*E) = f^*c(E). \tag{1.2.6}$$

Example 1.2.6. Denote by $L_{\mathbb{P}^n}$ the tautological line bundle over $\mathbb{CP}^n$. The natural inclusions

$$i_k : \mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}, \quad (z_1, \cdots, z_k) \mapsto (z_1, \cdots, z_k, 0)$$

induce inclusions $i_k : \mathbb{CP}^{k-1} \to \mathbb{CP}^k$ and tautological isomorphisms

$$L_{\mathbb{P}^{k-1}} \cong i_k^* L_{\mathbb{P}^k}.$$

We deduce that

$$c_1(L_{\mathbb{P}^n})\,|_{\mathbb{CP}^1} = c_1(L_{\mathbb{P}^1}).$$

We know that $H^2(\mathbb{CP}^n, \mathbb{C}) \cong \mathbb{R}$ and by Poincaré duality we can identify $H^2(\mathbb{CP}^n, \mathbb{C})$ with the dual of $H_2(\mathbb{CP}^n, \mathbb{C})$. This is a one-dimensional space with a canonical basis, namely the homology class carried by the oriented submanifold $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$. Thus, $H^2(\mathbb{CP}^n, \mathbb{C})$ carries a canonical basis usually denoted by H defined by

$$\langle H, [\mathbb{CP}^1]\rangle = 1.$$

We can write

$$c_1(L_{\mathbb{P}^n}) = xH$$

where

$$x = \langle c_1(L_{\mathbb{P}^n}), [\mathbb{CP}^1]\rangle = \int_{\mathbb{CP}^1} c_1(L_{\mathbb{P}^1}).$$

As shown in Exercise 1.4.8 the last integral in -1 so that

$$c_1(L_{\mathbb{P}^n}) = -H. \tag{1.2.7}$$

[6] See Exercise 1.4.13.

□

For a proof of the following result we refer to [16, Chap.8].

Proposition 1.2.7. *Suppose (E_i, h_i), $i = 0, 1$ are two hermitian vector bundles, $A_i \in \mathcal{A}_{E_i,h_i}$ and $f = 1 + a_1x + a_2x^2 + \cdots \in \mathbb{C}[[x]]$.. We denote by $A_0 \oplus A_1$ and $A_0 \otimes A_1$ the induced hermitian connections on $E_0 \oplus E_1$ and $E_0 \otimes E_1$ respectively. Then*

$$\mathbf{G}_f(A_0 \oplus A_1) = \mathbf{G}_f(A_0) \wedge \mathbf{G}_f(A_1),\ \ \mathbf{ch}(A_0 \oplus A_1) = \mathbf{ch}(A_0) + \mathbf{ch}(A_1),$$
$$\mathbf{ch}(A_0 \otimes A_1) = \mathbf{ch}(A_0) \wedge \mathbf{ch}(A_1).$$

In particular, we have

$$c(E_0 \oplus E_1) = c(E_0)c(E_1),\ \ \mathbf{ch}(E_0 \oplus E_1) = \mathbf{ch}(E_0) + \mathbf{ch}(E_1), \tag{1.2.8}$$
$$\mathbf{ch}(E_0 \otimes E_1) = \mathbf{ch}(E_0)\,\mathbf{ch}(E_1). \tag{1.2.9}$$

Remark 1.2.8. The identities (1.2.6), (1.2.7), (1.2.8) uniquely determine the Chern classes, [11, I§4]. □

Example 1.2.9. Suppose $L \to M$ is a hermitian line bundle. For any hermitian connection A we have

$$c(A) = 1 + \frac{\mathbf{i}}{2\pi}F(A),\ \ \mathbf{ch}(A) = \sum_{k\geq 0}\frac{1}{k!}\Big(\frac{\mathbf{i}}{2\pi}F(A)\Big)^k = e^{c_1(A)}.$$

□

1.2.4. Pontryagin classes. We now consider the case $G = O(n)$. We will have to separate the cases $n = 2k$ and $n = 2k + 1$ but we will discuss in detail only the n-even case. The Lie algebra of $O(n)$ is the space $\underline{o}(n)$ of skew-symmetric $n \times n$ matrices. From now on we assume $n := 2k$. We will denote by J the 2×2 matrix

$$J := \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right].$$

The Cartan subalgebra of $\underline{o}(n)$ is the subspace $\mathbf{Cartan}(\underline{o}(n))$ consisting of skew-symmetric matrices which have the quasi-diagonal form

$$\Theta(\lambda_1, \cdots, \lambda_n) = \lambda_1 J \oplus \cdots \oplus \lambda_k J,\ \ \lambda_i \in \mathbb{R}.$$

Every skew-symmetric matrix is conjugate with some element in the Cartan algebra. This element is in general non-unique. Observe that for every permutation $\varphi : \{1, 2, \cdots, k\} \circlearrowleft$ and every $\epsilon_1, \cdots, \epsilon_k \in \{\pm 1\}$ the matrix $\Theta(\lambda_1, \cdots, \lambda_k)$ is conjugate to $\Theta(\epsilon_1\lambda_{\varphi(1)}, \cdots, \epsilon_k\lambda_{\varphi(k)})$. In more modern terms, consider the Weyl group

$$W_{O(2k)} = S_k \times \{\pm 1\}^k$$

An element $(\varphi, \vec{\epsilon}) \in W_{O(2k)}$ acts on $\underline{o}(n)$ as above, and two elements in the Cartan algebra are conjugate if and only if they belong to the same orbit of this group action. Thus, any Ad-invariant function on $\underline{o}(n)$ is determined by its restriction to the Cartan subalgebra, which is a $W_{O(2k)}$-invariant function in the variables λ_i. In particular, and Ad-invariant polynomial on $\underline{o}(n)$ can be viewed as a symmetric polynomial in the variables $\lambda_1^2, \cdots, \lambda_k^2$, or equivalently, as a polynomial in the variables

$$p_j = \sum_{i_1 < \cdots < i_j} x_{i_1}^2 \cdots x_{i_j}^2,\ \ 1 \leq i_j \leq k,\ \ x_i = -\frac{\lambda_i}{2\pi}.$$

Observe that for every $\Theta(\vec{\lambda}) \in \mathbf{Cartan}(\underline{o}(n))$ we have

$$\det(\mathbb{1} - \frac{1}{2\pi}\Theta) = \prod_{i=1}^{k} \det(\mathbb{1} + x_j J) = \prod_{i=1}^{k}(1 + x_j^2) = \sum_{j=0}^{k} p_j.$$

There is a more convenient way of reformulating this fact. Note that we have a canonical inclusion

$$\underline{o}(n) \hookrightarrow \underline{u}(n).$$

For every $X \in \underline{o}(n)$ we denote by X^c its image in $\underline{u}(n)$. In plain terms, X^c is the same matrix as X but we think of it as acting on $\mathbb{C}^n$ rather than $\mathbb{R}^n$. Observe that Θ^c is $U(n)$-conjugate with

$$\mathrm{Diag}(\mathbf{i}\lambda_1, -\mathbf{i}\lambda_2, \cdots, \mathbf{i}\lambda_k, -\mathbf{i}\lambda_k),$$

so that

$$\det(\mathbb{1} - \frac{1}{2\pi}\Theta) = \det{}^{1/2}\Big(\mathbb{1} + (\frac{\mathbf{i}}{2\pi}\Theta^c)^2\Big).$$

where for every matrix X we set

$$\det{}^{1/2}(\mathbb{1} + X) = \det(\mathbb{1} + X)^{1/2},$$

with[7]

$$(1 + X)^{1/2} := \sum_{k \geq 0} \binom{1/2}{k} X^k = 1 + \frac{1}{2}X - \frac{1}{8}X^2 + \cdots, \quad \binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!},$$

We define

$$p \in \mathbb{C}[\underline{o}(n)^*]^{O(n)}, \;\; p(X) = \det{}^{1/2} p(\frac{\mathbf{i}}{2\pi}X^c), \;\; p(x) = 1 + x^2.$$

Let us point out an important fact. Given $X \in \underline{o}(n)$ we get $X^c \in \underline{u}(n)$. Then

$$\sum_{\ell=1}^{2k} c_\ell(X_c) = c(X^c) = \det(\mathbb{1} + \frac{\mathbf{i}}{2\pi}X^c) = \prod_{j=1}^{k} \det\begin{bmatrix} 1 + x_j & 0 \\ 0 & 1 - x_j \end{bmatrix}$$

$$= \prod_{j=1}^{k}(1 - x_j^2) = \sum_{j=1}^{k}(-1)^j p_j(X).$$

By identifying the homogeneous components we deduce

$$c_{2j-1}(X^c) = 0, \;\; p_j(X) = (-1)^j c_{2j}(X^c). \tag{1.2.10}$$

We can generate many more examples of Ad-invariant functions on $\underline{o}(n)$ by considering an *even* power series

$$f(x) = 1 + a_1x^2 + a_2x^4 + \cdots \in \mathbb{C}[[x^2]], \;\; a_0 = 1,$$

Then

$$\mathbf{G}_f(\vec{x}) = f(x_1)\cdots f(x_k) \in \mathbb{C}[[x_1, \cdots, x_k]], \;\; x_i = -\frac{\lambda_i}{2\pi}$$

is $W_{O(2k)}$-invariant. Moreover

$$\mathbf{G}_f(\vec{x}) = \det{}^{1/2} f(\frac{\mathbf{i}}{2\pi}\Theta^c),$$

and we obtain

$$\mathbf{G}_f \in \mathbb{C}[[\underline{o}(n)^*]]^{O(n)}, \; ; \; \mathbf{G}_f(X) = \det{}^{1/2} f(\frac{\mathbf{i}}{2\pi}X^c).$$

[7]We are not worried about convergence issues since the matrices used in geometry have nilpotent entries and all the formal power series reduce to polynomials.

Of particular interests are the functions[8]

$$L(x) = \frac{x}{\tanh x} = 1 + \sum_{k=1}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!} x^{2k} = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^2 + \cdots$$

and

$$\hat{A}(x) = \frac{x/2}{\sinh(x/2)} = 1 + \sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{2^{2k-1}(2k)!} B_{2k} x^{2k} = 1 - \frac{1}{24}x^2 + \frac{7}{2^7 \cdot 3^2 \cdot 5}x^2 + \cdots .$$

Then, we set $\mathbf{L} := \mathbf{G}_L$, $\hat{\mathbf{A}} := \mathbf{G}_{\hat{A}}$ and we get

$$\mathbf{L}(\vec{x}) = L(x_1) \cdots L(x_k) = 1 + \frac{1}{3}p_1 + \frac{1}{45}(7p_2 - p_1^2) + \cdots$$

and

$$\hat{\mathbf{A}}(\vec{x}) = \hat{A}(x_1) \cdots \hat{A}(x_2) = 1 - \frac{p_1}{24} + \frac{1}{2^7 \cdot 3^2 \cdot 5}(7p_1^2 - 4p_2) + \cdots$$

Suppose $E \to M$ is a real vector bundle equipped with a metric. Any connection compatible with this metric can be viewed as a connection on the principal bundle of orthonormal frames of E. Observe that the metric on E induces a hermitian metric on the complexification $E^c := E \otimes \mathbb{C}$ and any metric connection A on E induces a hermitian connection A^c on E^c. Denote by $F(A)$ the curvature of A.

$$p(A) = 1 + p_1(A) + p_2(A) + \cdots = \det\big(\mathbb{1} - \frac{1}{2\pi}F(A)\big)$$

$$= \det{}^{1/2}\big(\mathbb{1} - \frac{1}{4\pi^2}F(A^c) \wedge F(A^c)\big) = \det\big(\mathbb{1} - \frac{1}{8\pi^2}F(A^c) \wedge F(A^c) + \cdots\big)$$

Observe that as matrices with entries 2-forms we have $F(A) = F(A^c)$. The closed forms

$$p_j(A) \in \Omega^{4j}(M)$$

are called the *Pontryagin forms* associated to A. Note for example that

$$p_1(A) = -\frac{1}{8\pi^2} \operatorname{tr}\big(F(A) \wedge F(A)\big).$$

The cohomology classes determined by these forms are independent of the metric and the metric compatible connection A and therefore they are *topological invariants* of E. They are called the *Pontryagin classes* of E and they are denoted by $p_j(E)$. The identity (1.2.10) shows that

$$p_j(E) = (-1)^j c_{2j}(E \otimes \mathbb{C}).$$

The $\mathbf{L}$-genus and the $\hat{\mathbf{A}}$-genus of E are the cohomology classes $\mathbf{L}(E)$ and $\hat{\mathbf{A}}(E)$ carried by the closed forms

$$\mathbf{L}(A) = \det{}^{1/2}\left(\frac{\frac{\mathbf{i}}{2\pi}F(A)}{\tanh\big(\frac{\mathbf{i}}{2\pi}F(A)\big)} \right) = 1 + \frac{1}{3}p_1(A) + \cdots ,$$

$$\hat{\mathbf{A}}(A) = \det{}^{1/2}\left(\frac{\frac{\mathbf{i}}{4\pi}F(A)}{\sinh\big(\frac{\mathbf{i}}{4\pi}F(A)\big)} \right) = 1 - \frac{1}{24}p_1(A) + \cdots .$$

[8]In many places $L(x)$ is defined as $\frac{x/2}{\tanh x/2}$. We chose to stick to Hirzebruch's original definition, [11].

1.2.5. The Euler class. Consider now the group $SO(2k)$. It is the index two subgroup of $O(2k)$ consisting of orthogonal matrices with determinant 1. It is convenient to think of these matrices as orthogonal transformations of $\mathbb{R}^{2k}$ preserving the canonical orientation

$$\Omega := e_1 \wedge e_2 \wedge \cdots \wedge e_{2k},$$

where $e_1, \cdots, e_{2k}$ is the canonical orthonormal basis of $\mathbb{R}^{2k}$. We deduce that its Lie algebra $\underline{so}(2k)$ coincides with the Lie algebra $\underline{o}(2k)$. Any matrix $X \in \underline{so}(2k)$ will be $SO(2k)$-conjugate to a matrix in the Cartan algebra $\mathbf{Cartan}(\underline{o}(2k))$. However, two matrices in the Cartan algebra which are $O(2k)$-conjugate *need not be* $SO(2k)$-conjugate. For example, the matrix $J \in \underline{o}(2)$ is not $SO(2)$-conjugate to $-J$. To describe this phenomenon in more detail consider the group

$$W_{SO(2k)} = \Big\{ (\varphi, \vec{\epsilon}) \in W_{O(2k)};\ \ \epsilon_1 \cdots \epsilon_k = 1 \Big\}$$

Two matrices in the Cartan algebra $\mathbf{Cartan}(\underline{o}(2k))$ are $SO(2k)$-conjugate if and only if they belong to the same orbit of the Weyl group $W_{SO(2k)}$. We deduce that the polynomial functions on $\underline{o}(2k)$ which are invariant under the conjugations action of the smaller group $SO(2k)$ can be identified with the polynomial functions on the Cartan algebra invariant under the action of the subgroup $W_{SO(2k)}$ of $W_{O(2k)}$. It is therefore natural to expect that there are more functions invariant under $W_{SO(2k)}$ than function invariant under $W_{O(2k)}$.

This is indeed the case. We will describe one $W_{SO(2k)}$-invariant function which is not $W_{O(2k)}$-invaraint. For a complete description of the ring of $W_{SO(2k)}$-invariant polynomials we refer to [**16**, Chap. 8].

Given

$$\Theta(\vec{\lambda}) = \lambda_1 J \oplus \cdots \oplus \lambda_k J,\ \ \lambda_i \in \underline{so}(2k)$$

we set

$$\mathbf{e}(\Theta) := \prod_{i=1}^{n} x_i,\ \ x_i := -\frac{\lambda_i}{2\pi}$$

Clearly the polynomial function

$$\Theta \mapsto \mathbf{e}(\Theta)$$

is $W_{SO(2k)}$-invariant and thus it is the restriction of an invariant polynomial

$$\mathbf{e} \in \mathbb{C}[\underline{so}(2k)^*]^{SO(2k)}.$$

We would like to give a description of $\mathbf{e}(X)$ for any $X \in \underline{so}(2k)$. This will require the concept of *pfaffian*.

First of all, let us observe that the volume form Ω depends only on the orientation of $\mathbb{R}^{2k}$ and not on the choice of orthonormal basis $e_1, \cdots, e_{2k}$ compatible with the fixed orientation. To any skew-symmetric matrix $X \in \underline{so}(2k)$ we associate

$$\omega_X \in \Lambda^2(\mathbb{R}^{2k})^*,\ \ \omega_X(u, v) := g(Xu, v),$$

where $g(-,-)$ denotes the standard Euclidean metric on $\mathbb{R}^{2k}$. For example

$$\omega_{\Theta(\vec{\lambda})} = \sum_{j=1}^{k} \lambda_j e_{2j-1} \wedge e_{2j} = \lambda_1 e_1 \wedge e_2 + \cdots + \lambda_k e_{2k-1} \wedge e_{2k}.$$

The $2k$-form $\frac{1}{k!}\omega_X^k$ will be a scalar multiple of Ω, and we define the pfaffian to be exactly this scalar

$$\mathbf{Pfaff}(X)\cdot\Omega = \frac{1}{k!}\omega_X^k.$$

From its definition we deduce that the pffafian is invariant under $SO(2k)$-conjugation[9] Moreover

$$\mathbf{Pfaff}(\Theta(\vec{\lambda})) = \lambda_1\cdots\lambda_k.$$

More generally, if we express X as a $2k\times 2k$-matrix $X=(x_{ij})$, where

$$x_{ij} = g(e_i, Xe_j) = -g(Xe_i, e_j) = -\omega_X(e_i, e_j)$$

then

$$\omega_X = -\sum_{i<j} x_{ij} e_i\wedge e_j$$

and we conclude after a simple computation that

$$\mathbf{Pfaff}(X) = \frac{(-1)^k}{2^k\cdot k!}\sum_{\sigma\in S_{2k}}\epsilon(\sigma)x_{\sigma(1)\sigma(2)}\cdots x_{\sigma(2k-1)\sigma(2k)},$$

where S_n denotes the symmetric group on n-elements and $\epsilon(\sigma)$ denotes the signature of a permutation $\sigma\in S_n$. Hence

$$\mathbf{e}(X) = \mathbf{Pfaff}\big(-\frac{1}{2\pi}X\big).$$

Suppose $E\to M$ is an *oriented* rank $2k$ real vector bundle. Fix a metric g on E. Then any connection A on E compatible with g induces a connection on the principal $SO(2k)$-bundle of orthonormal frames of E *compatible with the orientation of* E. The *Euler form* determined by A is the closed $2k$-form

$$\mathbf{e}(A) = \mathbf{Pfaff}\big(-\frac{1}{2\pi}F(A)\big).$$

The cohomology class it determines is independent of the metric g and the connection A. It is a topological invariant of E called the *Euler class* of E and it is denoted by $\mathbf{e}(E)$.

[9] The only time we relied on an orthonormal basis in its description was in the definition of Ω which as pointed out, is independent of the choice of an oriented orthonormal basis.

1.3. Calculus on Riemann manifolds

Definition 1.3.1. A *Riemann manifold* is a pair (M, g) where M is a smooth manifold and g is a metric on the tangent bundle TM. g is called a *Riemann metric* on M.

If we choose local coordinates $(x^1, \cdots, x^n)$ near a point $p_0 \in M$ then the vectors $\partial_i = \frac{\partial}{\partial x^i}$ define a local frame of TM and the metric g is described near p_0 by the symmetric form

$$g_{ij}(x) = g_x(\partial_i, \partial_j), \;\; 1 \leq i, j \leq n.$$

The metric g induces metrics in the cotangent bundle and in all the tensor bundles

$$\mathcal{T}_s^r M = TM^{\otimes r} \otimes (T^*M)^{\otimes s}.$$

In particular it induces metrics in the exterior bundles $\Lambda^k T^*M$. When no confusion is possible we will continue to denote these induced metrics by g or $(\bullet, \bullet)$. For every section u of $\mathcal{T}_s^r M$ we set

$$|u|_g = \sqrt{g(u,u)} : M \to \mathbb{R}.$$

Fix a Riemann metric g on M. An orientation on M, that is a nowhere vanishing section of $\omega \in C^\infty(\det TM)$ canonically defines a *volume form* on M, i.e a nowhere vanishing form on M of top degree. This form, denoted by dV_g is uniquely determined by the following conditions.

$$dV_g(\omega) > 0 \;\; |dV_g|_g \equiv 1 \;\text{ on }\; M.$$

In local coordinates we have

$$dV_g = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots dx^n.$$

For every vector field X on M we denote by L_X the Lie derivative of a tensor field on M. In particular $L_X dV_g$ is a n-form on M and thus it is a multiple of dV_g

$$L_X(dV_g) = \lambda(X) dV_g.$$

Definition 1.3.2. The scalar $\lambda(X)$ is called the *divergence* of X with respect to the metric g. It is denoted by $\mathbf{div}_g\ X$.

Example 1.3.3. Suppose M is the vector space $\mathbb{R}^n$ equipped with the natural Euclidean metric g_0. The associated volume form is

$$dV_0 = dx^1 \wedge \cdots \wedge dx^n.$$

Given a vector field $X = \sum_i X^i \partial_i$ on $\mathbb{R}^n$ we have

$$L_X(dV_0) = (L_X dx^1) \wedge dx^2 \wedge \cdots \wedge dx^n + \cdots + dx^1 \wedge dx^2 \wedge \cdots \wedge (L_X dx^n)$$

Using Cartan formula

$$L_X = di_X + i_X d$$

where i_X denotes the contraction by X we deduce $L_X dx^j = d(i_X dx^j) = dX^j$. This shows that

$$L_X(dV_0) = \left(\sum_i \partial_i X^i \right) dV_0 \Longrightarrow \mathbf{div}_{g_0}\ X = \sum_i \partial_i X^i.$$

□

Proposition 1.3.4 (Divergence Formula)**.** *Suppose* (M, g) *is an oriented Riemann manifold. Then for every compactly supported smooth functions* $u, v : M \to \mathbb{R}$ *we have*

$$\int_M (L_X u) v dV_g = \int_M u(-L_X - \mathbf{div}_g X) v dV_g.$$

Proof We have

$$L_X(uvdV_g) = (L_X u)vdVg + u(L_X v)dV_H + uv\,\mathbf{div}_g(x)dV_g.$$

Using Cartan formula $L_X = i_X d + di_X$ again and observing that $d(uvdV_g) = 0$ since the form $uvdV_g$ is top dimensional we deduce

$$d\big(i_X(uvdV_g)\big) = (L_X u)vdV_g + u(L_X + \mathbf{div}_g(X))vdV_g.$$

Integrating over M (which is possible since all the above objects have compact support we deduce

$$\int_M d\big(i_X(uvdV_g)\big) = \int_M (L_X u)vdV_g + \int_M u(L_X + \mathbf{div}_g(X))vdV_g.$$

Stokes formula now implies that the integral in the left hand side is zero since the integrand is the exact differential of a compactly supported form.

□

The metric g is a section of $T^*M \otimes T^*M \cong \mathrm{Hom}(TM, T^*M)$ and thus we can regard it as a bundle morphism

$$TM \to T^*M.$$

This is an *isomorphism* called the *metric duality*. Thus, the metric associates to every vector field X a 1-form $X^\sharp$ called the *metric dual* of X. More concretely, $X^\sharp$ is the 1-form uniquely determined by the equality

$$X^\sharp(Y) = g(X, Y), \;\; \forall Y \in \mathrm{Vect}(M).$$

In local coordinates, if $X = \sum_i X^i \partial_i$ then

$$X^\sharp = \sum_i \left(\sum_j g_{ij} X^j \right) dx^i.$$

Conversely, to any 1-form α we can associate by metric duality a vector field on M which we denote by $\alpha^\sharp$. It is the vector field uniquely determined by the equality

$$\alpha(X) = g(\alpha^\sharp, X), \;\; \forall X \in \mathrm{Vect}(X).$$

In local coordinates, if $\alpha = \sum_i \alpha_i dx^i$ and if g^{ij} denotes the inverse of the matrix g_{ij} then

$$\alpha^\sharp = \sum_i \left(\sum_j g^{ij} \alpha_j \right) \partial_i.$$

In particular, the *gradient* of a function $f : M \to \mathbb{R}$ is the vector field dual to df

$$\mathbf{grad}_g\, f = (df)^\sharp.$$

In local coordinates we have

$$\mathbf{grad}_g\, f = \sum_i (\sum_j g^{ij} \partial_j f) \partial_i.$$

Definition 1.3.5. The *scalar Laplacian* on an oriented Riemann manifold is the operator

$$\Delta_M : C^\infty(M) \to C^\infty(M), \quad f \mapsto \Delta_M f = -\,\mathbf{div}(\mathbf{grad}\, f).$$

A Riemann metric together with an orientation define a more sophisticated type of duality.

Definition 1.3.6. Suppose (M, g) is an oriented Riemann manifold of dimension n. The *Hodge $*$-operator* is the linear operator

$$* = *_g : \Omega^\bullet(M) \to \Omega^{n-\bullet}(M)$$

uniquely determined by the requirement

$$\omega \wedge *\eta = g(\omega, \eta) dV_g, \quad \forall \omega, \eta \in \Omega^\bullet(M).$$

Example 1.3.7. Consider the Euclidean space $\mathbb{R}^n$ equipped with the natural metric and orientation defined by the n-form $dV_0 = dx^1 \wedge \cdots \wedge dx^n$. Then

$$*dx^1 = dx^1 \wedge \cdots \wedge dx^n, \quad *dx^1 \wedge dx^2 = dx^3 \wedge \cdots \wedge dx^n,$$

$$*(dx^1 \wedge \cdots \wedge dx^i) = dx^{i+1} \wedge \cdots \wedge dx^n.$$

□

The Hodge $*$-operator has a quasi-involutive behavior. More precisely,

$$*\,(*\alpha) = (-1)^{k(n-k)}\alpha, \quad \forall \alpha \in \Omega^k(M). \tag{1.3.1}$$

Using the Hodge $*$-operator we can define $\delta : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ by

$$\delta\omega = *d*\omega.$$

Proposition 1.3.8. *For any compactly supported forms $\omega \in \Omega^{k-1}(M)$ and $\eta \in \Omega^k(M)$ we have*

$$\int_M (d\omega, \eta) dV_g = \epsilon(n, k) \int_M (\omega, \delta\eta) dV_g,$$

where

$$\epsilon(n, k) = (-1)^{nk+n+1}.$$

For a proof we refer to [16].

Remark 1.3.9. Observe that if n is even then $\epsilon(n, k) = -1, \forall k$. □

We have the following fundamental result. Its proof can be found in any modern book of riemannian geometry, e.g. [7, 16].

Theorem 1.3.10. *Suppose (M, g) is a Riemann manifold. Then there exists a unique* metric *connection ∇ on TM satisfying the symmetry condition*

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad \forall X, Y \in \mathrm{Vect}(M).$$

We include here an explicit description of the Levi-Civita connection.

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\Big\{ & L_X g(Y,Z) - L_Z g(X,Y) + L_Y g(Z,X) \\ & - g(X,[Y,Z]) + g(Z,[X,Y]) + g(Y,[Z,X]) \Big\}. \end{aligned} \tag{1.3.2}$$

If we choose local coordinates $(x^1, \cdots, x^n)$ and we set $\nabla_i = \nabla_{\partial_i}$ then the Levi-Civita connection is completely determined by the *Christoffel symbols* Γ^k_{ij} defined by

$$\nabla_i \partial_j = \sum_k \Gamma^k_{ij} \partial_k$$

The symmetry of the condition translates into the equalities

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \quad \forall i, j, k.$$

Using (1.3.2) for $X = \partial_i$, $Y = \partial_j$, $Z = \partial_k$ we deduce that

$$\sum_\ell g_{\ell k} \Gamma^\ell_{ij} = \frac{1}{2}\{\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ki}\}.$$

If we denote by (g^{ij}) the inverse matrix of g_{ij} so that

$$\sum_j g^{ij} g_{jk} = \delta^i_k$$

then we deduce

$$\Gamma^m_{ij} = \frac{1}{2}\sum_k g^{mk}\{\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ki}\} \tag{1.3.3}$$

The *Riemann curvature* (or *tensor*) of a Riemann manifold is the curvature of the Levi-Civita connection. It is a section $R \in \Omega^2(\operatorname{End} TM)$. For every X, Y we get an endomorphism of $R(X,Y)$ of TM. In local coordinates we have

$$R(\partial_i, \partial_j)\partial_k = \sum_\ell R^\ell_{kij} \partial_\ell.$$

We set

$$R_{mkij} = \sum_\ell g_{m\ell} R^\ell_{kij} = g(\partial_m, R(\partial_i, \partial_j)\partial_k)$$

The Riemann tensor enjoys several symmetry properties.

$$R_{ijk\ell} = -R_{jik\ell}, \quad R_{ijk\ell} = R_{k\ell ij} \tag{1.3.4a}$$

$$R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0 \tag{1.3.4b}$$

$$(\nabla_i R)^j_{mk\ell} + (\nabla_\ell R)^j_{mik} + (\nabla_k R)^j_{m\ell i} = 0. \tag{1.3.4c}$$

The identity (1.3.4b) is called the *first Bianchi identity* while the (1.3.4c) is called the *second Bianchi identity*.

Using the Riemann tensor we can produce new tensors which contain partial information about the curvature. Given two linearly independent tangent vectors $X, Y \in T_pM$ we can define the *sectional curvature* at p along the 2-plane spanned by X, Y to be the scalar

$$K_p(X,Y) = \frac{(R(X,Y)Y, X)}{|X \wedge Y|}$$

where $|X \wedge Y|$ is the Gramm determinant

$$|X \wedge Y| = \begin{vmatrix} (X,X) & (X,Y) \\ (Y,X) & (Y,Y) \end{vmatrix}.$$

This determinant is the square of the area of the parallelogram spanned by X and Y.

The *Ricci curvature* is a symmetric tensor $\mathrm{Ric} \in C^\infty(T^*M)$ defined by

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\Big\{ Z \mapsto R(Z,X)Y) \Big) \}$$

In local coordinates

$$\mathrm{Ric} = \sum_{ij} \mathrm{Ric}_{ij}\, dx^i dx^j, \;\; \mathrm{Ric}_{ij} = \sum_k R^k_{jki}$$

The *scalar curvature* is the trace of the Ricci curvature

$$s = \sum_i g^{ij}\, \mathrm{Ric}_{ij}\,.$$

A vector field on a Riemann manifold is said to be parallel along a smooth path if it is parallel along that path with respect to the Levi-Civita connection. Suppose $\gamma : [0,1] \to M$ is a smooth path. If the tangent vector $\dot{\gamma}$ is parallel along γ then we say that γ is a *geodesic*. Formally this means that

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

Using local coordinates $(x^1, \cdots, x^n)$ in which γ is described by a smooth function $t \mapsto (x^1(t), \cdots, , x^n(t))$ we deduce from (1.1.8) that the functions $x^i(t)$ satisfy the *second order, nonlinear* system of differential equations

$$\frac{d^2x^i}{dt^2} + \sum_{j,k} \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \;\; 1 \leq i \leq n. \tag{1.3.5}$$

Observe that if $\gamma(t)$ is a geodesic then so is the rescaled path $t \mapsto \gamma(ct)$, where c is a real constant. Existence results for ordinary differential equations show that given a point $p \in M$, a vector $X \in T_pM$, there exists a geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0)$. Moreover any two such geodesics must coincide on their common interval of existence. We denote this unique geodesic by

$$t \mapsto \exp_p(tX).$$

$\exp_p(tX)$ is the point on the manifold M reached after t-seconds by the geodesic which starts at p and has initial velocity X. Observe that for every real constant c and any sufficiently small t we have

$$\exp_p(t \cdot (cX)) = \exp_p((ct) \cdot X).$$

We have the following result.

Theorem 1.3.11. *For every $p \in M$, there exists $r = r(p) > 0$ with the following properties.*

(a) For any tangent vector $X \in T_pM$ of length $|X|_{g_p} < r$ the geodesics $t \mapsto \exp_p(tX)$ exists for all $|t| \leq 1$. Denote by $\mathbb{B}_r(p) \subset T_pM$ the open ball of radius r.

(b) The map

$$\exp_p : \mathbb{B}_r(p) \to M, \;\; X \mapsto \exp_p(X)$$

is a diffeomorphism onto an open neighborhood of $p \in M$. We denote this open neighborhood of p by $B_r(p)$.

For a proof we refer to [**16**]. Exercise 1.4.15 probably explains the importance of this special choice of local coordinates.

The map $X \mapsto \exp_p(X)$ defined in a neighborhood of $0 \in T_pM$ is called the exponential map of (M, g) at p. The neighborhood $B_r(p)$ is called the *geodesic ball* of radius r centered at p. If we fix an orthonormal frame of T_pM we obtain Euclidean coordinates $\boldsymbol{x}^i$ on T_pM and via the exponential map coordinates on $B_p(r)$. The coordinates obtained in this fashion are called *normal coordinates near* p. We will continue to denote them by $(\boldsymbol{x}^i)$. In these coordinates, the Christofell symbols *vanish* at p.

1.4. Exercises for Chapter 1

Exercise 1.4.1. Two vector bundles over the same manifold B described by gluing cocycles $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(V)$ and $h_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(V)$ subordinated to the same open cover $\mathcal{U}$ are isomorphic if and only if they are *cohomologous*, i.e. there exist smooth maps

$$T_\alpha : U_\alpha \to \mathrm{Aut}(V)$$

such that for every α, β and every $u \in U_{\alpha\beta}$ the diagram below is commutative.

$$\begin{array}{ccc} V & \xrightarrow{T_\alpha(u)} & V \\ {\scriptstyle g_{\beta\alpha}(u)}\downarrow & & \downarrow{\scriptstyle h_{\beta\alpha}(u)} \\ V & \xrightarrow{T_\beta(u)} & V \end{array} \iff T_\beta(u) \cdot g_{\beta\alpha}(u) = h_{\beta\alpha}(u) \cdot T_\alpha(u).$$

□

Exercise 1.4.2. Recall that a refinement of an open cover $\mathcal{U} = (U_i)_{i \in I}$ is an open cover $\mathcal{U}' = (U'_\alpha)_{\alpha \in A}$ such that there exists a map $\varphi : A \to I$ with the property

$$U_\alpha \subset U_{\varphi(\alpha)}, \quad \forall \alpha \in A.$$

We write this $\mathcal{U}' \prec_\varphi \mathcal{U}$. Given a gluing cocycle g_{ij} subordinated to $\mathcal{U}$ then its restriction to $\mathcal{U}'$ is the gluing cocycle $g \mid_{\bullet\bullet}$ defined by.

$$g \mid_{\alpha\beta} = g_{\varphi(\alpha)\varphi(\beta)} \mid_{U_{\alpha\beta}}$$

Prove that the bundles $(g_{\bullet\bullet}, \mathcal{U}, W)$ and $(g'_{\bullet\bullet}, \mathcal{U}', W)$ are isomorphic if and only if there exist an open cover $\mathcal{V} \prec \mathcal{U}, \mathcal{U}'$ such that the restrictions of g and h to $\mathcal{V}$ are cohomologous. □ □

Exercise 1.4.3. Prove that for every vector bundle $E \to B$ the space of smooth sections $C^\infty(E)$ is infinite dimensional. □

Exercise 1.4.4. (a) Show that a metric on a real vector bundle $E \to M$ of rank m defines a canonical $O(m)$ structure on E, and conversely, a $O(m)$-structure on E defines a metric on E.

(b) Suppose that $E \to M$ is a rank r $\mathbb{K}$-vector bundle. Prove that a trivialization of $\det E$ defines a canonical $\mathrm{SL}_r(\mathbb{K})$-structure on E, and conversely, every $\mathrm{SL}_r(\mathbb{K})$-structure defines a trivialization of $\det E$. □

Exercise 1.4.5. Prove Proposition 1.1.20. □

Exercise 1.4.6. Suppose ∇^0 and ∇^1 are connections on the vector bundles $E_0, E_1 \to M$. They induce a connection ∇ on $E_1 \otimes E_0^* \cong \mathrm{Hom}(E_0, E_1)$. Prove that for every $X \in \mathrm{Vect}(M)$ and every bundle morphism $T : E_0 \to E_1$ the covariant derivative of T along X is the bundle morphism $\nabla_X T$ defined by

$$(\nabla_X T)s = \nabla^1_X(Ts) - T(\nabla^0_X s), \quad \forall s \in C^\infty(E_0).$$

In particular if $E_0 = E_1$ and $\nabla^0 = \nabla^1$ then we have

$$\nabla_X T = [\nabla^0_X, T],$$

where $[A, B] = AB - BA$ for any linear operators A and B. □

Exercise 1.4.7. Let ∇ be a connection on the vector bundle $E \to M$. Then the operator

$$d^\nabla : \Omega^\bullet(\mathrm{End}\, E) \to \Omega^{\bullet+1}(\mathrm{End}\, E)$$

satisfies

$$(d^\nabla)^2 u = F_\nabla \wedge u, \ \ \forall u \in \Omega^k(\mathrm{End}\, E),$$

and the **Bianchi identity**

$$d^\nabla F_\nabla = 0.$$

□

Exercise 1.4.8. (a) Construct a connection on the tautological line bundle over $\mathbb{CP}^1$ compatible with the natural hermitian metric.

(b) The curvature of the hermitian connection A you constructed in part (a) is a purely imaginary 2-form $F(A)$ on $\mathbb{CP}^1$. Show that

$$\int_{\mathbb{CP}^1} c_1(A) = \frac{\mathbf{i}}{2\pi}\int_{\mathbb{CP}^1} F(A) = -1.$$

(c) Prove that the tautological line bundle over $\mathbb{CP}^1$ cannot be trivialized. □

Exercise 1.4.9. Prove Proposition 1.1.25. □

Exercise 1.4.10. Suppose g is a metric on a vector bundle $E \to M$ and ∇ is a connection compatible with g. Prove that $F_\nabla \in \Omega^2(\mathrm{End}_h^- E)$.

□

Exercise 1.4.11. Suppose $g : \mathbb{R}^n \to \mathrm{GL}_r(\mathbb{K})$ is a smooth map. Prove that

$$dg^{-1} = -g^{-1} \cdot dg \cdot g,$$

i.e. for every smooth path $(-1, 1) \ni t \to \gamma(t) \in \mathbb{R}^n$ if we set $g_t = g(\gamma(t))$ we have

$$\frac{d}{dt} g_t^{-1} = -g_t^{-1} \cdot \frac{dg_t}{dt} \cdot g_t^{-1}.$$

□

Exercise 1.4.12. Suppose $E \to M$ is a rank two hermitian complex vector bundle and A^1, A^0 are two hermitian connections on E. Assume A^0 is flat, i.e. $F(A^0) = 0$. Describe the transgression $Tc_2(A^1, A^0)$ in terms of $C = A^1 - A^0$. The correspondence

$$\Omega^1(\mathrm{End}_h^- E) \ni C \mapsto Tc_2(A^0 + C, A^0)$$

is known as the *Chern-Simmons functional*. □

Exercise 1.4.13. Prove that the Chern classes are independent of the hermitian metric used in their definition. □

Exercise 1.4.14. Suppose (M, g) is a Riemann manifold and ∇ denotes the Levi-Civita connection on M. Prove that for every $\Omega \in \Omega^k(M)$ and every $X_0, X_1, \cdots, X_k \in \mathrm{Vect}(M)$ we have

$$d\omega(X_0, \cdots, \omega_k) = \sum_{i=0}^k (\nabla_{X_i}\omega)(X_0, \cdots, \hat{X}_i, \cdots, X_k)$$

where a hat indicates a missing entry. □

Exercise 1.4.15. Suppose $(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^n)$ are normal coordinates near a point p on a Riemann manifold. Denote by $\boldsymbol{g}_{ij}$ the coefficients of the Riemann metric in this coordinate system

$$g = \sum_{i,j} \boldsymbol{g}_{ij} d\boldsymbol{x}^i d\boldsymbol{x}^j,$$

and by $\boldsymbol{\Gamma}^i_{jk}$ the Christoffel symbols in this coordinate system. Set $r^2 := (\boldsymbol{x}^1)^2 + \cdots + (\boldsymbol{x}^n)^2$, $e_i = \frac{\partial}{\partial \boldsymbol{x}^i}$.

(a) Prove that near p we have the Taylor expansion

$$\boldsymbol{g}_{ij}(\boldsymbol{x}) = \delta_{ij} + \frac{1}{3} \sum_{k,\ell} R_{kij\ell} \boldsymbol{x}^k \boldsymbol{x}^\ell + O(r^3).$$

(b) If near p we write the volume form dV_g as $\rho(\boldsymbol{x}) d\boldsymbol{x}^1 \wedge \cdots \wedge d\boldsymbol{x}^n$ then we have the Taylor expansion

$$\rho(\boldsymbol{x}) = 1 - \frac{1}{6} \sum_{i,j} \mathrm{Ric}_{ij}\, \boldsymbol{x}^i \boldsymbol{x}^j + O(r^3).$$

□

Exercise 1.4.16. Suppose $E \to M$ is a complex vector bundle of rank r. Viewed as a *real* vector bundle it has rank $2r$ and it is equipped with a natural orientation. Show that

$$c_r(E) = \mathbf{e}(E).$$

□

Elliptic partial differential operators

2.1. Definition and basic constructions

2.1.1. Partial differential operators. Suppose E, F are smooth complex vector bundles over the same smooth manifold M of dimension n. We denote by $\mathbf{OP}(E,F)$ the space of $\mathbb{C}$-linear operators

$$L : C^\infty(E) \to C^\infty(F).$$

For every $f \in C^\infty(M)$ and any $L \in \mathbf{OP}(E,F)$ define $\mathbf{ad}(f)L \in \mathbf{OP}(E,F)$ by

$$\mathbf{ad}(f)Ps = [L, f]s = L(fs) - fL(s).$$

Observe that if $Q \in \mathbf{OP}(E,F)$, $P \in \mathbf{OP}(F,G)$ and $f \in C^\infty(M)$ we have

$$\mathbf{ad}(f)(PQ) = (\mathbf{ad}(f)P)Q + P(ad(f)Q). \tag{2.1.1}$$

We define inductively

$$\mathbf{PDO}^{(0)}(E,F) = \Big\{ L \in \mathbf{OP}(E,F);\ \ \mathbf{ad}(f)L = 0,\ \ \forall f \in C^\infty(M) \Big\},$$

$$\mathbf{PDO}^{(m)}(E,F) := \Big\{ L \in \mathbf{OP}(E,F);\ \ \mathbf{ad}(f)L \in \mathbf{PDO}^{(m-1)}(E,F),\ \ \forall f \in C^\infty(M) \Big\},$$

$$\mathbf{PDO}(E,F) = \bigcup_{m \geq 0} \mathbf{PDO}^{(m)}(E,F).$$

When $E = F$ we set $\mathbf{PDO}(E) = \mathbf{PDO}(E,E)$.

Definition 2.1.1. The elements of $\mathbf{PDO}(E,F)$ are called partial differential operators (p.d.o.'s) (from E to F). A partial operator $L \in \mathbf{PDO}(E,F)$ is said to have order m if it belongs to $\mathbf{PDO}^{(m)} \setminus \mathbf{PDO}^{(m-1)}$. We denote by $\mathbf{PDO}^m$ the set of p.d.o.'s of order m. □

We need to justify the above definition. We will do this via some basic examples.

Example 2.1.2. (a) Observe that $L \in \mathbf{PDO}^{(0)}(E,F)$ if and only if

$$L(fs) = fL(s), \ \ \forall f \in C^\infty(M), \ \ s \in C^\infty(E)$$

so that

$$\mathbf{PDO}^{(0)}(E,F) = \underline{Hom}(E,F).$$

(b) Assume $E = F = \underline{\mathbb{C}}_M$, $M = \mathbb{R}^n$. Then $\partial_i \in \mathbf{PDO}^{(1)}(\underline{\mathbb{C}}_M)$, $\forall i = 1, \cdots, n$. Indeed

$$\mathbf{ad}(f)\partial_i(u) = \partial_i(fu) - f(\partial_i u) = (\partial_i f)u$$

so that

$$\mathbf{ad}(f)\partial_i = (\partial_i f) \in \mathbf{PDO}^{(0)}.$$

Observe that $\mathbf{OP}(\underline{\mathbb{C}})$ is an algebra and (2.1.1) implies that for any $f \in C^\infty(M)$ the map

$$\mathbf{ad}(f) : \mathbf{OP}(\underline{\mathbb{C}}) \to \mathbf{OP}(\underline{\mathbb{C}})$$

is a derivation, i.e. it satisfies the product rule. This implies inductively that

$$\mathbf{PDO}^{(j)} \cdot \mathbf{PDO}^{(k)} \subset \mathbf{PDO}^{(j+k)},$$

i.e. $\mathbf{PDO}(\underline{\mathbb{C}})$ is a filtered algebra. In particular, for every multi-index $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$ the operator

$$\partial^{\vec{\alpha}} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

is a p.d.o. of order $|\vec{\alpha}| = \alpha_1 + \cdots + \alpha_n$.

(c) Suppose ∇ is a connection on the vector bundle E. Then $\nabla \in \mathbf{PDO}^{(1)}(E, T^*M \otimes E)$. Indeed given $f \in C\infty(M)$ and $s \in C^\infty(E)$ we have

$$\mathbf{ad}(f)\nabla(s) = \nabla(fs) - f(\nabla s) = df \otimes s$$

so that

$$\mathbf{ad}(f) = df \otimes \in \underline{Hom}(E, T^*M \otimes E) = \mathbf{PDO}^0(E, T^*M \otimes E).$$

Similarly, we can show that for every vector field X on M we have

$$\nabla_X \in \mathbf{PDO}^1(E).$$

(d) Consider the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ viewed as an operator $d \in \mathbf{OP}(\Lambda^k T^*M \otimes \mathbb{C}, \Lambda^{k+1}T^*M \otimes \mathbb{C})$. Then $d \in \mathbf{PDO}^1(\Lambda^k T^*M \otimes \mathbb{C}, \Lambda^{k+1}T^*M \otimes \mathbb{C})$. Indeed, given $f \in C^\infty(M)$ and $\omega \in \Omega^k(M)$ we have

$$\mathbf{ad}(f)d(\omega = d(f\omega) - fd\omega = df \wedge \omega + fd\omega - fd\omega = df \wedge \omega$$

so that

$$\mathbf{ad}(f) = df \wedge \in \underline{Hom}(\Lambda^k T^*M, \Lambda^{k+1}T^*M).$$

□

Lemma 2.1.3. *The p.d.o.'s are local. i.e. given $L \in \mathbf{PDO}^{(m)}(E,F)$ and $u \in C^\infty(E)$ we have*

$$\operatorname{supp} Lu \subset \operatorname{supp} u.$$

Proof We argue inductively. The result is true for $m = 0$. In general, for any open set $U \supset \operatorname{supp} u$ we choose a smooth function f such that such that $f \equiv 1$ on $\operatorname{supp} u$ and $f \equiv 0$ outside U. Then

$$u = fu.$$

Then

$$Lu = L(fu) = [L, f]u + fLu$$

so that

$$\operatorname{supp} Lu \subset U, \quad \forall U$$

□

Remark 2.1.4. One can show that an operator $L \in \mathbf{OP}(E, F)$ is local if and only if it is a p.d.o., [17, 18]. □

Using partitions of unity and the local nature of the p.d.o.'s we deduce that in order to understand the structure of these objects it suffices to understand the special case when M itself is a coordinate patch and the bundles E and F are trivial.

Suppose $L \in \mathbf{PDO}^m(E, F)$. Then for every $f_1, \cdots, f_m$ we have

$$\mathbf{ad}(f_1)\,\mathbf{ad}(f_2)\cdots\mathbf{ad}(f_m)L \in \underline{Hom}(E, F).$$

We denote this operator by $\mathbf{ad}(f_1, \cdots, f_m)$. Using the Jacobi identity for the commutators we deduce

$$\mathbf{ad}(f)\,\mathbf{ad}(g)L = [[L, g], f] = [[L, f], g] + \underbrace{[L, [g, f]]}_{=0} = \mathbf{ad}(g)\,\mathbf{ad}(f)L.$$

This shows that $ad(f_1, \cdots, f_m)L$ is symmetric and $\mathbb{C}$-multi-linear in the variables $f_1, \cdots, f_m$. Thus $ad(f_1, \cdots, f_m)L$ is uniquely determined by

$$\mathbf{ad}(f)^m L = \mathbf{ad}(\underbrace{f, \cdots, f}_{m})L.$$

via the polarization identity

$$\mathbf{ad}(f_1, \cdots, f_m)L = \frac{1}{m!}\frac{\partial^m}{\partial t_1 \cdots \partial t_m}\,\mathbf{ad}(t_1 f_1 + \cdots + t_m f_m)^m L.$$

Fix a point $p_0 \in M$ and denote by I_{p_0} the ideal of $C^\infty(M)$ consisting of functions vanishing at p_0. From the identity

$$\mathbf{ad}(fg)P = Pfg - fgP = [P, f]g + fPg - fgP = (\mathbf{ad}(f)P)g + f(\mathbf{ad}(g)P), \quad \forall P \in \mathbf{OP}(E, F)$$

we deduce that if $f_1 = gh$, $g, h \in I_{p_0}$ then

$$\mathbf{ad}(f_1, \cdots, f_m)L = \mathbf{ad}(gh)\underbrace{\mathbf{ad}(f_2, \cdots, f_m)L}_{:=P} = (\mathbf{ad}(g)P)h + g(\mathbf{ad}(h)P).$$

On the other hand $\mathbf{ad}(h)P$ is a zeroth order p.d.o. so that $(\mathbf{ad}(g)P)h = h\,\mathbf{ad}(g)P$. We conclude

$$\mathbf{ad}(f_1, \cdots, f_m)L = h\,\mathbf{ad}(g)P + g\,\mathbf{ad}(h)P.$$

Both $\mathbf{ad}(g)P$ and $\mathbf{ad}(h)P$ are bundle morphisms and for every section s of E we have

$$(ad(f_1, \cdots, f_m)L)s(p_0) = h(p_0)(ad(g)P)s(p_0) + g(p_0)(ad(h)P)(p_0) = 0$$

Hence $\mathbf{ad}(f_1, \cdots, f_m)L\ |_{p_0} = 0$ when one of the f_i belongs to $I_{p_0}^2$. This shows that we have a symmetric, m-linear map

$$\sigma(L) = \sigma_{p_0}(L) : \big(\, I_{p_0}/I_{p_0}^2\,\big)^m \to \mathrm{Hom}(E_{p_0}, F_{p_0}),$$

$$\big(\, I_{p_0}/I_{p_0}^2\,\big)^m \ni (\xi_1, \cdots, \xi_n) \mapsto \sigma(P)(\xi_1, \cdots, \xi_n) \mapsto \frac{1}{m!}\,\mathbf{ad}(f_1, \cdots, f_m)L|_{p_0}$$

$$f_i \in I_{p_0}, \quad f_i \equiv \xi_i \mod I_{p_0}^2.$$

This function is uniquely determined by

$$\sigma(L)(\xi) := \sigma(L)(\xi, \cdots, \xi)$$

To obtain a more explicit description of $\sigma(L)$ we need to use the following classical result whose proof is left as an exercise.

Lemma 2.1.5 (Hadamard Lemma)**.**

$$f \in I_{p_0}^2 \Longleftrightarrow f(p_0) = 0, \;\; df(p_0) = 0,$$

so that we have a natural isomorphism of vector spaces

$$T^*_{p_0}M \otimes \mathbb{C} \cong I_{p_0}/I_{p_0}^2.$$

Thus we have a linear map

$$\sigma(L) = \sigma_{p_0}(L) : \mathrm{Sym}^m\, T^*_{p_0}M \otimes \mathbb{C} \to \mathrm{Hom}(E_{p_0}, F_{p_0}).$$

It is called the *symbol* of the p.d.o. L at p_0.

Observe that if $L_0 \in \mathbf{OP}(E_0, E_1)$, $L_1 \in \mathbf{OP}(E_1, E_2)$ and $f \in C^\infty(M)$ then an iterated application of (2.1.1) yields the identity

$$\mathbf{ad}(f)^m(L_1L_0) = \sum_{j=0}^m \binom{m}{j}(\mathbf{ad}(f)^jL_1)(\mathbf{ad}(f)^{m-j}(L_0).$$

This shows that if $L_0 \in \mathbf{PDO}^{m_0}(E_0, E_1)$, $L_1 \in \mathbf{PDO}^{m_1}(E_1, E_2)$ then

$$\mathbf{ad}(f)^{m_0+m_1}(L_1L_0) = \binom{m_0+m_1}{m_0}\,\mathbf{ad}(f)^{m_1}(L_1)\,\mathbf{ad}(f)^{m_0}(L_0)$$

$$= \frac{(m_0+m_1)!}{m_0!m_1!}\,\mathbf{ad}(f)^{m_1}(L_1)\,\mathbf{ad}(f)^{m_0}(L_0)$$

and in particular, for every $p \in M$ and every $\xi \in T^*_pM$ we have

$$\sigma_p(L_1L_0) = \sigma_p(L_1)(\xi)\sigma_p(L_0)(\xi). \tag{2.1.2}$$

The symbols of a p.d.o. $L \in \mathbf{PDO}^m(E, F)$ can be put together to form a global geometric object

$$\sigma(L) \in \underline{Hom}(\mathrm{Sym}^m(T^*M), \mathrm{Hom}(E, F)) \cong \underline{Hom}(\mathrm{Sym}^m(T^*M) \otimes E, F)$$

$$\cong \mathbf{PDO}^0(\mathrm{Sym}^m(T^*M) \otimes E, F).$$

It is time to look at some simple examples.

Example 2.1.6. (a) Assume $M = \mathbb{R}^n$, $E = F = \underline{\mathbb{C}}_M$. Suppose for simplicity that $p_0 = 0$. Let $\xi = (\xi_1, \xi_2, \cdots, \xi_n) \in T_0^*M$ and $f \in C^\infty(M$ such that $f(0) = 0$ and $df(0) = \xi$., i.e.

$$\xi_i = \partial_i f(0), \ \ \forall i = 1, \cdots, n.$$

Then

$$\mathbf{ad}(f)\partial_i = (\partial_i f)$$

so that

$$\sigma_0(\partial_i) = (\partial_i f)(0) = \xi_i.$$

Using (2.1.2) we deduce

$$\sigma_0(\partial^\alpha)(\xi) = \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

(b) Suppose ∇ is a connection on the vector bundle E. Then for every $p \in M$ and every $\xi \in T_p^*M$ we have

$$\sigma_p(\nabla)(\xi) : E_{p_0} \to T_p^*M \otimes E, \ \ s \mapsto \xi \otimes s.$$

Indeed, as we have seen in Example 2.1.2 (c) we have

$$ad(f)\nabla = df \otimes$$

Now replace df with ξ. Similarly $\sigma_p(d) = \xi\wedge : \Lambda^k T_p^*M \to \Lambda^{k+1} T_p^*M$.

(c) Suppose ∇^E is a connection on the complex vector bundle $E \to M$ and ∇^M is a connection on T^*M. We obtain connection $\nabla = \nabla^{(k)}$ on $\operatorname{Sym}^k T^*M \otimes E$ and then a differential operator

$$S_{k+1}(\nabla) : \operatorname{Sym}^k T^*M \otimes E \to \operatorname{Sym}^{k+1} T^*M \otimes E$$

defined by the composition

$$C^\infty(\operatorname{Sym}^k T^*M \otimes E) \xrightarrow{\nabla^{(k)}} C^\infty(T^*M \otimes \operatorname{Sym}^k T^*M \otimes E) \to C^\infty(\operatorname{Sym}^{k+1} T^*M \otimes E)$$

where the last map is induced by the natural symmetrization projection

$$T^*M^{\otimes m} \to \operatorname{Sym}^m T^*M.$$

The symbol of this p.d.o. is a bundle morphism

$$\operatorname{Sym}^1 T^*M \otimes \operatorname{Sym}^k T^*M \otimes E \longrightarrow \operatorname{Sym}^{k+1} T^*M \otimes E.$$

It is the natural morphism induced by the natural multiplication $\operatorname{Sym}^1 T^*M \otimes \operatorname{Sym}^k T^*M \to \operatorname{Sym}^{k+1}$

If we define

$$S(\nabla^k) = S_k(\nabla) \cdots S_0(\nabla) : C^\infty(E) \to C^\infty(\operatorname{Sym}^k T^*M \otimes E)$$

then

$$S(\nabla^k) \in \mathbf{PDO}^{(k)}(E, \operatorname{Sym}^k T^*M \otimes E)$$

and its symbol is the identity morphism

$$\operatorname{Sym}^k T^*M \otimes E \to \operatorname{Sym}^k T^*M \otimes E.$$

Suppose we have $L \in \mathbf{PDO}^{(k)}(E, F)$. Then

$$\sigma(L) \in \mathbf{PDO}^0(\operatorname{Sym}^k T^*M \otimes E, F).$$

The composition $\sigma(L) \circ S(\nabla^k)$ is an operator in $\mathbf{PDO}^{(k)}(E, F)$ with the same symbol as L and in particular

$$L - \sigma(L) \circ S(\nabla^k) \in \mathbf{PDO}^{(k-1)}(E, F).$$

Iterating this construction we deduce that we can write L as a finite sum of "monomials" $P_1 \cdots P_m$ where each P_i is either a connection or a bundle morphism.

□

Finally we need to introduce the concept of formal adjoint of a p.d.o. For simplicity, we will discuss this concept in a more restricted geometric context. More precisely, we will assume that all our manifolds are *oriented*, equipped with Riemann metrics, and that all the bundles are equipped with hermitian metrics.

Let (M, g) be an oriented Riemann manifold. We denote by dV_g the induced Riemannian volume form. Assume E, F are complex vector bundles over M equipped with hermitian metrics $\langle -, - \rangle_E$ and $\langle -, - \rangle_F$. We will denote by $C_0^\infty(E)$ the space of compactly supported sections of E.

Definition 2.1.7. A formal adjoint for the p.d.o. $L \in \mathbf{PDO}(E, F)$ is a p.d.o. $L^* \in \mathbf{PDO}(F, E)$ such that

$$\int_M \langle Lu, v \rangle_F dV_g = \int_M \langle u, L^* v \rangle_E dV_g, \quad \forall u \in C_0^\infty(E), \quad v \in C_0^\infty(F).$$

□

The following result list some immediate consequences of the definition. Its proof is left as an exercise.

Proposition 2.1.8. *(a) A p.d.o. $L \in \mathbf{PDO}(E, F)$ has at most one formal adjoint. When it exists we have the equality*

$$L = (L^*)^*.$$

(b) If $L_0, L_1 \in \mathbf{PDO}(E, F)$ have formal adjoints then their sum has a formal adjoint and

$$(L_0 + L_1)^* = L_0^* + L_1^*.$$

(c) If $L_0 \in \mathbf{PDO}(E_0, E_1)$ and $L_1 \in \mathbf{PDO}(E_1, E_2)$ have formal adjoints, then their composition has a formal adjoint and

$$(L_1 L_0)^* = L_0^* L_1^*.$$

(d) Every zeroth order p.d.o. has a formal adjoint.

Here are a few fundamental examples.

Example 2.1.9. (a) Suppose $E \cong F \cong \underline{\mathbb{C}}_M$ are equipped with the canonical hermitian metric. For every vector field X on M the Lie derivative $L_X : C^\infty(M) \to C^\infty(M)$ is a first order p.d.o. From the divergence formula we deduce that for every $u, v \in C_0^\infty(\underline{\mathbb{C}}_M)$ we have

$$\int_M (L_X u) \cdot \bar{v} dV_g = \int_M u \cdot \overline{(-L_X - \mathbf{div}_g(X))v} dV_g$$

so that

$$L_X^* = -L_X - \mathbf{div}_g(X).$$

(b) Proposition 1.3.8 can be interpreted as stating that the formal adjoint of

$$d : \Omega^{k-1}(M) \to \Omega^k(M)$$

is ($n = \dim M$)

$$d^* = (-1)^{nk+n+1} * d * .$$

(c) Suppose $E \to M$ is a hermitian vector bundle and ∇ is a hermitian connection on E. Then for every vector field X on M we obtain a p.d.o. $\nabla_X \in \mathbf{PDO}(E)$. Given $u, v \in C_0^\infty(E)$ we have

$$L_X\langle u, v\rangle_E = \langle \nabla_X u, v\rangle_E + \langle u, \nabla_X v\rangle$$

Integrating over M and using the divergence formula again we deduce

$$\int_M \big(\langle \nabla_X u, v\rangle_E + \langle u, \nabla_X v\rangle \big) dV_g = \int_M 1 \cdot (L_X\langle u, v\rangle_E) dV_g = -\int_M \mathbf{div}_g(X)\langle u, v\rangle dV_g$$

so that

$$\nabla_X^* = -\nabla_X - \mathbf{div}_g(X).$$

(d) The above connection ∇, viewed as a p.d.o. has a formal adjoint $\nabla^* \in \mathbf{PDO}(T^*M \otimes E, E)$. We describe it in a local coordinate patch U where ∇ has the form

$$\nabla = \sum_i dx^i \otimes (\nabla_{\partial_i} + A_i), \quad A_i \in\in \underline{End}(E|_U).$$

Then

$$\nabla^* = \sum_i (\nabla_{\partial_i} + A_i)^*(dx^i \otimes)^*$$

The adjoint of $dx^i \otimes : C^\infty(E) \to C^\infty(T^*M \otimes E)$ is the contraction $(dx^i)^\sharp \lrcorner : C^\infty(T^*M \otimes E) \to C^\infty(E)$ by the vector field $(dx^i)^\sharp$, the metric dual of dx^i, which is

$$(dx^i)^\sharp = \sum_j g^{ij}\partial_j.$$

□

From the computations in Example 2.1.6 (c) and Example 2.1.9 (d) we deduce that every p.d.o. has a formal adjoint.

Proposition 2.1.10. *Suppose $L \in \mathbf{PDO}^m(E, F)$. Then for every $p \in M$ and every $\xi \in T_p^*M$ we have*

$$\sigma_p(\xi)(L^*) = (-1)^m \sigma_p(\xi)(L)^*.$$

Proof Observe that for every smooth function $f : M \to \mathbb{R}$ we have

$$\mathbf{ad}(f)L^* = (L^*f - fL^*) = (fL - Lf)^* = -(\mathbf{ad}(f)L)^*$$

so that

$$\mathbf{ad}(f_1, \cdots, f_m)L^* = (-1)^m(\mathbf{ad}(f_1, \cdots, f_m)L)^*.$$

□

Definition 2.1.11. A p.d.o. $L \in \mathbf{PDO}(E)$ is called *formally self-adjoint* or *symmetric* if

$$L = L^*.$$

There is a very simple way of constructing symmetric operators. Given $L \in \mathbf{PDO}(E, F)$ the operators

$$L^*L \in \mathbf{PDO}(E), \quad LL^* \in \mathbf{PDO}(F)$$

are symmetric.

When ∇ is a hermitian connection on E then we can form a symmetric second order p.d.o. $\nabla^*\nabla \in \mathbf{PDO}^2(E)$. It is usually known as the *covariant Laplacian* . Observe that

$$\sigma_p(\nabla^*\nabla)(\xi) = -\sigma_p(\nabla)(\xi)^*\sigma_p(\nabla)(\xi) = -|\xi|_g^2\mathbb{1}_E,$$

where $|\xi|_g$ denotes the length of $\xi \in T_p^*M$ with respect to the Riemann metric g on M.

Definition 2.1.12. (a) A *generalized Laplacian* on the hermitian bundle E over the oriented Riemann manifold (M, g) is a symmetric second order p.d.o. $L \in PDO^2(E)$ such that

$$\sigma_p(L)(\xi) = -|\xi|_g^2\mathbb{1}_E, \ \ \forall p \in M, \ \ \xi \in T_p^*M.$$

(b) A first order p.d.o. $D \in \mathbf{PDO}^1(E, F)$ is called a *Dirac type operator* if the operators D^*D and DD^* are generalized Laplacians. □

Example 2.1.13. Suppose (M, g) is an oriented Riemann manifold. Consider the *Hodge-DeRham operator*

$$d + d^* : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

It is a symmetric operator and

$$\sigma(d + d^*)(\xi) = \sigma(d)(\xi) - \sigma(d)(\xi)^*.$$

The symbol of d is $\sigma(d)(\xi) = e(\xi) = \xi\wedge : \Lambda^\bullet T^*M \to \Lambda^\bullet T^*M$, and its adjoint is $i(\xi) = -\xi^\sharp \lrcorner$ the contraction by the vector $\xi^\sharp$ metric dual to the covector ξ. We have a *Cartan formula*

$$e(\xi)i(\xi) + i(\xi)e(\xi) = |\xi|^2 \cdot \mathbb{1} \tag{2.1.3}$$

and using the identities $e(\xi)^2 = i(\xi)^2 = 0$ we deduce

$$\big(\, e(\xi) + i(\xi) \,\big)^2 = e(\xi)i(\xi) + i(\xi)e(\xi) = |\xi|^2 \cdot \mathbb{1}$$

so that

$$\sigma(d + d^*)(\xi)^2 = -|\xi|^2 \cdot \mathbb{1}.$$

Hence the Hodge-DeRham operator is a Dirac type operator.

□

Definition 2.1.14. An operator $L \in \mathbf{PDO}(E, F)$ is called *elliptic* if for all $p \in M$ and all $\xi \in T_p^*M \setminus 0$ the operator

$$\sigma_p(L)(\xi) : E_p \to F_p$$

is a linear isomorphism.

Example 2.1.15. (a) If L is elliptic iff L^* is also elliptic. If L_0, L_1 are elliptic then so is their composition L_1L_0 (when it makes sense). If $L, K \in \mathbf{PDO}(E, F)$ and the order of K is strictly smaller than the order of L then

$$\sigma(L) = \sigma(L + K)$$

so that L is elliptic iff $L + K$ is elliptic.

(b) Any generalized Laplacian is an elliptic operator.

(c) Any Dirac type operator is elliptic. In particular, the Hodge-DeRham operator is elliptic.

□

The next proposition shows that the generalized Laplacians are zeroth order perturbations of covariant Laplacians.

Proposition 2.1.16. ([4, Sec. 2.1], [8, Sec. 4.1.2]) *Suppose L is a generalized Laplacian on E. Then there exists a* unique hermitian *connection $\tilde{\nabla}$ on E and a* unique selfadjoint *endomorphism $\mathcal{R}$ of E such that*

$$L = \tilde{\nabla}^*\tilde{\nabla} + \mathcal{R} \tag{2.1.4}$$

We will refer to this presentation of a generalized Laplacian as the Weitzenböck presentation *of L.*

Proof Choose an arbitrary hermitian connection ∇ on E. Then $L_0 = \nabla^*\nabla$ is a generalized Laplacian so that $L - L_0$ is a first order operator which can be represented as

$$L - L_0 = A \circ \nabla + B$$

where

$$A : C^\infty(T^*M \otimes E) \to C^\infty(E)$$

is a bundle morphism and B is an endomorphism of E. We will regard A as an $\mathrm{End}\,(E)$-valued 1-form on M. Hence

$$L = \nabla^*\nabla + A \circ \nabla + B. \tag{2.1.5}$$

The connection ∇ induces a connection on $\mathrm{End}(E)$ which we continue to denote with ∇

$$\nabla : C^\infty(\mathrm{End}\,(E)) \to \Omega^1(\mathrm{End}\,(E)).$$

We define the *divergence* of A by

$$\mathbf{div}_g(A) := -\nabla^* A.$$

If (e_i) is a local synchronous frame at x_0 and, if $A = \sum_i A_i e^i$, then, at x_0, we have

$$\mathbf{div}_g(A) = \sum_i \nabla_i A_i.$$

Note that since $(L - L_0) = \sum_i A_i \nabla_i + B$ is formally selfadjoint we deduce

$$A_i^* = -A_i, \;\; \mathbf{div}_g(A) = B - B^*. \tag{2.1.6}$$

We seek a hermitian connection $\tilde{\nabla} = \nabla + C$, $C \in \Omega^1(\mathrm{End}\,(E))$ and an endomorphism $\mathcal{R}$ of E such that

$$\tilde{\nabla}^*\tilde{\nabla} + \mathcal{R} = \nabla^*\nabla + A \circ \nabla + B.$$

We set $C_i := e_i \lrcorner C$ so that we have the local description

$$\tilde{\nabla} = \sum_i e^i \otimes (\nabla_i + C_i), \;\; C_i^* = -C_i, \;\; \forall i.$$

We deduce that, at x_0

$$\tilde{\nabla}^*\tilde{\nabla} = -\sum_i (\nabla_i + C_i)(\nabla_i + C_i)$$

$(\langle C_i \rangle^2 := C_i C_i^* = -C_i^2)$

$$= -\sum_i \nabla_i^2 - \sum_i \nabla_i C_i - 2\sum_i C_i \nabla_i + \sum_i \langle C_i \rangle^2$$

$(\langle C \rangle^2 = \sum_i \langle C_i \rangle^2)$

$$= \nabla^*\nabla - 2C \circ \nabla - \mathbf{div}_g(C) + \langle C \rangle^2 = \nabla^*\nabla + A \circ \nabla + B - \mathcal{R}.$$

We deduce immediately that

$$C = -\frac{1}{2}A, \;\; \mathcal{R} = B - \frac{1}{2}\mathbf{div}_g(A) - \langle C \rangle^2 \overset{(2.1.6)}{=} \frac{1}{2}(B + B^*) - \frac{1}{4}\langle A \rangle^2. \tag{2.1.7}$$

The proposition is proved.

□

The connection $\tilde{\nabla}$ produced in the above proposition is called the *Weitzenböck connection* determined by L while $\mathcal{R}$ is called the *Weitzenböck remainder*.

2.1.2. Analytic properties of elliptic operators. We would like to describe some features of elliptic partial differential equations. We begin by introducing an appropriate functional framework.

Suppose $E \to M$ is a hermitian vector bundle over the connected oriented Riemann manifold (M, g). The volume form dV_g induces a (regular) Borel measure on M which we continue to denote by dV_g. A (possibly discontinuous) section $u \to E$ is called *measurable* if for any Borel set $B \subset E$ the preimage $u^{-1}(B)$ is a Borel subset of M. We denote by "$\doteq$" the almost everywhere (a.e.) equality of measurable sections.

Let $1 \leq p < \infty$. A measurable section $u : M \to E$ is called *p-integrable* if

$$\int_M |u|^p dV_g < \infty.$$

We denote by $L^p(E)$ the vector space of $\doteq$-classes of p-integrable spaces. It is a *Banach* space with respect to the norm

$$\|u\|_p = \|u\|_{p,E} = \left(\int_M |u|^p dV_g\right)^{1/p}.$$

We want to emphasize that this norm *depends* on the metric on M and in the noncompact case it is possible that different metrics induce non-equivalent norms. When $p = 2$ this is a *Hilbert space* with respect to the inner product

$$(u, v) = (u, v)_{L^2(E)} = \int_M \langle u, v\rangle_E dV_g.$$

A measurable section $u \to E$ is called *locally p-integrable* if for any compactly supported smooth function $\varphi : M \to \mathbb{C}$ the section φu is p-integrable. We denote by $L^p_{loc}(E)$ the vector space of $\doteq$-equivalence classes of locally p-integrable functions.

Suppose $E, F \to M$ are two hermitian vector bundles over the same connected, oriented Riemann manifold (M, g).

Definition 2.1.17. (a) Let $L \in \mathbf{PDO}(E, F)$, $u \in L^1_{loc}(E)$ and $v \in L^1_{loc}(F)$. We say that u is a *weak solution* of

$$Lu = v$$

or that $Lu = v$ *weakly* if for any $\varphi \in C_0^\infty(F)$ we have

$$\int_M \langle u, L^*\varphi\rangle_E dV_g = \int_M \langle v, \varphi\rangle_F dV_g.$$

(b) Suppose ∇ is a Hermitian connection on E. A locally integrable section $u \in L^1(E)$ is said to be *weakly differentiable* (with respect to ∇) if there exists $v \in L^1_{loc}(T^*M \otimes E)$ such that $\nabla u = v$ weakly, i.e.

$$\int_M \langle u, \nabla^*\varphi\rangle dV_g = \int_M \langle v, \varphi\rangle dV_g, \quad \forall \varphi \in C_0^\infty(T^*M \otimes E).$$

□

Suppose $E \to M$ is a hermitian bundle equipped with a hermitian connection. For every non-negative integer k and every $p \in [1, \infty)$ we denote by $L^{k,p}(E)$ the subspace of $L^p(E)$ consisting of sections u which are k-times weakly differentiable with respect to ∇ and their differentials $\nabla u, \cdots, \nabla^k u$ are p-integrable. This space is a Banach space with respect to the norm

$$\|u\|_{k,p} = \left(\sum_{j=0}^{k} \int_M |\nabla^j u|^p dV_g \right)^{1/p}.$$

For $p > 1$ they are *reflexive*. When $p = 2$ they are Hilbert spaces with respect to the obvious inner product. These Banach spaces are generically called the *Sobolev spaces of section*. We want to emphasize that the norms $\| - \|_{k,p}$ depend on the metric g on M, the metric h on E and the hermitian connection ∇ on E. To indicate this dependence we will sometime write $L^p(E, g, h, \nabla)$. The situation is much better in the compact case. For a proof of the following result we refer to [3, Chap.2].

Proposition 2.1.18. *(a) Suppose M is a compact, oriented manifold without boundary, and $E \to M$. For $i = 0, 1$ denote by g_i a Riemann metric on M, h_i a hermitian metric on E and ∇^i a connection on E compatible with h_i. Then for every $k \in \mathbb{Z}_{\geq 0}$ and every $p \in [1, \infty)$ we have an equality*

$$L^{k,p}(E, g_0, h_0, \nabla^0) = L^{k,p}(E, g_1, h_1, \nabla^1)$$

Moreover the two norms are equivalent, i.e. $\exists C > 0$ such that

$$\frac{1}{C}\|u\|_{k,p;g_0,h_0,\nabla^0} \leq \|u\|_{k,p;g_1,h_1,\nabla^1} \leq C\|u\|_{k,p;g_0,h_0,\nabla^0}, \quad \forall u \in L^{k,p}(E).$$

(b) The space $C^\infty(E)$ is dense in any Sobolev space $L^{k,p}(E)$. □

In the remainder of this section we will assume that the manifold M is *compact*, oriented without boundary. We set $n := \dim M$. In particular, the dependence of the Sobolev norms on the additional data will not be indicated in the notation.

The *conformal weight* of the Sobolev space $L^{k,p}(E)$ is the real number

$$w_n(k, p) = \frac{n}{p} - k.$$

Observe that if we regard a section u as a dimensionless quantity, then the volume form dV_g is measured in $meters^n$, $\nabla^k u$ is measured in $meters^{-k}$, and thus

$$\left(\int_M |\nabla^k u|^p dV_g \right)^{1/p}$$

is measured in $meters^{w_n(k,p)}$.

Denote by $C^k(E)$ the vector space of k-times differentiable functions with continuous differentials. It is a Banach space with respect to the norm

$$\|u\|_k = \sup_{x \in M} \sum_{j=0}^{k} |\nabla^j u(x)|$$

The conformal weight of C^k is $w_n(k) = -k$. We have the following fundamental result whose proof can be found in [3, Chap.2].

Theorem 2.1.19 (Sobolev Embedding)**.** *Suppose $E \to M$ is a hermitian vector bundle equipped with a Hermitian connection, and (M, g) is a compact, oriented Riemann manifold without boundary.*

(a) Let $k, m \in \mathbb{Z}_{\geq 0}$, $p, q \in [1, \infty)$. If

$$k \geq m \text{ and } w_n(k,p) \leq w_n(m,q) \Longleftrightarrow k \geq m \text{ and } \frac{n}{p} - k \leq \frac{n}{q} - m, \tag{2.1.8}$$

then $L^{k,p}(E) \subset L^{m,q}(E)$ and the natural inclusion is continuous, i.e.

$$\exists C > 0: \quad \|u\|_{m,q} \leq C\|u\|_{k,p}, \quad \forall u \in L^{k,p}(E).$$

(b) Let $k, m \in \mathbb{Z}_{\geq 0}$, $p \in [1, \infty)$. If

$$w_n(k,p) \leq -m \Longleftrightarrow \frac{n}{p} - k \leq -m, \tag{2.1.9}$$

then $L^{k,p}(E) \subset C^m(E)$ and the natural inclusion is continuous.

(c) If in (2.1.8) and in (2.1.9) we have strict inequalities then the corresponding inclusions are compact *operators, i.e. they map bounded sets to pre-compact subsets.*

We will frequently use the following special case of the Sobolev theorem.

Corollary 2.1.20. *Let $E \to M$ be as in Theorem 2.1.19.*

(a) If $\|u\|_{L^{m,2}(E)} < \infty$ and $m > k + \frac{n}{2}$ then there exists a k-times differentiable section $\hat{u}$ of E such that $u \doteq \hat{u}$.

(b) If $m > k$ then any sequence of sections of E bounded in the $L^{m,2}$-norm contains a subsequence convergent in the $L^{k,2}$-norm.

We can now state the central results of the theory of elliptic p.d.e.'s. For a proof we refer to [16, Chap.9].

Theorem 2.1.21 (The Fundamental Theorem of Elliptic P.D.O.s)**.** *Suppose $E, F \to M$ are hermitian vector bundles over the closed, oriented Riemann manifold M and $L \in \mathbf{PDO}^m(E, F)$ is an elliptic operator.*

(a) **(A priori estimate)** *Let $k \in \mathbb{Z}_{\geq 0}$, $1, p < \infty$. There exists a constant $C > 0$ such that for all $u \in L^{k+m,p}(E)$ we have*

$$\|u\|_{k+m,p} \leq C\Big(\|Lu\|_{k,p} + \|u\|_{0,p}\Big).$$

(b) **(Regularity)** *Let $k \in \mathbb{Z}_{\geq 0}$, $1, p < \infty$. Suppose $u \in L^p(E)$, $v \in L^{k,p}(F)$ and $Lu = v$ weakly. Then $u \in L^{k+m,p}(E)$.*

Corollary 2.1.22 (Weyl Lemma)**.** *Let E and L as above and $1 < p < \infty$. If $u \in L^p(E)$, $v \in C^\infty(F)$ and $Lu = v$ weakly then $u \in C^\infty(E)$. In particular if $u \in L^p(E)$ and $Lu = 0$ weakly then $u \in C^\infty(E)$.*

Proof

$$v \in C^\infty(F) \Longrightarrow v \in \bigcap_{k \geq 0} L^{k,p}(F) \Longrightarrow u \in \bigcap_{k \geq 0} L^{k+m,p}(E).$$

The Sobolev embedding theorem implies that

$$\bigcap_{k\geq 0} L^{k+m,p}(E) = C^\infty(E).$$

□

2.1.3. Fredholm index and Hodge theory. Suppose $E, F \in M$ are hermitian vector bundles over a closed, oriented Riemann manifold (M, g) and $L \in \mathbf{PDO}^m(E, F)$ is an elliptic operator of order m. Let

$$\ker L = \Big\{ u \in C^\infty(E);\ \ Lu = 0 \Big\}.$$

Weyl Lemma shows that a measurable section of E belongs to $\ker L$ if and only if it is p-integrable for some $p > 1$ and $Lu = 0$ weakly.

Proposition 2.1.23. $\ker L$ *is a finite dimensional vector space.*

Proof We first prove that $\ker L$ is a *closed* subspace of $L^2(E)$, i.e.

$$u_i \to u \in L^2(E),\ \ u_i \in \ker E,\ \ \forall n \Longrightarrow u \in \ker E.$$

Indeed

$$(u_i, L^*\varphi)_{L^2(F)} = \int_M \langle u_i, L^*\varphi\rangle dV_g = 0,\ \ \forall \varphi \in C_0^\infty(F).$$

Letting $i \to \infty$ we deduce

$$\int_M \langle u, L^*\varphi\rangle dV_g = 0\ \ \forall \varphi \in C_0^\infty(F).$$

so that $u \in \ker E$.

We will now show that any ball in $\ker E$ which is closed with respect to the L^2-norm must be compact in the topology of this norm. The desired conclusion will then follow from a classical result of F. Riesz, [6, Ch. VI] according to which a Banach space is finite dimensional if and only if it is locally compact.

Suppose $\{u_i\}$ is a L^2-bounded sequence in $\ker L$. From the a priori inequality we deduce

$$\|u_i\|_{m,2} \leq C\|u_0\|_{0,2}$$

we deduce that (u_i) is also bounded in the $L^{m,2}$-norm as well. Since the inclusion $L^{m,2} \hookrightarrow L^2$ is compact we deduce that the sequence (u_i) has a subsequence convergent in the L^2-norm.

□

Observe that L defines a bounded linear operator

$$L : L^{m,2}(E) \to L^2(F)$$

and we denote by $\mathbf{R}(L)$ its range.

Theorem 2.1.24 (Fredholm alternative)**.** *The range of* L *is a closed subspace of* $L^2(F)$*. More precisely*

$$\mathbf{R}(L) = (\ker L^*)^\perp,\ \ \mathbf{R}(L^*) = (\ker L)^\perp.$$

Proof The proof is based on the following important fact.

Lemma 2.1.25 (Poincaré Inequality)**.** *There exists $C > 0$ such that for all $u \in L^{m,2}(E) \cap (\ker L)^\perp$ we have*

$$\|u\|_{m,2} \leq C\|Lu\|_{0,2}.$$

Proof We argue by contradiction. Suppose that for every $k > 0$ there exists

$$u_k \in L^{m,2}(E) \cap (\ker L)^\perp : \quad \|u_k\|_{0,2} = 1, \quad \|u_k\|_{m,2} \geq k\|Lu_k\|_{0,2}$$

From the elliptic estimate we deduce that there exists $C > 0$ such that

$$\|u_k\|_{m,2} \leq C(\|Lu_k\|_{0,2} + \|u_k\|_{0,2}) = C(\|Lu_k\|_{0,2} + 1). \tag{2.1.10}$$

Hence

$$k\|Lu_k\|_{0,2} \leq C(\|Lu_k\|_{0,2} + 1).$$

so that

$$\|Lu_k\|_{0,2} \to 0, \quad \text{as} \quad k \to \infty.$$

Using this information in (2.1.10) we deduce that $\|u_k\|_{m,2} = O(1)$. Since the inclusion $L^{m,2} \hookrightarrow L^2$ is compact we deduce that a subsequence of u_k which we continue to denote by u_k converges strongly inL^2 to some u_∞. Since $\|u_k\|_{0,2} = 1$ and $u_k \in (\ker L)^\perp$ we deduce

$$\|u_\infty\|_{0,2} = 1, \quad u_\infty \in (\ker L)^\perp. \tag{2.1.11}$$

Set $v_k := Lu_k$. We know that $Lu_k = v_k$ weakly so that

$$\int_M \langle u_k, L^*\varphi\rangle dV_g = \int_M \langle v_k, \varphi\rangle DV_g, \quad \forall \varphi \in C_0^\infty(F).$$

We let $k \to \infty$ in the above equality and use the fact that $u_k \xrightarrow{L^2} u_\infty$, $v_k \xrightarrow{L^2} 0$ to conclude that

$$\int_M \langle u_\infty, L^*\varphi\rangle dV_g = 0, \quad \forall \varphi \in C_0^\infty(F).$$

Hence $Lu_\infty = 0$ weakly so that $u_\infty \in \ker L$. This contradicts (2.1.11) and concludes the proof of the Poincaré inequality.

□

Now we can finish the proof of the Fredholm alternative. Suppose we have a sequence $u_k \in L^{m,2}(E)$ such that $v_k = Lu_k$ converges in L^2 to some v_∞. We have to show that there exists $u_\infty \in L^{m,2}(E)$ such that $Lu_\infty = v_\infty$. We decompose

$$u_k = [u_k] + u_k^\perp, \quad [u_k] \in \ker L, \quad u_k^\perp \in (\ker L)^\perp.$$

Clearly $v_k = Lu_k^\perp$ and from the Poincaré inequality we deduce that

$$\|u_k^\perp\|_{m,2} \leq C\|v_k\|_{0,2}.$$

Since the sequence (v_k) converges in L^2 it must be bounded in this space so we conclude that

$$\|u_k^\perp\|_{m,2} = O(1).$$

Using again the fact that the inclusion $L^{m,2} \hookrightarrow L^2$ is compact we deduce that a subsequence of $u_k^\perp$ converges in L^2 to some u_∞. Since $Lu_k^\perp = v_k$ weakly we deduce

$$\int_M \langle u_k^\perp, L^*\varphi\rangle dV_g = \int_M \langle v_k, \varphi\rangle DV_g, \quad \forall \varphi \in C_0^\infty(F).$$

If we let $k \to \infty$ we deduce

$$Lu_\infty = v_\infty \ \text{ weakly}.$$

This proves that the range of L is closed. We still have to prove the equality

$$\mathbf{R}(L) = (\ker L^*)^\perp$$

Observe that if $v \in \mathbf{R}(L)$, there exists $u \in L^{m,2}(E)$ such that $Lu = v$ weakly. In particular, if $w \in \ker L^*$, then $w \in C^\infty(F)$ and

$$Lu = v \Longrightarrow 0 = \int_M \langle u, L^* w\rangle dV_g = \int_M \langle v, w\rangle dV_g \Longrightarrow v \in (\ker L^*) \perp .$$

Hence $\mathbf{R}(L) \subset (\ker L^*)^\perp$. Suppose conversely that $v \in (\ker L^*)^\perp$. Suppose $v \notin \mathbf{R}(L)$. Since $\mathbf{R}(L)$ is closed, the *Hahn-Banach theorem* implies the existence of $w \in L^2(F)$ such that

$$\langle w, v\rangle \neq 0, \ \ w \in \mathbf{R}(L)^\perp.$$

Hence

$$\langle w, (L^*)^* u\rangle = 0, \ \ \forall u \in L^{m,2}(E).$$

In particular

$$\langle w, (L^*)^* u\rangle = 0, \ \ \forall u \in C^\infty(E)$$

so that $L^* w = 0$ weakly, i.e. $w \in \ker L^*$. We have reached a contradiction since $\langle v, w'\rangle = 0$ for all $w' \in \ker L^*$. This concludes the proof of the Fredholm alternative.

□

Definition 2.1.26. The *Fredholm index* of an elliptic operator L between $\mathbb{K}$-vector bundles over a closed oriented manifold is the integer

$$\operatorname{ind}_{\mathbb{K}} L := \dim \ker_{\mathbb{K}} L - \dim_{\mathbb{K}} \ker L^* = \dim_{\mathbb{K}} \ker L - \dim_{\mathbb{K}} \operatorname{coker} L.$$

□

Proposition 2.1.27 (Finite dimensional Hodge theorem)**.** *Suppose*

$$0 \to V^0 \stackrel{D_0}{\longrightarrow} V^1 \stackrel{D_1}{\longrightarrow} \cdots \stackrel{D_{n-1}}{\longrightarrow} V^n \to 0$$

is a co-chain complex of finite dimensional $\mathbb{C}$-vector spaces and linear maps. Suppose each of the spaces V_i is equipped with a hermitian metric. Then for every $i = 0, 1, \cdots, n$ the induced map

$$\pi_i : \mathbb{H}^i(V^\bullet) := \ker D_i \cap \ker D_{i-1}^* \to H^i(V^\bullet, D_\bullet) = \ker D_i / \mathbf{R}(D_{i-1})$$

is an isomorphism. If we set $D = \oplus D_i : \oplus_i V^i \to \oplus_i V^i$ and $\Delta := (D + D^*)^2$ then

$$\mathbb{H}^\bullet(V^\bullet) := \oplus_i \mathbb{H}^i(V^\bullet) = \ker(D + D^*) = \ker \Delta.$$

In particular, the complex is acyclic if and only if $D + D^*$ is a linear isomorphism.

Proof Let us first prove that π_i is an isomorphism. We first prove it is injective.

Let $v \in \ker \pi_i$. Hence $D_i v = D_{i-1}^* v = 0$ and $v = 0 \in H^i(V^\bullet)$, i.e. there exists $u \in V^{i-1}$ such that $u = D_{i-1} v$. Hence

$$0 = D_{i-1}^* v = D_{i-1}^* D_i u = 0 \Longrightarrow 0 = \langle D_{i-1}^* D_i u, u\rangle = \langle D_{i-1} u, D_{i-1} u\rangle = |D_{i-1} u|^2 = |v|^2.$$

This shows that π_i is injective.

To prove the surjectivity we have to show that every $u \in \ker D_i$ is cohomologous to an element in $\ker D^*_{i-1}$. Let $v \in \ker D_i$. The cohomology class it determines can be identified with the affine subspace

$$C_v = \Big\{ v + D_{i-1}u;\ \ u \in V^{i-1} \Big\}.$$

We denote by $[v]$ the point on C_v closest to the origin (see Figure 1). *This point exists since V^{i-1} is finite dimensional.*

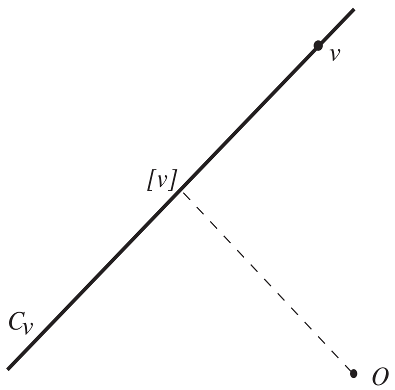


Figure 1. *Finding the harmonic representative of a cocycle.*

We claim that $D^*_{i-1}[v] = 0$. For every $u \in V^{i-1}$ we consider the function

$$f_u : \mathbb{R} \to [0, \infty),\ \ f_u(t) = \mathrm{dist}\,([v] + tD_{i-1}u, 0)^2 = |[v] + tD_{i-1}u|^2.$$

Since $[v] + tD_{i-1}u \in C_v$ we deduce

$$\mathrm{dist}\,([v], 0) \leq \mathrm{dist}\,([v] + tD_{i-1}u, 0),\ \ \forall t \Longrightarrow f_u(0) \leq f_u(t),\ \ \forall t.$$

Hence $f'_u(0) = 0$, i.e.

$$0 = \frac{d}{dt}|_{t=0}\ \langle [v] + tD_{i-1}u, [v] + tD_{i-1}u = 2\,\mathbf{Re}\langle [v], D_{i-1}u\rangle.$$

Hence

$$0 = \mathbf{Re}\langle [v], D_{i-1}u\rangle = \mathbf{Re}\langle D^*_{i-1}[v], u\rangle,\ \ \forall u \in V^{i-1}.$$

If in the above equality we take $u = D^*_{i-1}[v]$ we conclude $D^*_{i-1}[v] = 0$ which shows that π_i is a surjection.

The equality

$$\mathbb{H}^\bullet(V^\bullet) = \ker(D + D^*)$$

is simply a reformulation of the fact that π_i is an isomorphism.

If we let $\mathcal{D} = D + D^*$, then $\Delta = \mathcal{D}^2$ and thus $\ker \mathcal{D} \subset \ker \Delta$. Conversely, if $u \in \ker \Delta$ then

$$0 = \langle \Delta u, u\rangle = \langle \mathcal{D}^2 u, u\rangle = |\mathcal{D}u|^2$$

so that $\ker \mathcal{D} \subset \ker \Delta$.

□

Definition 2.1.28. Suppose $E^0, E^1, \cdots, E^N$ are hermitian vector bundles over the Riemann manifold $M, g)$ and $D_i : \mathbf{PDO}^1(E_i, E_{i-1})$ are first order p.d.o. such that

$$D_i D_{i-1} = 0, \ \ \forall i.$$

Then the cochain complex

$$0 \to C^\infty(E^0) \stackrel{D_0}{\longrightarrow} C^\infty(E^1) \to \cdots \to C^\infty(E^N) \to 0 \tag{2.1.12}$$

is called *elliptic* if for any $p \in M$ and any $\xi \in T_p^*M \setminus 0$ the complex of finite dimensional spaces

$$0 \to E_p^0 \stackrel{\sigma_p(D_0)(\xi)}{\longrightarrow} E_p^1 \to \cdots \to E_p^N \to 0 \tag{2.1.13}$$

is acyclic.

We set

$$E = \oplus_k E^k, \ \ D = \oplus_k D_k C^\infty(E) \to C^\infty(E), \ \ \mathcal{D} = D + D^*, \ \ \Delta = \mathcal{D}^2.$$

□

Applying the finite dimensional Hodge theory to the complex (2.1.13) we deduce that the complex $(C^\infty(E^\bullet), D_\bullet)$ is elliptic if and only if the operator $\mathcal{D}$ is elliptic.

Theorem 2.1.29 (Hodge). *Suppose*

$$0 \to C^\infty(E^0) \stackrel{D_0}{\longrightarrow} C^\infty(E^1) \to \cdots \to C^\infty(E^N) \to 0$$

is an elliptic complex. Then the natural map

$$\pi_i : \mathbb{H}^i(E^\bullet, D_\bullet) := \ker D_i \cap \ker D_{i-1}^* \to H^i(E^\bullet, D_\bullet) = \ker D_i / \mathbf{R}\, D_{i-1}$$

is an isomorphism.

(b) The spaces $\mathbb{H}^i(E^\bullet, D_\bullet)$ are finite dimensional and the Euler characteristic of the complex $(E^\bullet, D_\bullet)$ equals the Fredholm index of the elliptic operator

$$\mathcal{D} = D + D^* : C^\infty(E^{even}) \to C^\infty(E^{odd}).$$

(c) $\ker \mathcal{D} = \ker \Delta$.

Proof (a) We set $V^i = C^\infty(E^i)$. These spaces are equipped with the L^2-inner product but they are not complete with respect to this norm. We imitate the strategy used in the proof of Proposition 2.1.27. The only part of the proof that requires a modification is the proof of the surjectivity of π_i. In the finite dimensional case it was based on the existence of the element $[v]$, the point in the affine space C_v closest to the origin. A priori this may not exist[1] since in our case V^i is infinite dimensional and incomplete with respect to the L^2-norm. In the infinite dimensional case we will bypass this difficulty using the Fredholm alternative. Set $\| - \| := \| - \|_{0,2}$.

Observe first that $\mathbb{H}^i(E^\bullet, D_\bullet)$ is finite dimensional since it is a subspace of $\ker \mathcal{D}$ which is finite dimensional since $\mathcal{D}$ is elliptic. Let $v \in C^\infty(E^i)$ such that $Dv = 0$. We have to prove that $\exists [v] = v + Du$, $u \in C^\infty(E^{i-1})$ such that $D^*[v] = 0$. Denote by $[v]$ the L^2-orthogonal projection of v on $\mathbb{H}^i$. This projection exists since $\mathbb{H}^i$ is finite dimensional hence closed. We claim that $[v]$ is cohomologous to v.

[1]This is in essence the criticism Weierstrass had concerning Riemann's liberal use of the Dirichlet principle, i.e. the existence of a shortest element. A few decades later Hilbert and Weyl rehabilitated Riemann's insight and placed it on solid foundational ground.

By definition $v - [v] \perp \ker \mathcal{D}$ so that by the Fredholm alternative there exists $u \in L^{1,2}(E^\bullet$ such that

$$v - [v] = \mathcal{D}u = (D + D^*)u.$$

Since $v, [v]$ are smooth we deduce from Weyl's Lemma that u is smooth. Since $D(v - [v]) = 0$ we deduce $DD^*u = 0$ so that

$$0 = (DD^*u, u)_{L^2} = \|D^*u\|^2.$$

Hence $D^*u = 0$, i.e. $v - [v] = Du$ which shows that v and $[v]$ are cohomologous.

□

Example 2.1.30. Suppose (M, g) is a compact oriented Riemann manifold without boundary. Let $n := \dim M$. Then the DeRham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

is an elliptic complex (see Exercise 2.3.5). We denote its cohomology by $H^\bullet_{DR}(M)$. We set

$$\mathbb{H}^k(M, g) := \Big\{ \omega \in \Omega^k(M);\ \ d\omega = d^*\omega = 0 \Big\}.$$

The forms in $\mathbb{H}^k(M, g)$ are called *harmonic forms* with respect to the metric g.Hodge theorem implies that

$$\mathbb{H}^k(M, g) \cong H^k_{DR}(M) \cong H^k(M, \mathbb{R}).$$

This shows that once we fix a Riemann metric on M we have a canonical way of selecting a representative in each DeRham cohomology class, namely the unique harmonic form in that cohomology class. The above arguments shows that it is the form in the cohomology class with the shortest L^2-norm. One can show (see Exercise 2.3.5) that the Hodge $*$-operator

$$*_g : \Omega^k(M) \to \Omega^{n-k}(M)$$

induces an isomorphism

$$*_g : \mathbb{H}^k(M, g) \to \mathbb{H}^{n-k}(M, g).$$

In this case we have

$$\chi(M) = \mathrm{ind}_{\mathbb{R}}\Big(d + d^* : \Omega^{even}(M) \to \Omega^{odd}(M) \Big).$$

On the left-hand side we have a topological invariant while on the right-hand side we have an analytic invariant. This phenomenon is a manifestation of the Atiyah-Singer index theorem. □

2.2. Dirac operators

2.2.1. Clifford algebras and their representations. Suppose (M, g) is an oriented Riemann manifold, $E^+, E^- \to M$ are complex hermitian vector bundles and

$$D : C^\infty(E^+) \to C^\infty(E^-)$$

is a Dirac type operator. Recall that this means that the symmetric operators

$$D^*D : C^\infty(E^+) \to C^\infty(E^+), \quad DD^* : C^\infty(E^-) \to C^\infty(E^-)$$

are both generalized Laplacians. It is convenient to super-symmetrize this formulation. Set $E = E^+ \oplus E^-$ and define,

$$\mathcal{D} = \begin{bmatrix} 0 & D^* \\ D & 0 \end{bmatrix} : C^\infty(E) \to C^\infty(E).$$

Then

$$\mathcal{D}^* = \mathcal{D}, \quad \mathcal{D}^2 = \begin{bmatrix} D^*D & 0 \\ 0 & DD^* \end{bmatrix}.$$

We denote by $\boldsymbol{c}$ the symbol of $\mathcal{D}$. Observe that for every $x \in M$, and every $\xi \in T_x^*M$ we the linear map $\boldsymbol{c}(\xi) : E_\xi \to E_x$ satisfies

$$\boldsymbol{c}(\xi)^2 = -\boldsymbol{c}(\xi), \quad \boldsymbol{c}(\xi)^2 = -|\xi|_g^2 \mathbb{1}_E, \quad \boldsymbol{c}(\xi)E_x^\pm \subset E_x^\mp. \tag{2.2.1}$$

Thus, for fixed $x \in M$ we can view the symbol as a linear map $\boldsymbol{c} : T_x^*M \to \mathrm{End}(E_x)$ satisfying (2.2.1) for any $\xi \in T_xM$. Observe that

$$-|\xi + \eta|^2 = \boldsymbol{c}(\xi + \eta)^2 = \big\{ \boldsymbol{c}(\xi) + \boldsymbol{c}(\eta) \big\}^2 = \boldsymbol{c}(\xi)^2 + \boldsymbol{c}(\eta)^2 + \boldsymbol{c}(\xi)\boldsymbol{c}(\eta) + \boldsymbol{c}(\eta)\boldsymbol{c}(\xi)$$

$$= -|\xi|^2 - |\eta|^2 + \boldsymbol{c}(\xi)\boldsymbol{c}(\eta) + \boldsymbol{c}(\eta)\boldsymbol{c}(\xi).$$

Hence

$$|\xi|^2 + |\eta|^2 - \boldsymbol{c}(\xi)\boldsymbol{c}(\eta) - \boldsymbol{c}(\eta)\boldsymbol{c}(\xi) = |\xi + \eta|^2 = |\xi|^2 + |\eta|^2 + 2g(\xi, \eta)$$

so that

$$\boldsymbol{c}(\xi)\boldsymbol{c}(\eta) + \boldsymbol{c}(\eta)\boldsymbol{c}(\xi) = -2g(\xi, \eta), \quad \forall \xi, \eta \in T_x^*M. \tag{2.2.2}$$

Definition 2.2.1. Suppose (V, g) is a finite dimensional real Euclidean space. We define the *Clifford algebra* of (V, g) to be the associative $\mathbb{R}$-algebra with 1 generated by V and subject to the relations

$$u \cdot v + v \cdot u = -2g(u, v), \quad \forall u, v \in V.$$

Equivalently, it is the quotient of the tensor algebra $\bigoplus_{n \geq 0} V^{\otimes n}$ modulo the bilateral ideal generated by the set

$$\Big\{ u \cdot v + v \cdot u + 2g(u, v); \quad u, v \in V \Big\}.$$

We will denote this algebra by $\mathbf{Cl}(V, g)$. When no confusion is possible, we will drop the metric g from our notations. When (V, g) is the Euclidean metric space $\mathbb{R}^n$ equipped with the canonical metric g_{eucl} we write

$$\mathbf{Cl}_n := \mathbf{Cl}(\mathbb{R}^n, g_{eucl}).$$

□

We see that the symbol of the Dirac type operator $\mathcal{D}$ defines a representation of the Clifford algebra $\mathbf{Cl}(T_x^*M, g)$ on the complex Hermitian vector space E_x. We are thus forced to investigate the representations of a Clifford algebra. We need to introduce a bit of terminology.

For any elements a, b in an associative algebra A we define their anti-commutator by

$$\{a, b\} := ab + ba.$$

A *super-space* (or s-space) is a vector space E equipped with a $\mathbb{Z}/2$-grading, i.e. direct sum decomposition $E = E^+ \oplus E^-$. The elements in $E^\pm$ are called *even/odd*. . If $E = E^+ \oplus E^-$ is a s-space and $T \in \mathrm{End}(E)$ then we say that T is *even* (resp. *odd*) if

$$TE^\pm \subset E^\pm \ \ (\text{resp. } \ TE^\pm \subset E^\mp).$$

The even endomorphisms have the diagonal form

$$\left[\begin{array}{cc} A & 0 \\ 0 & B \end{array}\right]$$

and the odd endomorphisms have the anti-diagonal form

$$\left[\begin{array}{cc} 0 & C \\ D & 0 \end{array}\right].$$

We see that every endomorphism T decomposes in homogeneous components

$$T = T_{even} + T_{odd}$$

The *supertrace* (or *s-trace*) of an even endomorphism T of E is defined by

$$\mathrm{str}(T) = \mathrm{tr}(T\mid_{E^+}) - \mathrm{tr}(T\mid_{E^-}).$$

in general we set

$$\mathrm{str}\, T := \mathrm{str}\, T_{even}.$$

The *grading* of E is the operator

$$\gamma = \gamma_E = \mathbb{1}_{E^+} \oplus -\mathbb{1}_{E^-} \left[\begin{array}{cc} \mathbb{1}_{E^+} & 0 \\ 0 & -\mathbb{1}_{E^-} \end{array}\right].$$

Then

$$\mathrm{str}\, T = \mathrm{tr}(\gamma T).$$

A linear operator $T : E_0 \to E_1$ between two s-spaces is even iff $T(E_0^\pm) \subset E_1^\pm$ and odd iff $T(E_0^\pm) \subset E_1^\mp$.

A *super-algebra* over the field $\mathbb{K}$ is an associative $\mathbb{K}$-algebra $\mathcal{A}$ equipped with a $\mathbb{Z}/2$-grading, i.e. direct sum decomposition

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$$

such that

$$\mathcal{A}^+ \cdot \mathcal{A}^+ \subset \mathcal{A}^+, \ \ \mathcal{A}^+ \cdot \mathcal{A}^- \subset \mathcal{A}^-, \ \ \mathcal{A}^- \cdot \mathcal{A}^- \subset \mathcal{A}^+.$$

The elements of $\mathcal{A}^\pm$ are called even/odd. The elements in $\mathcal{A}^+ \cup \mathcal{A}^-$ are called homogeneous For $a \in \mathcal{A}^\pm$ we set

$$\mathrm{sign}(a) = \pm 1.$$

We see that if E is a $\mathbb{K}$-vector space then $\mathrm{End}_{\mathbb{K}}(E)$ is a s-algebra. We will use the notation $\widehat{\mathrm{End}}(E)$ to indicate the existence of a s-structure. . The supercommutator on a s-algebra is the bilinear map

$$[-,-]_s : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

defined by

$$[a, b]_s = ab - \operatorname{sign}(a)\operatorname{sign}(b)ba, \ \ \forall a, b \in \mathcal{A}^+ \cup \mathcal{A}^-.$$

More explicitely

$$[a^\pm, b^\pm]_s = [a^\pm, b^\pm], \ \ [a^\pm, b^\mp]_s = \{a^\pm, b^\mp\}, \forall a^\pm, b^\pm \in \mathcal{A}^\pm.$$

If $\mathcal{A}, \mathcal{B}$ are two s-algebras then their s-tensor product $\mathcal{A}\hat{\otimes}\mathcal{B}$ is defined by

$$\big(\mathcal{A}\hat{\otimes}\mathcal{B}\big)^+ = \mathcal{A}^+ \otimes \mathcal{B}^+ \oplus \mathcal{A}^- \oplus \mathcal{B}^-,$$

$$\big(\mathcal{A}\hat{\otimes}\mathcal{B}\big)^- = \mathcal{A}^+ \otimes \mathcal{B}^- \oplus \mathcal{A}^- \oplus \mathcal{B}^+,$$

and the product is defined by

$$(a_1 \otimes b_1) \cdot (a_1 \otimes b_2) = \operatorname{sign}(a_2)\operatorname{sign}(b_1)(a_1 a_2) \otimes (b_1 b_2),$$

for every homogeneous elements $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$.

If $\mathcal{A} = \widehat{\operatorname{End}}(E)$ then $\operatorname{str}([S, T]_s) = 0$, so that the supertrace is uniquely determined by the induced linear map

$$\operatorname{str} : \widehat{\operatorname{End}}(E)/[\widehat{\operatorname{End}}(E), \widehat{\operatorname{End}}(E)]_s \to \mathbb{K}.$$

A $\mathbb{Z}/2$-graded $\mathbb{K}$-module over the s-algebra $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ is a $\mathbb{K}$ super-space $E = E^+ \oplus E^-$ together with a morphism of s-algebras

$$\mathcal{A} \to \operatorname{End}(E^+ \oplus E^-).$$

Proposition 2.2.2. *Suppose (V, g) is a n-dimensional real Euclidean vector space. Then $\mathbf{Cl}(V, g)$ is a s-algebra and*

$$\dim_{\mathbb{R}} \mathbf{Cl}(V, g) = 2^n.$$

Proof Consider the isometry

$$\epsilon : V \to V, \ \ \epsilon(v) = -v.$$

It induces a morphism of algebras

$$\epsilon : \bigoplus_{k\geq 0} V^{\otimes k} \to \bigoplus_{k\geq 0} V^{\otimes k}, \quad \epsilon(v_1 \otimes \cdots \otimes v_k) = \epsilon(v_1) \otimes \cdots \otimes \epsilon(v_k) = (-1)^k v_1 \otimes \cdots \otimes v_k.$$

Clearly

$$\epsilon(u \otimes v + v \otimes u) = u \otimes v + v \otimes u$$

and since ϵ is an isometry we deduce that ϵ induces a morphism of algebras

$$\epsilon : \mathbf{Cl}(V, g) \to \mathbf{Cl}(V, g)$$

satisfying $\epsilon^2 = 1$. Define

$$\mathbf{Cl}^\pm(V, g) := \ker(\pm\mathbb{1} - \epsilon).$$

The decomposition $\mathbf{Cl}(V, g) = \mathbf{Cl}^+(V, g) \oplus \mathbf{Cl}^-(V, g)$ defines a structure of s-algebra on $\mathbf{Cl}(V, g)$.

Now choose an orthonormal basis $\{e_1, \cdots, e_n\}$ of V. Then in $\mathbf{Cl}(V, g)$ we have the equalities,

$$e_i^2 = -1, \ \ e_i e_j = -e_j e_i, \ \ \forall i \neq j.$$

For every ordered multi-index $I = (i_1 < \cdots < i_k)$ we set

$$e_I := e_{i_1} \cdots e_{i_k}, \ \ |I| = k.$$

We deduce that the collection $\{e_I\}$ spans $\mathbf{Cl}(V, g)$ so that

$$\dim_{\mathbb{R}} \mathbf{Cl}(V, g) \leq 2^n.$$

To prove the reverse inequality, we define for every $v \in V$ the endomorphism $\boldsymbol{c}(v)$ of $\Lambda^\bullet V$ by the equality

$$\boldsymbol{c}(v)\omega = (e(v) - i_{v^\sharp})\omega,$$

where $i_{v^\sharp}$ denotes the contraction with the metric dual $v^\sharp \in V^*$ of v. The Cartan formula implies

$$\boldsymbol{c}(v)^2 = -|v|^2$$

so that we have a morphism of algebras

$$\mathbf{Cl}(V, g) \to \mathrm{End}(\Lambda^\bullet V).$$

In particular we get a linear map

$$\sigma : \mathbf{Cl}(V, g) \to \Lambda^\bullet V, \;\; \mathbf{Cl}(V, g) \mapsto \boldsymbol{c}(x)1.$$

Observe that

$$\sigma(e_{i_1} \cdots e_{i_k}) = \boldsymbol{c}(e_{i_1}) \cdots \boldsymbol{c}(e_{i_k})1 = e_{i_1} \wedge \cdots \wedge e_{i_k}.$$

Since the collection $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ forms a basis of $\Lambda^\bullet V$ we deduce that σ is onto so that

$$\dim_{\mathbb{R}} \mathbf{Cl}(V, g) \geq \dim_{\mathbb{R}} \Lambda^\bullet V = 2^n.$$

In particular σ is a vector space isomorphism.

□

Definition 2.2.3. The vector space isomorphism $\sigma : \mathbf{Cl}(V, g) \to \Lambda^\bullet V$ is called the *symbol* map. □

Observe that the symbol map is an isomorphism of *super-spaces*. An orientation on V determines a canonical element Ω on $\det V$, the unique positively oriented element of length 1. In terms of an oriented orthonormal basis $(e_1, \cdots, e_n)$ we have

$$\Omega = e_1 \wedge \cdots \wedge e_n.$$

Using the symbol map we get an element

$$\Gamma := \sigma^{-1}(\Omega) = e_1 \cdots e_n$$

which satisfies the identities

$$e_i\Gamma = (-1)^{n-1}\Gamma e_i, \;\; \Gamma^2 = (-1)^{n(n+1)/2}. \tag{2.2.3}$$

We would like to investigate the structure of the $\mathbb{Z}/2$-graded complex $\mathbf{Cl}_{\mathbb{C}}(V)$-modules, or *Clifford modules*. A Clifford module is a pair (E, ρ), where E is a s-space and ρ is an even morphism of $\mathbb{Z}/2$*-graded algebras*

$$\rho : \mathbf{Cl}(V) \to \widehat{\mathrm{End}}(E).$$

The operation

$$\mathbf{Cl}(V) \times E \to E, \;\; (x, e) \mapsto \rho(x)e$$

is called the *Clifford multiplication by* x and we will denote it by $\boldsymbol{c}(x)e$. A *Clifford morphism* between the Clifford modules E_0, E_1 is a linear map $T : E_0 \to E_1$ which *super-commutes* with the Clifford action. We denote by $\widehat{\mathrm{Hom}}_{\mathbf{Cl}(V)}(E_0, E_1)$ the space of Clifford morphisms and by $\widehat{\mathrm{End}}_{\mathbf{Cl}(V)}(E)$ the s-algebra of Clifford endomorphisms of the Clifford module E.

Since we will be interested only in complex representations of $\mathbf{Cl}(V,g)$ we will study only the structure of the complexified Clifford algebra

$$\mathbf{Cl}_{\mathbb{C}}(V,g) := \mathbf{Cl}(V,g)\otimes_{\mathbb{R}}\mathbb{C}.$$

Set $V_c := V\otimes_{\mathbb{R}}\mathbb{C}$. The metric g on V extends by *complex* linearity to a $\mathbb{C}$- bilinear map

$$g_c : V_c\times V_c\to\mathbb{C}$$

Proposition 2.2.4. *Assume that* $n=\dim_{\mathbb{R}}V=2m$. *There exists an isomorphism of s-algebras*

$$\mathbf{Cl}_{\mathbb{C}}(V)\cong\widehat{\operatorname{End}}(\mathbb{S}_V),$$

where $\mathbb{S}_V$ *is complex s-space* $\mathbb{S}_n=\mathbb{S}_n^+\oplus\mathbb{S}_n^-$ *such that*

$$\dim_{\mathbb{C}}=\mathbb{S}_V^+=\dim_{\mathbb{C}}\mathbb{S}_V^-=2^{m-1}.$$

Proof Fix a complex structure on V, i.e. a skew-symmetric linear map $J:V\to V$ such that $J^2=-1$. We can find an orthonormal basis $e_1,f_1,\cdots,e_m,f_m$ of V such that

$$Je_i=f_i,\quad Jf_i=-e_i.$$

J extends to the complexification V_c since $J^2=-1$ we deduce that the eigenvalues of J on V_c are $\pm\mathbf{i}$. Denote by $V^{1,0}$ the $\mathbf{i}$-eigenspace of J and by $V^{0,1}$ the $-\mathbf{i}$-eigenspace so that

$$V_c=V^{1,0}\oplus V^{0,1}.$$

Note that V_c is equipped with an involution

$$w=v\otimes z\mapsto\bar{w}=v\otimes\bar{z}$$

and $\bar{V}^{1,0}=V^{0,1}$. Set

$$\varepsilon_j=\frac{1}{\sqrt{2}}(e_j-\mathbf{i}f_j)\in V^{1,0},\quad\bar{\varepsilon}_j=\frac{1}{\sqrt{2}}(e_j+\mathbf{i}f_j)\in V^{0,1}.$$

The collection (ε_j) is a $\mathbb{C}$-basis of $V^{1,0}$. Note that

$$g_c(\varepsilon_i,\bar{\varepsilon}_j)=g_c(\bar{\varepsilon}_j,\varepsilon_i)=\delta_{ij},\quad g_c(\varepsilon_i,\varepsilon_j)=g_c(\bar{\varepsilon}_i,\bar{\varepsilon}_j)=0,$$

$$e_j=\frac{1}{\sqrt{2}}(\varepsilon_j+\bar{\varepsilon}_j),\quad f_i=\frac{1}{\sqrt{2}}(\varepsilon_j+\bar{\varepsilon}_j)=\frac{1}{\sqrt{2}}(\varepsilon_j+\overline{\mathbf{i}\varepsilon_j}).$$

Define

$$\mathbb{S}_V:=\Lambda^{\bullet}V^{1,0}.$$

We want to define a representation of $\mathbf{Cl}_{\mathbb{C}}(V)$ on $\mathbb{S}_V$, that is, we need to produce a $\mathbb{C}$-linear map

$$\boldsymbol{c}:V_c\to\operatorname{End}_{\mathbb{C}}(\mathbb{S}_V)$$

such that

$$\boldsymbol{c}(v)^2=-g_c(v,v),\quad\forall v\in V_c.$$

Since $V_c=V^{1,0}\oplus V^{0,1}$ it suffices to describe how the elements in $V^{1,0}$ and the elements of $V^{0,1}$ act on $\mathbb{S}_n$. For every $w\in V^{1,0}$ we set

$$\boldsymbol{c}(w)=\sqrt{2}e(w),$$

where $e(w)$ denotes the exterior multiplication by w on $\Lambda^\bullet V^{1,0}$. For every $w \in V^{1,0}$ we have $\bar{w} = V^{0,1}$ and define the contraction

$$i(\bar{w}) : \Lambda^\bullet V^{1,0} \to \Lambda^{\bullet-1} V^{1,0}$$

by

$$i(\bar{w})(w_1 \wedge \cdots w_k) = g_c(\bar{w}, w_1) w_2 \wedge \cdots \wedge w_k - g_c(\bar{w}, w_2) w_1 \wedge w_3 \wedge \cdots \wedge w_k$$
$$+ \cdots + (-1)^{k-1} g_c(\bar{w}, w_k) w_1 \wedge \cdots \wedge w_{k-1}.$$

Now set

$$c(\bar{w}) = -\sqrt{2}\, i(\bar{w}).$$

We have the identity

$$c(w_0 + \bar{w}_1)^2 = -2\big(g_c(w_0, \bar{w}_1) + g_c(\bar{w}_1, w_0) \big), \;\; \forall w_0, w_1 \in V^{1,0}.$$

Using the identity

$$g_c(w, w) = 0 \;\; \forall w \in V^{1,0}$$

we deduce

$$g_c(w_0 + \bar{w}_1, w_0 + \bar{w}_1) = g_c(w_0, \bar{w}_1) + g_c(\bar{w}_1, w_0)$$

so that

$$c(w_0 + \bar{w}_1)^2 = -2 g_c(w_0 + \bar{w}_1, w_0 + \bar{w}_1), \;\; \forall w_0, w_1 \in V^{1,0}.$$

Hence we have produced a morphism of algebras

$$\boldsymbol{c} : \mathbf{Cl}_{\mathbb{C}}(V) \to \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet V^{1,0}).$$

The space $\Lambda^\bullet V^{1,0}$ is $\mathbb{Z}/2$-graded

$$\Lambda^\bullet V^{1,0} = \Lambda^{even} V^{1,0} \oplus \Lambda^{odd} V^{1,0}$$

and clearly $\boldsymbol{c}$ maps even/odd elements of $\mathbf{Cl}_{\mathbb{C}}(V)$ to even/odd elements of $\mathrm{End}_{\mathbb{C}}(\Lambda^\bullet V^{1,0})$. Note that

$$\boldsymbol{c}(\varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_k}) 1 = 2^{k/2} \varepsilon_{i_1} \wedge \varepsilon_{i_2} \wedge \cdots \wedge \varepsilon_{i_k}$$

so that

$$\boldsymbol{c}(\varepsilon_{i_1} \varepsilon_{i_2} \cdots \varepsilon_{i_k}) \neq 0 \in \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet V^{1,0}).$$

Hence the map $\boldsymbol{c} : \mathbf{Cl}_{\mathbb{C}}(V) \to \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet V^{1,0})$ is onto. Now observe that

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(\Lambda^\bullet V^{1,0}) = \big(\dim_{\mathbb{C}} \Lambda^\bullet V^{1,0}\big)^2 = (2^{\dim_{\mathbb{C}} V^{1,0}})^2 = 2^{2m} = 2^n = \dim_{\mathbb{C}} \mathbf{Cl}_{\mathbb{C}}(V),$$

which shows that $\boldsymbol{c}$ is an isomorphism.

□

The space $\mathbb{S}_V$ constructed above can be naturally regarded as a $\mathbb{Z}/2$-graded complex $\mathbf{Cl}(V)$-module.

Definition 2.2.5. Assume $\dim V$ is even. The $\mathbf{Cl}(V)$ complex **module** $\mathbb{S}_V$ constructed in Proposition 2.2.4 is called the space of complex spinors. The corresponding representation

$$\boldsymbol{c} : \mathbf{Cl}_{\mathbb{C}}(V) \to \widehat{\mathrm{End}}(\mathbb{S}_V)$$

is called the *complex spinorial representation*. □

We have shown that, if we forget the grading, the Clifford algebra $\mathbf{Cl}_{2m} \otimes \mathbb{C}$ is isomorphic to an algebra of matrices, $\mathrm{End}(\mathbb{S}_{2m})$ and the representations of such an algebra are well understood. Let us describe a simple procedure of constructing $\mathbb{Z}/2$-graded complex representations of $\mathbf{Cl}_{2m}$.

Suppose $W = W^+ \oplus W^-$ is a complex s-space. Denote by $\mathbb{S}_V \hat{\otimes} W$ the vector space $\mathbb{S}_{2m} \otimes W$ equipped with the $\mathbb{Z}/2$-grading

$$(\mathbb{S}_V \hat{\otimes} W)^+ = \mathbb{S}_V^+ \otimes W^+ \oplus \mathbb{S}_V^- \otimes W^-, \;\; (\mathbb{S}_V \hat{\otimes} W)^- = \mathbb{S}_V^+ \otimes W^- \oplus \mathbb{S}_V^- \otimes W^+.$$

We define the *complex spinorial representation twisted by* W to be

$$\boldsymbol{c}_W : \mathbf{Cl}_{\mathbb{C}}(V) \to \widehat{\mathrm{End}}_{\mathbb{C}}(\mathbb{S}_V \hat{\otimes} W), \;\; \boldsymbol{c}_W(x)(\psi \otimes w) = (\boldsymbol{c}(x)\psi) \otimes w, \;\; \forall \psi \in \mathbb{S}_V, \;\; w \in W.$$

Observe that each $w \in W$ defines a morphism of $\mathbf{Cl}_{\mathbb{C}}(V)$-**modules**

$$T_W(w) : \mathbb{S}_V \to \mathbb{S}_V \hat{\otimes} W, \;\; \psi \mapsto \psi \otimes w$$

and thus we get a linear map

$$T_W : W \to \widehat{\mathrm{Hom}}_{\mathbf{Cl}_{\mathbb{C}}(V)}(\mathbb{S}_V, \mathbb{S}_V \hat{\otimes} W).$$

Similarly every linear map $\Phi : W \to W$ defines a morphism of $\mathbf{Cl}_{\mathbb{C}}(V)$-**modules**

$$S_\Phi : \mathbb{S}_V \hat{\otimes} W \to \mathbb{S}_V \hat{\otimes} W, \;\; \psi \otimes w \mapsto \psi \otimes \Phi(w).$$

We obtain in this fashion a linear map

$$S : \widehat{\mathrm{End}}_{\mathbb{C}}(W) \to \widehat{\mathrm{End}}_{\mathbf{Cl}_{\mathbb{C}}(V)}(\mathbb{S}_V \hat{\otimes} W, \mathbb{S}_V \hat{\otimes} W).$$

The representation theory of algebras (see [13, Chap. XVII] or [22, Chap. 14]) implies the following result.

Proposition 2.2.6. *Suppose E is a $\mathbb{Z}/2$-graded $\mathbf{Cl}_{\mathbb{C}}(V)$-module, $\dim_{\mathbb{R}} V = 2m$. Then E is isomorphic as a $\mathbb{Z}/2$-graded $\mathbf{Cl}_{\mathbb{C}}(V)$-modules with the complex spinorial module twisted by the s-space*

$$W = \widehat{\mathrm{Hom}}_{\mathbf{Cl}_{\mathbb{C}}}(\mathbb{S}_V, E).$$

Moreover, we have an isomorphism of s-algebras

$$\mathbf{Cl}_{\mathbb{C}}(V) \hat{\otimes} \widehat{\mathrm{End}}_{\mathbb{C}}(W) \to \widehat{\mathrm{End}}_{\mathbb{C}}(E).$$

Via this isomorphism, we can identify $\widehat{\mathrm{End}}_{\mathbb{C}}(W)$ with the subalgebra of $\widehat{\mathrm{End}}_{\mathbb{C}}(E)$ consisting of endomorphism $T : E \to E$ supercommuting with the Clifford action

$$[T, \boldsymbol{c}(u)]_s = 0, \;\; \forall u \in \mathbf{Cl}_{\mathbb{C}}(V).$$

In other words

$$\widehat{\mathrm{End}}_{\mathbf{Cl}(V)}(E) \cong \widehat{\mathrm{End}}_{\mathbb{C}}(W).$$

The s-space $\mathrm{Hom}_{\mathbf{Cl}_{\mathbb{C}}}(\mathbb{S}_V, E)$ is called the *twisting space* of the Clifford module E and will be denoted by $E/\mathbb{S}$. We deduce from the above result that given a Clifford module E we can identify a Clifford endomorphism $L : E \to E$ with a linear map $L/\mathbb{S}$ on the twisting space $E/\mathbb{S}$, i.e.

$$\widehat{\mathrm{End}}_{\mathbf{Cl}(V)}(E) \cong \widehat{\mathrm{End}}_{\mathbb{C}}(E/\mathbb{S}), \;\; L \mapsto L/\mathbb{S}.$$

Definition 2.2.7. The *relative supertrace* of an endomorphism $L \in \widehat{\mathrm{End}}_{\mathbf{Cl}(V)}(E)$ of a $\mathbb{Z}/2$-graded complex $\mathbf{Cl}_{2m}$-module E is the scalar $\mathrm{str}_{E/\mathbb{S}} L$ defined as the supertrace of the linear operator $L/\mathbb{S}$,

$$\mathrm{str}_{E/\mathbb{S}} L := \mathrm{str}\, L/\mathbb{S}.$$

□

Suppose E is $\mathbf{Cl}_{\mathbb{C}}(V)$-module so that we can represent it as a twist of $\mathbb{S}_V$ with s-space W. We would like to relate the relative supertrace

$$\mathrm{str}^{E/\mathbb{S}} : \widehat{\mathrm{End}}_{\mathbb{C}}(W) \to \mathbb{C}.$$

to the absolute supertrace

$$\mathrm{str}^{E} : \widehat{\mathrm{End}}_{\mathbb{C}}(E) \to \mathbb{C}.$$

Suppose $F : E \to E$ is a linear map. By choosing a basis of W we can represent it as a matrix with coefficients in $\mathbf{Cl}_{\mathbb{C}}(V)$. Equivalently, we can regard F as an element of $\mathbf{Cl}_{\mathbb{C}}(V)\hat{\otimes}\widehat{\mathrm{End}}(W) = \widehat{\mathrm{End}}(\mathbb{S}_V)\hat{\otimes}\widehat{\mathrm{End}}(W)$ and we can write

$$F = \sum_{\ell} u_\ell \otimes F_\ell, \;\; u_\ell \in \widehat{\mathrm{End}}(\mathbb{S}_V), \;\; F_\ell \in \widehat{\mathrm{End}}(W).$$

We would like to compute $\mathrm{str}(F : E \to E)$. By linearity we have

$$\mathrm{str}(F) = \sum_{\ell} \mathrm{str}(u_\ell \otimes F_\ell).$$

Choose orthonormal bases $w_i^{\pm}$ in $W^{\pm}$ and orthonormal bases $\psi_j^{\pm}$ in $\mathbb{S}_V^{\pm}$. Define a metric on $\mathbf{Cl}_{\mathbb{C}}(V)$ by declaring the basis (e_I) orthonormal. Then

$$\langle (u_\ell \otimes F_\ell)(\psi_i^{\pm} \otimes w_j^{\pm}), \psi_i^{\pm} \otimes w_j^{\pm} \rangle = \langle u_\ell \psi_i^{\pm}, \psi_i^{\pm} \rangle \langle F_\ell w_j^{\pm}, w_j^{\pm} \rangle$$

It follows from this equality that

$$\mathrm{str}(u_\ell \otimes F_\ell) = \mathrm{str}\big(u_\ell : \mathbb{S}_V \to \mathbb{S}_V\big) \cdot \mathrm{str}(F_\ell : W \to W).$$

Thus we need to compute the supertrace of the action of an element in the Clifford algebra on the complex spinorial space. This supertrace is uniquely determined by the induced linear map

$$\mathbf{Cl}_{\mathbb{C}}(V)/[\mathbf{Cl}_{\mathbb{C}}(V), \mathbf{Cl}_{\mathbb{C}}(V)]_s = \widehat{\mathrm{End}}(\mathbb{S}_V)/[\widehat{\mathrm{End}}(\mathbb{S}_V), \widehat{\mathrm{End}}(\mathbb{S}_V)]_s \to \mathbb{C}.$$

It turns out that the space $\mathbf{Cl}_{\mathbb{C}}(V)/[\mathbf{Cl}_{\mathbb{C}}(V), \mathbf{Cl}_{\mathbb{C}}(V)]_s$ is quite small. Choose an orthonormal basis (e_i) of V. Fix $1 \le \ell \le \dim V$. Observe that for every multi-index $I = (1 \le i_1 < \cdots < i_{k-1} \le n)$ $i_j \neq \ell$ we have

$$[e_\ell, e_\ell e_I] = e_\ell^2 e_I - (-1)^k e_\ell e_I e_\ell = 2e_\ell^2 e_I = -2e_I \Longleftrightarrow e_I = [e_\ell, -\frac{1}{2} e_I]_s$$

By linearity we conclude that any monomial e_I, $|I| < \dim V$ is a s-commutator. Hence the only monomial e_I that could have nontrivial s-trace is $\Gamma = e_1 \cdots e_{2m}$. To compute its s-trace as a linear map on $\mathbb{S}_V$ we choose a complex structure J on V and an orthonormal basis $e_1, f_1, \cdots, e_m, f_m$ such that

$$f_i = Je_i, \;\; e_i = -Jf_i.$$

Consider as before

$$\varepsilon_j = \frac{1}{\sqrt{2}}(e_j - \mathbf{i} f_j) \in V^{1,0}, \;\; \bar{\varepsilon}_j = \frac{1}{\sqrt{2}}(e_j + \mathbf{i} f_j) \in V^{0,1}.$$

Then,

$$e_i = \frac{1}{\sqrt{2}}(\varepsilon_i + \bar{\varepsilon}_i), \ \ f_i = \frac{\mathbf{i}}{\sqrt{2}}(\varepsilon_i - \bar{\varepsilon}_i),$$

$$\boldsymbol{c}(e_j) = e(\varepsilon_j) - i(\bar{\varepsilon}_j), \ \ \boldsymbol{c}(f_j) = \mathbf{i}(e(\varepsilon_j) + i(\bar{\varepsilon}_j)),$$

and

$$\Gamma = \prod_{j=1}^{m} \boldsymbol{c}(e_j)\boldsymbol{c}(f_j) = \mathbf{i}^m \prod_{j=1}^{m} (e(\varepsilon_j) - i(\bar{\varepsilon}_j))(e(\varepsilon_j) + i(\bar{\varepsilon}_j)) = \mathbf{i}^m \prod_{s=1}^{m} \big(e(\varepsilon_j)i(\bar{\varepsilon}_j) - i(\bar{\varepsilon}_j)e(\varepsilon_j) \big).$$

For a multi-index $\varepsilon_J = \varepsilon_{j_1} \wedge \cdots \wedge \varepsilon_{j_k} \in \Lambda^k V^{1,0}$ we have

$$e(\varepsilon_j)i(\bar{\varepsilon}_j)\varepsilon_J = \begin{cases} \varepsilon_J & \text{if} \ \ j \in J \\ 0 & \text{if} \ \ j \notin J \end{cases}, \ \ i(\bar{\varepsilon}_j)e(\varepsilon_j)\varepsilon_J = \begin{cases} \varepsilon_J & \text{if} \ \ j \notin J \\ 0 & \text{if} \ \ j \in J \end{cases}.$$

Putting these two facts together we deduce

$$\big(e(\varepsilon_j)i(\bar{\varepsilon}_j) - i(\bar{\varepsilon}_j)e(\varepsilon_j) \big)\varepsilon_J = \begin{cases} \varepsilon_J & \text{if} \ \ j \in J \\ -\varepsilon_J & \text{if} \ \ j \notin J \end{cases}.$$

Hence

$$\Gamma\varepsilon_J = \underbrace{\mathbf{i}^m(-1)^{m-|J|}}_{:=\langle \varepsilon_J | \Gamma | \varepsilon_J \rangle} \varepsilon_J,$$

and thus

$$\operatorname{str}\Gamma = \sum_J (-1)^{|J|} \langle \varepsilon_J | \Gamma | \varepsilon_J \rangle = \sum_J (-1)^{|J|} \mathbf{i}^m (-1)^{m-|J|} = (-\mathbf{i})^m \sum_J 1 = (-2\mathbf{i})^m.$$

Let us summarize what we have proved so far.

Assume V is oriented. The orientation and the metric g determine a canonical section of $\det V$, the volume form Ω_g. For every $\omega \in \Lambda^\bullet V \otimes \mathbb{C}$ we denote by $[\omega]_k \in \Lambda^k V \otimes \mathbb{C}$ its degree k component. We then define $\langle \omega \rangle \in \mathbb{C}$ by the equality

$$[\omega]_n = \langle \omega \rangle \Omega_g.$$

We have thus established the following result.

Proposition 2.2.8. *Assume* $\dim V = 2m$. *If* $u \in \mathbf{Cl}_{\mathbb{C}}(V)$, W *is a s-space and* $F \in \widehat{\operatorname{End}}(W)$ *then*

$$\operatorname{str}(u \otimes F : \mathbb{S}_V \hat{\otimes} W \to \mathbb{S}_V \hat{\otimes} W) = (-2\mathbf{i})^m \langle \sigma(u) \rangle \operatorname{str} F.$$

In the above equality, both sides depend on a choice of an orientation on V. □

The above results have the following immediate consequence.

Corollary 2.2.9. *Suppose* (V, g) *is oriented and* $\dim_{\mathbb{R}} V = 2m$. *Then for any Clifford module* E *and any endomorphism of Clifford modules* $L : E \to E$ *we have*

$$\operatorname{str}^{E/\mathbb{S}} L = \frac{\mathbf{i}^m}{2^m} \operatorname{str}^E(\Gamma L),$$

where in the right-hand-side of the above equality we regard ΓL *as a morphism of* $\mathbb{C}$-vector spaces.

□

Remark 2.2.10. Let us say a few words about the odd dimensional case. If V is an odd dimensional vector space and $U := \mathbb{R} \oplus V$ then we have a natural isomorphism of algebras

$$\mathbf{Cl}(V) \to \mathbf{Cl}^{even}(U), \;\; \mathbf{Cl}^{even}(V) \oplus \mathbf{Cl}^{odd}(V) \ni x_0 \oplus x_1 \mapsto x_0 + e_0 x_1,$$

where e_0 denotes the canonical basic vector of the summand $\mathbb{R}$ of U. We can then prove that we have an isomorphism of algebras

$$\mathbf{Cl}_{\mathbb{C}}(V) \cong \mathrm{End}(\mathbb{S}_U^+) \oplus \mathrm{End}_{\mathbb{C}}(S_U^-).$$

For more details we refer to [14]. □

2.2.2. $Spin$ **and** $Spin^c$**.** Suppose (V, g) is an oriented finite dimensional Euclidean space. Recall that we have a vector space isomorphism

$$\sigma : \mathbf{Cl}(V) \to \Lambda^\bullet V$$

called the symbol map. Its inverse is called the *quantization map* and it is denote by $\mathfrak{q}$. Set

$$\underline{spin}(V) := \mathfrak{q}(\Lambda^2 V) \subset \mathbf{Cl}(V).$$

If $e_1, \cdots, e_n$ is an orthonormal basis of V then $\{e_i e_j; \;\; 1 \le i < j \le n\}$ is an orthonormal basis of $\underline{spin}(V)$. Observe that

$$[e_i e_j, e_k] = e_i e_j e_k - e_k e_i e_k = e_i(-2\delta_{jk} - e_k e_j) - e_k e_i e_k$$
$$= -2\delta_{jk} e_i - (-2\delta_{ik} - e_k e_i) e_j - e_k e_i e_k = -2\delta_{jk} e_i + 2\delta_{ik} e_j.$$

Hence

$$[\omega, v] \in V, \;\; \forall \omega \in \underline{spin}(V), \;\; v \in V.$$

Using the identity

$$[e_i e_j, e_j e_k] = [e_i e_k, e_k] e_\ell + e_j [e_i e_j, e_\ell]$$

we deduce

$$[e_i e_j, e_k e_\ell] \in \underline{spin}(V), \;\; \forall i < j, \;\; k < \ell,$$

which shows that $\underline{spin}(V)$ is a Lie algebra with respect to the commutator in $\mathbf{Cl}(V)$.

The Jacobi identity shows that we have morphism of Lie algebras

$$\tau : \underline{spin}(V) \to \mathrm{End}(V), \;\; \tau(\eta) v = [\eta, v]. \tag{2.2.4}$$

Observe that

$$g(\tau(e_i e_j) e_k, e_\ell) = -g(e_k, \tau(e_i e_j) e_\ell)$$

so that $\tau(\eta)$ is skew symmetric $\forall \eta \in \underline{spin}(V)$, i.e. $\tau(\eta) \in \underline{so}(V)$. Note that

$$\tau(e_i e_j) = 2X_{ij},$$

where X_{ij}

$$X_{ij} e_i = e_j, \;\; X_{ij} e_j = -e_i, \;\; X_{ij} e_k = 0, \;\; \forall k \neq i, j.$$

This implies that τ is injective. On the other hand

$$\dim_{\mathbb{R}} \underline{spin}(V) = \dim_{\mathbb{R}} \Lambda^2 V = \dim_{\mathbb{R}} \underline{so}(V)$$

so that τ is an isomorphism.

To every $A \in \underline{so}(V)$ we associate $\omega_A \in \Lambda^2 V$

$$\omega_A = \sum_{i<j} g(A e_i, e_j) e_i \wedge e_j.$$

Observe that

$$A = \sum_{i<j} g(Ae_i, e_j) X_{ij}$$

and thus

$$\tau^{-1}(A) = \sum_{i<j} g(Ae_i, e_j)\tau^{-1}(X_{ij}) = \frac{1}{2}\sum_{i<j} g(Ae_i, e_j)e_i e_j = \frac{1}{2}\,\mathfrak{q}(\omega_A). \tag{2.2.5}$$

Definition 2.2.11.

$$Spin(V) := \Big\{ u \in \mathbf{Cl}^{even}(V);\ \ u = v_1 \cdots v_{2k},\ \ v_i \in V,\ \ |v_i|_g = 1 \Big\}.$$

In particular,

$$Spin(n) := Spin(\mathbb{R}^n).$$

□

For any $e, v \in V$, $|e|_g = 1$ we have

$$eve^{-1} = -eve = -e(-2g(e,v) + ev) = v - 2g(e,v)e \in V.$$

Hence the map $\rho_e : V \to V$, $v \mapsto eve^{-1}$ is described by the orthogonal reflection in the hyperplane through the origin orthogonal to e. In particular, it is an orthogonal transformation of V with determinant -1. We deduce that we have a natural morphism of groups

$$\rho : Spin(V) \to SO(V),\ \ u \mapsto \rho_u,\ \ \rho_u(v) = u \cdot v \cdot u^{-1}.$$

Lemma 2.2.12. *The morphism ρ is surjective and*

$$\ker \rho \cong \{\pm 1\} \subset Spin(V).$$

Proof The surjectivity follows from the classical fact that any orthogonal transformation is a product of reflections. If $\eta \in \ker \rho$ then

$$\eta v = v \eta, \quad \forall v \in V$$

from which we conclude that u lies in the center of $\mathbf{Cl}(V)$. Choose an orthonormal basis $e_1, \cdots, e_n$ of V so we can write

$$\eta = \sum_I \eta_I e_I,\ \ u_I \in \mathbb{R},$$

and the sum is carried over all even dimensional ordered multi-indices I. Since η commutes with e_k the multi-indices I such that $\eta_I \neq 0$ cannot contain k. Since this happens for all k the above sum should contain only the empty multiindex for which $e_\emptyset = 1$. Hence η must be a scalar, $\eta \in \mathbb{R}$.

To show that $|\eta| = 1$ we consider the representation

$$\boldsymbol{c} : \mathbf{Cl}(V) \to \mathrm{End}(\Lambda^\bullet V).$$

The metric on V induces a metric on $\Lambda^\bullet V$ and thus for every $u \in \mathbf{Cl}(V)$ the linear map

$$\boldsymbol{c}(u) : \Lambda^\bullet V \to \Lambda^\bullet V$$

has a well defined norm $\|\boldsymbol{c}(u)\|$. Moreover

$$\|\boldsymbol{c}(u_1 u_2)\| \leq \|\boldsymbol{c}(u_1)\| \cdot \|\boldsymbol{c}(u_2)\|.$$

Observe that if $v \in V$ is a vector of length one then $\|\boldsymbol{c}(v)\| = 1$. We deduce that $\|\boldsymbol{c}(u)\| \leq 1$ for all $u \in Spin(V)$. In particular $\eta, \eta^{-1} \in Spin(V) \cap \mathbb{R}$ and we deduce

$$|\eta| = \|\boldsymbol{c}(\eta)\| \leq 1,\ \ |\eta^{-1}| = \|\boldsymbol{c}(\eta^{-1})\| \leq 1.$$

Hence $|\eta| = 1$. This completes the proof.

□

We have produced a $2:1$ continuous group morphism

$$\rho : Spin(V) \to SO(V).$$

This shows that ρ is a covering map.

Proposition 2.2.13. *$Spin(V)$ is connected if $\dim V > 1$ and simply connected if $\dim V > 2$.*

Proof Suppose $\dim V \geq 2$. Let us first show that for any $u, v \in V$, $|u| = |v| = 1$, there exists a smooth path $\gamma : [0,1] \to \mathbf{Cl}^{even}(V)$, $\gamma(0) = 1$, $\gamma(1) = uv$, $\gamma(t) \in Spin(V)$, for all $t \in [0,1]$. Choose $w \in W$ such that

$$|w| = 1, \;\; w \perp v, \;\; u, v \in \operatorname{span}(u, w).$$

Then we can write

$$v = (\cos\theta)(-u) + (\sin\theta)w \Longrightarrow uv = \cos\theta + (\sin\theta)uw.$$

Observe that

$$(uw)^2 = -1 \Longrightarrow \exp(\theta uv) = \cos\theta + (\sin\theta)uw.$$

Now observe that

$$\exp(t\theta uw) \in Spin(V)$$

since

$$\exp(t\theta uw) = \cos t + (\sin t)uw = (u \sin t/2 - w \cos t/2)(u \sin t/2 + w \cos t/2) \in Spin(V).$$

The above argument shows that every element $x \in Spin(V)$ can be written as a product

$$x = \exp(t_1 u_1 w_1) \cdots \exp(t_k u_k w_k), \;\; |u_i| = |w_i| = 1, \;\; u_i \perp w_i, \;\; t_i \in \mathbb{R},$$

and thus x lies in the same path component of $Spin(V)$ as 1.

Suppose $\dim V \geq 3$. Consider first the case $\dim V = 3$. Fix an orthonormal basis e_1, e_2, e_3 and set

$$f_1 = e_2 e_3, \;\; f_2 = e_3 e_1, \;\; f_3 = e_1 e_2.$$

Then $f_i^2 = -1$, $f_i f_j = -f_j f_i$, $i \neq j$. We deduce that

$$\mathbf{Cl}_3^{even} \cong \mathbb{H} = \text{the division ring of quaternions}.$$

We want to prove that $Spin(3)$ can be identified with the group of quaternions of norm 1. Suppose that

$$q = a + x, \;\; x = bf_1 + cf_2 + df_3 \neq 0, \;\; a^2 + b^2 + c^2 + d^2 = 1.$$

Then we can write

$$q = \cos\theta + \sin\theta y, \;\; y = \frac{1}{|x|}x.$$

and thus

$$q = \exp(\theta y), \;\; \theta y \in \underline{spin}(V).$$

Hence every quaternion of norm 1 can be written as the exponential of an element in $\underline{spin}(V)$. We can now see that every $z \in \underline{spin}(V)$ can be written as a product

$$z = uv, \;\; u, v \in V, \;\; u, v \in V \setminus 0, \;\; u \perp v.$$

More precisely if $z = af_1 + bf_2 + cf_3$ then we choose u, v such that $u \perp v$ and

$$u \times v = ae_1 + be_2 + ce_3 \in V,$$

where $\times$ denotes the cross product. Hence every unit quaternion can be written as an exponential $\exp(uv)$ where u, v are two nonzero orthogonal vectors in V. As we have seen before any such element belongs to $Spin(3)$. Hence $Spin(3)$ contains the group of unit quaternions. Conversely, every element in $Spin(3)$ can be written as a product of exponentials $\exp(uv)$ as above, i.e. as a product of unit quaternions. Hence $Spin(3)$ is contained in the group of unit quaternions.

This proves our claim and shows that $Spin(3)$ is simply connected. From the $2:1$ nontrivial cover $Spin(3) \to SO(3)$ we deduce that $SO(3) \cong \mathbb{RP}^3$ and $\pi_1(SO(3)) \cong \mathbb{Z}/2$.

Using the homotopy long exact sequence of the fibration $SO(n) \hookrightarrow SO(n+1) \twoheadrightarrow S^n$, $n \geq 3$ we obtain the exact sequence

$$0 = \pi_2(S^n) \to \pi_1(SO(n)) \to \pi_1(SO(n+1)) \to \pi_1(S^n) = 0.$$

Hence

$$\pi_1(SO(n)) \cong \pi_1(SO(3)) \cong \mathbb{Z}/2, \ \ \forall n \geq 3.$$

This implies that the covering $Spin(n) \to SO(n)$ is the universal covering of $Spin(n)$, and in particular $Spin(n)$ is simply connected.

□

Define

$$Spin^c(V) := Spin(V) \times S^1/(\mathbb{Z}/2),$$

where $\mathbb{Z}/2$ is identified with the subgroup $\{(1,1), (-1,-1)\} \subset Spin(V) \times S^1$. Observe that we have natural map

$$Spin(V) \to Spin^c(V)$$

and a short exact sequence

$$1 \to \mathbb{Z}/2 \to Spin^c(V) \xrightarrow{\rho^c} SO(V) \times S^1 \to 1,$$

where

$$Spin(V) \times S^1/(\mathbb{Z}/2) \ni [g, z] \stackrel{\rho^c}{\longmapsto} (\rho(g), z^2).$$

Suppose V is even dimensional, and J is a complex structure on V, i.e. a skew-symmetric operator such that $J^2 = -\mathbb{1}_V$. We denote by $U(V, J)$ the group of isometries of V which commute with J. We have a tautological morphisms

$$i : U(V, J) \hookrightarrow SO(V), \ \ \rho^c : Spin^c(V) \to SO(V) \times S^1 \twoheadrightarrow SO(V).$$

Proposition 2.2.14. *There exists a morphism*

$$\Phi = \Phi_{V,J} : U(V, J) \to Spin^c(V)$$

such that the diagram below is commutative.

$$\begin{array}{ccc} & & Spin^c(V) \\ & \overset{\Phi}{\nearrow} & \downarrow \rho^c \\ U(V,J) & \xrightarrow{\ i\ } & SO(V) \end{array} .$$

Sketch of proof We have a natural group morphism

$$\det : U(n) \to S^1, \;\; g \mapsto \det g$$

which induces an isomorphism

$$\det_* : \pi_1(U(1)) \to \pi_1(S^1) \cong \mathbb{Z}.$$

Consider the group morphism

$$\phi : U(V, J) \to SO(V) \times S^1, \;\; g \mapsto (i(g), \det(g)).$$

Observe that

$$\pi_1(SO(V) \times S^1) \cong \pi_1(SO(V)) \times \pi_1(S^1) \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z} & \text{if} \;\; \dim V > 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if} \;\; \dim V = 2 \end{cases}$$

Denote by ϕ_* the induced morphisms

$$\phi_* : \pi_1(U(V, J)) = \mathbb{Z} \to \mathbb{Z} = \pi_1(SO(V) \times S^1).$$

We have the following fact whose proof is left as an exercise.

Lemma 2.2.15. *The image of ϕ_* coincides with the image of*

$$\rho^c_* : \pi_1(Spin^c(V)) \to \pi_1(SO(V) \times S^1).$$

□

The above lemma implies that ϕ admits a unique lift $\Phi : U(V, J) \to Spin^c(V)$ such that $\Phi(\mathbb{1}) = [\mathbb{1}, \mathbb{1}]$. This is the morphism with the required properties.

□

2.2.3. Geometric Dirac operators. Suppose (M, g) is a compact oriented, n-dimensional Riemann manifold. We denote by $\mathbf{Cl}(M)$ the bundle over M whose fiber over $x \in M$ is the Clifford algebra $\mathbf{Cl}(T^*_x M, g)$.

To construct it, we first produce the principal $SO(n)$-bundle P_M of oriented, orthonormal frames of T^*M. Then observe that there is a canonical morphism

$$\rho : SO(n) \to \operatorname{Aut}(\mathbf{Cl}_n) = \text{the group of automorphism of the Clifford algebra } \mathbf{Cl}_n.$$

Then

$$\mathbf{Cl}(M) = P_M \times_\rho \mathbf{Cl}_n \, .$$

We will refer to $\mathbf{Cl}(M)$ as the *Clifford bundle* of (M, g). Note that we have a Clifford multiplication

$$\cdot : \mathbf{Cl}(M) \oplus \mathbf{Cl}(M) \to \mathbf{Cl}(M),$$

and a canonical inclusion

$$T^*M \hookrightarrow \mathbf{Cl}(M).$$

The symbol map

$$\mathbf{Cl}(V) \to \Lambda^\bullet V^*$$

induces an isomorphism of *vector bundles*

$$\sigma : \mathbf{Cl}(M) \to \Lambda^\bullet T^*M.$$

Definition 2.2.16. (a) A *s-bundle* over M is a vector bundle $E \to M$ together with a direct sum decomposition

$$E = E^+ \oplus E^-.$$

The grading of the s-bundle E is the endomorphism $\gamma = \mathbb{1}_{E^+} \oplus -\mathbb{1}_{E^-}$.
(b) A *Clifford bundle* (or $\mathbf{Cl}(M)$-module) is a *hermitian* s-bundle together with a morphism

$$\boldsymbol{c} : \mathbf{Cl}(M) \to \widehat{\mathrm{End}}(E)$$

which on each fiber is a morphism of s-algebras and such that for every $x \in M$, $\alpha \in T_x^*M \subset \mathbf{Cl}(M)_x$ the endomorphism

$$\boldsymbol{c}(\alpha) : E_x \to E_x$$

is (odd) and skew-symmetric. We will refer to $\boldsymbol{c}(-)$ as *the Clifford multiplication*.
(c) A *Dirac bundle* over M is a pair (E, ∇^E), where $E = E^+ \oplus E^-$ is a Clifford bundle and ∇^E is a hermitian connection on E which preserves the $\mathbb{Z}/2$ grading and it is compatible with the Clifford multiplication, i.e.

$$\forall X \in \mathrm{Vect}(M),\ \ \alpha \in \Omega^1(M),\ \ u \in C^\infty(E):\ \ \nabla^E_X(\boldsymbol{c}(\alpha)u) = \boldsymbol{c}(\nabla^g_X \alpha)u + \boldsymbol{c}(\alpha)\nabla^E_X u,$$

where ∇^g denotes the Levi-Civita connection on T^*M.

Suppose (E, ∇^E) is a Dirac bundle. Then $F_\nabla \in \Omega^2(\operatorname{End} E)$. Recall that we have an isomorphism

$$\widehat{\mathrm{End}}(E) \cong \mathbf{Cl}(M) \hat{\otimes} \operatorname{End}_{\mathbf{Cl}(M)}(E).$$

Hence we can view the curvature of ∇^E as a section of

$$F_\nabla \in \Omega^2(\mathbf{Cl}(M) \hat{\otimes} \operatorname{End}_{\mathbf{Cl}(M)}(E)).$$

On the other hand, the curvature R of the Levi-Civita connection is a section

$$R \in \Omega^2(\underline{so}(TM)),$$

where $\underline{so}(TM)$ denotes the space of skew-symmetric endomorphisms of TM. On the other hand we have a map

$$\delta : \underline{so}(TM) \xrightarrow{\ \sharp\ } \underline{so}(T^*M) \xrightarrow{\ \tau^{-1}\ } \mathbf{Cl}(M),$$

where $^\sharp : TM \to T^*M$ denotes the metric duality isomorphism. Via this isomorphism we can identify the curvature R with a section

$$\boldsymbol{c}(R) \in \Omega^2(\mathbf{Cl}(M)) \subset \Omega^2((\mathbf{Cl}(M) \hat{\otimes} \operatorname{End}_{\mathbf{Cl}(M)}(E)).$$

If we choose a local orthonormal frame (e_i) of TM and we denote by (e^i) the dual coframe then

$$R = \sum_{i<j} R(e_i, e_j) e^i \wedge e^j,\ \ R(e_i, e_j) \in \Gamma(\underline{so}(TM)),$$

and

$$\boldsymbol{c}(R)(e_i, e_j) = \frac{1}{2} \sum_{k<\ell} g(R(e_i, e_j)e_k, e_l)\boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell) = \frac{1}{4} \sum_{k,\ell} g(R(e_i, e_j)e_k, e_\ell)\boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell). \tag{2.2.6}$$

We set

$$F^{E/\mathbb{S}} := F - \boldsymbol{c}(R) \in \Omega^2((\mathbf{Cl}(M) \hat{\otimes} \operatorname{End}_{\mathbf{Cl}(M)}(E)).$$

We will refer to $F^{E/\mathbb{S}}$ as the *twisting curvature* of the Dirac bundle (E, ∇^E).

Proposition 2.2.17.

$$F^{E/\mathbb{S}} \in \Omega^2(\mathrm{End}_{\mathbf{Cl}(M)}(E)).$$

Proof We have to show that

$$\forall X, Y \in \mathrm{Vect}(M), \;\; \alpha \in \Omega^1(M): \;\; F^{E/\mathbb{S}}(X,Y)\boldsymbol{c}(\alpha) = \boldsymbol{c}(\alpha)F^{E/\mathbb{S}}(X,Y)$$

i.e.

$$[F^{E/\mathbb{S}}(X,Y), \boldsymbol{c}(\alpha)] = 0$$

so that $F^{E/\mathbb{S}}(X,Y)$ is a morphism of $\mathbf{Cl}(M)$-modules. We have

$$F_\nabla(X,Y) = [\nabla^E_X, \nabla^E_Y] - \nabla^E_{[X,Y]}.$$

$$[\nabla^E_Z, \boldsymbol{c}(\alpha)] = \boldsymbol{c}(\nabla^g_Z\alpha), \;\; \forall Z \in \mathrm{Vect}(M).$$

Hence

$$[\nabla^E_{[X,Y]}, \boldsymbol{c}(\alpha)] = \boldsymbol{c}(\nabla^g_{[X,Y]}\alpha)]$$

$$[[\nabla^E_X, \nabla^E_Y], \boldsymbol{c}(\alpha)] = [[\nabla^E_X, \boldsymbol{c}(\alpha)], \nabla^E_Y] + [\nabla^E_X, [\nabla^E_Y, \boldsymbol{c}(\alpha)]]$$
$$= [\boldsymbol{c}(\nabla^g_X\alpha), \nabla^E_Y] + [\nabla^E_X, \boldsymbol{c}(\nabla^g_Y\alpha)] = \boldsymbol{c}([\nabla^g_X, \nabla^g_Y]\alpha).$$

We deduce

$$[F_\nabla(X,Y), \boldsymbol{c}(\alpha)] = \boldsymbol{c}(R^\sharp(X,Y)\alpha).$$

On the other hand we have the following equality in $\mathbf{Cl}(M)$.

$$R^\sharp(X,Y)\alpha = \tau([\tau^{-1}R^\sharp(X,Y), \alpha])$$

$$\Longrightarrow (\boldsymbol{c}(R^\sharp(X,Y)\alpha) = [\boldsymbol{c}(R), \boldsymbol{c}(\alpha)] \in \mathrm{End}(E)$$

Hence we have

$$[F_\nabla(X,Y), \boldsymbol{c}(\alpha)] = [\boldsymbol{c}(R), \boldsymbol{c}(\alpha)] \Longleftrightarrow [F^{E/\mathbb{S}}, \boldsymbol{c}(\alpha)] = 0.$$

□

Let us now explain the process of *twisting* of a Dirac bundle which allows us to produce new Dirac bundles out of old ones.

Suppose (E, ∇^E) is a Dirac bundle and $W = W^+ \oplus W^-$ is a hermitian s-bundle equipped with a hermitian connection ∇^W compatible with the $\mathbb{Z}/2$-grading. The $\mathbb{Z}/2$-graded tensor product $E\hat{\otimes}W$ is bundle of Clifford module in a tautological way. Moreover ∇^E and ∇^W induce a connection on $E\hat{\otimes}W$ defined by

$$\nabla^{E\hat{\otimes}W} = \nabla^E \otimes \mathbb{1}_W + \mathbb{1}_E \otimes \nabla^W..$$

A simple computation shows that $\nabla^{E\hat{\otimes}W}$ is compatible with the Clifford multiplication. Hence $(E\hat{\otimes}W, \nabla^{E\hat{\otimes}W})$ is a Dirac bundle. We say that it was obtained from the Dirac bundle (E, ∇^E) by twisting with (W, ∇^W) we will denote it by $(E, \nabla^E)\hat{\otimes}(W, \nabla^W)$.

Observe that $\mathrm{End}(E \otimes W) \cong \mathrm{End}(E) \otimes \mathrm{End}(W)$ and with respect to this isomorphism we have

$$F^{E\otimes W} = F^E \otimes \mathbb{1}_W + \mathbb{1}_E \otimes F^W.$$

In particular

$$F^{(E\hat{\otimes}W)/\mathbb{S}} = F^{E/\mathbb{S}} + F^W. \tag{2.2.7}$$

Definition 2.2.18. Suppose (E, ∇^E) is a Dirac bundle. The *geometric Dirac operator* associated to (E, ∇) is the first order p.d.o. $\mathfrak{D}_E : C^\infty(E) \to C^\infty(E)$ defined by the compostion

$$C^\infty(E) \xrightarrow{\nabla^E} C^\infty(T^*M \otimes E) \xrightarrow{\boldsymbol{c}(-)} C^\infty(E),$$

where $\boldsymbol{c}(-)$ denotes the Clifford multiplication of a section on E with a 1-form. □

From the definition it follows that

$$\sigma(\mathfrak{D}_E) = \boldsymbol{c}(-).$$

Observe that the connection ∇^E preserves the grading, while the multiplication by a 1-form is odd, and thus maps ven/odd sections of E to odd/even sections. Hence

$$\mathfrak{D}_E C^\infty(E^\pm) \subset C^\infty(E^\mp)$$

In other words $\mathfrak{D}_E$ is an odd operator with respect to the $\mathbb{Z}/2$-grading

$$C^\infty(E) = C^\infty(E^+) \oplus C^\infty(E^-).$$

In particular, it has the block decomposition

$$\mathfrak{D}_E = \begin{bmatrix} 0 & \mathfrak{D}_{E^-} \\ \mathfrak{D}_{E^+} & 0 \end{bmatrix}.$$

Traditionally $\mathfrak{D}_{E^+}$ is denoted by $\not{\mathfrak{D}}_E$.

Proposition 2.2.19. *$\mathfrak{D}_E$ is symmetric, i.e.*

$$\mathfrak{D}_E^* = \mathfrak{D}_E.$$

Proof This is a local statement so we will work in local coordinates. Choose a local orthonormal frame e_i of TM and denote by e^i its dual coframe. Then

$$\mathfrak{D}_E = \sum_i \boldsymbol{c}(e^i)\nabla^E_{e_i}.$$

Hence

$$\mathfrak{D}_E^* = \sum_i (\boldsymbol{c}(e^i)\nabla^E_{e_i})^* = \sum_i (\nabla^E_{e_i})^* \boldsymbol{c}(e^i)^* = \sum_i (-\nabla^E_{e_i} - \mathbf{div}_g(e_i))(-\boldsymbol{c}(e^i))$$

$$= \sum_i \mathbf{div}_g(e_i)\boldsymbol{c}(e^i) + \sum_i \nabla^E_{e_i}\boldsymbol{c}(e^i) = \mathfrak{D}_E + \underbrace{\sum_i \mathbf{div}_g(e_i)\boldsymbol{c}(e^i) + \sum_i \boldsymbol{c}(\nabla^g_{e_i} e^i)}_{T}.$$

$T = \mathfrak{D}_E^* - \mathfrak{D}_E$ is a zero order operator so it suffices to understand its action on a fiber of E over an arbitrary point x_0 of M. If we assume the local frame e_i is synchronous at x_0, i.e.

$$\nabla^g_{e_i} e_j = 0 \ \text{ at } x_0$$

then

$$\nabla^g_{e_i} e^i = 0, \ \ \mathbf{div}_g\, e_i = 0 \Longrightarrow T = 0.$$

□

Since the symbol of $\mathfrak{D}_E$ is given by the Clifford multiplication we deduce that $\mathfrak{D}_E^2$ is a generalized Laplacian. We deduce that $\mathfrak{D}_E$ is indeed a Dirac type operator since $\mathfrak{D}_E^*\mathfrak{D}_E = \mathfrak{D}_E\mathfrak{D}_E^* = \mathfrak{D}_E^2$ is a generalized Laplacian. It can be described in the block form

$$\mathfrak{D}_E = \begin{bmatrix} 0 & \not{\mathfrak{D}}_E^* \\ \not{\mathfrak{D}}_E & 0 \end{bmatrix}.$$

Proposition 2.2.20 (Weitzenböck Formula)**.** *Suppose (E, ∇) is a Dirac bundle over the oriented Riemann manifold M, and $F^{E/\mathbb{S}} \in \mathrm{End}_{\mathbf{Cl}(M)}(E)$ is the twisting curvature. Then*

$$\mathfrak{D}_E^2 = (\nabla^E)^*\nabla^E + \frac{s(g)}{4} + \boldsymbol{c}(F^{E/\mathbb{S}}),$$

where $s(g)$ is the scalar curvature of the metric g, $\boldsymbol{c}(F^{E/\mathbb{S}})$ is the endomorphism of E defined locally by

$$\boldsymbol{c}(F^{E/\mathbb{S}}) = \sum_{i<j} F^{E/\mathbb{S}}(e_i, e_j)\boldsymbol{c}(e^i)\boldsymbol{c}(e^j),$$

(e_i) is a local orthonormal frame of TM and (e^i) is the dual coframe.

Proof The result is local. Assume the local orthonormal frame is synchronous at a point $x \in M$. We set $\nabla_i := \nabla_{e_i}$, we denote by $e_i \lrcorner$ the contraction by e_i so we have

$$(\nabla^E)^*\nabla^E = \Big(\sum_i e^i \otimes \nabla_i^E\Big)^* \sum_j e^j \otimes \nabla_j^E = \Big(\sum_i (-\nabla_i^E - \mathbf{div}_h(e_i))e_i \lrcorner\Big)\sum_j e^j \otimes \nabla_j^E$$

$(e_i \lrcorner e^j = \delta_i^j)$

$$= -\sum_i (\nabla_i^E)^2 - \sum_i \mathbf{div}_g(e_i)\nabla_i^E = -\sum_i (\nabla_i^E)^2 \;\text{ at } x_0.$$

On the other hand we have

$$\mathfrak{D}_E^2 = \sum_{i,j} \boldsymbol{c}(e^i)\nabla_i^E \boldsymbol{c}(e^j)\nabla_j^E = \sum_{i,j} \boldsymbol{c}(e^i)\boldsymbol{c}(e^j)\nabla_i^E\nabla_j^E + \sum_{i,j}\boldsymbol{c}(\nabla_i^E e^j)\nabla_j^E.$$

$$= -\sum_i (\nabla_i^E)^2 + \sum_{i\neq j} \boldsymbol{c}(e^i)\boldsymbol{c}(e^j)\nabla_i^E\nabla_j^E + \sum_{i,j}\boldsymbol{c}(\nabla_i^E e^j)\nabla_j^E$$

$$= -\sum_i (\nabla_i^E)^2 + \sum_{i<j} \boldsymbol{c}(e^i)\boldsymbol{c}(e^j)[\nabla_i^E, \nabla_j^E] + \sum_{i,j}\boldsymbol{c}(\nabla_i^E e^j)\nabla_j^E$$

(at x_0 we have $\mathbf{div}_g\, e_i = 0$, $[e_i, e_j] = 0$)

$$= (\nabla^E)^*\nabla^E + \sum_{i<j} \boldsymbol{c}(e^i)\boldsymbol{c}(e^j)F(e_i, e_j) = (\nabla^E)^*\nabla^E + c(F^{E/\mathbb{S}}) + \underbrace{\sum_{i<j} \boldsymbol{c}(R)(e_i, e_j)\boldsymbol{c}(e^i)\boldsymbol{c}(e^j)}_{:=T}.$$

On the other hand we have (see (2.2.6))

$$T = \sum_{i<j} \boldsymbol{c}(R)(e_i, e_j)\boldsymbol{c}(e^i)\boldsymbol{c}(e^j) = \frac{1}{4}\sum_{i<j}\Big(\sum_{k,\ell} \underbrace{g(R(e_i, e_j)e_k, e_\ell)}_{-R_{ijk\ell}} \boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell)\Big)\boldsymbol{c}(e^i)\boldsymbol{c}(e^j)$$

$(R_{ijk\ell} = -R_{jik\ell} = R_{k\ell ij})$

$$= -\frac{1}{8}\sum_{i,j,k,\ell} R_{k\ell ij}\boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell)\boldsymbol{c}(e^i)\boldsymbol{c}(e^j) = -\frac{1}{8}\sum_{i\neq j, k\neq \ell} R_{ijk\ell}\boldsymbol{c}(e^i)\boldsymbol{c}(e^j)\boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell)$$

Observe that $\boldsymbol{c}(e^i)\boldsymbol{c}(e^j)$ anticommutes with $\boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell)$ if the two sets $\{i,j\}$ and $\{j,k\}$ have exactly one element in common. Such pairs of anticommuting monomials do not contribute anything to the above sum due to the symmetry $R_{ijk\ell} = R_{k\ell ij}$ of the Riemann tensor. We can thus split the above sum into two parts

$$T = -\frac{1}{4}\sum_{i\neq j} R_{ijij}\big(\,\boldsymbol{c}(e^i)\boldsymbol{c}(e^j)\,\big)^2 - \frac{1}{8}\sum_{i,j,k,\ell \text{ distinct}} R_{ijk\ell}\boldsymbol{c}(e^i)\boldsymbol{c}(e^j)\boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell)$$

$(R_{ijk\ell} + R_{i\ell jk} + R_{ik\ell j} = 0$, $\boldsymbol{c}(e^j)\boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell) = \boldsymbol{c}(e^\ell)\boldsymbol{c}(e^j)\boldsymbol{c}(e^k) = \boldsymbol{c}(e^k)\boldsymbol{c}(e^\ell)\boldsymbol{c}(e^j))$

$$= \frac{1}{4}\sum_{i,j} R_{ijij} = \frac{s(g)}{4}.$$

□

Example 2.2.21. Let $E = \Lambda T^*M$. E is a $\mathbf{Cl}(M)$-module with Clifford multiplication by $\alpha \in \Omega^1(M)$ described by

$$\boldsymbol{c}(\alpha)\omega = \alpha\wedge\omega - i(\alpha^\sharp)\omega,\ \ \forall\omega\in\Omega^1(M),$$

where $i(\alpha^\sharp)$ denotes the contraction by the vector field g-dual to α. Clearly $\boldsymbol{c}(\alpha)$ is a skew-symmetric endomorphism of $\Lambda^\bullet T^*M$. The Levi-Civita connection induces a connection ∇^g on $\Lambda^\bullet T^*M$ which is compatible with the above Clifford multiplication. This shows that $(\Lambda T^*M, \nabla^g)$ is a Dirac bundle. The Dirac operator determined by this Dirac bundle is none other than the Hodge-Dolbeault operator. For a proof we refer to [16, Prop. 10.2.1].

□

2.3. Exercises for Chapter 2

Exercise 2.3.1. Prove Hadamard Lemma. □

Exercise 2.3.2. Suppose $M = \mathbb{R}^n$. Prove that any $L \in \mathbf{PDO}^{(m)}(\underline{\mathbb{C}}_M)$ has the form

$$L = \sum_{|\vec{\alpha}| \leq m} a_{\vec{\alpha}}(x)\partial^{\vec{\alpha}}, \quad a_{\vec{\alpha}} \in C^\infty(M).$$

Exercise 2.3.3. Prove Proposition 2.1.8. □

Exercise 2.3.4. Prove Cartan's formula (2.1.3).

□

Exercise 2.3.5. Suppose (M, g) is a compact oriented Riemann manifold without boundary. Let $n := \dim M$.

(a) Prove that the DeRham complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

is an elliptic complex.

(b) Let $\mathbb{H}^k(M, g) := \Big\{ \omega \in \Omega^k(M); \;\; d\omega = d^*\omega = 0 \Big\}$. Hodge theorem implies that

$$\mathbb{H}^k(M, g) \cong H^k_{DR}(M).$$

Prove that the Hodge $*$-operator $*_g : \Omega^k(M) \to \Omega^{n-k}(M)$ induces an isomorphism

$$*_g : \mathbb{H}^k(M, g) \to \mathbb{H}^{n-k}(M, g).$$

(c)**(Hodge decomposition)** Prove that we have a L^2-orthogonal decomposition

$$\Omega^k(M) = \mathbb{H}^k(M, g) + d\Omega^{k-1}(M) + d^*\Omega^{k+1}(M),$$

(d) The Levi-Civita connection on TM induces a connection $\hat{\nabla}$ on $\Lambda^\bullet T^*M$. Prove that the Laplacian

$$\Delta = (d + d^*)^2 : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

and the covariant Laplacian

$$\hat{\nabla}^*\hat{\nabla} : \Omega^*(M) \to \Omega^\bullet(M)$$

differ by a zero order term, i.e. an endomorphism of $\Lambda^\bullet T^*M$. □

Exercise 2.3.6. Let H be a complex Hilbert space. A bounded operator $L : H \to H$ is called Fredholm if both L and L^* have closed ranges and $\dim\ker L + \dim\ker L^* < \infty$. In this case the Fredholm index of L is

$$\operatorname{ind} L = \dim\ker L - \dim\ker L^*.$$

(a) Prove that L is Fredholm if and only if L admits a *parametrix* i.e. a bounded linear operator S such that $SL - \mathbb{1}$ and $LS - \mathbb{1}$ are compact.

(b)* $[0, 1] \ni t \mapsto L_t$ is a continuous family of Fredholm operators then $\operatorname{ind} L_t$ is independent of t.

(c) Show that if $L : H \to H$ is Fredholm and $K : H \to H$ is compact then $L + K$ is Fredholm and

$$\operatorname{ind}(L + K) = \operatorname{ind} L.$$

(d) Suppose $L_0, L_1 : H \to H$ are Fredholm. Construct a continuous family of Fredholm operators $A_t : H \oplus H \to H \oplus H$ such that $A_0 = L_0 \oplus L_1$, $A_1 = -\mathbb{1} \oplus L_1 L_0$. Conclude that

$$\operatorname{ind} L_0 L_1 = \operatorname{ind} L_0 \cdot \operatorname{ind} L_1.$$

Hint: For this exercise you need to know Fredholm-Riesz Theorem, [6, Chap. VI]. *If* $K : H \to H$ *is a compact operator then* $\mathbb{1} + K$ *is Fredholm and* $\operatorname{ind}(\mathbb{1} + K) = 0$.

Exercise 2.3.7. Prove Corollary 2.2.9. □

Exercise 2.3.8. Prove Lemma 2.2.15. □

Chapter 3

The Atiyah-Singer Index Theorem and the Atiyah-Bott fixed point formula

3.1. The index theorem and some basic applications

3.1.1. The statement of the index theorem. Suppose (M, g), is a compact, oriented, Riemann manifold without boundary. We denote by ∇^g the Levi-Civita connection on TM, and by $R = R_g \in \Omega^2(\mathrm{End}^g_-(TM))$ its curvature, i.e. the Riemann tensor. We form the $\hat{\mathbf{A}}$-genus form

$$\hat{\mathbf{A}}(M, g) = \det{}^{1/2}\left(\frac{\frac{\mathbf{i}}{4\pi}R_g}{\sinh\left(\frac{\mathbf{i}}{4\pi}R_g\right)}\right) \in \Omega^*(M).$$

This is a closed form whose cohomology class is independent of g and we denote by $\hat{\mathbf{A}}(M)$.

Suppose (E, ∇^E) is a Dirac bundle and $\mathfrak{D} : C^\infty(E^+) \to C^\infty(E^-)$ is the associated Dirac operator . We denote by $F^{E/\mathbb{S}} \in \mathrm{End}_{\mathbf{Cl}(M)}(E)$ the twisting curvature of E. Recall that we have a natural relative s-trace (see Definition 2.2.7)

$$\mathrm{str}_{E/\mathbb{S}} : \mathrm{End}_{\mathbf{Cl}(M)}(E) \to \underline{\mathbb{C}}_M.$$

This induces a map

$$\mathrm{str}^{E/\mathbb{S}} : \Omega^\bullet(\mathrm{End}_{\mathbf{Cl}(M)}(E)) \to \Omega^\bullet(M) \otimes \mathbb{C}.$$

we set

$$\mathbf{ch}^{E/\mathbb{S}}(E) := \mathrm{str}^{E/\mathbb{S}} \exp\left(\frac{\mathbf{i}}{2\pi}F^{E/\mathbb{S}}\right).$$

We will see a bit later that this is a closed form whose cohomology class depends only on the topology of E.

If we twist E by a s-bundle (W, ∇^W) then according to (2.2.7) we have

$$F^{E\hat{\otimes}W/\mathbb{S}} = F^{E/\mathbb{S}} \otimes \mathbb{1}_W + \mathbb{1}_E \otimes F^W.$$

where the curvature F^W of W has the direct sum decomposition

$$F^W = F^{W^+} \oplus F^{W^-}.$$

We deduce

$$\mathbf{ch}^{E\hat{\otimes}W/\mathbb{S}}(E\hat{\otimes}W) = \mathbf{ch}^{E/\mathbb{S}}(E/\mathbb{S})\big(\mathbf{ch}(\nabla^{W^+}) - \mathbf{ch}(\nabla^{W^-})\big). \tag{3.1.1}$$

We can now formulate the main result of these lectures, the celebrated *Atiyah-Singer index theorem*.

Theorem 3.1.1 (Atiyah-Singer)**.**

$$\operatorname{ind} \mathfrak{D}_E = \dim\ker \mathfrak{D}_E - \dim\ker \mathfrak{D}_E^* = \int_M \hat{\mathbf{A}}(M,g)\,\mathbf{ch}^{E/\mathbb{S}}(E/\mathbb{S}).$$

We will spend the remainder of this chapter elucidating the significance of the integrand in the Atiyah-Singer index theorem. Observe that the integrand on the right-hand side is a form of even degree so that the index of a geometric Dirac operator on an odd dimensional manifold must be zero. Therefor, in the sequel we will concentrate exclusively on even dimensional manifolds.

The theorem is true in a much more general context of elliptic operators but the formulation requires a rather long detour in topological K-theory. For the curious reader we refer to the magnificent papers [**1, 2**].

3.1.2. The Hodge-DeRham operators. Suppose (M, g) is a compact, oriented Riemann even dimensional manifold without boundary. Set $2m := \dim M$. We set $E := \Lambda^\bullet T^*M \otimes \mathbb{C}$ and we denote by ∇^E the connection on E induced by the Levi-Civita connection. As explained in Example 2.2.21 E is a Clifford bundle and (E, ∇^g) is a Dirac bundle with associated Dirac operator

$$\mathfrak{D}_E = d + d^* : \Omega^\bullet(M) \to \Omega^\bullet(M).$$

The bundle E has an obvious $\mathbb{Z}/2$-grading

$$E^\pm = \Lambda^{even/odd} T^*M$$

and $\mathfrak{D}_E$ is odd with respect to this grading, i.e.

$$\mathfrak{D}_E C^\infty(E^\pm) \subset C^\infty(E^\mp).$$

Hodge theorem shows that the index of

$$\mathfrak{D}_E : C^\infty(E^+) \to C^\infty(E^-)$$

is precisely the Euler characteristic of M. The Atiyah-Singer index formula shows that

$$\chi(M) = \int_M \hat{\mathbf{A}}(M,g)\,\mathbf{ch}^{E/\mathbb{S}}(E/\mathbb{S}).$$

Let us analyze the integrand in the right-hand-side of the above equality. We first need to understand the twisting curvature $F^{E/\mathbb{S}}$

$$F^{E/\mathbb{S}} = F^E - \boldsymbol{c}(R) \in \Omega^2(\operatorname{End}_{\mathbf{Cl}(M)}(E)).$$

Fix $x \in M$ choose a local orthonormal frame e_i of TM near x. Set $V_x := T_x^*M$. We denote by e^i the dual coframe. We assume additionally that (e_i) is synchronous at x. Set

$$R_{ijk\ell} := g\big(e_i, R_g(e_k, e_\ell)e_j\big) = -g\big(R_g(e_k, e_\ell)e_i, e_j\big) \Longleftrightarrow R_g(e_k, e_\ell)e_j = \sum_i R_{ijk\ell}e_i.$$

For every $i < j$ the curvature F^E of ∇^E induces a skew-symmetric endomorphism

$$F_x^E(e_i, e_j) \in \operatorname{End}(E_x).$$

We want to describe this endomorphism in terms of the components $R_{ijk\ell}$. For every ordered multi-index $I = (i_1 < \cdots < i_\alpha)$ we set

$$e^I := e^{i_1} \wedge \cdots \wedge e^{i_\alpha}.$$

The collection $\{ e^I;\ I \}$ defines a local orthonormal frame of E near x and thus we only need to understand

$$F_x^E(e_k, e_\ell)e^I(x).$$

Setting $\nabla_i = \nabla_{e_i}$ we have

$$F^E(e_k, e_\ell)e^I = \big([\nabla_k^E, \nabla_\ell^E] - \nabla^E_{[e_k, e_\ell]}\big)e^I$$

Since (e_i) is synchronous at x we deduce that at x

$$\nabla^E_{[e_k, e_\ell]}e^I = 0.$$

We deduce that at x we have

$$F^E(e_k, e_\ell)e^I = (F^E(e_k, e_\ell)e^{i_1}) \wedge \cdots \wedge e^{i_\alpha} + \cdots e^{i_1} \wedge \cdots \wedge (F^E(e_k, e_\ell)e^{i_\alpha}).$$

Let us observe that

$$F^E(e_k, e_\ell)e^a = \sum_i R_{iak\ell}e^i. \tag{3.1.2}$$

We denote by $\varepsilon_i \in \operatorname{End}(E_x)$ the exterior multiplication by e^i and by $\iota_j \in \operatorname{End}(E_x)$ the contraction by e_j. We can then rewrite (3.1.2) as

$$F^E(e_k, e_\ell)e^a = \Big(\sum_{i,j} R_{ijk\ell}\varepsilon_i\iota_j\Big)e^a.$$

Moreover

$$\varepsilon_i\iota_j e^I = (\varepsilon_i\iota_j e^{i_1}) \wedge e^{i_2} \wedge \cdots \wedge e^{i_\alpha} + \cdots + e^{i_1} \wedge \cdots e^{i_{\alpha-1}} \wedge (\varepsilon_i\iota_j e^{i_\alpha})$$

and we deduce

$$F^E(e_k, e_\ell)e^I = \Big(\sum_{i,j} R_{ijk\ell}\varepsilon_i\iota_j\Big)e^I. \tag{3.1.3}$$

For every $j = 1, 2, \cdots, 2m$ we set

$$\beta_j := \varepsilon_j + \iota_j, \ \ \boldsymbol{c}_i = \varepsilon^i - \iota_i = \boldsymbol{c}(e^i).$$

Observe that

$$\{\beta_i, \beta_j\} = -\{\boldsymbol{c}_i, \boldsymbol{c}_j\} = 2\delta_{ij},$$
$$\{\boldsymbol{c}_i, \beta_j\} = (\varepsilon_i - \iota_i)(\varepsilon_j + \iota_j) + (\varepsilon_j + \iota_j)(\varepsilon_i - \iota_i) = 0.$$

This shows that $\beta_i \in \operatorname{End}(E)$ s-commute with the Clifford action so that

$$\beta_i \in \widehat{\operatorname{End}}_{\mathbf{Cl}_{\mathbb{C}}(V_x)}(E_x).$$

Now observe that

$$2\varepsilon_i = \boldsymbol{c}_i + \beta_i, \quad -2\iota_j = \boldsymbol{c}_j - \beta_j \quad \varepsilon_i\iota_j = -\frac{1}{4}(\boldsymbol{c}_i + \beta_i)(\boldsymbol{c}_j - \beta_j)$$

Using this in (3.1.3) we deduce

$$F^E(e_k, e_\ell)e^I = -\frac{1}{4}\sum_{i,j} R_{ijk\ell}(\boldsymbol{c}_i + \beta_i)(\boldsymbol{c}_j - \beta_j)e^I. \tag{3.1.4}$$

The sum in the right-hand-side can be further simplified. We have

$$\sum_{i,j} R_{ijk\ell}(\boldsymbol{c}_i + \beta_i)(\boldsymbol{c}_j - \beta_j) = \sum_{i,j} R_{ijk\ell}\big(\boldsymbol{c}_i\boldsymbol{c}_j - \beta_i\beta_j + \beta_i\boldsymbol{c}_j - \boldsymbol{c}_i\beta_j\,\big).$$

Using the symmetry $R_{ijk\ell} = -R_{jik\ell}$ and the s-commutativity $\{\boldsymbol{c}_i, \beta_j\} = 0$ we deduce

$$F^E(e_k, e_\ell)e^I = -\frac{1}{4}\sum_{i,j} R_{ijk\ell}(\boldsymbol{c}_i\boldsymbol{c}_j - \beta_i\beta_j)e^I$$

$$= \underbrace{\frac{1}{4}\Big(\sum_{i,j} g(\,R_g(e_k, e_\ell)e_i, e_j)\boldsymbol{c}^i\boldsymbol{c}^j\,\Big)}_{\boldsymbol{c}(R)(e_k,e_\ell)}\,e^I - \frac{1}{4}\Big(\sum_{i,j} g(\,R_g(e_k, e_\ell)e_i, e_j)\beta_i\beta_j\,\Big)e^I. \tag{3.1.5}$$

This implies

$$F^{E/\mathbb{S}}(e_k, e_\ell) = \frac{1}{4}\sum_{i,j} R_{ijk\ell}\beta_i\beta_j = -\frac{1}{4}\sum_{i,j} g(R_g(e_k, e_\ell)e^i, e^j)\beta_i\beta_j. \tag{3.1.6}$$

We now turn to the investigation of the s-trace

$$\mathrm{str}^{E/\mathbb{S}} : \widehat{\mathrm{End}}_{\mathbf{Cl}_{\mathbb{C}}(V_x)}(E_x) \to \mathbb{C}.$$

For every skew-symmetric endomorphism R of V_x we define $\beta_R \in \widehat{\mathrm{End}}_{\mathbf{Cl}_{\mathbb{C}}(V_x)}(E_x)$ by

$$\beta_R = -\frac{\mathbf{i}}{8\pi}\sum_{i,j} g(Re^i, e^j)\beta_i\beta_j.$$

Lemma 3.1.2.

$$\mathrm{str}^{E/\mathbb{S}}\exp\beta_R = \frac{\mathbf{Pfaff}(-\frac{1}{2\pi}R)}{\hat{\mathbf{A}}(-\frac{1}{2\pi}R)}$$

Proof β_R is independent of the orthonormal basis (e^i) of V_x. Chose an oriented orthonormal frame $\{e^i, f^i;\ \ 1 \le i \le m\}$ with respect to which A is quasi-diagonal

$$Re^i = \lambda_i f^i, \quad Rf^i = -\lambda_i f^i.$$

Then

$$\beta_R = -\frac{\mathbf{i}}{4\pi}\sum_{i=1}^m \lambda_i \underbrace{\beta(e^i)\beta(f^i)}_{:=B_i}$$

Observe that $[B_i, B_j] = 0$ for all $i \neq j$ so that

$$\exp(\beta_R) = \prod_{i=1}^m \exp(\frac{-\mathbf{i}\lambda_i}{4\pi}B_i).$$

Now observe that

$$B_i^2 = \beta(e^i)\beta(f^i)\beta(e^i)\beta(f^i) = -\beta(e^i)^2\beta(f^i)^2 = -1$$

so that

$$\exp(zB_i) = \cos z + (\sin z)B_i,$$

and

$$\exp(\beta_R) = \prod_{j=1}^m \Big(\cos\frac{\mathbf{i}\lambda_j}{4\pi} + B_j \sin\frac{-\mathbf{i}\lambda_j}{4\pi}\Big).$$

Using the identities

$$\cos(iz) = \cosh z, \quad \sinh z = \mathbf{i}\sin(-\mathbf{i}z)$$

we deduce

$$\exp(\beta_R) = \prod_{j=1}^m \Big(\cosh(\frac{\lambda_j}{4\pi}) - \mathbf{i}B_j\sinh(\frac{\lambda_j}{4\pi})\Big).$$

Let

$$\Gamma = \prod_{j=1}^m \underbrace{\boldsymbol{c}(e^j)\boldsymbol{c}(f^j)}_{:=C_j}$$

Using Corollary 2.2.9 we deduce

$$\operatorname{str}^{E/\mathbb{S}} \exp\beta_R = \frac{\mathbf{i}^m}{2^m}\operatorname{str}^E(\Gamma\exp(\beta_R)).$$

Now observe that

$$\Gamma\exp(\beta_R) = \prod_{j=1}^m C_j\Big(\cosh(\frac{\lambda_j}{4\pi}) - \mathbf{i}B_j\sinh(\frac{\lambda_j}{4\pi})\Big).$$

Set

$$T_j := \frac{\mathbf{i}}{2}C_j\Big(\cosh(\frac{\lambda_j}{4\pi}) - \mathbf{i}B_j\sinh(\frac{\lambda_j}{4\pi})\Big), \quad V_j := \operatorname{span}_{\mathbb{C}}(e^j, f^j), \quad E_j := \Lambda^\bullet V_j.$$

Note that we can view T_j as an operators on E_j. Using the isomorphism of s-algebras

$$E \cong \widehat{\bigotimes}_j E_j$$

We can identify $\frac{\mathbf{i}^m}{2^m}\Gamma\exp\beta_A$ with the tensor product $T_1\hat{\otimes}\cdots\hat{\otimes}T_m$ and we deduce

$$\operatorname{str}^E\frac{\mathbf{i}^m}{2^m}\Gamma\exp\beta_R = \prod_j \operatorname{str}^{E_j} T_j.$$

Observe that

$$T_j = \frac{\mathbf{i}}{2}\cosh(\frac{\lambda_j}{4\pi})C_j + \frac{1}{2}\sinh(\frac{\lambda_j}{4\pi})C_jB_j.$$

Observe that $1, e^j, f^j, e^j\wedge f^j$ is an orthonormal basis of E_j and we have

$$B_j1 = e^j\wedge f^j = C_j1, \quad B_je^j\wedge f^j = -1 = C_je^j\wedge f^j$$

$$B_je^j = -f^j = -C_je^j, \quad B_jf^j = -e^j = -C_jf^j$$

We deduce

$$C_jB_j = -\mathbb{1}_{E_j^{even}} + \mathbb{1}_{E_j^{odd}}.$$

Hence

$$\operatorname{str}^{E_j} C_jB_j = -4,\ \ \operatorname{str}_{E_j} C_j = 0,\ \ \operatorname{str}_{E_j} T_j = -2\sinh(\frac{\lambda_j}{4\pi}) = 2\sinh(-\frac{\lambda_j}{4\pi})$$

$(x_j := -\frac{\lambda_j}{2\pi})$

$$= x_j\frac{\sinh(\frac{x_j}{2})}{\frac{x_j}{2}} = \frac{x_j}{\hat{\mathbf{A}}(x_j)}.$$

□

From (3.1.6) and Lemma 3.1.2 we deduce

$$\operatorname{str}^{E/\mathbb{S}}(\frac{\mathbf{i}}{2\pi}F^{E/\mathbb{S}}) = \frac{\mathbf{Pfaff}(-\frac{1}{2\pi}R_g)}{\hat{\mathbf{A}}(-\frac{1}{2\pi}R_g)} = \frac{\mathbf{e}(M,g)}{\hat{\mathbf{A}}(M,g)},$$

where $\mathbf{e}\,(M,g)$ denotes the Euler form determined by the Levi-Civita connection on TM. Using this in the index theorem we obtain the following result.

Theorem 3.1.3 (Gauss-Bonnet-Chern)**.** *For every compact oriented, even dimensional Riemann manifold (M,g) we have*

$$\chi(M) = \int_M \mathbf{e}(M,g).$$

The bundle E is equipped with another $\mathbb{Z}/2$-grading induced by the Hodge $*$-operator

$$* : \Lambda^{\bullet}T^*M \to \Lambda^{2m-\bullet}T^*M.$$

Recall that for any $\alpha \in \Omega^p(M)$ we have (see (1.3.1))

$$*(*\alpha) = (-1)^{p(2m-p)}\alpha = (-1)^p\alpha.$$

Define

$$\mu(m,p) = p(p-1) + m$$

$$\gamma_p = \mathbf{i}^{\mu(m,p)}* : \Lambda^pT^*M\otimes\mathbb{C} \to \Lambda^{2m-p}T^*M\otimes\mathbb{C},\ \ \gamma := \oplus_p\gamma_p \in \operatorname{End}(E).$$

Observe that

$$\mu(m,p) + \mu(m,2m-p) = p(p-1) + (2m-p)(2m-p-1) + 2m$$

$$= 4m^2 + 2m + p(p-1) - 2m(2p+1) + p(p+1) = 2p^2 \mod 4.$$

Since $i^{2p^2} = (-1)^p$ we deduce $\gamma^2 = \mathbb{1}_E$ and the ±1-eigenspaces of γ define a $\mathbb{Z}/2$-grading on E. Moreover a simple computation left as an exercise shows that[1]

$$\gamma = \mathbf{i}^m\boldsymbol{c}(dV_g) = \mathbf{i}^m\boldsymbol{c}(\Gamma), \tag{3.1.7}$$

where for a local, oriented, orthonormal frame $e^1,\cdots,e^{2m}$ of T^*M we have

$$\Gamma = e^1\cdots e^{2m} \in \mathbf{Cl}(M).$$

We deduce

$$\boldsymbol{c}(\alpha)\gamma + \gamma\boldsymbol{c}(\alpha) = 0,\ \ \forall\alpha\in\Omega^1(M).$$

This shows that we can interpret E equipped with this new $\mathbb{Z}/2$-grading as a new bundle Clifford bundle. We will denote it by $\mathcal{E} = \mathcal{E}^+\oplus\mathcal{E}^-$. Since the bundle and the Clifford action has not changed

[1] This explains the weird choice of $\mu(m,p)$.

it is clear that $\mathcal{E}$ is a Dirac bundle with associated geometric Dirac operator $d + d^*$. This induces an elliptic operator

$$D = (d + d^*) : C^\infty(\mathcal{E}^+) \to C^\infty(\mathcal{E}^-).$$

We would like to compute its index. Observe that

$$\ker D = \Big\{ \alpha \in \Omega^\bullet(M) \otimes \mathbb{C};\ \ \gamma\alpha = \alpha,\ \ d\alpha = d^*\alpha = 0 \Big\},$$

$$\ker D^* = \Big\{ \alpha \in \Omega^\bullet(M) \otimes \mathbb{C};\ \ \gamma\alpha = -\alpha,\ \ d\alpha = d^*\alpha = 0 \Big\}.$$

To compute $\operatorname{ind}_{\mathbb{C}} D = \dim_{\mathbb{C}} \ker D - \dim_{\mathbb{C}} \ker D^*$ we will use the Poincaré-Hodge duality. Denote by $\mathcal{H}^p(M, g)$ the space of complex valued g-harmonic $(m + p)$-forms,

$$\mathcal{H}^p(M, g) := \ker(d + d^*) \cap \Omega^{m+p}(M) \otimes \mathbb{C} \cong H^{m-p}(M, \mathbb{C}).$$

Then γ defines an isomorphism $\gamma : \mathcal{H}^{-p}(M, g) \to \mathcal{H}^p(M, g)$. We get a decomposition into γ-invariant subspaces

$$\ker(d + d^*) = \bigoplus_{p=0}^{m} K_p,\ \ K_0 = \mathcal{H}^0(M, g) \cong H^m(M, \mathbb{C}),$$

$$K_p = \mathcal{H}^{-p}(M, g) \oplus \mathcal{H}^p(M, g) \cong H^{m-p}(M, \mathbb{C}) \oplus H^{m+p}(M, \mathbb{C}).$$

We deduce

$$\dim_{\mathbb{C}} \ker D = \sum_{p\geq 0} \dim \ker_{\mathbb{C}}(\mathbb{1}_{K_p} - \gamma),\ \ \dim_{\mathbb{C}} \ker D = \sum_{p\geq 0} \dim_{\mathbb{C}} \ker(\mathbb{1}_{K_p} + \gamma).$$

For $p > 0$ we have another involution ϵ on K_p

$$\epsilon_p = \mathbb{1}_{\mathcal{H}^{-p}} \oplus -\mathbb{1}_{\mathcal{H}^p}.$$

Note that $\gamma|_{K_p}$ anticommutes with ϵ_p. This implies that ϵ_p induces an isomorphism

$$\epsilon_p : \ker(\mathbb{1}_{K_p} - \gamma) \to \ker(\mathbb{1}_{K_p} + \gamma)$$

so that

$$\dim_{\mathbb{C}} \ker D - \dim_{\mathbb{C}} \ker D^* = \dim_{\mathbb{C}} \ker(\mathbb{1}_{K_0} - \gamma) - \dim_{\mathbb{C}} \ker(\mathbb{1}_{K_0} + \gamma).$$

Observe that K_0 is the complexification of the real vector space of real valued g-harmonic m-forms and as such it its equipped with a $\mathbb{R}$-*linear* involution, the *conjugation.* We will denote this operator by C. We will compute

$$\dim_{\mathbb{R}} \ker(\mathbb{1}_{K_0} - \gamma) - \dim_{\mathbb{R}} \ker(\mathbb{1}_{K_0} + \gamma) = 2 \operatorname{ind}_{\mathbb{C}} D.$$

At this point we have to consider two cases.

1. m *is odd.* Observe that for every complex valued m-form α we have

$$\gamma C\alpha = \mathbf{i}^{m^2} * \bar{\alpha} = (-1)^m \overline{\mathbf{i}^{m^2} * \alpha} = -C\gamma\alpha.$$

This shows that C defines an isomorphism of *real vector spaces*

$$C : \ker(\mathbb{1}_{K_0} - \gamma) \to \ker(\mathbb{1}_{K_0} + \gamma)$$

which shows that in this case

$$\operatorname{ind}_{\mathbb{C}} D = 0.$$

2. *m is even.* Then $\gamma\,|_{K_0} = \mathbf{i}^{m^2} * = *$ and it particular γ commutes with the conjugation. Denote by $\mathbb{H}^m(M,g)$ the space of real g-harmonic m-forms on M so that

$$K_0 = \mathbb{H}^m(M,g)\otimes\mathbb{C}.$$

We deduce

$$\dim_{\mathbb{C}}\ker(\mathbb{1}_{K_0}\pm\gamma) = \dim_{\mathbb{R}}(\mathbb{1}_{\mathbb{H}^m(M,g)}\pm\gamma).$$

The vector space $\mathbb{H}^m(M,g)$ is equipped with a symmetric bilinear form

$$\mathbb{H}^m(M,g)\times\mathbb{H}^m(M,g)\ni(u,v)\mapsto I(u,v)=\int_M u\wedge v.$$

Moreover

$$I(u,\gamma v) = (u,v)_{L^2}\Longrightarrow I(u,v) = (u,\gamma v)_{L^2}.$$

The eigenvalues of I (with respect to an orthonormal basis of $\mathbb{H}^m$) are thus the same as the eigenvalues of γ, i.e. ±1. Thus the difference between the dimension of the 1-eigenspace of γ and the dimension of the -1-eigenspace of γ is the signature of the symmetric bilinear form I. The Poincaré duality shows that I is precisely the intersection form of the manifold M and the signature of I is a topological invariant, namely the signature $\operatorname{sign}(M)$ of M. We conclude

$$\operatorname{ind}_{\mathbb{C}} D = \operatorname{sign}(M)$$

To express the index as an integral quantity we need to find an explicit description of the $\operatorname{str}^{\mathcal{E}/\mathbb{S}}(\frac{\mathbf{i}}{2\pi}F^{\mathcal{E}/\mathbb{S}})$.

Observe that $F^{\mathcal{E}/\mathbb{S}} = F^{E/\mathbb{S}}$. The only difference between this situation and the Gauss-Bonnet situation encountered earlier is in the choice of gradings.

Lemma 3.1.4. *Using the same notations as in Lemma 3.1.2 we have*

$$\operatorname{str}^{\mathcal{E}/\mathbb{S}}\exp\beta_R = 2^m\frac{\hat{\mathbf{L}}(-\frac{1}{4\pi}R_g)}{\hat{\mathbf{A}}(-\frac{1}{2\pi}R_g)},$$

where we recall that $\mathbf{L}$ *denotes the genus determined by* $L(x)=\frac{x}{\tanh x}$.

Proof Using Corollary 2.2.9 we deduce

$$\operatorname{str}^{\mathcal{E}/\mathbb{S}}\exp\beta_R = \frac{\mathbf{i}^m}{2^m}\operatorname{str}^{\mathcal{E}}\Gamma\exp\beta_R = \frac{\mathbf{i}^m}{2^m}\operatorname{tr}^{\mathcal{E}}\gamma\Gamma\exp\beta_R.$$

Using the equality (3.1.7) we deduce that $\mathbf{i}^m\Gamma\gamma=\mathbb{1}_{\mathcal{E}}$ so that

$$\operatorname{str}^{\mathcal{E}/\mathbb{S}}\exp\beta_R = \frac{1}{2^m}\operatorname{tr}^{E}\exp\beta_R.$$

To compute this trace we choose as in the proof of Lemma 3.1.2 an oriented, orthonormal basis $\{E^1,f^1,\cdots,e^m,f^m\}$ of V_x such that

$$Re^j=\lambda_jf^j,\ \ Rf^j=-\lambda f^j.$$

We deduce again that

$$\exp\beta_R=\prod_{j=1}^m\Big(\cosh(\frac{\lambda_j}{4\pi})-\mathbf{i}B_j\sinh(\frac{\lambda_j}{4\pi})\Big)$$

Set again

$$V_j:=\operatorname{span}_{\mathbb{C}}(e^j,f^j),\ \ E_j:=\Lambda^\bullet V_j.$$

We deduce

$$\operatorname{tr}^{\mathcal{E}} \exp \beta_R = \prod_{j=1}^{m} \operatorname{tr}^{E_j} \Big(\cosh(\frac{\lambda_j}{4\pi}) - \mathbf{i} B_j \sinh(\frac{\lambda_j}{4\pi}) \Big)$$

Since $\operatorname{tr}^{E_j} B_j = 0$ we deduce

$$\operatorname{str}^{\mathcal{E}/\mathbb{S}} \exp \beta_R = \frac{1}{2^m} \prod_{j=1}^{m} \cosh(\frac{\lambda_j}{4\pi}) \dim E_j = 2^m \cosh(-\frac{1}{4\pi} R) = 2^m \frac{\hat{\mathbf{L}}(-\frac{1}{4\pi} R_g)}{\hat{\mathbf{A}}(-\frac{1}{2\pi} R_g)}.$$

□

Putting together the facts obtained so far we obtain the following important result.

Theorem 3.1.5 (Hirzebruch signature theorem). *Suppose (M, g) is a compact, oriented Riemann manifold without boundary such that $\dim M = 4k$. Then*

$$\operatorname{sign}(M) = 2^{2k} \int_M \mathbf{L}(-\frac{1}{4\pi} R_g) = \int_M \mathbf{L}(-\frac{1}{2\pi} R_g).$$

In particular when $\dim M = 4$ we obtain

$$\operatorname{sign}(M) = \frac{1}{3} \int_M p_1(M) = -\frac{1}{24\pi^2} \int_M \operatorname{tr}(R_g \wedge R_g),$$

where $R_g \in \Omega^2(\operatorname{End} TM)$ denotes the Riemann curvature tensor, and $p_1(M)$ denotes the first Pontryagin class of the tangent bundle of M.

Example 3.1.6. We would like to discuss an amusing consequence of the signature theorem. The Poincaré duality shows that the Betti numbers of a compact, connected, oriented n-dimensional manifold M satisfy the symmetry conditions

$$b_k(M) = b_{n-k}(M).$$

If we form the Poincaré polynomial of M

$$P_M(t) = 1 + b_1(M)t + \cdots + b_{n-1}(M)t^{n-1} + t^n$$

then we see that the coefficients of this polynomial are symmetrically distributed. It is more convenient to consider the polynomial

$$Q_M(t) = t^{-n/2} + b_1(M)t^{-n/2+1} + \cdots + t^{n/2}.$$

The Poincaré duality then shows that

$$Q_M(1/t) = Q_M(t).$$

For example

$$Q_{S^4} = t^{-2} + t^2, \;\; Q_{\mathbb{CP}^2} = t^{-2} + 1 + t^2, \;\; Q_{S^{2m}} = t^{-m} + t^m.$$

Observe that

$$Q_{\mathbb{CP}^2} - Q_{S^4} = 1.$$

We can ask if for every $m > 0$ we can find an oriented manifold X of dimension $2m$ such that

$$Q_X - Q_{S^{2m}} = 1. \tag{3.1.8}$$

Let us point out that if $m = 2k+1$ so that $n = 4k+2$, then the intersection form on the middle cohomology group $H^{2k+1}(X, \mathbb{R})$ is skew-symmetric, and non-degenerate according to the Poincaré duality. In particular the middle Betti number $b_{2k+1}(X)$ must be even so that

$$1 \neq b_{2k+1}(X) + b_{2k}(X)(t + t^{-1}) + \cdots + b_1(X)(t^{2k} + t^{-2k}) = Q_X - Q_{S^{4k+2}}.$$

Thus the "equation" (3.1.8) does not have a solution when m is odd. We can refine our question and ask if it has a solution for every even m. For the smallest possible choice of m the answer is positive and $X = \mathbb{CP}^2$ is such a solution. We want to show that for $m = 6$ we cannot find a solution either, but for different other reasons.

Suppose X is a 12-dimensional manifold "solving" the equation (3.1.8). This means

$$Q_X = t^{-6} + 1 + t^6 \Longleftrightarrow b_k(X) = \begin{cases} 0 & \text{if} \quad k \neq 0, 6, 12 \\ 1 & \text{if} \quad k = 0, 6, 12 \end{cases}$$

In particular $H^k(X, \mathbb{R})$ for $k \neq 0, 6, 12$. From the signature theorem we deduce

$$\operatorname{sign}(X) = \int_X \mathbf{L}_{12}(X)$$

where $\mathbf{L}_{12}$ denotes the degree 12 part of the $\mathbf{L}$-genus. We have (see [11])

$$\mathbf{L}_{12}(X) = \frac{2 \cdot 31}{3^3 \cdot 5 \cdot 7}\big(p_3(X) - 13p_2(X)p_1(X) + 2p_1^3(X)\big).$$

The Pontryagin classes $p_1(X) \in H^4(X, \mathbb{R})$ and $p_2(X) \in H^8(X, \mathbb{R})$ vanish so that

$$\operatorname{sign}(X) = \frac{2 \cdot 31}{3^3 \cdot 5 \cdot 7}\int_X p_3(X).$$

On the other hand[2]

$$\int_X p_3(X) \in \mathbb{Z}$$

and we deduce that the signature of X must be divisible by 62. On the other hand, the signature of X is the signature of the intersection form on the *one-dimensional space* $H^6(X, \mathbb{R})$ so that this signature can only be ± 1. We reached a contradiction!

For more examples of this nature we refer to J. P. Serre, "*Travaux de Hirzebruch sur la topologie des variétés*", Séminaire Bourbaki 1953/54, n° **88**. □

3.1.3. The Hodge-Dolbeault operators. Suppose M is a connected manifold. An *almost complex structure* on M is a an endomorphism $J : TM \to TM$ such that

$$J^2 = -\mathbb{1}.$$

An *almost complex* manifold is a manifold equipped with an almost complex structure. The existence of an almost complex structure imposes restrictions on the manifold.

Proposition 3.1.7. *Suppose (M, J) is an almost complex manifold. Then $n = \dim \mathbb{R}$ is even $n = 2m$ and the tangent bundle TM admits a* $\mathrm{GL}(m, \mathbb{C})$*-structure. More precisely if we denote by ρ the canonical inclusion*

$$\mathrm{GL}_m(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2m}(\mathbb{R})$$

then there exists a principal $\mathrm{GL}_m(\mathbb{C})$*-bundle $P \to M$ such that*

$$TM \cong P \times_\rho \mathbb{R}^{2m}.$$

[2] For a proof of this fact we refer to [15].

From Example 1.1.13(h) and the above proposition we deduce that an almost complex manifold is orientable. There is a canonical way of choosing an orientation of TM. To describe it we need to indicate a basis of $\det T_xM$ at some point $x \in M$. We do this by choosing a basis $e_1, f_1, \cdots, e_m, f_m$ of T_xM adapted to J, i.e.

$$Je_k = f_k, \;\; Jf_k = -e_k, \;\; \forall k = 1, \cdots, m. \tag{3.1.9}$$

Then the canonical orientation is determined by $e_1 \wedge f_1 \wedge \cdots \wedge e_m \wedge f_m \in \det T_xM$. One can check easily that if $\tilde{e}_1, \tilde{f}_1, \cdots \tilde{e}_m, \tilde{f}_m$ is another basis adapted to J then we can find a *positive* scalar c such that

$$\tilde{e}_1 \wedge \tilde{f}_1 \wedge \cdots \wedge \tilde{e}_m \wedge \tilde{f}_m = c\big(e_1 \wedge f_1 \wedge \cdots \wedge e_m \wedge f_m\big),$$

so that this orientation is independent of the choice of adapted basis. We will refer to this as the *complex orientation*.

Example 3.1.8. Any complex manifold (i.e. a manifold which is described by charts with holomorphic transition maps) carries a natural almost complex structure. An almost complex structure produced in this fashion is called *integrable* □

If (M, J) is an almost complex manifold, we define a structure on $C^\infty(M, \mathbb{C})$-module on $\mathrm{Vect}(M)$ by setting

$$(u + \mathbf{i}v) \cdot X = uX + vJX, \;\; \forall u, v \in C^\infty(M, \mathbb{R}), \;\; X \in \mathrm{Vect}(M).$$

By duality we get an operator $J^\vee : T^*M \to T^*M$ satisfying $(J^\vee)^2 = -\mathbb{1}$. The complexified cotangent bundle $T^*M^c := T^*M \otimes \mathbb{C}$ admits a decomposition

$$T^*M^c = T^*M^{1,0} \oplus T^*M^{0,1}, \;\; T^*M^{1,0} = \ker(\mathbf{i} - J^\vee), \;\; T^*M^{0,1} = \ker(-\mathbf{i} - J^\vee).$$

In particular, for every k we have a decomposition

$$\Lambda^k T^*M^c = \bigoplus_{p+q=k} \underbrace{\Lambda^p T^*M^{1,0} \oplus \Lambda^q T^*M^{0,1}}_{:=\Lambda^{p,q}T^*M}.$$

We set

$$\Omega^{p,q}(M) := C^\infty(\Lambda^{p,q}T^*M).$$

The elements of $\Omega^{p,q}(M)$ are called (p,q)-*forms* on M. The bundle

$$\det_{\mathbb{C}} TM^{0,1} = \Lambda^{0,m}TM \cong \Lambda^{m,0}T^*M, \;\; 2m = \dim_{\mathbb{R}} M$$

is called the *canonical line bundle* of the almost complex manifold M and it is denoted by K_M. It is a complex line bundle and its sections are $(m, 0)$-forms on M.

For any $\alpha \in \Omega^{p,q}(M) \subset \Omega^k(M) \otimes \mathbb{C}$, $k = p + q$, we have

$$d\alpha \in \bigoplus_{p'+q'=k+1} \Omega^{p',q'}(M).$$

In particular $d\alpha$ will have a component in $\Omega^{p+1,q}(M)$ which we denote by $\partial\alpha$ and a component in $\Omega^{p,q+1}(M)$ which we denote by $\bar{\partial}$.

For a proof of the following result we refer to [12, IX,§2].

Proposition 3.1.9 (Nirenberg-Newlander)**.** *Suppose (M, J) is an almost complex manifold. Then the following conditions are equivalent.*

(a) The almost complex structure is integrable.

(b) For every p, q and every $\alpha \in \Omega^{p,q}$ we have

$$d\alpha = \partial\alpha + \bar{\partial}\alpha.$$

An *almost Hermitian structure* on M is a pair (g, J), where g is a Riemann metric and J is an almost complex structure such that $J^* = -J$, i.e. J is an orthogonal endomorphisms. To any almost Hermitian (g, J) structure we can associate a 2-form

$$\omega \in \Omega^2(M),\ \ \omega(X, Y) = g(JX, Y),\ \ \forall X, Y \in \mathrm{Vect}(M).$$

The metric g defines a Hermitian metric $h : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to C^\infty(M, \mathbb{C})$ on TM by setting

$$h(X, Y) := g(X, Y) - \mathbf{i}\omega(X, Y) \in C^\infty(M, \mathbb{C}),\ \ \forall X, Y \in \mathrm{Vect}(M).$$

One can check that

$$h(aX, bY) = a\bar{b} \cdot h(X, Y)\ \ \forall a, b \in C^\infty(M, \mathbb{C}),\ \ X, Y \in \mathrm{Vect}(M). \tag{3.1.10}$$

We can run the above arguments in reverse and deduce the following fact.

Proposition 3.1.10. *Suppose (M, g) is an almost complex manifold. Suppose $\omega \in \Omega^2(M)$ is a 2-form adapted to J i.e.*

$$\omega(X, JX) > 0,\ \ \omega(X, JY) = \omega(Y, JX),\ \ \forall X, Y \in \mathrm{Vect}(M) \setminus 0.$$

Then $g(X, Y) := \omega(X, JY)$ defines an almost Hermitian structure on (M, J) with associated 2-form ω.

Example 3.1.11. The standard almost Hermitian structure. Consider the Euclidean vector space $\mathbb{R}^{2m} = \mathbb{R}^m \oplus \mathbb{R}^m$ equipped with the almost complex structure

$$J = \begin{bmatrix} 0 & -\mathbb{1}_{\mathbb{R}^m} \\ \mathbb{1}_{\mathbb{R}^m} & 0 \end{bmatrix}.$$

Denote by $e_1, \cdots, e_m$ the canonical basis of the first summand $\mathbb{R}^m$ in $\mathbb{R}^m \oplus \mathbb{R}^m$ and set $f_k = Je_k$. The basis $e_1, f_1, \cdots, e_k, f_k$ is orthonormal and we denote by $e^1, f^1, \cdots, e^k, f^k$ the dual basis of $(\mathbb{R}^{2m})^*$. We regard e^j, f^j as functions of $\mathbb{R}^{2m}$. The Euclidean metric has the description

$$g = \sum_k (e^k \otimes e^k + f^k \otimes f^k)$$

The associated 2-form satisfies

$$\omega(e_i, f_j) = \delta_{ij},\ \ \omega(e_i, e_j) = \omega(f_i, f_j) = 0$$

so that

$$\omega = e^1 \wedge f^1 + \cdots + e^k \wedge f^k.$$

We set

$$\varepsilon^k := \frac{1}{\sqrt{2}}(e^k + \mathbf{i}f^k),\ \ \bar{\varepsilon}^k = \frac{1}{\sqrt{2}}(e^k - \mathbf{i}f^k).$$

Then the associated hermitian metric h has the form

$$h = 2\sum_k \varepsilon^k \otimes \bar{\varepsilon}^k = \sum_k (e^k \otimes e^k + f^k \otimes f^k) - \mathbf{i}\sum_k e^k \wedge f^k.$$

We deduce

$$g = \mathbf{Re}\, h, \quad \omega = -\,\mathbf{Im}\, h = \mathbf{i}\sum_k \varepsilon^k \wedge \bar{\varepsilon}^k.$$

□

Definition 3.1.12. (a) An almost Hermitian structure (g, J) on M is called *almost Kähler* if the associated 2-form is closed .

(b) An almost Kähler structure (g, J) is called *Kähler* if the almost complex structure is integrable. □

We have the following sequence of implications

$$\text{Kähler} \Longrightarrow \text{almost Kähler} \Longrightarrow \text{almost Hermitian} \Longrightarrow \text{almost complex.}$$

For a proof of the following result we refer to [12, IX§4].

Proposition 3.1.13. *Suppose (M, g, J) is an almost Kähler manifold. Denote by ∇^g the Levi-Civita connection on TM. Then (M, g, J) is Kähler if and only if $\nabla^g J = 0$, i.e.*

$$\nabla^g_X(JY) = J(\nabla^g_X Y), \ \ \forall X, Y \in \mathrm{Vect}(M).$$

Example 3.1.14. (a) **(The standard (Euclidean) Kähler metric)** The vector space $\mathbb{C}^n$ equipped with the natural complex structure and hermitian metric h is a Kähler manifold. If we denote by $z^k = x^k + \mathbf{i}y^k$ the natural complex coordinates, and we set

$$e_k := \frac{\partial}{\partial x^k}, \quad f_k := \frac{\partial}{\partial y^k}$$

then we have

$$e^k = dx^k, \quad f^k = dy^k, \quad \varepsilon^k = \frac{1}{\sqrt{2}}dz^k, \quad \bar{\varepsilon}^k = \frac{1}{\sqrt{2}}d\bar{z}^k$$

so that

$$h = \sum_k dz^k \otimes d\bar{z}^k, \quad \omega = \frac{\mathbf{i}}{2}\sum_k dz^k \wedge d\bar{z}^k.$$

We set

$$\partial_{z^k} = \frac{1}{2}(\partial_{x^k} - \mathbf{i}\partial_{y^k}), \quad , \partial_{\bar{z}^k} = \frac{1}{2}(\partial_{x^k} + \mathbf{i}\partial_{y^k}).$$

Then

$$\partial = \sum_k dz^k \wedge \partial_{z^k}, \quad \bar{\partial} = \sum_k \bar{z}^k \wedge \partial_{\bar{z}^k}.$$

(b) Suppose Σ is a compact oriented Riemann surface equipped with a Riemann metric on M. Then the Hodge $*$-operator induces an operator

$$* : T^*\Sigma \to T^*\Sigma, \quad *^2 = -1.$$

By duality this induces an almost complex structure on Σ. We obtain in this fashion an almost Hermitian structure $(g, *)$ on Σ. The associated 2-form is the volume form dV_g which must be closed since its differential is zero due to dimensional constraints. We deduce that this structure is almost Kähler. Dimensional constraints imply

$$d = \partial + \bar{\partial}$$

so that by Proposition 3.1.9 this structure is also Kähler.

(c)**(The Fubini-Study metric)** Consider the projective space $\mathbb{CP}^n$. Recall that this is defined as a quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ with respect to the natural action of $\mathbb{C}^*$. Set $Z = (z^0, \cdots, z^n)$, and

$$|Z|^2 = \sum_{k=0}^{n} |z^k|^2, \;\; \omega := \frac{\mathbf{i}}{2\pi} \partial\bar{\partial} \log |Z|^2 = \Omega^{1,1}(\mathbb{C}^{n+1} \setminus 0).$$

For every holomorphic function f defined on an open set $U \subset \mathbb{C}^{n+1} \setminus 0$ we have

$$\log |f|^2 |Z|^2 = \log |f|^2 + \log |Z|^2 = \log(f\bar{f}) + \log |Z|^2.$$

and a simple computation shows that

$$\partial\bar{\partial} \log(f\bar{f}) = 0.$$

In particular, this shows that for $z^k \neq 0$ if we set

$$\vec{\zeta}_k = (z^0/z^k, \cdots, z^{k-1}/z^k, z^{k+1}/z^k, \cdots, z^n/z^k)$$

we have

$$\omega_0 = \frac{\mathbf{i}}{2\pi} \partial\bar{\partial} \log(1 + |\vec{\zeta}_k|^2).$$

The vector $\vec{\zeta}_k$ defines local coordinates on the region

$$U_k = \Big\{ [z^0, \cdots, z^n] \in \mathbb{CP}^n; \;\; z^k \neq 0 \Big\}.$$

The above equality shows that on the overlap $U_j \cap U_k$ we have

$$\frac{\mathbf{i}}{2\pi} \partial\bar{\partial} \log(1 + |\vec{\zeta}_k|^2) = \frac{\mathbf{i}}{2\pi} \partial\bar{\partial} \log(1 + |\vec{\zeta}_j|^2)$$

so that the collection of forms $\frac{\mathbf{i}}{2\pi} \partial\bar{\partial} \log(1 + |\vec{\zeta}_k|^2)$ defines a global $(1,1)$-form on $\mathbb{CP}^n$. This is called the *Fubini-Study form*. We will denote it by Ω_{FS}

Observe that Ω_{FS} is closed and it is invariant with respect to the action of $U(n+1)$ on $\mathbb{CP}^n$. If we write generically $\vec{\zeta} = (\zeta^1, \cdots, \zeta^n)$ and

$$\Omega_{FS} = \frac{\mathbf{i}}{2\pi} \partial\bar{\partial} \log\big(1 + \sum_j |\zeta^j|^2 \big)$$

we deduce that

$$\Omega_{FS} = \frac{\mathbf{i}}{2\pi(1 + |\vec{\zeta}|^2)^2} \left((1 + |\vec{\zeta}|^2) \sum_j d\zeta^j \wedge d\bar{\zeta}^j - \Big(\sum_j \bar{\zeta}^j d\zeta^j \Big) \wedge \Big(\sum_k \zeta^k d\bar{\zeta}^k \Big) \right) \tag{3.1.11}$$

Observe that at the point $P_0 \in \mathbb{CP}^n$ with coordinates $\vec{\zeta} = (1, 0, \cdots, 0$ we have

$$\Omega_{P_0} := \Omega_{FS} \mid_{T_{P_0}\mathbb{CP}^n} = \frac{\mathbf{i}}{4\pi} \Big(d\zeta^1 \wedge d\bar{\zeta}^1 + 2 \sum_{k>1} d\zeta^k \wedge d\bar{\zeta}^k \Big).$$

In particular, arguing as in (a) we deduce that for every $X, Y \in T_{P_0}\mathbb{CP}^n \setminus 0$

$$\Omega_{FS}(X, \mathbf{i}X) > 0, \;\; \Omega_{FS}(X, \mathbf{i}Y) = \Omega_{FS}(Y, \mathbf{i}X).$$

Using Proposition 3.1.10 we deduce that Ω_{FS} defines an almost Kähler structure on $\mathbb{CP}^n$. Since the underlying almost complex structure is integrable we deduce that this structure is Kähler. It is known as the *Fubini-Study structure* .

(d) Any complex submanifold M of a Kähler manifold X has a natural Kähler structure induced from the structure on X. In particular, any complex submanifold of $\mathbb{CP}^n$ has a natural Kähler structure induced by the Fubini-Study theorem. Chow's Theorem (see [**10**, Chap.I,§3]) implies that every complex submanifold of $\mathbb{CP}^n$ is algebraic, i.e. is can be described as the vanishing locus of a finite collection of homogeneous polynomials. Thus the projective algebraic manifolds admit Kähler structures.

□

Suppose M is a complex manifold with induced almost complex structure $J : TM \to TM$. The complexified tangent bundle $TM^C = TM \otimes \mathbb{C}$ admits a decomposition

$$TM^c = TM^{1,0} \oplus TM^{0,1}, \;\; TM^{1,0} = \ker(\mathbf{i} - J), \;\; TM^{0,1} = \ker(-\mathbf{i} - J).$$

Moreover is equipped with an involution

$$TM^c \to TM^c, \;\; v \longmapsto \bar{v}$$

which is $\mathbb{R}$-linear and maps $TM^{1,0}$ to $TM^{0,1}$. We have the following result whose proof is left as an exercise.

Proposition 3.1.15. *The complex manifold M admits a Kähler structure if and only if it admits a* positive, closed, $(1,1)$-form*, i.e. a closed form $\omega \in \Omega^{1,1}(M)$ such that*

$$\omega(v, \bar{v}) :=, \;\; \forall v \in T_xM^c \setminus 0, \;\; x \in M.$$

In this case the Riemann metric on TM is defined by

$$g(X, Y) = \omega(X, \mathbf{i}Y), \;\; \forall X, Y \in \mathrm{Vect}(M).$$

□

Definition 3.1.16. A *holomorphic structure* on a rank r complex vector bundle $\pi : E \to M$ over a complex manifold M is a trivializing cover (U_α) together with local trivializations

$$\Psi_\alpha : E\,|_{U_\alpha} \to \underline{\mathbb{C}}_{U_\alpha}$$

such that the transition maps

$$\Psi_\beta \circ \Psi_\alpha^{-1} : \underline{\mathbb{C}}_{U_{\alpha\beta}} \to \underline{\mathbb{C}}_{U_{\alpha\beta}}$$

are biholomorphic. A holomorphic vector bundle is a pair

(vector bundle E, holomorphic structure on E).

Two holomorphic bundles over the same complex manifold are isomorphic if there exists a *biholomorphic* bundle isomorphism between them. □

Example 3.1.17. (a) If M is a complex manifold then the trivial line bundle $\underline{\mathbb{C}}_M$ admits a trivial holomorphic structure. A holomorphic line bundle isomorphic to the trivial line bundle is called holomorphically trivial. We want warn the reader that there exist complex line bundles can be trivialized topologically but *cannot be trivialized holomorphically* .

(b) If M is a complex manifold then the bundles $\Lambda^{p,q}T^*M$ are equipped with natural holomorphic structures.

(c) A holomorphic line bundle over a complex manifold is uniquely determined by an open cover (U_α) and a holomorphic gluing cocycle

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathbb{C}^*.$$

We deduce that the tautological line bundle over $\mathbb{CP}^n$ is equipped with a natural holomorphic structure.

(d) All the tensorial operations on bundles transform holomorphic vector bundles to holomorphic vector bundles. Similarly, the pullback of a holomorphic vector bundle via a holomorphic map is a holomorphic vector bundle.

We denote by $\mathrm{Pic}\,(M)$ the collection of isomorphism classes of holomorphic line bundles over the complex manifold M. The tensor product induces a group structure on $\mathrm{Pic}\,(M)$ with identity element $\underline{\mathbb{C}}_M$ and inverse $L^{-1} := L^*$. This group is known as the *Picard group* of M.

□

Definition 3.1.18. Suppose M is a complex manifold and $E \to M$ is a complex vector bundle. We set

$$\Omega^{p,q}(E) := C^\infty(\Lambda^{p,q}T^*M \otimes_{\mathbb{C}} E).$$

□

Proposition 3.1.19. *Suppose $E \to M$ is a rank r holomorphic vector bundle over a complex manifold M. Then E is equipped with a canonical CR (Cauchy-Riemann) operator, i.e. a $\mathbb{C}$-linear operator*

$$\bar{\partial}_E : \Omega^{0,0}(E) \to \Omega^{0,1}(E)$$

such that for every smooth function $f : M \to \mathbb{C}$ and every smooth section u of E we have

$$\bar{\partial}_E(fu) = (\bar{\partial}f) \otimes u + f(\bar{\partial}_E u).$$

□

Proof Suppose that the bundle E has the gluing description

$$E = (U_{\bullet\bullet}, g_{\bullet\bullet}, \mathrm{GL}_r(\mathbb{C}))$$

where the maps $g_{\bullet\bullet} : U_{\bullet\bullet}\, \mathrm{GL}_r(\mathbb{C}) \subset \mathbb{C}^{r^2}$ are holomorphic. Then a smooth section u of E is defined by a collection of smooth maps $u_\alpha : U_\alpha \to \mathbb{C}^r$ satisfying the gluing conditions

$$u_\beta(x) = g_{\beta\alpha}(x) \cdot u_\alpha(x), \;\; \forall \alpha, \beta, \;\; x \in U_{\alpha\beta}.$$

Define

$$v_\alpha = \bar{\partial} u_\alpha.$$

Observe that on the overlap $U_{\alpha\beta}$ we have

$$v_\beta = \bar{\partial} u_\beta = \bar{\partial}(g_{\beta\alpha} u_\alpha) = (\bar{\partial} g_{\beta\alpha}) u_\alpha + g_{\beta\alpha} \bar{\partial} u_\alpha.$$

Since $g_{\beta\alpha}$ is holomorphic we deduce $\bar{\partial} g_{\beta\alpha} = 0$ and thus

$$v_\beta = g_{\beta\alpha} \bar{\partial} u_\alpha = g_{\beta\alpha} v_\alpha.$$

Hence the collection (v_α) defines a global section v of $T^*M^{0,1} \otimes E$ and we set

$$\bar{\partial}_E u := v.$$

This definition implies immediately that $u \mapsto \bar{\partial}_E u$ is a CR operator.

□

Definition 3.1.20. Suppose $E \to M$ is a holomorphic vector bundle over the complex manifold M. A smooth section u of E defined over the open subset $V \subset M$ is called *holomorphic* if $\bar{\partial}_E u = 0$. We denote by $\mathcal{O}(V, E)$ the space of holomorphic sections of $E|_V$. □

Iterating the construction in Proposition 3.1.19 we obtain for every $p \in \mathbb{Z}_{\geq 0}$ a sequence

$$0 \to \Omega^{p,0}(E) \xrightarrow{\bar{\partial}_E} \Omega^{p,1}(E) \to \cdots \to \Omega^{p,q}(E) \xrightarrow{\bar{\partial}_E} \Omega^{p,q+1}(E) \to \cdots . \tag{3.1.12}$$

From the definition of $\bar{\partial}_E$ it follows that $\bar{\partial}_E^2 = 0$ so that (3.1.12) is a cochain complex. It is known as the pt-th *Dolbeault complex* of E. We will denote it by $\Omega^{p,\bullet}(E), \bar{\partial}_E)$ the cohomology groups of this complex are denoted by

$$H^{p,q}_{\bar{\partial}}(E) = \frac{\ker\big(\bar{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)\big)}{\text{Range}\,\big(\bar{\partial}_E : \Omega^{p,q-1}(E) \to \Omega^{p,q}(E)\big)}.$$

Observe that $\bar{\partial}_E$ is a first order p.d.o, and for every $x \in M$ and every $\xi \in T^*M$ we have

$$\sigma_E(\xi) := \sigma(\bar{\partial}_E)(\xi) = \xi^{0,1}\wedge : \Lambda^{p,q}T^*_xM \otimes E_x \to \Lambda^{p,q+1}T^*_xM \otimes E,$$

where $\xi^{0,1}$ denotes the $T^*_xM^{0,1}$ component of ξ with respect to the canonical decomposition

$$T^*_xM^c \cong T^*_xM^{1,0} \oplus T^*_xM^{0,1}.$$

Lemma 3.1.21. *The p-th Dolbeault complex is an elliptic complex, i.e. for every $x \in M$ and every $\xi \in T^*_xM \setminus 0$ the symbol complex*

$$0 \to \Lambda^{p,0}T^*_xM \otimes E \xrightarrow{\sigma_E(\xi)} \Lambda^{p,1}T^*_xM \otimes E \to \cdots \to \Lambda^{p,q}T^*_xM \otimes E \xrightarrow{\sigma_E(\xi)} \Lambda^{p,q}T^*_xM \otimes E \to \cdots$$

is acyclic.

The proof is left as an exercise. Using the above lemma and the general Hodge Theorem 2.1.29 we deduce the following result.

Theorem 3.1.22 (Hodge)**.** *Suppose M is a compact complex manifold and $E \to M$ is a holomorphic line bundle. Then the cohomology groups of the p-th Dolbeault complex are finite dimensional. Moreover, for any hermitian metric h on TM and any hermitian metric h_E on E we have*

$$H^{p,q}_{\bar{\partial}}(E) \cong \Big\{\alpha \in \Omega^{p,q}(E);\;\; \bar{\partial}_E\alpha = \bar{\partial}^*_E\alpha = 0\Big\},$$

where $\bar{\partial}^*_E$ denotes the formal adjoint of $\bar{\partial}_E$ with respect to h and h_E.

We set

$$h^{p,q}(E) := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(E)$$

We will refer to these numbers as the *holomorphic Betti numbers* of E. These numbers are invariants of the holomorphic structure on E. If we vary the holomorphic structure while keeping the topological structure on E fixed these numbers could change. When $E = \underline{\mathbb{C}}_M$ we set

$$h^{p,q}(M) := h^{p,q}(\underline{\mathbb{C}}_M)$$

and we will refer to these as the *holomorphic Betti numbers of M*. We define the *holomorphic Poincaré polynomials*

$$\mathcal{H}^p_E(t) = \sum_q h^{p,q}t^q,\;\; \mathcal{H}_E(s,t) = \sum_p \mathcal{H}^p_E(t)s^p = \sum_{p,q} h^{p,q}(E)s^pt^q.$$

When $E = \underline{\mathbb{C}}_M$ we write $\mathcal{H}_M$ instead of $\mathcal{H}_{\underline{\mathbb{C}}_M}$.

To relate the Dolbeault complex with geometric Dirac operators we need to discuss another important concept.

Definition 3.1.23. Suppose $E \to M$ is a complex vector bundle over the complex manifold M. Then for every connection ∇ on E we define $\bar{\partial}_\nabla$ as the composition

$$\bar{\partial}_\nabla : C^\infty(E) \stackrel{\nabla}{\longrightarrow} C^\infty(T^*M^c \otimes E) \longrightarrow C^\infty(T^*M^{0,1} \otimes E).$$

We will refer to $\bar{\partial}_\nabla$ as the CR operator defined by the connection ∇. □

Proposition 3.1.24 (Chern)**.** *Suppose $E \to M$ is a holomorphic vector bundle over the complex manifold M. Then for every hermitian metric h on E there exists a unique hermitian connection ∇^h on E satisfying*

$$\bar{\partial}_{\nabla^h} = \bar{\partial}_E.$$

The connection ∇^h is known as the Chern connection determined by h.

Proof For every vector field $X \in C^\infty(TM^c)$ we denote by $\overline{X}$ its conjugate, by $X^{1,0}$ and $X^{0,1}$ its $(1,0)$ and respectively $(0,1)$-components. Suppose $X \in C^\infty(TM^c)$, $u, v \in C^\infty(E)$. Then for every hermitian connection ∇ on E we have

$$L_X h(u,v) = h(\nabla_X u, v) + h(u, \nabla_{\overline{X}} v)$$

since $h(-,-)$ is conjugate linear in the second variable. In particular

$$L_{X^{1,0}} h(u,v) = h(\nabla_{X^{1,0}} u, v) + h(u, \nabla_{X^{0,1}} v).$$

We deduce that

$$\partial h(u,v) = h((\nabla - \bar{\partial}_\nabla) u, v) + h(u, \bar{\partial}_\nabla v).$$

Hence

$$h(\nabla u, v) = \partial h(u,v) + h(\bar{\partial}_\nabla u, v) - h(u, \bar{\partial}_\nabla v).$$

This shows that ∇ is completely determined by the associated CR operator, and thus establishes the uniqueness claim. To prove the existence we use the last equality as a guide and define

$$h(u, \nabla^h v) = \partial h(u,v) + h(\bar{\partial}_E u, v) - h(u, \bar{\partial}_E v). \tag{3.1.13}$$

One can show that this defines indeed a hermitian connection on E.

□

Example 3.1.25. (a) Suppose M is a complex manifold and h is a Hermitian metric on TM. The metric h induces hermitian metrics on all the holomorphic bundles $\Lambda^{p,q}T^*M$. If the Levi-Civita is compatible with the complex structure on TM, i.e. if M is Kähler then the Levi-Civita connection induces hermitian connections on all these holomorphic bundles. Moreover, these induced connections are exactly the Chern connections determined by the corresponding metrics and holomorphic structures.

(b) Suppose $E \to M$ is a holomorphic vector bundle and (e_a) is a local *holomorphic* frame of E. We set

$$h_{ab} = h(e_a, e_b).$$

If (z^j) is local holomorphic coordinate system, using (3.1.13) we deduce

$$h(\nabla^h_{z^j} e_a, e_b) = \frac{\partial h_{ab}}{\partial z^j}.$$

If we write

$$\nabla^h_{z^j} e_a = \sum_c \Gamma^c_{ja} e_c$$

then we deduce

$$\sum_c \Gamma^c_{ja} h_{cb} = \frac{\partial h_{ab}}{\partial z^j}$$

so in matrix notation we can write

$$h \cdot \Gamma_j = \frac{\partial h}{\partial z^j} \Longleftrightarrow \Gamma_j = h^{-1}\frac{\partial h}{\partial z^j}. \tag{3.1.14}$$

The connection 1-form with respect to this frame is then

$$\Gamma = \sum_j \Gamma_j dz^j = h^{-1}\partial h.$$

The curvature is then given by

$$F = d\Gamma + \Gamma \wedge \Gamma = d(h^{-1}\partial h) + h^{-1}\partial h \wedge h^{-1}\partial h.$$

Using the identity

$$d(h^{-1}\partial h) = \partial(h^{-1}\partial h) + \bar\partial(h^{-1}\partial h) = -h^{-1}\partial h \wedge h^{-1}\partial h + \bar\partial\Gamma = -\Gamma\wedge\Gamma + \bar\partial\Gamma$$

we deduce

$$F = \bar\partial\Gamma = -h^{-1}\bar\partial h \wedge h^{-1}\partial h + h^{-1}\bar\partial\partial h \in \Omega^{1,1}(\mathrm{End}(E)). \tag{3.1.15}$$

□

Suppose M is a compact Kähler manifold with underlying Riemann metric g . We denote by ∇^g the hermitian connections induced by the Levi-Civita connection on $\Lambda^{\bullet,\bullet}T^*M$. Suppose $E \to M$ is a holomorphic vector bundle equipped with a hermitian metric. We denote by ∇^E the corresponding Chern connection.

As explained in §2.2.1, the hermitian vector bundle $\Lambda^{0,\bullet}T^*M$ is a bundle of Clifford modules in a natural way, where the Clifford multiplication is given by

$$\boldsymbol{c}(\alpha) = \sqrt{2}\big(\alpha^{0,1}\wedge - \alpha^{1,0}\lrcorner\big),\ \ \forall \alpha \in \Omega^1(M)\otimes\mathbb{C}.$$

where

$$\alpha^{1,0}\lrcorner\,\beta = g^{\mathsf{v}}_c(\alpha^{1,0},\beta),\ \ \forall\beta \in \Omega^1(M)\otimes\mathbb{C},$$

and g^{v}_c denotes the extension by complex bilinearity of the Riemann metric g^{v} on T^*M to a symmetric bilinear form on $T^*M\otimes\mathbb{C}$. Let us point out, that for every $x \in M$ the $\mathbf{Cl}(T^*_xM)$-module $\Lambda^{0,\bullet}T^*_xM$ is isomorphic to the *dual* of the complex spinor module $\mathbb{S}_{T^*_xM}$. In particular, this shows that the Clifford multiplication by a *real* 1-form is skew-hermitian. Tautologically, this Clifford multiplication is compatible with the Levi-Civita connection. We conclude that $(\Lambda^{0,\bullet}, \boldsymbol{c}, \nabla^g)$ is a Dirac bundle.

Proposition 3.1.26. *The geometric Dirac operator $\mathfrak{D}$ determined by the Dirac bundle $(\Lambda^{0,\bullet}, \boldsymbol{c}, \nabla^g)$ is equal to*

$$\sqrt{2}(\bar\partial + \bar\partial^*) : \Omega^{0,\bullet}(M) \to \Omega^{0,\bullet}(M).$$

Proof Fix a point $\mathbf{p}_0 \in M$. Since M is *Kähler* we can choose normal coordinates x^k, y^k near $\mathbf{p}_0$ such that

$$x^k(\mathbf{p}_0) = y^k(\mathbf{p}_0) = 0, \quad J\partial_{x^k} = \partial_{y^k}, \ \forall k.$$

Set

$$e_k = \partial_{x^k}, \ f_k = \partial_{y^k}, \ e^k = dx^k, \ f^k = dy^k, \ z^k = x^k + \mathbf{i}y^k.$$

$$\varepsilon^k = \frac{1}{\sqrt{2}}dz^k, \ \bar{\varepsilon}^k = \frac{1}{\sqrt{2}}d\bar{z}^k.$$

$$\varepsilon_k = \frac{1}{\sqrt{2}}\big(e_k - \mathbf{i}f_k\big) = \sqrt{2}\partial_{z^k}, \ \bar{\varepsilon}_k = \sqrt{2}\partial_{\bar{z}^k}$$

Then

$$\mathfrak{D} = \sqrt{2}\sum_k\big(\,\bar{\varepsilon}^k \wedge \nabla^g_{\bar{\varepsilon}_k} - \varepsilon^k \lrcorner \nabla^g_{\varepsilon_k}\,\big).$$

$$\bar{\partial} = \sum_k d\bar{z}^k \wedge \partial_{\bar{z}^k} = \sum_k \bar{\varepsilon}^k \wedge \partial_{\bar{\varepsilon}_k}.$$

We denote by $o(1)$ any bundle morphisms T such that $T(\mathbf{p}_0) = 0$. Since (x^k, y^k) are normal coordinates at $\mathbf{p}_0$ we deduce the following identities

$$\mathbf{div}_g(e_k) = \mathbf{div}_g(f_k) = o(1), \ \nabla^g_{\varepsilon_k} = \partial_{\varepsilon_k} + o(1), \ \nabla^g_{\bar{\varepsilon}_k} = \partial_{\bar{\varepsilon}_k} + o(1)$$

so that

$$(\nabla^g_{\bar{\varepsilon}_k})^* = \partial^*_{\bar{\varepsilon}_k} + o(1) = -\partial_{\varepsilon_k} + o(1) = -\nabla^g_{\varepsilon_k} + o(1) \Longrightarrow \partial^*_{\bar{\varepsilon}_k} = -\nabla^g_{\varepsilon_k} + o(1).$$

Using the equalities

$$\nabla_{e_i} f_j = \nabla_{f_j} e_i = 0 \ \text{ at } \mathbf{p}_0, \ \forall i, j,$$

we deduce

$$(\bar{\varepsilon}^k \wedge \partial_{\bar{\varepsilon}_k})^* = (\partial_{\bar{\varepsilon}_k})^*((\bar{\varepsilon}^k\wedge)^* = \big(-\nabla^g_{\varepsilon_k} + o(1)\,\big)\big(\varepsilon_k\lrcorner\,\big) = -\varepsilon_k \lrcorner \nabla^g_{\varepsilon_k} + o(1).$$

This implies that

$$\mathfrak{D} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*) + o(1).$$

The proposition now follows from the fact that the point p_0 was chosen arbitrarily.

□

We can twist this Dirac bundle with any other complex Hermitian vector bundle W equipped with hermitian connection A and we deduce that the corresponding geometric Dirac operator is

$$\mathfrak{D}_W = \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*) : \Omega^{0,\bullet}(W) \to \Omega^{0,\bullet}(W).$$

In particular if we tensor with $\Lambda^{p,0}T^*M \otimes E$, where $\Lambda^{p,0}T^*M$ is equipped with the Levi-Civita connection and E is equipped with the Chern connection we deduce that the geometric Dirac operator associated to the Dirac bundle $\Lambda^{p,\bullet}T^*M \otimes E$ is

$$\mathfrak{D}_{E,p} = \sqrt{2}(\bar{\partial}_E + \bar{\partial}_E^*).$$

In particular, we deduce that

$$\operatorname{ind}\mathfrak{D}_{E,p} = \sum_{q\geq 0}(-1)^q h^{p,q}(E) =: \chi_p(E).$$

When E is the trivial line bundle we set

$$\chi_p(E) =: \chi_p(M) \sum_{q\geq 0}(-1)^q h^{p,q}(M).$$

Theorem 3.1.27 (Riemann-Roch-Hirzebruch)**.** *Suppose (M,g) is a Kähler manifold, $\dim_{\mathbb{R}} M = 2m$, and $E \to M$ is a holomorphic vector bundle equipped with a hermitian metric. Then*

$$\chi_0(E) = \int_M \mathbf{td}\,(M)\cdot \mathbf{ch}\,(E),$$

where $\mathbf{td}\,(M)$ denotes the Todd genus of $TM^{1,0}$ and $\mathbf{ch}\,(E)$ denotes the Chern character of E.

Proof We consider first the case when E is the (holomorphically) trivial line bundle. We have to show that

$$\chi_0(M) = \sum_{q\geq 0}(-1)^q h^{0,q}(M) = \int_M \mathbf{td}\,(M).$$

Consider the Dirac bundle $(\mathcal{E},\nabla) = (\Lambda^{0,*}T^*M, \nabla^g)$. We denote by R the Riemann curvature tensor and by $F^{\mathcal{E}}$ the curvature of ∇. Fix a point $\mathbf{p}_0\in M$, normal coordinates (x^k,y^k) at $\mathbf{p}_0$ and define as before

$$e_k = \partial_{x^k},\ \ , f_k = \partial_{y^k},\ \ e^k = dx^k,\ \ f^k = dy^k,\ \ z^k = x^k + \mathbf{i}y^k,\ \ 1\leq k\leq m.$$

We set $e_{i+m} := f_i$. The twisting curvature of ∇ is

$$F^{\mathcal{E}/\mathbb{S}} = F^{\mathcal{E}}(X,Y) - \boldsymbol{c}(R) \in \Omega^2(\mathrm{End}(\mathcal{E})),$$

where according to (2.2.6) we have

$$\boldsymbol{c}(R)(X,Y) = \frac{1}{4}\sum_{1\leq k,\ell\leq 2m} g\big(R(X,Y)e_k, e_\ell\big)\boldsymbol{c}(e_k)\boldsymbol{c}(e_\ell),\ \ \forall X,Y\in \mathrm{Vect}(M). \tag{3.1.16}$$

We need to better understand the nature of these quantities. We begin with the curvature $F^{\mathcal{E}}$. Set as before

$$\varepsilon_k = \frac{1}{\sqrt{2}}(e_k - \mathbf{i}f_k),\ \ \bar{\varepsilon}_k = \frac{1}{\sqrt{2}}(e_k + \mathbf{i}f_k),\ \ \varepsilon^k = \frac{1}{\sqrt{2}}dz^k,\ \ \bar{\varepsilon}^k = \frac{1}{\sqrt{2}}d\bar{z}^k. \tag{3.1.17}$$

For every ordered multi-index $I = (i_1,\cdots,i_k)$ we set

$$\bar{\varepsilon}^I = \bar{\varepsilon}^{i_1}\wedge\cdots\wedge\bar{\varepsilon}^{i_k}.$$

For every $X,Y\in\mathrm{Vect}(M)$ and $u: M\to\mathbb{C}$ we set

$$F^{\mathcal{E}}(uX,Y) = F^{\mathcal{E}}(X,uY) = uF^{\mathcal{E}}(X,Y)\in \mathrm{End}(\mathcal{E}).$$

Then

$$F^{\mathcal{E}} = \sum_{k<\ell} F^{\mathcal{E}}(\varepsilon_k,\varepsilon_\ell)\varepsilon^k\wedge\varepsilon^\ell + \sum_{k<\ell} F^{\mathcal{E}}(\bar{\varepsilon}_k,\bar{\varepsilon}_\ell)\bar{\varepsilon}^k\wedge\bar{\varepsilon}^\ell + \sum_{k,\ell} F^{\mathcal{E}}(\varepsilon_k,\bar{\varepsilon}_\ell)\varepsilon^k\wedge\bar{\varepsilon}^\ell.$$

The identity (3.1.15) implies that $F^{\mathcal{E}}\in\Omega^{1,1}(\mathrm{End}\,\mathcal{E})$ so that the first two terms above vanish. Hence

$$F^{\mathcal{E}} = \sum_{k,\ell} F^{\mathcal{E}}(\varepsilon_k,\bar{\varepsilon}_\ell)\varepsilon^k\wedge\bar{\varepsilon}^\ell.$$

The curvature $F^{\mathcal{E}}$ is induced from the Riemann curvature tensor and if for simplicity we set $F^{\mathcal{E}}(-) = F^{\mathcal{E}}(\varepsilon_k, \bar{\varepsilon}_\ell)$ then

$$F^e(-)\bar{\varepsilon}^i = R(-)\bar{\varepsilon}^i = \sum_j g_c(R(-)\bar{\varepsilon}^i, \varepsilon^j)\bar{\varepsilon}^j = \Big(\sum_{s,j} g_c(R(-)\varepsilon_s, \bar{\varepsilon}_j)\cdot e(\bar{\varepsilon}^j)\cdot i(\varepsilon_s)\Big)\bar{\varepsilon}^i$$

In general we have

$$F^e(-)\bar{\varepsilon}^I = \Big(\sum_{s,j} g_c(R(-)\varepsilon_s, \bar{\varepsilon}_j)\cdot e(\bar{\varepsilon}^j)\cdot i(\varepsilon_s)\Big)\bar{\varepsilon}^I.$$

For simplicity set

$$R_{k\bar{\ell}} = g(R(-)\varepsilon_k, \bar{\varepsilon}_\ell),\ \ C^{k\ell} = \boldsymbol{c}(\varepsilon^k)\boldsymbol{c}(\varepsilon^\ell),\ \ C^{k\bar{\ell}} = \boldsymbol{c}(\varepsilon^k)\boldsymbol{c}(\bar{\varepsilon}^\ell),\ \ \text{etc.}$$

Since $\boldsymbol{c}(\bar{\varepsilon}^k) = \sqrt{2}e(\bar{\varepsilon}^k)$, $\boldsymbol{c}(\varepsilon^\ell) = -\sqrt{2}i(\varepsilon^\ell)$

$$F^{\mathcal{E}} = -\frac{1}{2}\sum_{k,\ell} R_{k\bar{\ell}}C^{\bar{\ell}k} = \frac{1}{2}\sum_k R_{k\bar{k}} + \frac{1}{2}\sum_{k\neq\ell} R_{k\bar{\ell}}C^{k\bar{\ell}}.$$

To describe the term $\boldsymbol{c}(R)$ let us observe that the expression in the right-hand-side of (3.1.16) is independent of the dual pair of bases $\{(e_i),\ \ (e^i)\}$ of $TM\otimes\mathbb{C}$ and $T^*M\otimes\mathbb{C}$. We would like to express everything in terms of the bases

$$\{(\varepsilon_j, \bar{\varepsilon}_k),\ \ (\varepsilon^j, \bar{\varepsilon}^k)\}.$$

A few cancellations take place. To express them we define every $X\in \mathrm{Vect}(X)$

$$X^{1,0} = \frac{1}{2}(\mathbb{1} - \mathbf{i}J)X,\ \ X^{0,1} = \frac{1}{2}(\mathbb{1} + \mathbf{i}J)X$$

so that

$$X = X^{1,0} + X^{0,1},\ \ JX^{1,0} = \mathbf{i}X^{1,0},\ \ JX^{0,1} = -\mathbf{i}X^{0,1},$$

$$g_c(X^{1,0}, Y^{1,0}) = g_c(X^{0,1}, Y^{0,1}) = 0,\ \ g_c(X^{1,0}, Y^{0,1}) + g_c(X^{0,1}, Y^{1,0}) = 2g(X,Y).$$

Since the Levi-Civita connection is compatible with J we have

$$R(-)J = JR(-),\ \ (R(-)X)^{1,0} = R(-)X^{1,0},\ \ (R(-)X)^{0,1} = R(-)X^{0,1}.$$

Writing for simplicity R instead of $R(X,Y)$ and using the equalities

$$R_{k\bar{\ell}} = -R_{\bar{\ell}k},\ \ \forall k,\ell,\ \ C^{k\bar{\ell}} = -C^{\bar{\ell}k},\ \ \forall k\neq\ell$$

we deduce

$$\boldsymbol{c}(R)(X,Y) = \frac{1}{4}\sum_{1\le k,\ell\le m}(R_{\bar{k}\ell}C^{\bar{k}\ell} + R_{k\bar{\ell}}C^{k\bar{\ell}}) = \frac{1}{2}\sum_{k\neq\ell} R_{k\bar{\ell}}C^{k\bar{\ell}}.$$

Hence

$$F^{\mathcal{E}/\mathbb{S}} = \frac{1}{2}\sum_k R_{k\bar{k}}.$$

The quantity $\sum_k R_{k\bar{k}}$ is precisely the curvature of $\det T^*M^{0,1} \cong \det TM^{1,0} \cong K_M^{-1}$. Using the decomposition

$$TM\otimes\mathbb{C} \cong TM^{1,0}\oplus TM^{0,1}$$

and the compatibility of the Levi-Civita connection with the complex structure we deduce a decomposition of the Riemann tensor

$$R = R^{1,0}\oplus R^{0,1},$$

and we have

$$F^{\mathcal{E}/\mathbb{S}} = \frac{1}{2}\operatorname{tr} R^{1,0}.$$

Recall

$$\hat{\mathrm{A}}\,(x) = \frac{x}{e^{x/2} - e^{-x/2}},\quad \operatorname{td}(x) = \frac{x}{1-e^{-x}} = e^{x/2}\hat{\mathrm{A}}\,(x).$$

We deduce

$$\prod_k \operatorname{td}(x_k) = \exp\Big(\frac{1}{2}\sum_k x_k\Big)\prod_k \hat{\mathrm{A}}\,(x)$$

so that

$$\mathbf{td}\,(M) = \operatorname{td}\,(\frac{\mathbf{i}}{2\pi}R^{1,0}) = \exp\Big(\frac{1}{2}\operatorname{tr}\frac{\mathbf{i}}{2\pi}R^{1,0}\Big)\hat{\mathbf{A}}\,(M) = \hat{\mathbf{A}}\,(M)\cdot\mathbf{ch}\,(\mathcal{E}/\mathbb{S}).$$

The general case when we twist the Hodge-Dolbeault operator with a holomorphic complex bundle follows from (3.1.1). This concludes the proof of the Riemann-Roch-Hirzebruch theorem.

□

Let us look at a few special cases. Suppose Σ is a Riemann surface of genus $g(\Sigma)$ equipped with a Riemann metric h. As explained in Example 3.1.14(b) this induces a Kähler structure on Σ. Given a holomorphic line bundle $L \to \Sigma$ equipped with a Hermitian metric we obtain a Hodge-Dolbeault operator

$$\bar{\partial}_L : \Omega^{0,0}(\Sigma) \to \Omega^{0,1}(\Sigma).$$

Then

$$\operatorname{ind}\bar{\partial}_L = \int_\Sigma \mathbf{td}\,(\Sigma)\cdot\mathbf{ch}\,(L).$$

We have

$$\mathbf{td}\,(\Sigma) = 1 + \frac{1}{2}c_1(\Sigma) + \cdots,\quad \mathbf{ch}\,(L) = 1 + c_1(L) + \cdots$$

so that the degree 2 part of $\mathbf{td}\,(\Sigma)\cdot\mathbf{ch}\,(L)$ is $\frac{1}{2}c_1(\Sigma) + c_1(L)$. Hence

$$\operatorname{ind}\bar{\partial}_L = \frac{1}{2}\int_\Sigma c_1(\Sigma) + \int_\Sigma c_1(L).$$

Observe that $c_1(\Sigma) = \mathbf{e}(\Sigma)$ so the Gauss-Bonnet theorem implies

$$\frac{1}{2}\int_\Sigma c_1(\Sigma) = 1 - g(\Sigma).$$

The integer $\int_\Sigma c_1(L)$ is called the *degree* of L and it is denoted by $\deg L$. . We obtain the classical *Riemann-Roch formula*

$$h^{0,0}(L) - h^{0,1}(L) = 1 - g(\Sigma) + \deg L.$$

Suppose (M,h) is a Kähler surface (complex dimension 2) and $L \to M$ is a holomorphic line bundle. Then

$$\chi_{hol}(L) = h^{0,0}(L) - h^{0,1}(L) + h^{0,2}(L) = \int_M \mathbf{td}\,(M)\cdot\mathbf{ch}\,(L).$$

Writing for simplicity $c_k = c_k(M)$ we have

$$\mathbf{td}\,(M) = 1 + \frac{c_1}{2} + \frac{1}{12}(c_1^2 + c_2) + \cdots,\quad \mathbf{ch}\,(L) = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \cdots$$

so that the degree 4 part of $\mathbf{td}\,(M)\cdot\mathbf{ch}\,(L)$ is

$$\frac{1}{12}(c_1^2+c_2)+\frac{1}{2}c_1c_1(L)+\frac{1}{2}c_1(L)^2.$$

Let us now describe a convention frequently used in algebraic geometry, namely that in computations involving characteristic classes we will replace $c_1(E)$ with E for any complex line bundle E. Now observe that

$$c_1(M)=c_1(\det T^M1,0),\ \ \det TM^{1,0}\cong(\det T^*M1,0)^*\cong K_M^*.$$

Thus, $c_1(M)=-c_1(K_M)$ and instead of $c_1(M)$ we will write $-K_M$. Also, we will write the integration $\int_M$ as a Kronecker pairing $\langle -,[M]\rangle$. We deduce

$$\chi_{hol}(L)=\frac{1}{12}\langle K_M^2+c_2,[M]\rangle-\frac{1}{2}\langle K_M\cdot L,[M]\rangle+\frac{1}{2}\langle L^2,[M]\rangle$$

Now observe that $c_2(M)=\mathbf{e}(M)$ so the Gauss-Bonnet theorem implies

$$\langle c_2(M),[M]\rangle=\chi_{top}(M).$$

We deduce

$$\chi_{hol}(L)=\frac{1}{12}\chi_{top}(M)+\frac{1}{12}\langle K_M^2,[M]\rangle+\frac{1}{2}\langle L(L-K_M),[M]\rangle. \tag{3.1.18}$$

This can be further simplified using Hirzebruch signature theorem. Observe that

$$p_1(M)=-c_2(TM\otimes\mathbb{C})$$

On the other hand

$$1+c_1(TM\otimes\mathbb{C})+c_2(TM\otimes\mathbb{C})=c(TM\otimes\mathbb{C})=c(TM^{1,0})c(TM^{0,1})$$

$$=c(TM^{1,0})\cdot c((TM^{1,0})^*)=(1+c_1(M)+c_2(M))(1-c_1(M)+c_2(M))=1-K_M^2+2c_2(M)$$

Hence

$$p_1(M)=K_M^2-2c_2(M)$$

so that

$$\langle K_M^2,[M]\rangle=2\langle c_2(M),[M]\rangle+\langle p_1(M),[M]\rangle.$$

The Hirzebruch signature theorem implies

$$\langle p_1(M),[M]\rangle=3\,\mathrm{sign}(M),$$

while by Gauss-Bonnet we have

$$\langle c_2(M),[M]\rangle=\chi_{top}(M).$$

Hence

$$\langle K_M^2,[M]\rangle=2\chi_{top}(M)+3\,\mathrm{sign}(M).$$

Using this information in (3.1.18) we deduce

$$\chi_{hol}(L)=\frac{1}{4}\big(\chi_{top}(M)+\mathrm{sign}(M)\big)+\frac{1}{2}\langle L(L-K_M),[M]\rangle. \tag{3.1.19}$$

If in the above equality we choose L to be the trivial line bundle we obtain the *Noether theorem*

$$h^{0,0}(M)-h^{0,1}(M)+h^{0,2}(M)=\frac{1}{4}\big(\chi_{top}(M)+\mathrm{sign}(M)\big). \tag{3.1.20}$$

3.1.4. The $spin$ Dirac operators. We would like to present what is arguably the most important example of geometric Dirac operator. This operator generates in a certain sense all the other examples of geometric Dirac operators. This will require a topological detour in the world of $spin$ structures. We will use the basic facts about the $spin$ group proved in §2.2.2.

Suppose (M, g) is a compact connected, oriented Riemann manifold of (real) dimension n. The tangent bundle TM can be described by a $SO(n)$ gluing cocycle

$$\big(U_\alpha,\ \ g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)\,\big).$$

We regard this cocycle as defining the principal bundle of oriented orthonormal frames of TM. Consider the double cover

$$\rho : Spin(n) \to SO(n),\ \ \ker \rho = \{\pm 1\}.$$

The manifold M is called *spinnable* if the principal bundle of oriented orthonormal frames of TM can be given a $Spin(n)$-structure, i.e. there exists a gluing cocycle

$$\big(U_\alpha,\ \ \tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)\,\big)$$

such that the diagram below is commutative

$$\begin{array}{ccc} & & Spin(n) \\ & \overset{\tilde{g}_{\alpha\beta}}{\nearrow} & \downarrow \rho \\ U_{\alpha\beta} & \xrightarrow[g_{\alpha\beta}]{} & SO(n) \end{array} .$$

A lift as above is called a *spin structure*. Spin structures may not exist due to the possible presence of *global* topological obstructions. To understand their nature we try a naive approach.

Assume that the open cover $\mathfrak{U} = (U_\alpha)$ is *good*, i.e. all the overlaps $U_{\alpha\beta\cdots\gamma}$ are contractible. Such covers can be constructed easily by choosing U_α to be geodesically convex. Since $U_{\alpha\beta}$ is contractible, each of the maps $g_{\alpha\beta}$ admits lifts to $Spin(n)$. Pick one such lift $\tilde{g}_{\alpha\beta}$ for every $U_{\alpha\beta} \neq \emptyset$. Assume $\tilde{g}_{\beta\alpha} = \tilde{g}_{\alpha\beta}^{-1}$. We have to check whether such a random choice does indeed produce a $Spin(n)$-cycle, i.e.

$$\epsilon_{\alpha\beta\gamma} := \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = 1.$$

All we can say at this moment is

$$\epsilon_{\alpha\beta\gamma} \in \ker \rho = \{\pm 1\}.$$

Let us observe that $\epsilon_{\alpha\beta\gamma}$ itself satisfies a cocycle condition

$$\epsilon_{\beta\gamma\delta} \cdot \epsilon_{\beta\delta\alpha} \cdot \epsilon_{\beta\alpha\gamma} \cdot \epsilon_{\gamma\alpha\delta}.$$

$$= \tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\delta}\underbrace{\tilde{g}_{\delta\beta} \cdot \tilde{g}_{\beta\delta}}_{=1}\tilde{g}_{\delta\alpha}\underbrace{\tilde{g}_{\alpha\beta} \cdot \tilde{g}_{\beta\alpha}}_{=1}\tilde{g}_{\alpha\gamma}\tilde{g}_{\gamma\beta} \cdot \tilde{g}_{\gamma\alpha}\tilde{g}_{\alpha\delta}\tilde{g}_{\delta\gamma}$$

$$= \tilde{g}_{\beta\gamma}\underbrace{\tilde{g}_{\gamma\delta} \cdot \tilde{g}_{\delta\alpha} \cdot \tilde{g}_{\alpha\gamma}}_{=\epsilon_{\gamma\delta\alpha}}\tilde{g}_{\gamma\beta} \cdot \tilde{g}_{\gamma\alpha}\tilde{g}_{\alpha\delta}\tilde{g}_{\delta\gamma}$$

(use the fact that $\epsilon_{\gamma\delta\alpha} \in \ker \rho$ is in the center of $Spin(n)$)

$$= \epsilon_{\gamma\delta\alpha} \cdot \underbrace{\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\beta}}_{=1} \cdot \tilde{g}_{\gamma\alpha}\tilde{g}_{\alpha\delta}\tilde{g}_{\delta\gamma} = \tilde{g}_{\gamma\alpha}\tilde{g}_{\alpha\delta}\tilde{g}_{\delta\gamma} \cdot \epsilon_{\gamma\delta\alpha}$$

$$= \tilde{g}_{\gamma\alpha} \cdot \tilde{g}_{\alpha\delta} \cdot \underbrace{\tilde{g}_{\delta\gamma} \cdot \tilde{g}_{\gamma\delta}}_{=1} \cdot \tilde{g}_{\delta\alpha} \cdot \tilde{g}_{\alpha\gamma} = 1.$$

If we identify $\{\pm 1\}$ with the group $(\mathbb{Z}/2, +)$ we see that a choice of lifts $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)$ produces a collection $\epsilon_{\alpha\beta\gamma} \in \mathbb{Z}/2$, one element for each triplet (α, β, γ) such that $U_{\alpha\beta\gamma} \neq \emptyset$ satisfying the cocycle condition

$$\epsilon_{\beta\gamma\delta} + \epsilon_{\alpha\gamma\delta} + \epsilon_{\alpha\beta\delta} + \epsilon_{\alpha\beta\gamma} = 0, \;\; \forall U_{\alpha\beta\gamma\delta} \neq \emptyset. \tag{3.1.21}$$

Let us rephrase this in a more intuitive way using basic facts of Čech cohomology. For more information on this important concept we refer to [5, 11, 20].

First, let associate to the cover $\mathcal{U}$ a simplicial complex $\mathcal{N}(\mathcal{U})$ called the *nerve* of the cover. For every $q \geq 0$ the q-simplices of $\mathcal{N}(\mathcal{U})$ correspond to the collections

$$\{U_{\alpha_0}, \cdots, U_{\alpha_q}\} \subset \mathcal{U} \;\text{ such that }\; \bigcap_{k=0}^{q} U_{\alpha_k} \neq \emptyset.$$

We denote by $\mathcal{N}_q(\mathcal{U})$ the collection of q-simplices of the nerve. We denote by $C_q(\mathcal{U})$ the free $\mathbb{Z}$-module generated by the collection $\{\sigma \in \mathcal{N}_q(\mathcal{U})\}$. We set

$$C^q(\mathcal{U}, \mathbb{Z}/2) := \operatorname{Hom}(C_q(\mathcal{U}), \mathbb{Z}/2).$$

The collection $\epsilon_{\alpha\beta\gamma}$ can be viewed a function

$$\epsilon : \mathcal{N}_2(\mathcal{U}) \to \mathbb{Z}/2, \;\; \sigma = [\alpha, \beta, \gamma] \mapsto \epsilon(\sigma) := \epsilon_{\alpha\beta\gamma}$$

We extend it by linearity to a morphism

$$\epsilon \in \operatorname{Hom}(C_2(\mathcal{U}), \mathbb{Z}/2) = C^2(\mathcal{U}, \mathbb{Z}/2).$$

We have a boundary operator

$$\partial : C_q(\mathcal{U}) \to C_{q-1}(\mathcal{U}), \;\; \partial[\alpha_0, \alpha_1, \cdots, \alpha_q] = \sum_{k=0}^{q} (-1)^k [\alpha_0, \cdots, \hat{\alpha}_k, \cdots, \alpha_q],$$

where a hat indicates a missing entry. This operator satisfies

$$\partial^2 = 0.$$

Using this operator we define a coboundary operator

$$\delta : C^q(\mathcal{U}, \mathbb{Z}/2) \to C^{q+1}(\mathcal{U}, \mathbb{Z}/2),$$

$$(\delta\eta)(\sigma) := \eta(\partial\sigma), \;\; \forall \eta \in C^q(\mathcal{U}, \mathbb{Z}/2), \;\; \sigma \in C_{q+1}(\mathcal{U}, \mathbb{Z}/2).$$

This operator satisfies

$$\delta^2 = 0.$$

The cocycle condition (3.1.21) can be rewritten as

$$\delta\epsilon = 0.$$

We denote by $H^q(\mathcal{U}, \mathbb{Z}/2)$ the cohomology groups of the cochain complex $(C^\bullet(\mathcal{U}, \mathbb{Z}/2)$. They are known as the *Čech cohomology groups* of the cover $\mathcal{U}$. Given two lifts

$$\tilde{g}_{\alpha\beta}, \hat{g}_{\alpha\beta} : U_{\alpha\beta} \to Spin(n)$$

of $g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)$ we set

$$\kappa_{\alpha\beta} := \tilde{g}_{\alpha\beta} \cdot \hat{g}_{\alpha\beta}^{-1} \in \ker(Spin(n) \to SO(n)) \cong \mathbb{Z}/2.$$

We regard $\kappa_{\alpha\beta}$ as an element $\kappa \in C^1(\mathcal{U}, \mathbb{Z}/2)$. If we denote by $\tilde{\epsilon}$ the cocycle corresponding to $\tilde{g}_{\bullet\bullet}$ and by $\hat{\epsilon}$ the cocycle corresponding to $\hat{g}_{\bullet\bullet}$ we deduce

$$\tilde{\epsilon}_{\alpha\beta\gamma} - \hat{\epsilon}_{\alpha\beta\gamma} = \kappa_{\beta\gamma} - \kappa_{\alpha\gamma} + \kappa_{\alpha\beta}, \quad \forall[\alpha, \beta, \gamma] \in \mathcal{N}_2(\mathcal{U}).$$

We can rewrite the last equality as

$$\tilde{\epsilon} - \hat{\epsilon} = \delta\kappa.$$

Thus the cocycles $\tilde{\epsilon}$ and $\hat{\epsilon}$ are Čech cohomologous and thus determine a cohomology class

$$w_2(\mathcal{U}) \in H^2(\mathcal{U}, \mathbb{Z}/2).$$

This is called the *second Stiefel-Whitney class* of the cover $\mathcal{U}$.

A theorem of Leray ([5, Thm.15.8]) shows that for every good cover $\mathcal{U}$ of M there exists a natural isomorphism

$$I_{\mathcal{U}} : H^q(\mathcal{U}, \mathbb{Z}/2) \to H^q(M, \mathbb{Z}/2),$$

where the group in the right-hand-side denotes the singular cohomology with $\mathbb{Z}/2$-coefficients. Additionally, one can show that the image of $w_2(\mathcal{U})$ in $H^2(M, \mathbb{Z}/2)$ via $I_{\mathcal{U}}$ is *independent of the good cover*. We thus obtain a cohomology class $w_2(M) \in H^2(M, \mathbb{Z}/2)$ called the second Stiefel-Whitney class of M.

If the manifold M is spinnable, and $\tilde{g}_{\bullet\bullet} : U_{\bullet\bullet} \to Spin(n)$ is a gluing *cocycle* covering $g_{\bullet\bullet}$ then the associated cocycle $\epsilon_{\alpha\beta\gamma}$ is trivial and therefore $w_2(M) = 0$. Conversely, if $w_2(M) = 0$ then one can show (see [14, II§2]) M is spinnable. Two $spin$ structures described by lifts $\tilde{g}_{\alpha\beta}$ and $\tilde{h}_{\alpha\beta}$ are called *isomorphic* if there exists a collection of continuous maps

$$\epsilon_\alpha : U_\alpha \to \ker(Spin(n) \to SO(n))$$

such that for every $x \in U_{\alpha\beta}$ we have a commutative diagram

$$\begin{array}{ccc} Spin(n) & \xrightarrow{\epsilon_\beta} & Spin(n) \\ {\scriptstyle \tilde{g}_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \tilde{h}_{\alpha\beta}} \\ Spin(n) & \xrightarrow[\epsilon_\alpha]{} & Spin(n) \end{array} \iff \epsilon_\alpha \tilde{g}_{\alpha\beta} = \tilde{h}_{\alpha\beta}\epsilon_\beta.$$

We denote by $Spin(M)$ the set of isomorphisms classes of $spin$ structures on M. A $spin$ manifold is a manifold M together with a choice of $\lambda \in Spin(M)$.

Observe that given a $spin$-structure λ defined by the lift $\tilde{g}_{\bullet\bullet}$ and a cohomology class $c \in H^1(M, \mathbb{Z}/2)$ described by the Čech cocycle $\epsilon_{\bullet\bullet}$ we can produce a new $spin$ structure $c \cdot \lambda$ defined by the lift

$$\hat{g}_{\bullet\bullet} := \epsilon_{\bullet\bullet} \cdot \tilde{g}_{\bullet\bullet}.$$

The isomorphism class of $\hat{g}_{\bullet\bullet}$ depends only on the isomorphism class of $\tilde{g}_{\bullet\bullet}$ and the cohomology class of $\epsilon_{\bullet\bullet}$. In other words, we have produced a map

$$H^1(M, \mathbb{Z}/2) \times Spin(M) \to Spin(M), \quad (c, \lambda) \mapsto c \cdot \lambda$$

which satisfies the obvious relation

$$(c_1 + c_2) \cdot \lambda = c_1 \cdot (c_2 \cdot \lambda).$$

In other words we have produced a left action of $H^1(M, \mathbb{Z}/2)$ on $Spin(n)$ and one can check (see [14, II§2]) that this action is *free and transitive*. We say that $Spin(M)$ is a $H^1(M, \mathbb{Z}/2)$-*torsor*. In particular there exists a *non-canonical* bijection

$$H^1(M, \mathbb{Z}/2) \to Spin(M).$$

Let us summarize the results established so far.

Proposition 3.1.28. *Suppose M is a compact, oriented, connected smooth manifold. Then M is spinnable iff $w_2(M) = 0$. If this is the case then there exists a free and transitive action of $H^1(M, \mathbb{Z}/2)$ on $Spin(M)$.*

Example 3.1.29. So far we have produced arguments that $spin$ structures might not exist. Let us describe a few instances when $spin$ structures do exists. Suppose M is a smooth, compact, oriented, connected manifold. The universal coefficients theorem implies

$$\begin{aligned} H^q(M, \mathbb{Z}/2) &\cong \operatorname{Hom}(H_q(M, \mathbb{Z}), \mathbb{Z}/2) \oplus \operatorname{Ext}(H_{q-1}(B, \mathbb{Z}), \mathbb{Z}/2) \\ &\cong H^q(M, \mathbb{Z}) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}(H^{q+1}(M, \mathbb{Z}), \mathbb{Z}/2). \end{aligned}$$

We deduce that if $b_2(M) = b_1(M) = 0$ and $H_2(M, \mathbb{Z})$ and $H_1(M, \mathbb{Z})$ have no 2-torsion than $H^2(M, \mathbb{Z}/2) = 0$ and thus M is spinnable. In particular it admits a unique $spin$ structure. For example, the lens spaces $L(p, q)$ with p odd satisfy these conditions.

If the tangent bundle of M is trivializable, then any trivialization of M defines a $spin$ structure on M. It is known that the tangent bundle of a compact, connected oriented 3-manifold is trivializable and thus such manifolds are spinnable. Similarly, a compact Lie group admits a canonical $spin$-structure induced by the natural trivialization.

There are subtler conditions which imply $w_2(M) = 0$. We list without proof a few of them.

Suppose M is a compact, simply connected 4-manifold without boundary. Then M is spinnable iff the intersection form of M is even, i.e.

$$c \cdot c = 0 \mod 2, \ \ \forall c \in H_2(M, \mathbb{Z})/\text{Torsion}.$$

Equivalently, if we represent the intersection form of M as a unimodular symmetric matrix I_M, then the intersection form is even iff all the diagonal elements of I_M are even. For example the intersection form of $M = S^2 \times S^2$ with respect to the canonical basis

$$c_1 = [S^2 \times \{*\}], \ \ c_2 = [\{*\} \times S^2]$$

is given by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus the intersection form is even. The manifold $S^2 \times S^2$ is spinnable and in fact it admits a unique $spin$ structure.

The complex projective plane $\mathbb{CP}^2$ is simply connected, $b_2(M) = 1$ and the intersection form is given by the 1×1 matrix $[1]$. This shows that $\mathbb{CP}^2$ is not spinnable.

Recall that we have a canonical morphism

$$i_2 : H^\bullet(M, \mathbb{Z}) \to H^\bullet(M, \mathbb{Z}/2)$$

which sits in a long exact sequence

$$\cdots \to H^{q-1}(M, \mathbb{Z}/2) \stackrel{\beta}{\longrightarrow} H^q(M, \mathbb{Z}) \stackrel{2\times}{\longrightarrow} H^q(M, \mathbb{Z}) \stackrel{i_2}{\longrightarrow} H^q(M, \mathbb{Z}/2) \to \cdots,$$

where β is the Bockstein morphism. One can prove (see [14, Example D.6]) that if M is an almost complex manifold then

$$c_1(M) = w_2(M) \mod 2 \Longleftrightarrow i_2(c_1) = w_2. \tag{3.1.22}$$

In particular if $H_1(M, \mathbb{Z})$ has no 2-torsion then $H^1(M, \mathbb{Z}/2) = 0$, $\beta = 0$ and thus $i_2(c_1) = 0$ iff there exists $x \in H^2(M, \mathbb{Z})$ such that

$$2x = c_1(M).$$

Using this fact one can prove (see [11, §22]) that any smooth complex hypersurface in $\mathbb{CP}^{n+1}$ defined by a degree d homogeneous complex polynomial is spinnable off $d+n$ is even. In particular a quartic in $\mathbb{CP}^3$ (degree 4 hypersurfaces) are spinnable. These quartics are also known as $K3$ *hypersurfaces*. The degree 5 hypersurfaces in $\mathbb{CP}^4$ (also known as *Calabi-Yau hypersurfaces*) are also spinnable. □

Suppose (M, g) is a smooth, compact,connected, oriented Riemann manifold without boundary, and λ is a $spin$ structure on M. Assume $\dim_{\mathbb{R}} M = 2m$. Denote by $\pi : P \to M$ the principal $SO(2m)$-bundle of oriented orthonormal frames of TM. The $spin$ structure λ produces a $Spin(2m)$-principal bundle $\tilde{\pi} : \tilde{P}_\lambda \to M$ and the natural morphism $\rho : Spin(2m) \to SO(2m)$ induces a smooth map $\rho : \tilde{P}_\lambda \to P$ such that the diagram below is commutative

$$\begin{array}{ccc} \tilde{P}_\lambda & \xrightarrow{\rho} & P \\ & \searrow^{\tilde{\pi}} & \downarrow^{\pi} \\ & & M \end{array}$$

and for every $x \in M$ the restriction $\rho : \tilde{\pi}^{-1}(x) \to \pi^{-1}(x)$ is $2 : 1$.

Fix an isomorphism

$$\varphi : \mathbf{Cl}_{2m} \otimes \mathbb{C} \to \operatorname{End}(\mathbb{S}_{2m}$$

and denote by $\varphi : Spin(2m) \subset \mathbf{Cl}_{2m} \to \operatorname{Aut}(\mathbb{S}_{2m})$ the induced complex spinor representation and set

$$\mathbb{S}_\lambda := \tilde{P}_\lambda \times_\varphi \mathbb{S}_{2m}.$$

We say that $\mathbb{S}_\lambda$ is the *complex spinor bundle* associated to the $spin$ structure λ. Note that it is equipped with a natural $\mathbb{Z}/2$-grading

$$\mathbb{S}_\lambda = \mathbb{S}_\lambda^+ \oplus \mathbb{S}_\lambda^-.$$

Proposition 3.1.30. *Any $Spin(2m)$-invariant hermitian metric on $\mathbb{S}_{2m}$ induces on $\mathbb{S}_\lambda$ a natural structure of Dirac bundle whose twisting curvature is trivial.*

Proof The $Spin(2m)$-invariant hermitian metric on $\mathbb{S}_{2m}$ will equip $\mathbb{S}_\lambda$ with a hermitian metric. We need to produce a hermitian connection on $\mathbb{S}_\lambda$ and a Clifford multiplication on $\mathbb{S}_\lambda$ which is compatible with both the metric and the connection.

Fix a good cover $\mathcal{U} = (U_\bullet)$ of M and a gluing cocycle

$$g_{\bullet\bullet} : U_{\bullet\bullet} \to SO(2m)$$

describing (TM, g). The $spin$ structure λ picks a lift

$$\tilde{g}_{\bullet\bullet} : U_{\bullet\bullet} \to Spin(2m)$$

of $g_{\bullet\bullet}$. The Levi-Civita connection on TM is described by a collection of 1-forms

$$A_\bullet \in \Omega^1(U_\bullet) \otimes \underline{so}(2m)$$

satisfying the transition rules

$$A_\beta = g_{\beta\alpha} A_\alpha g_{\beta\alpha}^{-1} - dg_{\beta\alpha} \cdot g_{\beta\alpha}^{-1} = \mathrm{Ad}(g_{\beta\alpha}) A_\alpha - dg_{\beta\alpha} \cdot g_{\beta\alpha}^{-1}.$$

The representation $\rho : Spin(2m) \to SO(2m)$ induces an isomorphism of Lie algebras

$$\rho_* : \underline{spin}(2m) \to \underline{so}(2m)$$

described explicitly in (2.2.4). Set

$$\tilde{A}_\alpha := \rho_*^{-1}(A_\alpha) \in \Omega^1(U_\alpha) \otimes \underline{spin}(2m).$$

Then the collection $(\tilde{A}_\bullet)$ satisfies the transition rules

$$\tilde{A}_\beta = \mathrm{Ad}(\tilde{g}_{\beta\alpha}) \tilde{A}_\alpha - d\tilde{g}_{\beta\alpha} \cdot \tilde{g}_{\beta\alpha}^{-1}. \tag{3.1.23}$$

The derivative of φ at $1 \in Spin(2m)$ induces a morphism of Lie algebras

$$\varphi_* : \underline{spin}(2m) \to \underline{u}(\mathbb{S}_{2m}) = \text{skew-hermitian endomorphisms of } \mathbb{S}_{2m}$$

and we set

$$B_\bullet := \varphi_*(\tilde{A}_\bullet).$$

The transition rules (3.1.23) imply that the collection $B_\bullet$ defines a connection $\tilde{\nabla}^g$ on $\mathbb{S}_\lambda$ compatible with the hermitian metric and the $\mathbb{Z}/2$-grading.

To produce a Clifford multiplication we first describe TM as a subbundle

$$\boldsymbol{c} : TM \hookrightarrow \mathrm{End}(\mathbb{S}_\lambda)$$

such that for every

$$\boldsymbol{c}(X)^2 = -|X|_g^2 \cdot \mathbb{1}_{\mathbb{S}_\lambda}, \quad \boldsymbol{c}(X)^* = -\boldsymbol{c}(X), \quad \forall X \in \mathrm{Vect}(M). \tag{3.1.24}$$

Observe that the spinor representation φ induces a representation on $\mathrm{End}(\mathbb{S}_{2m})$

$$\varphi_\flat : Spin(2m) \to \mathrm{Aut}(\mathrm{End}(\mathbb{S}_{2m})), \quad \varphi_\flat(g)T = \varphi(g) T \varphi(g)^{-1}, \quad \forall g \in Spin(2m), \quad T \in \mathrm{End}(\mathbb{S}_{2m}).$$

Observe that

$$\varphi_\flat(\pm 1) = \mathbb{1}$$

so this representation factors through a representation of $SO(2m)$, i.e. there exists

$$[\varphi_\flat] : SO(2m) \to \mathrm{Aut}\big(\mathrm{End}(\mathbb{S}_{2m})\big)$$

such that the diagram below is commutative

$$\begin{array}{ccc} Spin(2m) & \stackrel{\varphi_\flat}{\longrightarrow} & \mathrm{Aut}(\mathrm{End}(\mathbb{S}_{2m})) \\ {\scriptstyle\rho}\downarrow & \nearrow{\scriptstyle [\varphi_\flat]} & \\ SO(2m) & & \end{array} .$$

We have and inclusion

$$\boldsymbol{c} : \mathbb{R}^{2m} \hookrightarrow \mathbf{Cl}_{2m} \stackrel{\varphi}{\longrightarrow} \mathrm{End}(\mathbb{S}_{2m}).$$

and we know that any vector space isomorphism $\mathbf{Cl}_{2m} \to \mathbf{Cl}_{2m}$ induced by an orthogonal changes of basis in $\mathbb{R}^{2m}$ leaves the subspace $\mathbb{R}^{2m} \hookrightarrow \mathbf{Cl}_{2m}$ invariant. Identifying $\mathbf{Cl}_{2m} \otimes \mathbb{C}$ with $\mathrm{End}(\mathbb{S}_{2m})$ via φ and denoting by $\mathrm{Aut}_V(U)$ the group of vector space isomorphisms

$$T : U \to U \quad \text{such that} \quad T(V) \subset V$$

we deduce the above diagram can be refined to a commutative diagram

$$\begin{array}{ccc} Spin(2m) & \xrightarrow{\varphi_\flat} & \mathrm{Aut}_{\mathbb{R}^{2m}}\big(\,\mathrm{End}(\mathbb{S}_{2m})\,\big) \\ \rho \downarrow & \overset{[\varphi_\flat]}{\nearrow} & \uparrow j \\ SO(2m) & \xrightarrow[i]{} & \mathrm{Aut}(\mathbb{R}^{2m}) \end{array} .$$

Now observe that

$$\mathrm{End}(\mathbb{S}_\lambda) \cong \tilde{P}_\lambda \times_{\varphi_\flat} \mathrm{End}(\mathbb{S}_{2m})$$

and since $\mathbb{R}^{2m}$ is a $\varphi_\flat$-invariant subspace of $\mathrm{End}(\mathbb{S}_{2m})$ we deduce from the above diagram that we can view

$$TM \cong \tilde{P}_\lambda \times_{i\circ\rho} \mathbb{R}^{2m}$$

as a subbundle of $\mathrm{End}(\mathbb{S}_{2m})$. We denote by $\boldsymbol{c} : TM \hookrightarrow \mathrm{End}(\mathbb{S}_{2m})$ the inclusion. Since all the above constructions are invariant under the various symmetry groups we deduce that $\boldsymbol{c}$ satisfies tautologically the conditions (3.1.24). In particular, the Clifford multiplication

$$\boldsymbol{c} : TM \to \mathrm{End}(\mathbb{S}_\lambda)$$

must also be $\tilde{\nabla}^g$-covariant constant. Now define a Clifford multiplication

$$\boldsymbol{c} : T^*M \to \mathrm{End}(\mathbb{S}_\lambda)$$

using the metric duality $T^*M \xrightarrow{\sharp} TM$. Finally, let us prove that the twisting curvature of $\tilde{\nabla}^g$ is trivial.

Fix an oriented local orthonormal frame (e_i) of TM and denote by (e^i) the dual coframe. Let R be the curvature of the Levi-Civita connection on TM. For every $X, Y \in \mathrm{Vect}(M)$ we identify $R(X,Y) \in \underline{so}(TM)$ with the section of $\Lambda^2 TM$

$$\omega_R = \sum_{i<j} g(R(X,Y)e_i, e_j) e_i \wedge e_j.$$

Then, using (2.2.5) we deduce

$$\rho_*^{-1} R(X,Y) = \frac{1}{2} \sum_{i<j} g(R(X,Y)e_i, e_j) e_i e_j = \frac{1}{4} \sum_{i,j} g(R(X,Y)e_i, e_j) e_i e_j.$$

The curvature $\tilde{R}$ of $\tilde{\nabla}^g$ is described by

$$\tilde{R}(X,Y) = \varphi_*\big(\rho_*^{-1} R(X,Y)\,\big) = \frac{1}{4} \sum_{i,j} g(R(X,Y)e_i, e_j) \boldsymbol{c}(e_i) \boldsymbol{c}(e_j)$$

$$= \frac{1}{4} \sum_{i,j} g(R(X,Y)e_i, e_j) \boldsymbol{c}(e^i) \boldsymbol{c}(e^j) = \boldsymbol{c}(R).$$

This shows that

$$F^{\mathbb{S}_\lambda/\mathbb{S}} = 0.$$

□

We denote by

$$\mathfrak{D}_\lambda : C^\infty(\mathbb{S}_\lambda^+) \to C^\infty(\mathbb{S}_\lambda^-)$$

the geometric Dirac operator determined by the above Dirac bundle. We will refer to it as the *spin Dirac operator* associated to a Riemannian $spin$ manifold (M, g, λ). Using the above proposition we deduce from the index theorem the following result.

Theorem 3.1.31 (Atiyah-Singer)**.**

$$\operatorname{ind}_{\mathbb{C}} \mathfrak{D}_\lambda = \int_M \hat{\mathbf{A}}(M).$$

Suppose M is a spinnable 4-manifold. Then for every $spin$ structure $\lambda \in Spin(M)$ we have

$$\operatorname{ind}_{\mathbb{C}} \mathfrak{D}_\lambda = -\frac{1}{24}\int_M p_1(M).$$

Using the Hirzebruch signature theorem we deduce

$$\operatorname{ind}_{\mathbb{C}} \mathfrak{D}_\lambda = -\frac{1}{8}\operatorname{sign}(M).$$

Corollary 3.1.32. *The signature of any smooth* 4*-dimensional manifold is divisible by* 8*.*

Suppose M is a simply connected smooth 4-manifold with even intersection form I_M. Then M is spinnable, and in fact it admits a unique $spin$ structure. We denote by $\mathfrak{D}$ the associated $spin$ Dirac operator. We deduce

$$\operatorname{sign}(M = -8 \operatorname{ind}_{\mathbb{C}} \mathfrak{D}.$$

One can prove this divisibility result much faster relying on purely elementary result (see e.g.[20]) but the above equality will allow us to prove a stronger result concerning the symmetric, even unimodular bilinear forms which are intersection forms of some *smooth spin* 4-manifold. We will need the following fact.

Proposition 3.1.33.

$$\operatorname{ind}_{\mathbb{C}} \mathfrak{D}_\lambda \in 2\mathbb{Z}.$$

Proof The proof relies on a concrete description on $Spin(4)$ and $\mathbb{S}_4$. Consider again the division ring of quaternions

$$\mathbb{H} = \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}.$$

It is equipped with an involution

$$\mathbb{H} \ni q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto \bar{q} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

such that

$$q \cdot \bar{q} = |q|^2 = a^2 + b^2 + c^2 + d^2.$$

Recall that we have identified $Spin(3)$ with the group of unit quaternions. We want to prove that

$$Spin(4) \cong Spin(3) \times Spin(3) \cong SU(2) \times SU(2).$$

Let

$$G = \{\vec{q} = (q_1, q_2) \in \mathbb{H} \times \mathbb{H};\ \ |q_1| = |q_2| = 1\} \cong Spin(3) \times Spin(3).$$

We have a natural representation

$$\rho : G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}), \ \ \rho(q_1, q_2)h = q_1 h \bar{q}_2, \ \ \forall (q_1, q_2) \in G, \ \ h \in \mathbb{H}.$$

Observe that

$$|q_1 h \bar{q}_2| = |h|$$

so that $\rho(q_1, q_2)$ is an isometry of $\mathbb{H}$. Since G is connected we deduce that we have a morphisms

$$\tau : G \to SO(\mathbb{H}) \cong SO(4).$$

One can check that $\ker \rho = \{\pm 1\}$ and we deduce that ρ is a nontrivial double cover of $SO(4)$ and thus

$$G \cong Spin(4).$$

In fact one can check (see [14, I§4]) that

$$\mathbf{Cl}_4 \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^2),$$

where $\mathbb{H}^2$ is regarded as a *right* $\mathbb{H}^2$ module. Then $Spin(4)$ can be identified with the diagonal subgroup (see Exercise 3.3.8)

$$Spin(4) \cong \big\{ \mathrm{Diag}(q_1, q_2) \in \mathrm{End}_{\mathbb{H}}(\mathbb{H}^2); \ \ |q_1| = |q_2| = 1 \big\}.$$

The induced complex spinor representation is then the tautological one (see Exercise 3.3.8)

$$\varphi : Spin(4) \hookrightarrow \mathrm{End}_{\mathbb{H}}(\mathbb{H}^2) \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}^2).$$

More precisely

$$\varphi(q_1, q_2) \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} q_1 h_1 \\ q_2 h_2 \end{bmatrix}.$$

Moreover

$$\mathbb{S}_4^+ = \mathbb{H} \oplus 0, \ \ \mathbb{S}_4^- = 0 \oplus \mathbb{H}.$$

For every $x \in \mathbb{H}$ we denote by $L_x : \mathrm{End}_{\mathbb{R}}(\mathbb{H}^2)$ (resp. $R_x \in \mathrm{End}_{\mathbb{R}}(\mathbb{H}^2)$) the left (resp. right) multiplication by x. Observe that $R_{\mathbf{i}}^2 = -\mathbb{1}$ so that $R_{\mathbf{i}}$ induces a complex structure on $\mathbb{H}$ and

$$\varphi(q_1, q_2) \circ R_{\mathbf{i}} = R_{\mathbf{i}} \circ \varphi(q_1, q_2), \ \ \forall (q_1, q_2) \in G.$$

In other words the linear maps $\varphi(q_1, q_2)$ are complex linear with respect to the complex structure induced by $R_{\mathbf{i}}$. Similarly we have

$$\varphi(q_1, q_2) \circ R_{\mathbf{j}} = R_{\mathbf{j}} \circ \varphi(q_1, q_2), \ \ \forall (q_1, q_2) \in G.$$

This shows that $\mathbb{S}_4^{\pm}$ has a canonical structure of *right* $\mathbb{H}$-module, the complex structure is induced from the inclusion $\mathbb{C} \hookrightarrow \mathbb{H}$ and that the $\mathbb{R}$-linear endomorphisms $\varphi(q_1, q_2)$ are morphisms of right $\mathbb{H}$-modules. Equivalently, this means that $\mathbb{S}_4^{\pm}$ has a $Spin(4)$-invariant structure of right $\mathbb{H}$-module.

If (M, g, λ) is a *spin* 4-manifold, then $\mathbb{S}_\lambda^{\pm}$ have natural structures of right $\mathbb{H}$-modules. These are covariant constant with respect to $\tilde{\nabla}^g$ and moreover, from the description

$$\mathbf{Cl}_4 \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^2)$$

we deduce

$$[\boldsymbol{c}(\alpha), R_{\mathbf{k}}] = [\boldsymbol{c}(\alpha), R_{\mathbf{k}}] = [\boldsymbol{c}(\alpha), R_{\mathbf{k}}] = 0, \ \ \forall \alpha \in \Omega^1(M).$$

This implies that $\ker \mathfrak{D}_\lambda$ and $\ker \mathfrak{D}_\lambda^*$ are right $\mathbb{H}$-modules and in particular

$$\mathrm{ind}_{\mathbb{C}} \mathfrak{D} = 2 \, \mathrm{ind}_{\mathbb{H}} \mathfrak{D} \in 2\mathbb{Z}.$$

□

Corollary 3.1.34 (Rohlin)**.** *If M is a compact, oriented, simply connected smooth 4-manifold without boundary and even intersection form then*

$$\operatorname{sign}(M) \in 16\mathbb{Z}.$$

Remark 3.1.35. Three decades after Rohlin proved this result, M. Freedman has shown that there exists a compact, oriented, simply connected *topological* 4-manifold M without boundary whose intersection form is even and

$$\operatorname{sign}(M) = 8.$$

Rohlin's result shows that such a manifold cannot admit any *smooth* structure!!! □

3.1.5. The $spin^c$ Dirac operators. Suppose (M, g) is a compact connected, oriented Riemann manifold of (real) dimension n. The tangent bundle TM can be described by a $SO(n)$ gluing cocycle

$$\big(U_\alpha,\;\; g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)\,\big).$$

We regard this cocycle as defining the principal bundle of oriented orthonormal frames of TM.

Identify $\mathbb{Z}/2$ with the multiplicative group $\{\pm 1\}$. Recall that $Spin^c(n)$ is the Lie group

$$Spin^c(n) \cong \Big(Spin(n) \times S^1\Big)/\mathbb{Z}/2$$

where $\mathbb{Z}/2$ acts diagonally on $Spin(n) \times S^1$

$$t \cdot (g, s) = (tg, ts),\;\; \forall g \in Spin(n),\;\; s \in S^1,\;\; t \in \mathbb{Z}/2.$$

Consider the group morphism

$$\rho^c : Spin^c(n) \to SO(n).$$

A *$spin^c$ structure* on M is a gluing cocycle

$$\tilde{g}_{\bullet\bullet} : U_{\bullet\bullet} \to Spin^c(n)$$

such that

$$\rho^c(\tilde{g}_{\alpha\beta}) = g_{\alpha\beta},\;\; \forall \alpha, \beta,$$

i.e. the diagram below is commutative

$$\begin{array}{ccc} & & Spin^c(n) \\ & \overset{\tilde{g}_{\alpha\beta}}{\nearrow} & \downarrow \rho^c \\ U_{\alpha\beta} & \underset{g_{\alpha\beta}}{\longrightarrow} & SO(n) \end{array} .$$

Spin structures may not exist due to possible presence of *global* topological obstructions. To understand their nature we follow the same approach used in the description of *spin* structures. Assume that $\mathcal{U}$ is a good open cover, i.e. all the overlaps are contractible Over each $U_{\alpha\beta}$ we choose arbitrarily

$$\tilde{g}_{\alpha\beta} = [\hat{g}_{\alpha\beta}, z_{\alpha\beta} = \exp(\pi \mathbf{i} \theta_{\alpha\beta})] \in Spin^c(n),$$

$$\hat{g}_{\alpha\beta} : U_{\alpha\beta} : U_{\alpha\beta} \to Spin(n),\;\; \theta_{\alpha\beta} \in C^\infty(U_{\alpha\beta}, \mathbb{R}),\;\; \rho(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}.$$

Assume

$$\tilde{g}_{\alpha\beta} = \tilde{g}_{\beta\alpha}^{-1},\;\; \tilde{g}_{\alpha\alpha} \equiv \mathbb{1},\;\; \theta_{\alpha\beta} = -\theta_{\beta\alpha}.$$

Denote by K the kernel of $\rho : Spin(n) \to SO(n)$ and by K^c the kernel of $\rho^c : Spin^c(n) \to SO(n)$

$$K = (\mathbb{Z}/2 \times S^1)/\mathbb{Z}/2 \cong S^1.$$

Observe that K^c lies in the center of $Spin^c(n)$. We hope that

$$\tilde{g}_{\gamma\alpha} = \tilde{g}_{\gamma\beta}\tilde{g}_{\beta\alpha} \Longleftrightarrow \mathbb{1} \equiv \tilde{g}_{\alpha\gamma}\tilde{g}_{\gamma\beta}\tilde{g}_{\beta\alpha}.$$

If choose the lifts $\tilde{g}_{\alpha\beta}$ carelessly all we could say is

$$\rho_c(\tilde{g}_{\alpha\gamma}\tilde{g}_{\gamma\beta}\tilde{g}_{\beta\alpha}) \equiv \mathbb{1}.$$

We set

$$\epsilon_{\gamma\beta\alpha} = \hat{g}_{\alpha\gamma}\hat{g}_{\gamma\beta}\hat{g}_{\beta\alpha}, \;\; c_{\gamma\beta\alpha} = z_{\alpha\gamma}z_{\gamma\beta}z_{\beta\alpha} \in S^1.$$

Since $\rho_c(\tilde{g}_{\alpha\gamma}\tilde{g}_{\gamma\beta}\tilde{g}_{\beta\alpha}) \equiv \mathbb{1}$ we deduce

$$\epsilon_{\gamma\beta\alpha} \in K \subset S^1.$$

For $\tilde{g}_{\bullet\bullet}$ to be a gluing cocycle we need

$$c_{\gamma\beta\alpha} = \epsilon_{\gamma\beta\alpha} \in \mathbb{Z}/2 = \exp(\pi \mathbf{i}\mathbb{Z}) \subset S^1.$$

In particular we deduce

$$c^2_{\gamma\beta\alpha} \equiv 1 \Longleftrightarrow z^2_{\gamma\alpha} = z^2_{\gamma\beta}z^2_{\beta\alpha}$$

i.e. $(z^2_{\bullet\bullet})$ is a S^1-gluing cocycle for some complex line bundle $L \to M$. We set

$$\Theta_{\gamma\beta\alpha} = 2\Big(\theta_{\gamma\beta} + \theta_{\beta\alpha} + \theta_{\alpha\gamma}\Big)$$

The equality $c^2_{\gamma\beta\alpha} \equiv 1$ implies

$$\Theta_{\gamma\beta\alpha} \in \mathbb{Z}.$$

Note also that the image of $\frac{1}{2}\Theta_{\gamma\beta\alpha}$ in $\frac{1}{2}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Z}/2$ coincides with $\epsilon_{\gamma\beta\alpha}$.

As in the previous subsection we denote by $\mathcal{N}_q(\mathcal{U})$ the collection of q-simplices of the nerve of the open cover $\mathcal{U}$. We denote by $C_q(\mathcal{U})$ the free $\mathbb{Z}$- module generated by the collection $\{\sigma \in \mathcal{N}_q(\mathcal{U})\}$. For every Abelian group G we set

$$C^q(\mathcal{U}, G) := \mathrm{Hom}(C_q(\mathcal{U}), G).$$

Then

$$\epsilon_{\gamma\beta\alpha} \in C^2(\mathcal{U}, \mathbb{Z}/2), \;\; \Theta_{\gamma\beta\alpha} \in C^2(\mathcal{U}, \mathbb{Z}).$$

We deduce as before that the above Čech cochains are in fact Čech cocycles. The cohomology class of the cocycle $(\epsilon_{\gamma\beta\alpha})$ is the second Stieffel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}/2)$ of the manifold M, while the cohomology class of the cocycle $(\Theta_{\gamma\beta\alpha})$ is the first Chern class $c_1(L) \in H^2(M, \mathbb{Z})$ of the complex line bundle $L \to M$ defined by the gluing cocycle $z^2_{\bullet\bullet}$ (see [**10**, Chap.1] for a proof of this general fact). Thus, the existence of a *spin*^{c} structure implies the existence of an integral cohomology class $c \in H^2(M, \mathbb{Z})$ such that

$$c \mod 2 = w_2(M) \in H^2(M, \mathbb{Z}/2).$$

Arguing in reverse one can prove the following result (see Exercise 3.3.9).

Proposition 3.1.36. *The manifold (M, g) admits spin^c structures if and only if $w_2(M)$ is the mod 2 reduction of an integral cohomology class.* □

Two $spin^c$ structures described by lifts $\tilde{g}_{\alpha\beta}$ and $\tilde{h}_{\alpha\beta}$ are called *isomorphic* if there exists a collection of continuous maps

$$k_\alpha : U_\alpha \to K^c = \ker(Spin^c(n) \to SO(n))$$

such that for every $x \in U_{\alpha\beta}$ we have a commutative diagram

$$\begin{array}{ccc} Spin^c(n) & \xrightarrow{k_\beta} & Spin^c(n) \\ {\scriptstyle \tilde{g}_{\alpha\beta}}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{h}_{\alpha\beta}} \\ Spin^c(n) & \xrightarrow[k_\alpha]{} & Spin^c(n) \end{array} \iff k_\alpha \tilde{g}_{\alpha\beta} = \tilde{h}_{\alpha\beta} k_\beta.$$

We denote by $Spin^c(M)$ the set of isomorphisms classes of $spin^c$ structures on M. A $spin^c$ manifold is a manifold M together with a choice of $\sigma \in Spin(M)$.

Denote by $\mathrm{Pic}_t(M)$ the topological Picard group, i.e. the space of isomorphisms classes of complex line bundles over M. To a $spin^c$ structure σ over M given by the gluing cocycle $\tilde{g}_{\alpha\beta} = [\hat{g}_{\alpha\beta}, z_{\alpha\beta} = \exp(\pi \mathbf{i} \theta_{\alpha\beta})]$ we associate a complex line bundle $\det \sigma$ given by the gluing cocycle $(z^2_{\alpha\beta})$. One can show that this induces a map

$$\det : Spin^c \to \mathrm{Pic}_t(M), \;\; \sigma \mapsto \det \sigma.$$

The image of this map consists of line bundles $L \to M$ such that

$$c_1(L) \mod 2 = w_2(M).$$

Note that $\mathrm{Pic}_t(M)$ is a group with respect to $\otimes$. Moreover, the first Chern class induces an isomorphism

$$c_1 : (\mathrm{Pic}_t(M), \otimes) \to H^2(M, \mathbb{Z}).$$

Proposition 3.1.37. *There exists a natural free and transitive action of* $\mathrm{Pic}_t(M)$ *on* $Spin^c(M)$

$$\mathrm{Pic}_t(M) \times Spin^c(M) \to Spin^c(M), \;\; (L, \sigma) \mapsto L \cdot \sigma$$

satisfying

$$\det(L \cdot \sigma) = L^2 \otimes \det \sigma.$$

Sketch of proof. Consider a $spin^c$ structure σ given by the gluing cocycle $\tilde{g}_{\alpha\beta} = [\hat{g}_{\alpha\beta}, z_{\alpha\beta} = \exp(\pi \mathbf{i} \theta_{\alpha\beta})]$ and a line bundle L given by the gluing cocycle $\zeta_{\alpha\beta}$. We define $L \cdot \sigma$ to be the $spin^c$ structure given by the gluing cocycle $[\hat{g}_{\alpha\beta}, z_{\alpha\beta}\zeta_\alpha \beta]$.

We let the reader check that this action is well defined and free, i.e.

$$[\tilde{g}_{\alpha\beta} = [\hat{g}_{\alpha\beta}, z_{\alpha\beta}] \cong [[\hat{g}_{\alpha\beta}, z_{\alpha\beta}\zeta_\alpha \beta] \iff (\zeta_{\alpha\beta}) \cong (1).$$

The line bundle associated to $L \cdot \sigma$ is given by the gluing cocycle $(\zeta^2_{\alpha\beta} z^2_{\alpha\beta})$ so that

$$\det(L \otimes \sigma) \cong L^2 \otimes \det \sigma.$$

To prove that the action is transitive consider two $spin^c$ structures σ_0, σ_1 given by gluing cocycles

$$\sigma_0 \to \tilde{g}_{\alpha\beta} = [\hat{g}_{\alpha\beta}, z_{\alpha\beta}], \;\; \sigma_1 \to \tilde{h}_{\alpha\beta} = [\hat{h}_{\alpha\beta}, v_{\alpha\beta}].$$

we can arrange so that

$$\hat{g}_{\gamma\beta}\hat{g}_{\beta\alpha}\hat{g}_{\alpha\gamma} = \hat{h}_{\gamma\beta}\hat{h}_{\beta\alpha}\hat{\alpha\gamma}$$

Then $\sigma_1 = L \cdot \sigma_0$ where L is the line bundle given by the gluing cocycle

$$\zeta_{\alpha\beta} = v_{\alpha\beta}/z_{\alpha\beta}.$$

□

The results in the above proposition is often formulated by saying that $Spin^c(M)$ is a $\mathrm{Pic}_t(M)$-*torsor* or $H^2(M,\mathbb{Z})$-torsor.

Example 3.1.38. (a) A spinnable manifold M admits $spin^c$ structures. In fact, to any $spin$ structure $\epsilon \in Spin(M)$ there corresponds a canonical $spin^c$ structure $\sigma(\epsilon)$ such that $\det\sigma(\epsilon)$ is trivial. We thus have a natural map

$$Spin(M) \to Spin^c(M), \;\; \epsilon \mapsto \sigma(\epsilon)$$

We denote by β the Bockstein morphism

$$\beta : H^1(M,\mathbb{Z}/2) \to H^2(M,\mathbb{Z}).$$

We know that $Spin(M)$ is a $H^1(M,\mathbb{Z}/2)$-torsor. For every $\lambda \in H^1(M,\mathbb{Z}/2)$ we have

$$\sigma(\lambda\epsilon) = \beta(\lambda)\cdot\sigma(\epsilon), \;\; \forall\epsilon \in Spin(M).$$

Observe that if $Spin^c(M) \neq \emptyset$ then $Spin(M) \neq \emptyset$ if and only if for any (or for some) $spin^c$ structure σ on M there exists $L \in \mathrm{Pic}_t(M)$ such that $L^2 \cong \det\sigma$. We will denote by $\sqrt{\det\sigma}$ the collection of such line bundles. Hence

$$Spin(M) \neq \emptyset \Longleftrightarrow \forall(\exists)\sigma \in Spin^c(M) : \sqrt{\det\sigma} \neq \emptyset.$$

Given a $spin^c$ structure σ on M we can identify the image of $Spin(M)$ in $Spin^c(M)$ with the collection of $spin^c$ structures

$$\Big\{L^{-1}\cdot\sigma \in Spin^c(M);\;\; L^2 = \det\sigma\Big\}.$$

Since the compact oriented manifolds of dimension ≤ 3 are spinnable we deduce that any such manifold admits $spin^c$ structure.

(b) A result of Hirzebruch-Hopf shows that any compact, oriented smooth 4-manifolds admits $spin^c$ structures.

(c) Using the identity (3.1.22) in the previous subsection we deduce that any almost complex manifold admits $spin^c$ structures. In fact we can be much more precise. Suppose (M,J) is an almost complex manifold and g is a Riemann metric compatible with J. Then $\dim_{\mathbb{R}} M = 2m$ and the tangent bundle is described by a gluing cocycle

$$g_{\alpha\beta} : U_{\alpha\beta} \to U(m) \stackrel{i}{\hookrightarrow} SO(2m).$$

Using Proposition 2.2.14 in §2.2.2 we deduce that there exists a smooth group morphism

$$\Phi_m : U(m) \to Spin^c(2m)$$

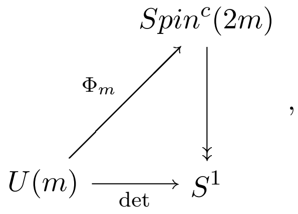
such that the diagram below is commutative.

$$\begin{array}{ccc} & & Spin^c(2m) \\ & \stackrel{\Phi_m}{\nearrow} & \downarrow \rho^c \\ U(m) & \stackrel{i}{\hookrightarrow} & SO(2m) \end{array}.$$

Then

$$\tilde{g}_{\alpha\beta} = \Phi_m(g_{\alpha\beta})$$

defines a $spin^c$ structure on M called the $spin^c$ structure associated to an almost complex structure. We will denote it by $\sigma_{\mathbb{C}}$. Observe that we have a commutative diagram



where we recall that the vertical arrow is given by $[\tilde{g}, z] \mapsto z^2$. This shows that the line bundle associated to $\sigma_{\mathbb{C}}$ is given by the gluing cocycle $\det g_{\alpha\beta}$. It is therefore isomorphic to

$$\det{}_{\mathbb{C}}(TM, J) \cong K_M^{-1}.$$

Hence

$$\det \sigma_{\mathbb{C}} \cong K_M^{-1}.$$

We deduce that an almost complex manifold is spinnable iff $\sqrt{K_M} \neq \emptyset$ and we can bijectively identify the $spin$ structures with the square roots of the canonical line bundle. □

Suppose (M, g) is a compact oriented Riemann manifold of even dimension $\dim_{\mathbb{R}} M = 2m$. Assume the tangent bundle is defined by a gluing cocycle

$$g_{\bullet\bullet} : U_{\bullet\bullet} \to SO(2m)$$

Fix a $spin^c$-structure $\sigma \in Spin^c(M)$ described by a gluing cocycle

$$\tilde{g}_{\bullet\bullet} = [\hat{g}_{\bullet\bullet}, z_{\bullet\bullet}] : U_{\bullet\bullet} \to Spin^c(2m).$$

We denote by P_σ the principal $Spin^c(2m)$ bundle determined by this cocycle so that

$$TM \cong P_\sigma \times_{\rho^c} \mathbb{R}^{2m}.$$

$Spin^c(2m)$ can be naturally viewed as a subgroup in $\mathbf{Cl}_{2m} \otimes \mathbb{C} \subset \operatorname{End}_{\mathbb{C}}(\mathbb{S}_{2m})$ and as such we have representations

$$\varphi^c_{\pm} : Spin^c(2m) \to \operatorname{Aut}_{\mathbb{C}}(\mathbb{S}^{\pm}_{2m}), \;\; \varphi^c \cong \varphi^c_+ \oplus \varphi^c_-.$$

Define

$$\mathbb{S}_\sigma := P_\sigma \times_{\varphi^c} \mathbb{S}_{2m}.$$

As in the previous section we see that $\mathbb{S}_\sigma$ has a natural structure of $\mathbf{Cl}(T^*M)$-module. Moreover, if we fix a $Spin^c(2m)$-invariant metric on $\mathbb{S}_{2m}$ then the induced Clifford multiplication

$$\boldsymbol{c} : T^*M \to \operatorname{End}_{\mathbb{C}}(\mathbb{S}_\sigma)$$

is odd and skew-symmetric with respect to the induced metric on $\mathbb{S}_\sigma$.

Suppose that the Levi-Civita connection on TM is described by a collection

$$A_\alpha \in \Omega^1(U_\alpha) \otimes \underline{so}(2m), \;\; A_\beta = g_{\beta\alpha} A_\alpha g_{\beta\alpha}^{-1} - dg_{\beta\alpha} \cdot g_{\beta\alpha}^{-1}.$$

We denote by $\rho_* : \underline{spin}(2m) \to \underline{so}(2m)$ the differential of $\rho : Spin(2m) \to SO(2m)$ described explicitly in (2.2.4) and set

$$\hat{A}_\alpha := \rho_*^{-1}(A_\alpha) \in \Omega^1(U_\alpha) \otimes \underline{spin}(2m).$$

Then the collection $(\tilde{A}_\bullet)$ satisfies the transition rules

$$\hat{A}_\beta = \hat{g}_{\beta\alpha}\hat{A}_\alpha\hat{g}_{\beta\alpha}^{-1} - d\hat{g}_{\beta\alpha}\cdot\hat{g}_{\beta\alpha}^{-1}. \tag{3.1.25}$$

Observe that although $\hat{g}_{\bullet\bullet}$ is only defined up to a $\pm 1 \in \ker\rho$, this ambiguity is lost in the above equality. Consider a connection B on the line bundle defined by the cocycle $(z_{\bullet\bullet}^2)$. It can be described by a collection

$$B_\alpha \in \Omega^1(U_\alpha)\otimes \underline{u}(1): \quad B_\beta = B_\alpha - 2\frac{dz_{\beta\alpha}}{z_{\beta\alpha}} \Longleftrightarrow \frac{1}{2}B_\beta = \frac{1}{2}B_\alpha - \frac{dz_{\beta\alpha}}{z_{\beta\alpha}}.$$

We deduce that the collection

$$\tilde{A}_\alpha = (\hat{A}_\alpha, \frac{1}{2}B_\alpha) \in \Omega^1(U_\alpha)\otimes \underline{spin}^c(2m)$$

satisfies the gluing conditions

$$\tilde{A}_\beta = \tilde{g}_{\beta\alpha}\tilde{A}_\alpha\tilde{g}_{\beta\alpha}^{-1} - d\tilde{g}_{\beta\alpha}\cdot\tilde{g}_{\beta\alpha}^{-1}$$

and thus defines a connection on P_σ. In particular it induces a connection on $\mathbb{S}_\sigma$ which we denote by $\nabla^{\sigma,B}$. As in the previous subsection one can verify that $(\mathbb{S}_\sigma, \nabla^{\sigma,B})$ is a Dirac bundle. We denote by

$$\not{\mathfrak{D}}_{\sigma,B} : C^\infty(\mathbb{S}_\sigma^+) \to C^\infty(\mathbb{S}_\sigma^-)$$

the associated geometric Dirac operator.

Theorem 3.1.39 (Atiyah-Singer)**.**

$$\operatorname{ind}_{\mathbb{C}}\not{\mathfrak{D}}_{\sigma,B} = \int_M \hat{\mathbf{A}}(M)\wedge\exp(\frac{\mathbf{i}}{4\pi}F_B) = \big\langle \hat{\mathbf{A}}(M)\exp(\frac{1}{2}c_1(\det\sigma)), [M]\big\rangle,$$

where we denoted by $\langle -,-\rangle : H^*(M,\mathbb{R})\times H_*(M,\mathbb{R}) \to \mathbb{R}$ *the Kronecker pairing.*

3.2. The heat kernel of generalized Laplacians

3.2.1. Spectral theory of generalized Laplacians.

3.2.2. The heat kernel and the McKean-Singer formula.

3.3. Exercises for Chapter 3

Exercise 3.3.1. Prove (3.1.1). □

Exercise 3.3.2. Prove (3.1.7). □

Exercise 3.3.3. Prove Proposition 3.1.7. □

Exercise 3.3.4. Prove the identity (3.1.10). □

Exercise 3.3.5. Prove the identity (3.1.11). □

Exercise 3.3.6. Prove Proposition 3.1.15. □

Exercise 3.3.7. Prove Lemma 3.1.21.

□

Exercise 3.3.8. (a) Show that we have an isomorphism of $\mathbb{Z}/2$-graded algebras

$$\mathbf{Cl}_4 \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^2).$$

(b) Equip $\mathbb{H}^2$ with the complex structure defined by $R_{\mathbf{i}}$ so as a complex vector space we have $\mathbb{H}^2 \cong \mathbb{C}^4$. Prove that

$$\mathbf{Cl}_4 \otimes \mathbb{C} \cong \mathrm{End}_{\mathbb{C}}(\mathbb{H}^2).$$

(c) Show that $Spin(4) \subset \mathbf{Cl}_4$ can be identified via the isomorphism $\mathbf{Cl}_4 \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^2)$ with the diagonal subgroup

$$\big\{\mathrm{Diag}(q_1, q_2);\ \ |q_1| = |q_2| = 1\big\}.$$

□

Exercise 3.3.9. Prove Proposition 3.1.36 □

Chapter 4

The proof of the Index Theorem

4.1. Would I really have enough time do do it?

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