Three-dimensional Seiberg-Witten theory

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*Notes for a mini-course on SW-theory at the Université Joseph Fourier. Available electronically at *http://www.nd.edu/~lnicolae/grenoble.pdf*

1 The Seiberg-Witten Monopoles

§1.1 ***Spin^c* structures and Dirac operators on 3-manifolds.** An *admissible* 3-manifold is a smooth, compact, orientable 3-manifold M, such that ∂M is either empty or a disjoint union of tori. Equivalently, this means that $\chi(M) = 0$.

A *direction* on an admissible 3-manifold is a smooth function $\tau : M \to [-1, 1]$ such that (see Figure 1)

$$\partial M = \tau^{-1}(\{\pm 1\}), \;\; d\tau(x) \neq 0, \;\; \forall x \in \partial M.$$

Two directions τ_1, τ_2 are called *equivalent* if $\tau_1 = \tau_2$ near ∂M. Note that all directions on a closed manifold are equivalent. A *directed* 3-manifold is a pair (M, τ), where M is an admissible 3-manifold and τ is an equivalence class of directions on it. We set

$$\partial^\tau_\pm M := \tau^{-1}(\{\pm 1\}).$$

Note that the components $\partial^\tau_\pm M$ depend only on the equivalence class of the direction τ.

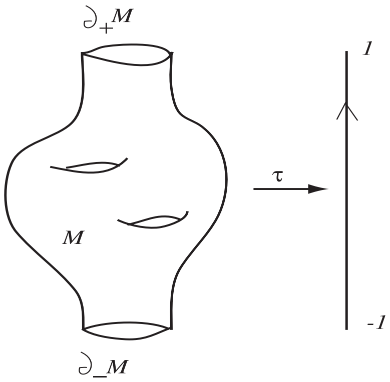

Figure 1: *A directed manifold*

Suppose (M, τ) is a directed 3-manifold. Since the Euler characteristic is trivial there exist nowhere vanishing vector fields V on M such that

$$V \lrcorner\, d\tau > 0, \;\; \text{near } \partial M.$$

We call such vector fields *admissible*. We see that admissible vector fields point outwards on $\partial_+ M$ and inwards along $\partial_- M$.

Definition 1.1. *Two admissible vector fields V_0, V_1 on (M, τ) are called* homologous *if there exists a smooth family $\tilde{V}_s$ of vector fields on M such that the following hold.*

- $V_i = \tilde{V}_i$*, $i = 0, 1$.*
- $\tilde{V}_s \lrcorner\, d\tau > 0$ *near ∂M, $\forall s$.*
- *There exists an open ball $B \subset M$ such that for any s $\tilde{V}_s|_{M\setminus B}$ is nowhere zero.*

The homology class of an admissible vector field U is denoted by $[U]$. A smooth Euler structure *on M is a homology class of admissible vector fields. We denote by $\mathfrak{Eul}_s(M, \tau)$ the set smooth Euler structures.*

Observe that we have a natural bijection

$$\mathfrak{Eul}_s(M, \tau) \to \mathfrak{Eul}_s(M, -\tau), \;\; \mathfrak{e} = [V] \mapsto \bar{\mathfrak{e}} = [-V].$$

Suppose U, V are two admissible vector fields on (M, τ). There is only one obstruction to them being homologous, and is given by an element

$$[U/V] \in H^2((M, \partial M) \times (I, \partial I); \pi_2(S^2)) \cong H^2(M, \partial M; \mathbb{Z}) \cong H_1(M, \mathbb{Z}).$$

Conversely, given an element $h \in H_1(M, \mathbb{Z})$ and an admissible vector field V there exists an admissible vector field, unique up to homology such that

$$h = [U/V].$$

We set $[U] := h \cdot [V]$. We have produced a free and transitive action

$$H_1(M, \mathbb{Z}) \times \mathfrak{Eul}_s(M, \tau) \to \mathfrak{Eul}_s(M), \quad (h, \mathfrak{e}) \mapsto h \cdot \mathfrak{e}.$$

In other words, $\mathfrak{Eul}_s(M, \tau)$ is a $H_1(M, \mathbb{Z})$-torsor.

Convention *In the sequel we will denote multiplicatively the group operation on $H_1(M, \mathbb{Z})$.*

Suppose g is an *admissible* Riemann metric on (M, τ), which means that near the boundary g is product like,

$$g = d\tau^2 + g_\partial, \quad g_\partial := g\,|_{\partial M}\ .$$

Suppose V is a nowhere vanishing vector field on M outward pointing along ∂M. Assume

$$|V|_g := 1, \quad V \equiv \partial_\tau, \quad \text{near} \ \ \partial M.$$

Denote by $\mathfrak{e}$ the associated smooth Euler structure. V determines a real line sub-bundle $\langle V \rangle \subset TM$. We denote by $\langle V \rangle^\perp \subset TM$ the orthogonal plane sub-bundle.

Fix an orientation on TM and orient $\langle V \rangle^\perp$ by the rule

$$\mathbf{or}\,(TM) = V \wedge \mathbf{or}\,\langle V \rangle^\perp.$$

Thus $\langle V \rangle^\perp$ has an $SO(2) \cong U(1)$-structure and we can think of it as a complex line bundle. We denote it by $\det \mathfrak{e}$. Along the boundary we have an isomorphism of *oriented* 2-plane bundles

$$\mathfrak{t}(\mathfrak{e})_\pm : \det \mathfrak{e}\,|_{\partial_\pm^\tau M} \to \pm T(\partial_\pm^\tau M).$$

The boundary of M is a union of tori, and the tangent bundle of a torus T^2 admits a *canonical* trivialization induced by any orientation preserving diffeomorphism $T^2 \to S^1 \times S^1$. Thus $\det \mathfrak{e}$ has a canonical trivialization along the boundary and thus it has a relative first Chern class

$$c(\mathfrak{e}) := c_1(\det \mathfrak{e}, \tau) \in H^2(M, \partial M; \mathbb{Z}) \cong H_1(M; \mathbb{Z}).$$

Proposition 1.2 (Turaev).

$$c(h \cdot \mathfrak{e}) = h^2 \cdot c(\mathfrak{e}).$$

For a proof we refer to [13, §3.2]. Now define

$$\mathbb{S} = \mathbb{S}_\mathfrak{e} := \underline{\mathbb{R}}_M \oplus TM = (\underline{\mathbb{R}} \oplus \langle V \rangle) \oplus \det \mathfrak{e}.$$

$\underline{\mathbb{R}} \oplus \langle V \rangle$ is trivial oriented 2-plane bundle and thus we can identify it with the trivial complex line bundle $\underline{\mathbb{C}}_M$. Thus the bundle $\mathbb{S}_\mathfrak{e}$ can be identified with a rank 2 complex Hermitian vector bundle.

$$\mathbb{S}_\mathfrak{e} \cong \underline{\mathbb{C}}_M \oplus \det \mathfrak{e}.$$

We define a Clifford multiplication map

$$\boldsymbol{c} = \boldsymbol{c}_{\mathfrak{e}} : TM \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_{\mathfrak{e}}),$$

$$T_xM = \langle V\rangle \oplus \langle V\rangle^{\perp} \ni (tV, \phi) \mapsto \boldsymbol{c}(t,\phi) := \left[\begin{array}{cc} -\mathbf{i}t\cdot\bullet & -\langle \bullet, \phi\rangle_{\mathbb{S}} \\ \bullet\cdot\phi & \mathbf{i}t\cdot\bullet \end{array}\right] \in \mathrm{End}_{\mathbb{C}}(\mathbb{S}_x).$$

More precisely, if $(z,\psi) \in \mathbb{C}_x \oplus \det \mathfrak{e}_x$ then

$$\boldsymbol{c}(tV,\phi)\left[\begin{array}{c} z \\ \psi \end{array}\right] = \left[\begin{array}{c} -\mathbf{i}tz - \langle \psi,\phi\rangle_{\mathbb{S}} \\ z\phi + \mathbf{i}t\psi \end{array}\right].$$

The Clifford multiplication map produces a linear isomorphism $\boldsymbol{c} : TM \to \underline{su}(\mathbb{S}_{\mathfrak{e}})$, satisfying the identity

$$-\mathrm{tr}\,\boldsymbol{c}(X)^2 = \mathrm{tr}\,(\boldsymbol{c}(X)\boldsymbol{c}(X)^*) = 2g(X,X),\ \ \forall X \in \mathrm{Vect}\,(M).$$

This construction satisfies the requirements of a Clifford multiplication since

$$\boldsymbol{c}(X)^2 = \frac{1}{2}\mathrm{tr}\,(\boldsymbol{c}(X)^2)\mathbf{1}_{\mathbb{S}} = -g(X,X)\mathbf{1}_{\mathbb{S}},\quad \forall X \in \mathrm{Vect}\,(M).$$

Definition 1.3. *(a) A* relative geometric *spin^c*-structure *on (M,τ) is a triple $\sigma = (\mathbb{S}, \boldsymbol{c}, \mathfrak{t}_{\pm})$ with the following properties.*

- $\mathbb{S}$ *is a rank 2-complex Hermitian vector bundle.*
- $\boldsymbol{c} : TM \to \mathrm{End}_{\mathbb{C}}(\mathbb{S})$ *is a $\mathbb{R}$-linear isomorphism onto $\underline{su}(\mathbb{S})$ satisfying*

$$\boldsymbol{c}(X)^2 = \frac{1}{2}\mathrm{tr}\,(\boldsymbol{c}(X)^2)\mathbf{1}_{\mathbb{S}},\quad \forall X \in \mathrm{Vect}\,(M),$$

- $\mathfrak{t}_{\pm}$ *is an isomorphism of oriented real vector bundles*

$$\mathfrak{t}_{\pm} : \mathbb{S}|_{\partial_{\pm}M} \to \mathbb{C} \oplus \pm T(\partial_{\pm}M).$$

We denote by $Spin^c_{geom}(M,\tau)$ the set of geometric spin^c structures.

$\boldsymbol{c}$ is called the Clifford multiplication map of the spin^c structure σ and $\mathbb{S}$ is called the bundle of complex spinors associated to σ. We will denote it by $\mathbb{S}_\sigma$. We set $\det\sigma := \det \mathbb{S}_\sigma$. Observe that

$$g(X,Y) = -\frac{1}{2}\mathrm{tr}\,\Big(\boldsymbol{c}(X)\boldsymbol{c}(Y) + \boldsymbol{c}(Y)\boldsymbol{c}(X)\Big),\ \ X,Y \in \mathrm{Vect}\,(M)$$

defines a Riemann metric on M.
(b) Two relative geometric spin^c-structures σ_0 and σ_1 are called isomorphic if there exists an isometry $\mathbb{S}_{\sigma_0} \to \mathbb{S}_1$ which commutes with the Clifford multiplications and the boundary trivializations. A topological *relative spin^c structure is an isomorphism class of geometric spin^c structures.*

Denote by $Spin^c(M,\tau)$ the set of topological relative *spin*-structures on M. We have a natural projection

$$Spin^c_{geom}(M,\tau) \to Spin^c(M,\tau),\ \ \sigma \mapsto [\sigma]$$

The topological type of $\mathbb{S}_\sigma$ and $\det\sigma$ depends only on the isomorphism class of σ. A *geometric spin^c* structure σ is completely determined by the following three data.

- Its topological type, $[\sigma]$.
- A Riemann metric g on M.

- A hermitian metric h on $\det \sigma$.

There exists a bijection

$$Spin^c_{geom}(M,\tau) \to Spin^c_{geom}(M,-\tau), \;\; \sigma = (\mathbb{S}_\sigma, \boldsymbol{c}, \mathfrak{t}_\pm) \mapsto \bar\sigma = (\mathbb{S}^*_\sigma, \boldsymbol{c}^*_\sigma, \Xi \circ \mathfrak{t}_\mp),$$

where $\Xi : \mathbb{C} \oplus \pm T(\partial_\pm M) \to \mathbb{C} \oplus \pm T(\partial_\pm M)$ is the map

$$\Xi(z,v) = (\bar z, v).$$

Observe that $\boldsymbol{c}_{\bar\sigma} = \boldsymbol{c}^*_\sigma = -\boldsymbol{c}_\sigma$. Note that we have a metric induced complex conjugate linear isomorphism

$$I_\sigma : \mathbb{S}_\sigma \to \mathbb{S}^*_\sigma = \mathbb{S}_{\bar\sigma}. \tag{1.1}$$

It satisfies the property

$$I_\sigma(\boldsymbol{c}_\sigma(X)\psi) = \boldsymbol{c}_{\bar\sigma}(X)(I_\sigma\psi), \;\; \forall \psi \in \Gamma(\mathbb{S}_\sigma)$$

that is the diagram below is commutative

$$\begin{tikzcd} \mathbb{S}_\sigma \arrow[r, "I_\sigma"] \arrow[d, "\boldsymbol{c}_\sigma(\bullet)"'] & \mathbb{S}_{\bar\sigma} \arrow[d, "\boldsymbol{c}_{\bar\sigma}(\bullet)"] \\ \mathbb{S}_\sigma \arrow[r, "I_\sigma"] & \mathbb{S}_{\bar\sigma} \end{tikzcd}$$

This induces a bijection $Spin^c(M,\tau) \to Spin^c(M,-\tau)$, $\sigma \longleftrightarrow \bar\sigma$.

Denote by $\mathrm{Pic}_t(M,\tau)$ the set of isomorphism classes of pairs $(L,\mathfrak{t})$, where $L \to M$ is a complex line bundle on M and $\mathfrak{t} : L|_{\partial M} \to \underline{\mathbb{C}}_{\partial M}$ is a trivialization of L along the boundary of M. The tensor product of such pairs induces a group structure on $\mathrm{Pic}_t(M,\partial M)$ and we have a group isomorphism

$$c_1 : \mathrm{Pic}_t(M,\tau) \to H^2(M,\partial M;\mathbb{Z}) \cong H_1(M,\mathbb{Z}), \;\; (L,\mathfrak{t}) \mapsto c_1(L,\mathfrak{t}).$$

For every $\sigma \in Spin^c(M,\tau)$ we set

$$c(\sigma) = c_1(\det\sigma, \mathfrak{t}) \in H^2(M,\partial M;\mathbb{Z}) \cong H-1(M,\mathbb{Z}).$$

We get a $\mathrm{Pic}_t(M,\tau)$-action on $Spin^c(M,\tau)$

$$\mathrm{Pic}_t(M,\tau) \times Spin^c(M,\tau) \to Spin^c(M,\tau), \;\; (L,\mathfrak{t};\sigma) \mapsto (L,\mathfrak{t}) \otimes \sigma,$$

$$(L,\mathfrak{t}) \otimes (\mathbb{S}, \boldsymbol{c}, \mathfrak{t}_\pm) = (L \otimes \mathbb{S}, \boldsymbol{c}, \mathfrak{t} \otimes \mathfrak{t}_\pm).$$

We have an isomorphism

$$\det(L \otimes \sigma) \cong L^{\otimes 2} \otimes \det\sigma.$$

Proposition 1.4 (Turaev). *Suppose (M,τ) is a directed admissible 3-manifold. The correspondence*

$$\mathfrak{Eul}_s(M,\tau) \ni \mathfrak{e} \mapsto \sigma(\mathfrak{e}) = (\mathbb{S}_\mathfrak{e}, \boldsymbol{c}_\mathfrak{e}, \mathfrak{t}(\mathfrak{e})_\pm) \in Spin^c(M,\tau)$$

is a $H_1(M,\mathbb{Z})$-equivariant bijection. Moreover

$$\sigma(\bar{\mathfrak{e}}) \cong \overline{\sigma(\mathfrak{e})}, \;\; \forall \mathfrak{e} \in \mathfrak{Eul}_s(M,\tau)$$

Fix a *spin*^{c} structure $\sigma \in Spin^c(M,\tau)$, a Riemann metric g on M and a Hermitian metric h on $\det\sigma$. We obtain in this fashion a geometric *spin*^{c} structure $(\mathbb{S}_\sigma, \boldsymbol{c}, \mathfrak{t}_\pm)$. Using the metric duality $TM \to T^*M$ we obtain a Clifford multiplication

$$\boldsymbol{c} : T^*M \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_\sigma).$$

This further extends to a linear map

$$\boldsymbol{c} : \Lambda^* T^*M \to \mathrm{End}_{\mathbb{C}}(\mathbb{S}_\sigma)$$

defined by

$$\boldsymbol{c}(e^1 \wedge \cdots \wedge e^k) = \boldsymbol{c}(e^1)\cdots\boldsymbol{c}(e^k),$$

where $1 \le k \le 3$, $e^1, \cdots e^k \in T^*_xM$, $g(e^i, e^j) = \delta_{ij}$, $\forall 1 \le i,j \le k$. We have the following identities

$$\boldsymbol{c}(dV_g) = -\mathbf{1}_{\mathbb{S}_\sigma} \Longleftrightarrow \boldsymbol{c}(\alpha) = \boldsymbol{c}(*_g\alpha),\ \ \forall \alpha \in \Omega^1(M),$$

where dV_g denotes the volume form induced by the metric g and the chosen orientation on M. The Levi-Civita connection ∇^g on TM together with a hermitian connection A on $\det\sigma$ define a hermitian connection ∇^A on $\mathbb{S}_\sigma$ as follows.

Fix a local orthonormal frame (e_i) of TM defined on an open set U, and denote by e^i the dual coframe. Then ∇^g has the form

$$\nabla^g = d + \sum_i e^i \otimes \Gamma_i,$$

where Γ_i is a local section of $\underline{so}(TM)$= skew-symmetric endomorphisms of TM. We can identify Γ_i with a local vector field $\hat{\Gamma}_i$ on M via the correspondence

$$\hat{\Gamma}_i \times_g V = \Gamma_i(V),\ \ \forall V \in T_xM,\ \ x \in U,$$

where $\times_g : T_xM \times T_xM \to T_xM$ denotes the cross product in the 3-dimensional Euclidean space (T_xM, g).

A connection A on $\det\sigma$ has the local description $A := d + \mathbf{i}\sum_k a_k e^k$, where a_k are real valued functions. Then the induced connection ∇^A on $\mathbb{S}_\sigma$ has the local description

$$\nabla^A = d + \sum_k e^k \otimes \Big(\boldsymbol{c}(\hat{\Gamma}_k) + \frac{\mathbf{i}}{2} a_k \mathbf{1}_{\mathbb{S}_\sigma}\Big).$$

The connection ∇^A induces a first order partial differential operator p.d.o.

$$\mathfrak{D}_A : C^\infty(\mathbb{S}_\sigma) \to C^\infty(\mathbb{S}_\sigma),$$

defined by the composition

$$\begin{tikzcd} C^\infty(\mathbb{S}_\sigma) \arrow[r, "\nabla^A"] \arrow[dr, "\mathfrak{D}_A"'] & C^\infty(T^*M \otimes \mathbb{S}_\sigma) \arrow[d, "\boldsymbol{c}(\star)\bullet"] \\ & C^\infty(\mathbb{S}_\sigma) \end{tikzcd}.$$

$\mathfrak{D}_A$ is elliptic, symmetric,

$$\mathfrak{D}_A^* = \mathfrak{D}_A,$$

and satisfies the *Weitzenböck identity*

$$\mathfrak{D}_A^2\psi = (\nabla^A)^*\nabla^A\psi + \frac{s_g}{4}\psi + \frac{1}{2}\mathbf{c}(F_A)\psi,\ \ \forall \psi \in C^\infty(\mathbb{S}_\sigma), \tag{1.2}$$

where s_g denotes the scalar curvature of the metric g and $F_A \in \Omega^2(M) \otimes \mathbf{i}\mathbb{R}$ denotes the curvature of A.

§1.2 The Seiberg-Witten equations. Suppose (M, τ) is an oriented, directed 3-manifold. To formulate the Seiberg-Witten equations we need to choose some additional data.

- A relative $spin^c$-structure $\sigma \in Spin^c(M, \tau)$.
- An admissible metric g on M.
- A hermitian metric h on $\det \sigma = \det \mathbb{S}_\sigma$.
- A co-closed 1-form η.
- A real valued function $\mu : M \to \mathbb{R}$.

Definition 1.5. *Denote by* $\mathcal{U} = \mathcal{U}(M, \tau, \sigma)$ *the space of* 4*-uples* (g, h, η, μ) *as above.*

Fix a parameter $u = (g, u, \eta, \mu) \in \mathcal{U}$. The metrics g and h induce a geometric $spin^c$ structure $(\mathbb{S}_\sigma, \boldsymbol{c}, \mathfrak{t}_\pm)$. Denote by $\langle \bullet, \bullet \rangle$ the hermitian metric on $\mathbb{S}_\sigma$. To every spinor $\psi \in \Gamma(\mathbb{S}_\sigma)$ we associate a traceless symmetric endomorphism of $\mathbb{S}_\sigma$

$$q(\psi) \in \mathrm{End}_{\mathbb{C}}(\mathbb{S}_\sigma), \quad q(\psi)\phi = \langle \phi, \psi \rangle \phi - \frac{|\psi|^2}{2}\phi.$$

Thus $\mathbf{i}q(\bullet) \in \underline{su}(\mathbb{S}_\sigma)$. $q(\psi)$ satisfies the important identities

$$\langle q(\psi)\psi, \psi \rangle = \frac{1}{2}|\psi|^2 = |q(\psi)|^2_{\mathrm{End}}. \tag{1.3}$$

. Using the Clifford multiplication isomorphism

$$\boldsymbol{c} : \Omega^1(M) \to \underline{su}(\mathbb{S}_\sigma),$$

we obtain for every spinor ψ a purely imaginary 1-form

$$Q_\sigma(\psi) := \boldsymbol{c}^{-1}(q(\psi)) \in \mathbf{i}\Omega^1(M).$$

Denote by $\mathcal{A}_\sigma$ the affine space of hermitian connections on $\det \sigma$ and form the *configuration space*

$$\mathcal{C}_\sigma := \Gamma(\mathbb{S}_\sigma) \times \mathcal{A}_\sigma$$

The configuration space is an affine space, and the tangent space to $\mathcal{C}_\sigma$ at a configuration $\mathsf{S}_0 := (\psi_0, A_0)$ is

$$T_{\mathsf{S}_0}\mathcal{C}_\sigma = \Gamma(\mathbb{S}_\sigma) \times \mathbf{i}\Omega^1(M).$$

We indicate the sections of the tangent bundle $T\mathcal{C}_\sigma$ by dots, e.g. $\dot{\mathsf{S}} = (\dot{\psi}, \mathbf{i}\dot{a})$. The metrics on M and $\mathbb{S}_\sigma$ induce a real valued L^2-metric on $T\mathcal{C}_\sigma$

$$\Big\langle (\dot{\psi}_1, \mathbf{i}\dot{a}_1), (\dot{\psi}_2, \mathbf{i}\dot{a}_2) \Big\rangle = \int_M \Big(\mathfrak{Re}\langle \dot{\psi}_1, \dot{\psi}_2 \rangle + g(\dot{a}_1, \dot{a}_2) \Big) dV_g.$$

In particular we have a natural identification $T^*\mathcal{C}_\sigma \longleftrightarrow T\mathcal{C}_\sigma$. We have a section $\mathcal{SW} = \mathcal{SW}_{\sigma,u}$ of $T\mathcal{C}_\sigma$ defined by

$$\mathcal{SW}(\psi, A) = \Big(\mathfrak{D}_A + \mu\psi, \ \frac{1}{2}Q_\sigma(\psi) - (*_g F_A + \mathbf{i}\eta) \Big).$$

Definition 1.6. *Let* $u \in \mathcal{U}(M, \tau, \sigma)$. *A configuration* $\mathsf{S} = (\psi, A) \in \mathcal{C}_\sigma$ *is called a* (σ, u)-**Seiberg-Witten monopole** *if it satisfies the* **Seiberg-Witten equations**

$$\mathcal{SW}_{\sigma,u}(\mathsf{S}) = 0 \Longleftrightarrow \left\{ \begin{array}{rcl} \mathfrak{D}_A\psi + \mu\psi & = & 0 \\ \boldsymbol{c}(*_g F_A + \mathbf{i}\eta) & = & \frac{1}{2}q(\psi) \end{array} \right. .$$

We will denote by $\mathcal{Z}_{\sigma,u}$ *the space of* (σ, u)*-monopoles*

$$\mathcal{Z}_{\sigma,u} := \mathcal{SW}^{-1}_{\sigma,u}(0).$$

The Seiberg-Witten equations have a variational interpretation. More precisely, $\mathcal{SW}$ is the (formal) L^2-gradient of a certain energy functional. We have the following result whose proof can be found in [11, p. 179].

Proposition 1.7. *Fix $A_0 \in \mathcal{A}_\sigma$ and define $\mathcal{E} = \mathcal{E}_{\sigma,u,A_0} : \mathcal{C}_\sigma \to \mathbb{R}$*

$$\mathcal{E}(\psi, A) = \frac{1}{2}\int_M (A - A_0) \wedge (F_A + F_{A_0}) + \frac{1}{2}\int_M \langle \mathfrak{D}_A\psi + \mu\psi, \psi\rangle dV_g + \mathbf{i}\int_M (A - A_0) \wedge *_g\eta.$$

Then

$$\mathcal{SW}_{\sigma,u} = \nabla^{L^2}\mathcal{E}_{\sigma,u,A_0}$$

that is for every $\mathsf{S} \in \mathcal{C}_\sigma$ and every compactly supported $\dot{\mathsf{S}}$ we have

$$\frac{d}{dt}|_{t=0}\, \mathcal{E}(\mathsf{S} + t\dot{\mathsf{S}}) = \Big\langle \mathcal{SW}_{\sigma,u}(\mathsf{S}), \dot{\mathsf{S}}\Big\rangle.$$

There is an infinite dimensional group rendering equivariant the above constructions. It is the gauge group $\mathcal{G}_\sigma$ consisting of maps

$$\gamma : M \to S^1 := \{z \in \mathbb{C};\ \ |z| = 1\}$$

such that $\gamma\, |_{\partial M}$ is homotopic to the identity. The gauge group acts on $\mathcal{C}_\sigma$ according to the prescription

$$\gamma \cdot (\psi, A) = \big(\gamma \cdot \psi, A - 2\frac{d\gamma}{\gamma}\big).$$

Since S^1 is an Eilenberg-MacLane space $K(\mathbb{Z}, 1)$ we deduce that the group of components of $\mathcal{G}_\sigma$ is isomorphic to $H^1(M, \partial M; \mathbb{Z})$. We thus get a surjective group morphism

$$\deg : \mathcal{G}_\sigma \to H^1(M, \partial M; \mathbb{Z}).$$

More explicitly, for every gauge transformation γ, its degree is the integral cohomology class represented by the closed 1-form

$$\frac{1}{2\pi}\gamma^*(d\theta) = \frac{1}{2\pi\mathbf{i}}\frac{d\gamma}{\gamma}.$$

Every co-closed 1-form η, the 2-form $*\eta$ is closed and thus determines a cohomology class $[*\eta] \in H^2(M, \mathbb{R})$. Set

$$\delta_{\sigma,\eta} : \mathcal{G}_\sigma \to \mathbb{R},\ \ \delta_{\sigma,\eta}(\gamma) = \int_M \deg\gamma \wedge (8\pi^2 c(\sigma) - 4\pi[*\eta]).$$

We say that $\delta_{\sigma,\eta}$ is the (σ, η)-*defect* of the gauge transformation γ. The image of δ is a discrete subgroup of $\mathbb{R}$. A simple computation shows that

$$\mathcal{E}_{\sigma,u}(\gamma \cdot \mathsf{S}) = \mathcal{E}_{\sigma,u}(\mathsf{S}) - \delta_{\sigma,\eta}(\gamma),\ \ \forall \mathsf{S} \in \mathcal{C}_\sigma,\ \ \gamma \in \mathcal{G}_\sigma. \tag{1.4}$$

The above identity implies that $\mathcal{SW}_\sigma$ is a $\mathcal{G}_\sigma$-equivariant section of $T\mathcal{C}_\sigma$. In particular, the Seiberg-Witten equations are $\mathcal{G}_\sigma$-invariant, i.e. the set $\mathcal{Z}_\sigma$ of SW monopoles is $\mathcal{G}_\sigma$-invariant. We denote by $\mathfrak{M}_{\sigma,u}$ the set of $\mathcal{G}_\sigma$-orbits of monopoles,

$$\mathfrak{M}_{\sigma,u} := \mathcal{Z}_{\sigma,u}/\mathcal{G}_\sigma.$$

We introduce the following important *gauge invariant* subclasses of configuration.

• The *irreducible configurations*, $\mathcal{C}_\sigma^* = \{(\psi, A) \in \mathcal{C}_\sigma;\ \ \psi \not\equiv 0\}$.
• The *reducible configurations*, $\mathcal{C}_\sigma^0 := \mathcal{C}_\sigma \setminus \mathcal{C}_\sigma^*$.

The group $\mathcal{G}_\sigma$ acts freely on $\mathcal{C}_\sigma^*$, while the stabilizer of a reducible configuration $\mathsf{S} = (0, A)$ is the group St_{S} consisting of constant gauge transformations. We set

$$\mathcal{Z}_{\sigma,u}^* = \mathcal{Z}_{\sigma,u} \cap \mathcal{C}_\sigma^*,\ \ \mathcal{Z}_{\sigma,u}^0 = \mathcal{Z}_{\sigma,u} \cap \mathcal{C}_\sigma^0,$$

$$\mathfrak{M}_{\sigma,u}^* = \mathcal{Z}_{\sigma,u}^*/\mathcal{G}_\sigma,\ \ \mathfrak{M}_{\sigma,u}^0 = \mathcal{Z}_{\sigma,u}^0/\mathcal{G}_\sigma.$$

We have a natural bijection

$$\mathcal{U}(M, \tau, \sigma) \longleftrightarrow \mathcal{U}(M, -\tau, \bar{\sigma})$$

given by

$$u = (g, h, \eta, \mu) \longleftrightarrow \bar{u} = (g, h^*, -\eta, \mu)$$

Consider the natural bijection

$$\mathcal{I}_\sigma := \mathcal{C}_\sigma \longleftrightarrow \mathcal{C}_{\bar{\sigma}},\ \ \mathsf{S} = (\psi, A) \longleftrightarrow \bar{\mathsf{S}} = (I_\sigma(\psi), A^*)$$

where $I_\sigma : \mathbb{S}_\sigma \to \mathbb{S}_{\bar{\sigma}} = \mathbb{S}_\sigma^*$ is the metric duality described in (1.1). This map satisfies

$$\overline{\gamma \cdot \mathsf{S}} = \mathcal{I}_\sigma(\gamma \cdot \mathsf{S}) = \bar{\gamma} \cdot \bar{\mathsf{S}} = \bar{\gamma} \cdot \mathcal{I}_\sigma(\mathsf{S}),\ \ \mathcal{E}_{\sigma,u,A_0} = \mathcal{E}_{\bar{\sigma},\bar{u},A_0^*} \circ \mathcal{I}_\sigma$$

This map thus induces bijections

$$\mathfrak{M}_{\sigma,u} \stackrel{\mathcal{I}_\sigma}{\longleftrightarrow} \mathfrak{M}_{\bar{\sigma},\bar{u}},\ \ \mathfrak{M}_{\sigma,u}^* \stackrel{\mathcal{I}_\sigma}{\longleftrightarrow} \mathfrak{M}_{\bar{\sigma},\bar{u}}^*.$$

§1.3 Some concrete computations. We discuss below the nature of Seiberg-Witten monopoles in some special cases. In the sequel we assume M is a closed, oriented 3-manifold.

§1.3.1 A vanishing result

Proposition 1.8. *Fix a spin^c structure σ and a Riemann metric g on M, and a hermitian metric h on σ. We choose the parameter u such that $\mu = 0$, $\eta = 0$. If the scalar curvature s_g of g is positive then there are no irreducible (σ, u)-monopoles.*

Proof Suppose (ψ, A) is a (σ, g)-monopole. Then

$$0 = \mathfrak{D}_A\psi = \mathfrak{D}_A^2\psi = (\nabla^A)^*\nabla^A\psi + \frac{s_g}{4}\psi + \frac{1}{2}\boldsymbol{c}(F_A)\psi = 0.$$

On the other hand $\boldsymbol{c}(F_A) = \boldsymbol{c}(*F_A) = \frac{1}{2}q(\psi)$ so we get

$$(\nabla^A)^*\nabla^A\psi + \frac{s_g}{4}\psi + \frac{1}{4}q(\psi)\psi = 0.$$

A simple computation shows that

$$\langle q(\psi)\psi, \psi\rangle_{\mathbb{S}_\sigma} = \frac{1}{4}|\psi|^4$$

and we deduce

$$0 = \int_M \Big(\langle(\nabla^A)^*\nabla^A\psi, \psi\rangle + \frac{s_g}{4}|\psi|^4 + \frac{1}{16}|\psi|^4\Big)dV_g$$

Integrating by parts the first term we deduce

$$\int_M \Big(|\nabla^A\psi|^2 + \frac{s_g}{4}|\psi|^4 + \frac{1}{16}|\psi|^4\Big)dV_g = 0.$$

This implies $\psi \equiv 0$ so that (ψ, A) is reducible. $\square$

§1.3.2 Monopoles on $S^1 \times \Sigma$.

Suppose M is a product $M = S^1 \times \Sigma$, where Σ is a closed oriented Riemann surface. Denote by $\pi_M : M \to \Sigma$ the canonical projection. We equip it with a product metric $g_M = d\theta^2 + g_\kappa$, where $d\theta$ is the standard angular form on S^1, and g_κ is a metric on Σ of constant sectional curvature κ. From the Gauss-Bonnet formula we deduce

$$\kappa \cdot \mathrm{vol}_\kappa(\Sigma) = 2\pi\chi(\Sigma). \tag{1.5}$$

Denote by K_Σ the canonical line bundle of Σ. We set

$$\mathcal{K} = \mathcal{K}_M := \pi_M^*(K_\Sigma).$$

$\mathcal{K}_M$ is equipped with a natural hermitian metric. The dual vector field ∂_θ defines a *spin*^{c} structure σ_0 on M. Moreover

$$\det(\sigma_0) \cong \mathcal{K}_M^{-1}, \;\; \mathbb{S}_{\sigma_0} \cong \underline{\mathbb{C}}_M \oplus \mathcal{K}_M^{-1}.$$

From the Künneth formula we obtain an injection

$$\pi_M^* : \mathrm{Pic}_t(\Sigma) \to \mathrm{Pic}_t(M).$$

Given a complex line bundle $L \to M$ we obtain a *spin*^{c} structure $\sigma_L = L \otimes \sigma_0$ on M. Observe that

$$\bar{\sigma}_L = L^* \otimes \bar{\sigma}_0 = (\mathcal{K} - L) \otimes \sigma_0 = \sigma_{\mathcal{K}-L}$$

A hermitian metric h_L on L induces a hermitian metric on

$$\mathbb{S}_L := \mathbb{S}_{\sigma_L} \cong \mathbb{S}_{\sigma_0} \otimes L \cong L \oplus L \otimes \mathcal{K}_M^{-1}.$$

In particular, every spinor $\psi \in \Gamma(\mathbb{S}_L)$ decomposes as

$$\psi = \alpha \oplus \beta, \;\; \alpha \in \Gamma(L), \;\; \beta \in \Gamma(L \otimes \mathcal{K}_M^{-1}).$$

The Levi-Civita connection on Σ induces a hermitian connection on K_Σ^{-1} with curvature

$$F_0 = \mathbf{i} c dV_\Sigma, \;\; c \in \mathbb{R}$$

where

$$\chi(\Sigma) = \deg K_\Sigma^{-1} = \frac{\mathbf{i}}{2\pi}\int_\Sigma F_0 - \frac{c}{2\pi}\mathrm{vol}_\kappa(\Sigma).$$

Using (1.5 we deduce

$$F_0 = -\mathbf{i}\kappa dV_\Sigma \Longrightarrow F_{A_0} = -\mathbf{i}\kappa\pi^* dV_\Sigma.$$

Denote $*_M$ the Hodge $*$-operator on M. We extend it by *complex linearity* to complex valued forms. Observe that

$$|F_{A_0}(x)|^2_{g_M} = \kappa^2, \;\; \forall x \in M, \;\; dV_M = d\theta \wedge \pi^* dV_\Sigma \Longrightarrow *_M F_{A_0} = -\mathbf{i}\kappa d\theta.$$

For brevity, we will denote by φ the angular form $d\theta$,

$$\varphi := d\theta.$$

Using the isomorphism

$$T^*M \cong \underline{\mathbb{R}}_M\langle\varphi\rangle \oplus \mathcal{K}_M \Longleftrightarrow T^*M \otimes \underline{\mathbb{C}} \oplus \mathcal{K}_M \oplus \mathcal{K}_M^{-1}$$

we obtain an isomorphism of complex vector bundles

$$T^*M \otimes L \cong L \oplus L \otimes \mathcal{K}_M \oplus L \otimes \mathcal{K}_M^{-1}.$$

Accordingly, any hermitian connection B on L decomposes

$$\nabla^B : \Gamma(L) \to \Gamma(L) \oplus \Gamma(L \otimes \mathcal{K}_M) \oplus \Gamma(L \otimes \mathcal{K}_M^{-1}), \quad \alpha \mapsto \nabla^B_\theta \alpha \oplus \partial_B \alpha \oplus \bar{\partial}_B \alpha.$$

On the other hand hermitian connection B on L induces a hermitian connection $A = A(B) = A_0 + B^{\otimes 2}$ on $\det \sigma_L = \mathcal{K}_M^{-1}$. We will use the less rigorous but more intuitive notation

$$A = A_0 \dot{+} 2B.$$

For every $\alpha \oplus \beta \in \Gamma(\mathbb{S}_L)$ we have (see [10] for a proof)

$$\mathfrak{D}_{A_0 \dot{+} 2B} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\mathbf{i}\nabla^B_\theta & \bar{\partial}^*_B \\ \bar{\partial}_B & \mathbf{i}\nabla^B_\theta \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad F_{A_0 \dot{+} 2B} = F_{A_0} + 2F_B,$$

$$Q(\alpha \oplus \beta) = \frac{\mathbf{i}}{2}(|\alpha|^2 - |\beta|^2)\varphi + \frac{1}{\sqrt{2}}(\bar{\alpha}\beta - \frac{1}{\sqrt{2}}\alpha\bar{\beta}) \in \Gamma(\underline{\mathbb{C}} \oplus \mathcal{K}_M^{-1} \oplus \mathcal{K}_M)$$

Every complex valued 2-form Ξ on M has a decomposition

$$\Xi = \Xi^h dV_\Sigma + d\theta \wedge \Xi^\perp, \quad \Xi^\perp = \partial_\theta \lrcorner\, \Xi \in \Gamma(\mathcal{K}_M \otimes \mathbb{C}).$$

$\Xi^\perp$ further decomposes

$$\Xi^\perp = \Xi^{1,0} \oplus \Xi^{0,1} \in \Gamma(\mathcal{K}_M) \oplus \Gamma(\mathcal{K}_M^{-1})$$

Observe that

$$*_M(d\theta \wedge \Xi^\perp) = -\mathbf{i}\Xi^{1,0} + \mathbf{i}\Xi^{0,1}.$$

For every real number t we denote by $u_t \in \mathcal{U}(M, \sigma_L)$ the parameter

$$u_t = (g, h_L, \eta = t\varphi, \mu = 0),$$

The Seiberg-Witten equations satisfied by a (σ_L, u_t) monopole $\mathsf{S} = (\psi, A) = (\alpha, \beta; A_0 \dot{+} 2B)$ can be rewritten as

$$\left\{ \begin{array}{rcl} -\mathbf{i}\nabla^B_\theta + \bar{\partial}^*_B \beta & = & 0 \\ \bar{\partial}_B \alpha + \mathbf{i}\nabla^B_\theta \beta & = & 0 \\ 2F^h_B + \mathbf{i}(t - \kappa) & = & \frac{\mathbf{i}}{2}(|\alpha|^2 - |\beta|^2) \\ 2\mathbf{i}F^{0,1}_B & = & \frac{1}{\sqrt{2}}\bar{\alpha}\beta \\ 2\mathbf{i}F^{1,0}_B & = & \frac{1}{\sqrt{2}}\alpha\bar{\beta} \end{array} \right. . \tag{1.6}$$

Denote by $\mathfrak{M}_{L,t}$ the set of gauge equivalence classes of (σ_L, u_t)-monopoles. For every nonnegative integer d we denote by $S^d(\Sigma)$ the symmetric product of d-copies of Σ. For $d < 0$ we set $S^d\Sigma = \emptyset$.

Proposition 1.9 (Mũnoz). *Set*

$$\deg_\varphi(L) := \int_M \frac{\varphi}{2\pi} \wedge c_1(L) \in \mathbb{Z}.$$

(i) Suppose $t \gg 0$. Then $\mathfrak{M}_{L,t} \neq \emptyset$ if and only if $\deg_\varphi(L) \geq 0$ and there exists $\hat{L} \to \Sigma$ such that $L \cong \pi^*\hat{L}$. In this case there are no reducible monopoles and there exists a natural bijection

$$\Psi_{L,t} \to \mathcal{D}_t(L) := S^{\deg_\varphi(L)}\Sigma.$$

(ii) Suppose $t \ll 0$. Then $\mathfrak{M}_{L,t} \neq \emptyset$ if and only if $\deg_\varphi(L) \leq \deg_\varphi \mathcal{K}_M = \deg K_\Sigma$ and there exists $\hat{L} \to \Sigma$ such that $L \cong \pi^* \hat{L}$. In this case there exists a natural bijection

$$\Psi_{L,t} : \mathfrak{M}_{L,t} \to \mathcal{D}_t(L) : S^{\deg K_\Sigma - \deg_\varphi(L)}.$$

(iii) For any line bundle $L \to M$ which is the pullback of a line bundle on Σ there exists $T_L > 0$ such that for all $|t| > T_L$ we have an equality

$$\mathcal{D}_t(L) = \mathcal{D}_{-t}(\mathcal{K} - L).$$

and a commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_{L,t} & \xrightarrow{\Psi_{L,t}} & \mathcal{D}_t(L) \\ \mathcal{I}_{\sigma_L} \Big\downarrow & & \Big\downarrow \mathbf{1} \\ \mathfrak{M}_{\mathcal{K}-L,-t} & \xrightarrow{\Psi_{\mathcal{K}-L,-t}} & \mathcal{D}_{-t}(\mathcal{K}_M - L) \end{array} .$$

Sketch of proof. For every hermitian connection B on L we consider the partial differential operators $\Gamma(\mathbb{S}_L) \to \Gamma(\mathbb{S}_L)$

$$V_B = \left[\begin{array}{cc} -\mathbf{i}\nabla^B_\theta & 0 \\ 0 & \mathbf{i}\nabla^B_\theta \end{array} \right], \;\; H_B := \left[\begin{array}{cc} 0 & \bar{\partial}^*_B \\ \bar{\partial}_B & 0 \end{array} \right].$$

Observe that both operators are formally self-adjoint, V_B involves only derivatives in the vertical direction, H_B involves derivatives only in horizontal directions and

$$\mathfrak{D}_{A_0 \dot{+} 2B} = V_B + H_B.$$

These two operators satisfy the following fundamental identity (see [10])

$$\{V_B, H_B\} := V_B H_B + H_B V_B = \sqrt{2} \left[\begin{array}{cc} 0 & \mathbf{i}F^{1,0}_B \otimes \\ \mathbf{i}F^{0,1}_B \otimes & 0 \end{array} \right], \tag{1.7}$$

where we identify $\mathcal{K}_M \otimes_{\mathbb{C}} \mathcal{K}_M^{-1} \cong \underline{\mathbb{C}}_M$. When $(\psi, A) = (\alpha, \beta; A_0 \dot{+} 2B)$ is a monopole we deduce from (1.6) that

$$\{V_B, H_B\} = \frac{1}{2} \left[\begin{array}{cc} 0 & \alpha\bar{\beta} \\ \bar{\alpha}\beta & 0 \end{array} \right]. \tag{1.8}$$

Form the equality $(V_B + H_B)\psi = 0$ we deduce

$$0 = (V_B + H_B)^2 \psi = (V_B^2 + H_B^2 + \{V_B, H_B\})\psi$$

Taking the L^2-inner product of the above equality with ψ and using the symmetry of V_B and H_B we deduce

$$0 = \|V_B\psi\|^2_{L^2} + \|H_B\psi\|^2_{L^2} + \Big\langle \{V_B, H_B\}\psi, \psi \Big\rangle_{L^2} \stackrel{(1.8)}{=} \|V_B\psi\|^2_{L^2} + \|H_B\psi\|^2_{L^2} + \int_M |\alpha|^2 |\beta|^2 dV_M.$$

Hence we deduce

$$\nabla^B_\theta \alpha = 0, \;\; \nabla^B_\theta \beta = 0, \tag{1.9a}$$

$$\bar{\partial}_B\alpha = 0, \;\; \bar{\partial}_B^*\beta = 0 \tag{1.9b}$$

$$\alpha \otimes \bar{\beta} = \partial_\theta \wedge F_B = 0. \tag{1.9c}$$

From the third equation in (1.6) and (1.9c) we deduce the equality

$$F_B = F_B^h \wedge dV_\Sigma = \frac{\mathbf{i}}{4}(|\alpha|^2 - |\beta|^2 - \frac{t-\kappa}{2})dV_\Sigma$$

Since α and β are covariant constant along the fibers of $S^1 \times \Sigma \to \Sigma$ we can regard $|\alpha|^2$ and $\beta|^2$ as functions on Σ. We deduce

$$\frac{\mathbf{i}}{2\pi}\int_M \varphi \wedge F_B = -\frac{1}{4}\int_\Sigma (|\alpha|^2 - |\beta|^2 - \frac{t-\kappa}{2})dV_\Sigma$$

so that

$$\int_\Sigma (|\alpha|^2 - |\beta|^2)dV_\Sigma = -\frac{2\mathbf{i}}{\pi}\int_M \varphi \wedge F_B + \frac{(t-\kappa)}{2}\mathrm{vol}\,(\Sigma) = -8\pi\int_M \frac{\varphi}{2\pi} \wedge \frac{\mathbf{i}}{2\pi}F_B + \frac{(t-\kappa)}{2}\mathrm{vol}\,(\Sigma).$$

Hence

$$\int_\Sigma (|\alpha|^2 - |\beta|^2)dV_\Sigma = -8\pi \deg_\varphi(L) + \frac{t\mathrm{vol}\,(\Sigma)}{2} - \pi\chi(\Sigma).$$

This shows that

$$t\mathrm{vol}\,(\Sigma) > 16\pi \deg_\varphi(L) + 2\pi\chi(\Sigma) \Longrightarrow \alpha \neq 0$$

$$t\mathrm{vol}\,(\Sigma) < 16\pi \deg_\varphi(L) + 2\pi\chi(\Sigma) \Longrightarrow \beta \neq 0.$$

Using (1.9a) we deduce that the line bundle L admits a nontrivial section which is B-covariant constant along the fibers of π_M. Using (1.9c) and the unique continuation principle applied to the solutions of

$$(V_B + H_B)(\alpha \oplus 0) = (V_B + H_B)(0 \oplus \beta) = 0$$

we deduce that either $\alpha \equiv 0$ or $\beta \equiv 0$. Thus

$$t \gg 0 \Longrightarrow \alpha \neq 0, \;\; \beta \equiv 0,$$

$$t \ll 0 \Longrightarrow \alpha \equiv 0, \;\; \beta \neq 0.$$

These facts imply that there exist a hermitian line bundle $\hat{L} \to \Sigma$, a hermitian connection $\hat{B}$ on $\hat{L}$, and a gauge transformation $\gamma : M \to S^1$ such that

$$L \cong \pi^*\hat{L}, \;\; \gamma \cdot \nabla^B \gamma^{--1} = \pi^*(\nabla^{\hat{B}}).$$

Assume $t \gg 0$. (The case $t \ll 0$ is completely analogous.) Since α is covariant constant along the fibers of π_M there exists a sections $\hat{\alpha} \in \Gamma(\hat{L})$ along Σ which pull-back to α. The hermitian connection defines a *holomorphic* structure on $\hat{L} \to \Sigma$ and α is a nontrivial holomorphic section satisfying

$$\int_\Sigma |\hat{\alpha}|^2 dV_\Sigma = -8\pi \deg_\varphi L + \pi\big(t + \chi(\Sigma)\big) = -8\pi \deg \hat{L} + \pi\big(t + \chi(\Sigma)\big). \tag{1.10}$$

In particular

$$\deg_\varphi(L) = \deg \hat{L} \geq 0.$$

The map $\Psi_{L,t}$ associates to the monopole $(\alpha, \beta; A_0 \dot{+} 2B)$ the divisor associated to the holomorphic section $\hat{\alpha}$. It is easy to see that if two monopoles S, S' are gauge equivalent then

$$\Psi_{L,t}(\mathsf{S}) = \Psi_{L,t}(\mathsf{S}').$$

Conversely an effective divisor determines a holomorphic line bundle and a holomorphic section, unique up to a nonzero multiplicative constant. The identity (1.10) determines this constant up to multiplication by a complex number of norm 1. We get a $1-1$ map

$$\Psi_{L,t} : \mathfrak{M}_{L,t} \to (L).$$

Conversely, given an effective divisor D on Σ of degree $\deg D = \deg_\varphi(L)$ we can produce a hermitian line bundle $\hat{L}$ on Σ such that

$$\deg \hat{L} = \deg D =, \ \ \pi^* \hat{L} \cong L,$$

a hermitian connection $\hat{B}$ and a section $\hat{\alpha}$ of $\hat{L}$ such that $\bar{\partial}_{\hat{B}} \hat{\alpha} = 0$, and satisfying (1.10). We obtain by pullback a monopole

$$\mathsf{S}_D := (\pi^* \alpha, \pi^* \beta; A_0 \dot{+} 2\pi^* \hat{B})$$

such that

$$\Psi_{L,t}(\mathsf{S}_D) = D.$$

This requires solving a Kazhdan-Warner type equation. □

Remark 1.10. The above argument extends to the more general case of Seifert manifolds. The major obstacle in this case is the lack of an identity of the type (1.7). We refer to [8, 10] for different ways of dealing with this issues.

2 The Seiberg-Witten invariants of closed manifolds

In the sequel we will show that an appropriate count of point in $\mathfrak{M}_{\sigma,u}$ yields a topological invariant of the pair (M,σ). We will first show that for each parameter u the moduli space $\mathfrak{M}_{\sigma,u}$ has a natural structure of compact ringed space. In fact, if all the parameter u is real analytic then $\mathfrak{M}_{\sigma,u}$ has a natural structure of real analytic space. For generic u it consists of finitely many points, and a certain signed count of these pints will yield the sought for invariant.

§2.1 The Seiberg-Witten moduli spaces Suppose M is a closed, oriented 3-manifold. In this case all directions are equivalent so we will not keep track of them. Fix a *spin*^{c} structure $\sigma \in Spin^c(M)$, a parameter $u = (g,h,\eta,\mu) \in \mathcal{U}(M,\sigma)$ and a point $p_0 \in M$. Fix a *smooth* reference hermitian connection B_u on $\det\sigma$. To produce additional structures on $\mathfrak{M}_{\sigma,u}$ we first need to introduce additional structures on the configuration space. We will organize this space as a Hilbert manifold using Sobolev spaces $L^{k,2}$ = "k derivatives in L^2". The fixed reference connection produces an identification

$$\Gamma(\mathbb{S}_\sigma) \oplus \mathbf{i}\Omega^1(M) \ni (\psi, \mathbf{i}a) \longleftrightarrow (\psi, B_u + \mathbf{i}a) \in \mathcal{C}_\sigma.$$

For $k \geq 1$ define the configuration space $\mathcal{C}_\sigma = \mathcal{C}^k_\sigma$ to consist of pairs (ψ, A) such that

$$\|\psi\|_{k,2} + \|A - B_u\|_{k,2} < \infty.$$

It is obviously a Hilbert manifold. Re-define the gauge group by setting

$$\mathcal{G} = \mathcal{G}^k := \Big\{\gamma \in L^{k+1,2}(M,\mathbb{C});\ \ |\gamma(x)| = 1,\ \ \forall x \in M\Big\}.$$

The Sobolev embedding theorem shows that all the functions in $L^{k,2}(M,\mathbb{C})$ are at least C^1 and one can show easily that $\mathcal{G}$ is a Hilbert-Lie group, and that the action

$$\mathcal{G} \times \mathcal{C}_\sigma \ni (\gamma; \psi, \mathbf{i}a) \mapsto \gamma \cdot (\psi, \mathbf{i}a) = (\gamma \cdot \psi, \mathbf{i}a - 2\frac{d\gamma}{\gamma}).$$

is smooth. Denote by $\mathcal{X} = \mathcal{X}^k = T_1\mathcal{G}$ the Lie algebra of this group,

$$\mathcal{X} \cong L^{k+1,2}(M, \mathbf{i}\mathbb{R}).$$

We will denote the elements of $\mathcal{X}$ by the symbols X, Y etc. The exponential map

$$\exp : \mathcal{X} \ni \mathsf{X} = \mathbf{i}f \mapsto \exp(\mathsf{X}) = \exp(\mathbf{i}f) \in \mathcal{G}$$

maps $\mathcal{X}$ onto the identity component ${}_1\mathcal{G}$ of $\mathcal{G}$. Every configuration $\mathsf{S} = (\psi, A = B_u + \mathbf{i}a)$ defines a linear map

$$\mathcal{L}_{\mathsf{S}} : \mathcal{X} \to T_{\mathsf{S}}\mathcal{C}_\sigma,\ \ \mathcal{L}_{\mathsf{S}}(\mathbf{i}f) = \frac{d}{dt}\exp(\mathbf{i}tf) \cdot \mathsf{S} = (\mathbf{i}f\psi, -2\mathbf{i}df).$$

We denote by $\mathcal{X}_{\mathsf{S}}$ the subspace $\mathcal{L}_{\mathsf{S}}(\mathcal{X}) \subset T_{\mathsf{S}}\mathcal{C}_\sigma$. This is the tangent space at S to the orbit of $\mathcal{G} \cdot \mathsf{S}$. Set

$$\mathcal{X}^\perp_{\mathsf{S}} := \Big\{\dot{\mathsf{S}} \in T_{\mathsf{S}}\mathcal{C}_\sigma;\ \ \langle \dot{\mathsf{S}}, \mathcal{L}_{\mathsf{S}}(\mathsf{X})\rangle_{L^2} = 0,\ \ \forall \mathsf{X} \in \mathcal{X}\Big\} = \ker(\mathcal{L}^*_{\mathsf{S}} : T_{\mathsf{S}}\mathcal{C}_\sigma \to \mathcal{X}),$$

where

$$\mathcal{L}^*_{\mathsf{S}} : C^\infty(\mathbb{S}_\sigma \oplus \mathbf{i}T^*M) \to C^\infty(M, \mathbf{i}\mathbb{R})$$

is the formal adjoint of $\mathcal{L}_{\mathsf{S}=(\psi,A)}$ defined by

$$C^\infty(\mathbb{S}_\sigma \oplus \mathbf{i}T^*M) \ni \dot{\psi} \oplus \mathbf{i}\dot{a} \mapsto \mathcal{L}_{(\psi,A)}(\dot{\psi} \oplus \mathbf{i}\dot{a}) = -2\mathbf{i}d^*\dot{a} - \mathbf{i}\mathfrak{Im}\,\langle \psi, \dot{\psi}\rangle \in C^\infty(M, \mathbf{i}\mathbb{R}).$$

The group $\mathcal{G}$ is not connected. In fact, its group of components $[\mathcal{G}]$ is isomorphic to $H^1(M,\mathbb{Z})$. We have a natural epimorphism

$$[\bullet]:\mathcal{G}\to H^1(M,\mathbb{Z}),\ \ \gamma\mapsto[\gamma]=\frac{1}{2\pi}\gamma^*(d\theta).$$

The metric g and the point p_0 produces a natural splitting of this epimorphism

$$H^1(M,\mathbb{Z})\ni\lambda\mapsto\gamma_\lambda=\gamma_{\lambda,g,p_0}\in\mathcal{G},\ \ \gamma_\lambda(p)=\exp(2\pi\mathbf{i}\int_{p_0}^p[\lambda]_g),$$

where $[\lambda]_g$ denotes the unique g-harmonic representative of the class λ. Here we used the fact that the natural map $H^1(M,\mathbb{Z})\to H^1(M,\mathbb{R})$ is $1-1$.

We identify $S^1\subset\mathbb{C}^*$ with the subgroup of constant gauge transformation

$$S^1\ni z\mapsto\gamma_z\in\mathcal{G},\ \ \gamma_z(p)=z,\ \ \forall p\in M.$$

The point p_0 determines a natural splitting of the inclusion $z\mapsto\gamma_z$ via the evaluation map

$$\mathbf{ev}_{p_0}:\mathcal{G}\to S^1,\ \ \gamma\mapsto\gamma(p_0).$$

We define the group of *gauge transformations based at* p_0 by

$$\mathcal{G}_{p_0}:=\ker\mathbf{ev}_{p_0}.$$

$\mathcal{G}_{p_0}$ is a closed Lie subgroup of $\mathcal{G}$. We thus have isomorphisms

$$\mathcal{G}=\mathcal{G}_{p_0}\times S^1\cong{}_1\mathcal{G}\times H^1(M,\mathbb{Z})\cong{}_1\mathcal{G}_{p_0}\times S^1\times H^1(M,\mathbb{Z}).$$

For every configuration $\mathsf{S}\in\mathcal{C}_\sigma$ we define the *slice* of the $\mathcal{G}$-action at S to be the affine subspace

$$\mathcal{S}_{\mathsf{S}}=\mathsf{S}+\mathcal{X}_{\mathsf{S}}^\perp\subset\mathcal{C}_\sigma.$$

The subgroup $S^1\times[\mathcal{G}]\cong S^1\times H^1(M,\mathbb{Z})$ acts on $\mathcal{C}_\sigma$ by

$$(z,\lambda)\cdot(\psi,A)=\big(z\cdot\gamma_\lambda\cdot\psi,A-4\pi\mathbf{i}[\lambda]_g\big).$$

Set $\mathsf{S}_0:=(0,B_u)$. Then

$$\mathcal{X}_{\mathsf{S}_0}^\perp=\Big\{(\dot\psi,\mathbf{i}\dot a)\in T_{\mathsf{S}_0}\mathcal{C}_\sigma;\ \ d^*\dot a=0\Big\},$$

so that

$$\mathcal{S}_{\mathsf{S}_0}=\Big\{(\psi,A)\in\mathcal{C}_\sigma;\ \ d^*(A-B_u)=0\Big\}.$$

Observe that $\mathcal{S}_{\mathsf{S}_0}$ is invariant under the above action of $S^1\times[\mathcal{G}]$. Moreover, the quotient $\mathcal{S}_{\mathsf{S}_0}/[\mathcal{G}]$ is a smooth Hilbert manifold equipped with an S^1-action and the natural projection $\mathcal{S}_{\mathsf{S}_0}\to\mathcal{S}_{\mathsf{S}_0}/[\mathcal{G}]$ is a Galois covering with automorphism group $[\mathcal{G}]$.

Proposition 2.1 (Existence of global slices). *Every $\mathcal{G}^1_{p_0}$-orbit intersects $\mathcal{S}_{\mathsf{S}_0}$ exactly once.*

Proof The Lie algebra of $\mathcal{G}_{p_0}$ is

$$T_1\mathcal{G}_{p_0}=\Big\{\mathbf{i}f\in L^{k+1,2}(M,\mathbf{i}\mathbb{R});\ \ f(p_0)=0\Big\}$$

and the exponential map $\exp : T_1\mathcal{G}^1_{p_0} \to \mathcal{G}_{p_0}$ is onto. Thus we can represent every element $\gamma \in {}_1\mathcal{G}_{p_0}$ in the form

$$\gamma = \exp(\mathbf{i}f), \;\; f(p_0) = 0.$$

Suppose $\mathsf{S} = (\psi, B_u + \mathbf{i}a) \in \mathcal{C}_\sigma$. To show that ${}_1\mathcal{G}_{p_0} \cdot \mathsf{S} \cap \mathcal{S}_{\mathsf{S}_0}$ consists of a single point we need to show that there exists a unique $\mathbf{i}f \in T_1\mathcal{G}^1_{p_0}$ such that

$$\exp(\mathbf{i}f) \cdot \mathsf{S} - \mathsf{S}_0 \in \mathcal{X}^{\perp}_{\mathsf{S}_0} \Longleftrightarrow d^*(a - 2df) = 0.$$

Using the Hodge decomposition of a with respect to the metric a we can write

$$a = [a]_g + d\alpha + d^*\beta, \;\; \alpha \in \Omega^0(M), \;\; \beta \in \Omega^2(M)$$

and where $[a]_g$ denotes the g-harmonic part of a. The function α is unique up to an additive constant. We fix that constant by requiring that $\alpha(p_0) = 0$. The equation $d^*(a - 2f) = 0$ is then equivalent to

$$\Delta_M(2f - \alpha) = 0$$

$2f - \alpha$ is thus a harmonic function and hence it must be constant. Since $f(p_0) = \alpha(p_0) = 0$ we deduce $2f = \alpha$. Hence $\gamma = \exp(2\mathbf{i}\alpha)$ is the unique based gauge transformation which maps S to the slice $\mathcal{S}_{\mathsf{S}_0}$.

□

We deduce that there exists a S^1-equivariant bijection

$$\Phi_{\mathsf{S}_0} : \mathcal{S}_{\mathsf{S}_0}/[\mathcal{G}] \to \mathcal{C}_\sigma/\mathcal{G}_{p_0} =: \mathcal{B}_{\sigma,p_0}.$$

We use this bijection to transport the Hilbert-manifold structure on $\mathcal{S}_{\mathsf{S}_0}/[\mathcal{G}]$ to a Hilbert manifold structure on $\mathcal{B}_{\sigma,p_0}$. This manifold is equipped with a residual S^1-action and we set

$$\mathcal{B}_\sigma = \mathcal{B}_{\sigma,p_0}/S^1 \cong \mathcal{C}_\sigma/\mathcal{G}.$$

Proposition 2.2 (Compactness). *For every $u = (g, h, \eta, \mu) \in \mathcal{U}_\sigma$ the set of gauge equivalence classes of monopoles $\mathfrak{M}_{\sigma,u} \subset \mathcal{B}_\sigma$ is compact.*

Proof To prove this key fact we use a trick of Kronheimer-Mrowka,[4]. Suppose (ψ, A) is a (σ, u)-monopole, i.e.

$$\left\{ \begin{array}{rcl} \mathfrak{D}_A\psi + \mu\psi &=& 0 \\ \boldsymbol{c}(*_g F_A + \mathbf{i}\eta) &=& \frac{1}{2}q(\psi) \end{array} \right. .$$

From the first equality we deduce

$$\mathfrak{D}_A^2\psi - \mu^2\psi = 0 \Longrightarrow (\nabla^A)^*\nabla^A\psi + \frac{s(g)}{4}\psi + \frac{1}{2}\boldsymbol{c}(F_A)\psi - \mu^2\psi = 0.$$

From the second equality we get

$$(\nabla^A)^*\nabla^A\psi + \frac{s(g)}{4}\psi + \frac{1}{4}q(\psi)\psi - \frac{1}{2}\boldsymbol{c}(\mathbf{i}\eta)\psi - \mu^2\psi = 0.$$

On the other hand, we have a Kato (pointwise) inequality

$$\Delta_M|\psi|^2 \leq 2\langle(\nabla^A)^*\nabla^A\psi, \psi\rangle = -\Big\langle \frac{s(g)}{4}\psi + \frac{1}{4}q(\psi)\psi - \frac{1}{2}\boldsymbol{c}(\mathbf{i}\eta)\psi - \mu^2\psi, \psi \Big\rangle$$

$$= (\mu^2 - \frac{s}{4})|\psi|^2 + \frac{1}{2}\langle \boldsymbol{c}(\mathbf{i}\eta)\psi, \psi\rangle - \frac{1}{4}\langle q(\psi)\psi,\psi\rangle \leq \underbrace{(\mu^2 - \frac{s}{4} + |\eta|_g)}_{=:w} |\psi|^2 + \frac{1}{8}|\psi|^4$$

If we set $f := |\psi|^2$ we deduce that the nonnegative function f satisfies the differential inequality

$$\Delta_M f \leq f(w + \frac{1}{8}f).$$

If x_0 is a maximum point of f then $(\Delta_M f)|_{x_0} \geq 0$ and we deduce

$$0 \leq f(x_0)(w(x_0) - f(x_0)).$$

This implies

$$\|\psi\|_\infty^2 = \max_{x\in M} |\psi(x)|^2 \leq \max_{x\in M} w(x) =: K(u). \tag{2.1}$$

Using Proposition 2.1 we deduce that we can move (ψ, A) along its $\mathcal{G}$-orbit until it intersects $\mathcal{S}_{\mathsf{S}_0}$. Thus we can assume that (ψ, A) has the form

$$(\psi, A) = (\psi, B_u + \mathbf{i}da), \quad d^*a = 0.$$

Using gauge transformations in $[\mathcal{G}]$ we can even arrange that

$$\|[a]_g\|_{L^2} \leq \Lambda(u) := \sup_\alpha \inf_\lambda \Big\{ \|[\alpha]_g - 4\pi[\lambda]_g\|_{L^2};\ \alpha \in H^1(M,\mathbb{R}),\ \ \lambda \in H_1(M,\mathbb{Z})\Big\}.$$

We deduce that

$$\|\mathfrak{D}_A\psi\|_\infty \leq \|\mu\|_\infty K_u,\ \ d^*a = 0,\ \ \|da\|_\infty = \|F_A\|_\infty \leq C = C(u),\ \ [a]_g \leq \Lambda_u.$$

An elliptic bootstrap applied to the above inequalities implies that for every positive integer ℓ there exists a constant $C = C(\ell, u)$ such that

$$\|\psi\|_{\ell,2} + \|a\|_{\ell,2} \leq C(\ell, u).$$

The compactness result now follows from the compactness of the embeddings

$$L^{\ell,2} \hookrightarrow L^{k,2},\ \ \ell > k \geq 0.$$

$\square$

Definition 2.3. *The set* $\mathfrak{M}_{\sigma,u} \subset \mathcal{B}_\sigma$ *with the induced topology is called the* Seiberg-Witten moduli space *corresponding to* (σ, u).

The decomposition $\mathcal{C}_\sigma = \mathcal{C}_\sigma^0 \cup \mathcal{C}_\sigma^*$ is $\mathcal{G}$-equivariant, and we get corresponding decompositions into reducible and irreducible parts,

$$\mathcal{B}_\sigma = \mathcal{B}_\sigma^0 \cup \mathcal{B}_\sigma^*,\ \ \mathfrak{M}_\sigma = \mathfrak{M}_{\sigma,u}^0 \cup \mathfrak{M}_{\sigma,u}^*.$$

$\mathcal{B}_\sigma^*$ is a smooth Hilbert manifold while $\mathcal{B}_\sigma^0$ is homeomorphic to the quotient

$$\frac{\text{co-closed 1-forms on } M}{4\pi H^1(M,\mathbb{Z})}.$$

We have a natural projection

$$W : \mathcal{U}_\sigma \to H^2(M,\mathbb{R}),\ \ W(g,h,\eta,\mu) = [*\eta]_g.$$

Define the σ-*wall*

$$\mathcal{W}_\sigma := W^{-1}(2\pi c(\sigma)) = \Big\{(g,h,\eta,\mu) \in \mathcal{U}_\sigma;\ \ [*\eta]_g = 2\pi[c(\sigma)]_g\Big\}.$$

Proposition 2.4 (Existence of reducible monopoles).

$$\mathfrak{M}^0_{\sigma,u} \neq \emptyset \iff u \in \mathcal{W}_\sigma.$$

Moreover, when $u \in \mathcal{W}_\sigma$ we have

$$\mathfrak{M}^0_{\sigma,u} \cong_{C^0} H^1(M,\mathbb{R})/4\pi H^1(M,\mathbb{Z}).$$

Proof Let $u = (g,h,\eta,\mu) \in \mathcal{U}_\sigma$. A reducible (σ,u)-monopole is a configuration of the form $(0, B_u + \mathbf{i}a)$ satisfying

$$F_{B_u+\mathbf{i}a} = -\mathbf{i} * \eta \Longrightarrow F_{B_u} + \mathbf{i}da = -\mathbf{i} * \eta. \tag{2.2}$$

If this equation has solutions we deduce that

$$[F_{B_u}]_g = -\mathbf{i}[*\eta]_g \Longrightarrow [c(\sigma)]_g = \frac{\mathbf{i}}{2\pi}[F_{B_u}]_g = \frac{1}{2\pi}[*\eta]_g$$

so that $u \in \mathcal{W}_\sigma$.

Conversely, if $u \in \mathcal{W}_\sigma$ then $[F_{B_u}]_g = -\mathbf{i}[*\eta]_g$ and using the Hodge decomposition of the *closed* 2-form F_{B_u},

$$F_{B_u} = [F_{B_u}]_g + \mathbf{i}d\alpha_0 \quad \alpha_0 \in \Omega^1(M),$$

we can rewrite (2.2) as

$$\mathbf{i}d\alpha_0 + \mathbf{i}da = 0 \Longrightarrow da = -d\alpha_0$$

The solutions of the last equation have the form

$$a = -\alpha_0 + \text{closed 1-form}$$

The conclusion of the proposition now follows by factoring out the $\mathcal{G}$ action.

□

Observe that the wall $\mathcal{W}_\sigma$ is a codimension $b_1(M)$ submanifold of $\mathcal{U}_\sigma$. In particular when M is a rational homology sphere, i.e. $b_1(M) = 0$ we have $\mathcal{W}_\sigma = \mathcal{U}_\sigma$.

Corollary 2.5. *If M is a rational homology 3-sphere then for every spin^c-structure σ and every $u \in \mathcal{U}_\sigma$ there exists a unique gauge orbit of reducible (σ,u)-monopoles.*

When $b_1(M) = 1$ the wall $\mathcal{W}_\sigma$ is a connected hypersurface of $\mathcal{U}_\sigma$. Its complement in the space of parameters $\mathcal{U}_\sigma$ consists of two components called the *chambers*.

Proposition 2.6. *Fix a Riemann metric g_0 on M. For every positive constant C there exists a compact convex set $P = P(C) \subset H^2(M,\mathbb{R})$ such that if $u = (g,h,\eta,\mu) \in \mathcal{U}_\sigma$ satisfies*

$$\sup_{x\in M} |\eta(x)|_g + \sup_{x\in M} |\mu(x)| + \|g\|_{C^2(M,g_0)} < C, \;\; \mathfrak{M}_{\sigma,u} \neq 0$$

then $c(\sigma) \in P(C)$.

Proof. Using (2.1) and the equality

$$\boldsymbol{c}(*_g F_A + \mathbf{i}\eta) = q(\psi)$$

we deduce that there exists a constant K depending continuously on C such that

$$\|F_A\|_{L^2(M,g_0)} \leq K \Longrightarrow \|[F_A]_{g_0}\|_{L^2(M,g_0)}.$$

The last inequality describes a compact convex set in $H^2(M,\mathbb{R})$.

$\square$

Let us summarize the things we have proved so far. We know that for every parameter $u \in \mathcal{U}_\sigma$ the moduli space $\mathfrak{M}_{\sigma,u}$ is a compact metric space. Moreover, if the parameter u lies on the wall, then the closed subspace consisting of reducibles is homeomorphic to a torus of dimension $b_1(M)$. We now shift our investigation towards the local properties of the moduli space.

For every $\mathsf{S} \in \mathcal{C}_\sigma$ we denote by $[\mathsf{S}]$ its image in $\mathcal{B}_\sigma$. The corresponding map

$$\mathcal{C}_\sigma \ni \mathsf{S} \mapsto [\mathsf{S}] \in \mathcal{B}_\sigma$$

is continuous. Denote by $\mathbf{St}_\mathsf{S}$ the stabilizer of $\mathsf{S} \in \mathcal{C}_\sigma$

$$\mathbf{St}_\mathsf{S} \cong \begin{cases} \{1\} & \text{if } \mathsf{S} \text{ is irreducible} \\ S^1 & \text{if } \mathsf{S} \text{ is reducible} \end{cases}$$

As we have seen the slice $\mathcal{S}_\mathsf{S}$ is $\mathbf{St}_\mathsf{S}$-invariant. The next result is a variation of Proposition 2.1.

Proposition 2.7 (Local slices). *For every $\mathsf{S} \in \mathcal{C}^k_\sigma$ there exists $R = R_\mathsf{S} > 0$ such that*

$$\mathcal{N} = \mathcal{N}_\mathsf{S} = \{\mathsf{S} + \dot{\mathsf{S}};\ \ \dot{\mathsf{S}} \in \mathcal{X}^\perp_\mathsf{S},\ \ |\dot{\mathsf{S}}\|_{k,2} < r\} \subset \mathcal{S}_\mathsf{S}$$

is a $\mathbf{St}_\mathsf{S}$-invariant neighborhood of S whose image $[\mathcal{N}_\mathsf{S}]$ in $\mathcal{B}_\sigma$ is a neighborhood of S and the $\mathbf{St}_\mathsf{S}$-invariant projection $\mathcal{N}_\mathsf{S} \to [\mathcal{N}_\mathsf{S}]$ descends to a homeomorphism

$$\Phi_\mathsf{S} : \mathcal{N}_\mathsf{S}/\mathbf{St}_\mathsf{S} \to [\mathcal{N}_\mathsf{S}].$$

Suppose $\mathsf{S} \in \mathcal{C}_\sigma$ is a (σ, u)-monopole. To find all the orbits of monopoles very close to $[\mathsf{S}]$ we need to find the very small solutions $\dot{\mathsf{S}}$ of the

$$\mathcal{SW}_{\sigma,u}(\mathsf{S} + \dot{\mathsf{S}}) = 0,\ \ \dot{\mathsf{S}} \in \mathcal{X}^\perp_\mathsf{S},\ \ \|\dot{\mathsf{S}}\|_{2,2} \ll R_\mathsf{S}.$$

This is equivalent to

$$\begin{cases} \mathcal{SW}_{\sigma,u}(\mathsf{S} + \dot{\mathsf{S}}) & = & 0 \\ \mathcal{L}^*_\mathsf{S}\dot{\mathsf{S}} & = & 0 \\ \|\dot{\mathsf{S}}\|_{k,2} & \ll & R_\mathsf{S} \end{cases}. \tag{2.3}$$

Observe that if $\mathsf{S}' \in \mathcal{S}_\mathsf{S}$ is sufficiently close to S then the orbit $\mathcal{G} \cdot \mathsf{S}'$ is orthogonal to $\mathcal{S}_{\mathsf{S}'}$ and thus it is "almost" orthogonal to $\mathcal{S}_\mathsf{S}$. Equivalently the subspaces $\mathcal{X}^\perp_\mathsf{S}$ and $\mathcal{X}^\perp_{\mathsf{S}'}$ are not very far apart (see Figure 2). On the other hand, the vector field $\mathcal{SW}$ is L^2-orthogonal to the orbits of the $\mathcal{G}$-action. This implies that the vector $\mathcal{SW}(\mathsf{S}') \in \mathcal{X}^\perp_{\mathsf{S}'}$ is very close to being tangent to $\mathcal{S}_\mathsf{S}$, i.e. the component of $\mathcal{SW}(\mathsf{S})$ along $\mathcal{X}^\perp_\mathsf{S}$ is much larger than the component along $\mathcal{X}_\mathsf{S}$. More precisely, we have the following result.

Lemma 2.8. *There exists $r = r_\mathsf{S} \in (0, R_\mathsf{S})$ such that for any $\dot{\mathsf{S}} \in \mathcal{X}^\perp_\mathsf{S}$ satisfying $\|\dot{\mathsf{S}}\|_{k,2} < r$ we have*

$$\mathcal{SW}(\mathsf{S} + \dot{\mathsf{S}}) \in \mathcal{X}_\mathsf{S} \Longleftrightarrow \mathcal{SW}(\mathsf{S} + \dot{\mathsf{S}}) = 0$$

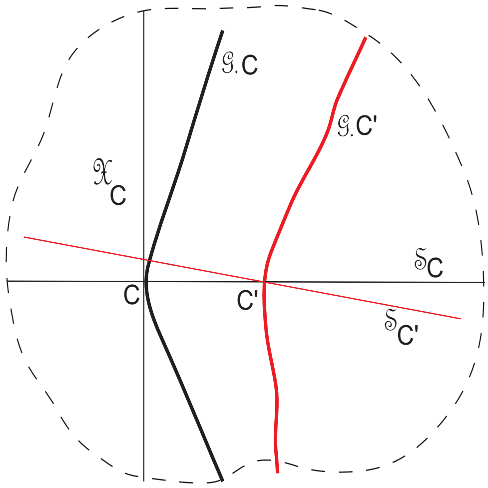

Figure 2: *The gauge orbits and a slice of the $\mathcal{G}$-action.*

The equation (2.3) is thus equivalent to

$$\begin{cases} \mathcal{SW}_{\sigma,u}(\mathsf{S}+\dot{\mathsf{S}})+\mathcal{L}_{\mathsf{S}}(\mathsf{X}) & = & 0 \\ \mathcal{L}_{\mathsf{S}}^*\dot{\mathsf{S}} & = & 0 \\ \dot{\mathsf{S}}\in T_{\mathsf{S}}\mathcal{C}_\sigma & |\dot{\mathsf{S}}\|_{k,2}\ll r_{\mathsf{S}} & \\ \mathsf{X}\in\mathcal{X}=T_1\mathcal{G} & & \end{cases}. \tag{2.4}$$

To understand this equation we first study its linearization at S.

$$\begin{cases} \underline{\mathcal{SW}}_{\mathsf{S}}(\dot{\mathsf{S}})+\mathcal{L}_{\mathsf{S}}(\mathsf{X}) & = & 0 \\ \mathcal{L}_{\mathsf{S}}^*\dot{\mathsf{S}} & = & 0 \\ \dot{\mathsf{S}}\in T_{\mathsf{S}}\mathcal{C}_\sigma & |\dot{\mathsf{S}}\|_{k,2}\ll r_{\mathsf{S}} & \\ \mathsf{X}\in\mathcal{X} & & \end{cases},$$

where $\underline{\mathcal{SW}}_{\mathsf{S}}:=D_{\mathsf{S}}\,\mathcal{SW}_{\sigma,u}$. Consider now the linear first order partial differential

$$\mathcal{T}_{\mathsf{S}}:T_{\mathsf{S}}\mathcal{C}_\sigma^k\oplus\mathcal{X}^k\to T_{\mathsf{S}}\mathcal{C}_\sigma^{k-1}\oplus\mathcal{X}^{k-1},\ \ \mathcal{T}_{\mathsf{S}}\cdot\begin{bmatrix}\dot{\mathsf{S}}\\ \mathsf{X}\end{bmatrix}=\begin{bmatrix}\underline{\mathcal{SW}}_{\mathsf{S}} & \mathcal{L}_{\mathsf{S}}\\ \mathcal{L}_{\mathsf{S}}^* & 0\end{bmatrix}\cdot\begin{bmatrix}\dot{\mathsf{S}}\\ \mathsf{X}\end{bmatrix}.$$

More precisely, for $\dot{\mathsf{S}}=(\dot\psi,\mathbf{i}\dot a)$ and $\mathsf{X}=\mathbf{i}f$ we have

$$\mathcal{T}_{\mathsf{S}}\begin{bmatrix}\dot\psi\\ \mathbf{i}\dot a\\ \mathbf{i}\end{bmatrix}=\underbrace{\begin{bmatrix}\mathfrak{D}_A+\mu & 0 & 0\\ 0 & -*d & d\\ 0 & d^* & 0\end{bmatrix}}_{\mathcal{T}_{\mathsf{S}}^0}\cdot\begin{bmatrix}\dot\psi\\ \mathbf{i}\dot a\\ \mathbf{i}f\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\mathbf{c}(\mathbf{i}\dot a)\psi-\frac{\mathbf{i}}{2}f\psi\\ \frac{1}{2}\dot q(\psi,\dot\psi)\\ \frac{\mathbf{i}}{2}\mathfrak{Im}\,\langle\psi,\dot\psi\rangle\end{bmatrix} \tag{2.5}$$

where

$$\dot q(\psi,\dot\psi)=\frac{1}{2}\frac{d}{dt}Q(\psi+t\dot\psi).$$

$\mathcal{T}_{\mathsf{S}}$ is a formally self-adjoint, Fredholm operator. We denote by $\mathcal{P}_{\mathsf{S}}$ the zeroth order operator in the right hand side of (2.5). Observe that if S is reducible then $\mathcal{P}_{\mathsf{S}}=0$. In this case

$$\ker\mathcal{T}_{\mathsf{S}}=\ker\mathcal{T}_{\mathsf{S}}^0\cong\ker(\mathfrak{D}_A+\mu)\oplus H^1(M,\mathbb{R})\oplus H^0(M,\mathbb{R}).$$

The component $H^1(M,\mathbb{R})$ coincides with the tangent space at S to the reducible component of $\mathfrak{M}_{\sigma,u}$, while the component $H^0(M,\mathbb{R})$ corresponds to the Lie algebra of the stabilizer $\mathbf{St}_{\mathsf{S}}$.

We have the following genericity result whose proof can be found e.g. in [2, 6]

Theorem 2.9. *There exists a generic set $\mathcal{U}_\sigma^* \subset \mathcal{U}_\sigma$ with the following properties.*

(a) If $u \in \mathcal{U}_\sigma^*$, and S is an irreducible (σ,u)-monopole then $\ker \mathcal{T}_{\mathsf{S}} = 0$.
(b) If $b_1(M) = 0$, $u = (g,h,\eta,\mu) \in \mathcal{U}_\sigma^*$, and $\mathsf{S} = (0,A)$ is a reducible (σ,u)-monopole then

$$\ker(\mathfrak{D}_A + \mu) = 0.$$

(c) $u \in \mathcal{U}_\sigma^\dagger \Longleftrightarrow \bar{u} \in \mathcal{U}_\sigma^\dagger$.

We set

$$\mathcal{U}_\sigma^\dagger = \begin{cases} \mathcal{U}_\sigma^* \setminus \mathcal{W}_\sigma & \text{if} \quad b_1(M) > 0 \\ \mathcal{U}_\sigma^* & \text{if} \quad b_1(M) = 0 \end{cases}$$

The previous discussion implies the following result.

Corollary 2.10. *Let $u \in \mathcal{U}_\sigma^\dagger$. Then $\mathfrak{M}_{\sigma,u}$ consists of finitely many $\mathcal{G}$-orbits of monopoles. If $b_1(M) > 0$ all these monopoles are irreducible, while if $b_1(M) = 0$, then $\mathfrak{M}_{\sigma,u}^0$ consists of exactly one point.*

§2.2 Spectral flows of paths of selfadjoint operators with compact resolvent. Suppose $(A_t)_{t\in[0,1]}$ is a family of self-adjoint operators on a real Hilbert space H, with the following properties.

- For every t the operator $R_t := (\mathbf{i} - A_t)^{-1}$ on $H \otimes \mathbb{C}$ is compact. In particular, the spectrum of A_t is discrete, and consists only of eigenvalues with finite multiplicities.
- The bounded operator R_t depends analytically on t.
- All the operators A_t have the same domain.

As explained in [3, Thm. 3.9, Chap. VII, §4.5], we can find real analytic maps

$$\lambda_n : [0,1] \to \mathbb{R}, \quad n \in \mathbb{Z}$$

such that for every $t \in [0,1]$ the spectrum of A_t *with multiplicities included*, coincides with the collection $\{\lambda_n(t); \; n \in \mathbb{Z}\}$. For every $t \in [0,1]$ we define

$$\mathbf{sf}_t^+(A_\bullet) = \text{ the number of indices } n \text{ such that } \lambda_n(t) = 0 \text{ but } \lambda_n(t') < 0 \text{ for } t' \nearrow t,$$

$$\mathbf{sf}_t^-(A_\bullet) = \text{ the number of indices } n \text{ such that } \lambda_n(t) = 0 \text{ but } \lambda_n(t') < 0 \text{ for } t' \searrow t.$$

The *local spectral flow* at the moment t is then

$$\mathbf{sf}_t(A_\bullet) := \mathbf{sf}_t^+(A_\bullet) - \mathbf{sf}_t^-(A_\bullet).$$

Define the *spectral flow* of the family $A_\bullet$ to be the integer

$$\mathbf{sf}\,(A_\bullet) = \sum_{t\in[0,1]} \mathbf{sf}_t(A_\bullet) = \sum_{0<t<0} \Big(\mathbf{sf}_t^+(A_\bullet) - \mathbf{sf}_t^-(A_\bullet)\Big) + \mathbf{sf}_{t=1}^+(A_\bullet) - \mathbf{sf}_{t=0}^-(A_\bullet)$$

Loosely speaking, the spectral flow is the difference between the number of times the eigenvalues $\lambda_n(t)$ change the sign from negative to positive and the number of times the eigenvalues $\lambda_n(t)$ change the sign from positive to negative (see Figure 3).

For more information about the spectral flow we refer to the original source [1].

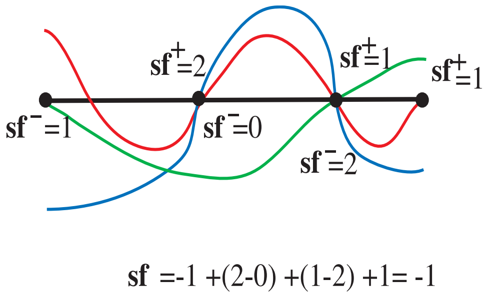

Figure 3: *Computing a spectral flow.*

§2.3 Seiberg-Witten invariants Suppose $u \in \mathfrak{U}_\sigma^\dagger$. Fix as before a smooth reference configuration $\mathsf{S}_0 = (0, B_u)$. For every pair of configurations $\mathsf{S}_1, \mathsf{S}_2 \in \mathcal{C}_\sigma$ we set

$$m(\mathsf{S}_2, \mathsf{S}_1) = \mathbf{sf}\left((1-t)\mathfrak{T}_{\mathsf{S}_1} + \mathfrak{T}_{\mathsf{S}_2},\ \ 0 \le t \le 1\right).$$

This integer depends only on the gauge equivalence class of the pair, i.e.

$$m(\mathsf{S}_2, \mathsf{S}_1) = m(\gamma \cdot \mathsf{S}_2, \gamma \cdot \mathsf{S}_1),\ \ \forall \gamma \in \mathcal{G}.$$

Note that

$$m(\mathsf{S}'', \mathsf{S}) = m(\mathsf{S}'', \mathsf{S}') + m(\mathsf{S}', \mathsf{S}),\ \ \forall \mathsf{S}, \mathsf{S}', \mathsf{S}'' \in \mathcal{C}_\sigma.$$

For every (σ, u) monopole S we now set

$$\epsilon(\mathsf{S}) := (-1)^{m(\mathsf{S}, \mathsf{S}_0)} = \pm 1.$$

One can show that $\epsilon(\mathsf{S})$ depends only on the $\mathcal{G}$-equivalence class of S, and it is independent of the choice of S_0. Indeed, for a different choice $\mathsf{S}_0' = (0, B_u')$ we have

$$m(\mathsf{S}, \mathsf{S}_0') = m(\mathsf{S}, \mathsf{S}_0) + m(\mathsf{S}_0, \mathsf{S}_0') = m(\mathsf{S}, \mathsf{S}_0) \bmod 2.$$

We set

$$\mathbf{sw}_M(\sigma, u) := \sum_{[\mathsf{S}] \in \mathfrak{M}^*_{\sigma,u}} \epsilon([\mathsf{S}]) \in \mathbb{Z}.$$

It is not difficult to see that

$$\mathbf{sw}_M(\bar{\sigma}, \bar{u}) = \mathbf{sw}_M(\sigma, u). \tag{2.6}$$

Theorem 2.11. *Suppose $b_1(M) > 0$ so that $\mathfrak{U}_\sigma \neq \mathcal{W}_\sigma$. If $u_0, u_i \in \mathfrak{U}_\sigma^\dagger$ lie in the same path component of $\mathfrak{U}_\sigma \setminus \mathcal{W}_\sigma$ then*

$$\mathbf{sw}_M(\sigma, u_0) = \mathbf{sw}_M(\sigma, u_1).$$

Idea of proof. For every smooth path $\hat{u} : [0,1] \to \mathfrak{U}_\sigma \setminus \mathcal{W}_\sigma$ such that $\hat{u}(i) = u_i$, $i = 0, 1$, we form

$$\widehat{\mathfrak{M}}_{\sigma,\hat{u}} = \Big\{([\mathsf{S}], t) \in \mathcal{B}_\sigma \times [0,1];\ \ \mathcal{SW}_{\sigma,\hat{u}(t)}(\mathsf{S}) = 0\Big\}$$

Since $\hat{u}(t) \in \mathfrak{U}_\sigma \setminus \mathcal{W}_\sigma$ we deduce that $\widehat{\mathfrak{M}}_{\sigma,\hat{u}}$ lies inside the smooth part $\mathcal{B}^*_\sigma \times [0,1]$. A genericity argument based on the Sard-Smale theorem shows that we can choose the path $\hat{u}$ so that $\widehat{\mathfrak{M}}_{\sigma,\hat{u}}$ is smooth one-dimensional manifold with boundary such that

$$\partial \widehat{\mathfrak{M}}_{\sigma,\hat{u}} = \mathfrak{M}_{\sigma,u_0} \cup \mathfrak{M}_{\sigma,u_1}.$$

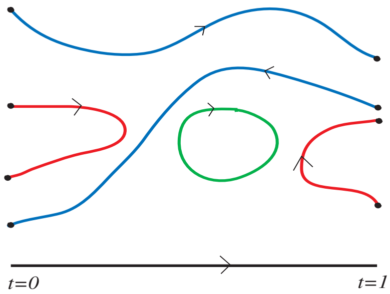

Figure 4: $\widehat{\mathfrak{M}}_{\sigma,\hat{u}}$ *is an oriented cobordism between* $\mathfrak{M}_{\sigma,u_0}$ *and* $\mathfrak{M}_{\sigma,u_1}$.

One can then show that $\widehat{\mathfrak{M}}_{\sigma,\hat{u}}$ is equipped with a natural orientation. It thus defines a 1-chain in $\mathcal{B}^*_\sigma \times [0,1]$ and the tricky part is to show that we have the more refined equality of 0-chains (see Figure 4)

$$\partial\widehat{\mathfrak{M}}_{\sigma,\hat{u}} = \sum_{\mathsf{S}'\in\mathfrak{M}_{\sigma,u_1}} \epsilon([\mathsf{S}'])[\mathsf{S}'] - \sum_{[\mathsf{S}']\in\mathfrak{M}_{\sigma,u_0}} \epsilon([\mathsf{S}])[\mathsf{S}].$$

For more details on how to prove this identity we refer to [11, Lemma 2.3.4].

□

The wall $\mathcal{W}_\sigma$ is a codimension $b_1(M)$-submanifold of $\mathfrak{U}_\sigma$ so that if $b_1(M) > 1$ its complement is path connected. Theorem 2.11 implies the following result.

Corollary 2.12. *Suppose* $b_1(M) > 1$. *Then the integer* $\mathbf{sw}_M(\sigma, u)$ *is independent of the generic parameter* $u \in \mathfrak{U}^\dagger_\sigma \setminus \mathcal{W}_\sigma$. *It is therefore a topological invariant of the pair* (M,σ). *We will denote it by* $\mathbf{sw}_M(\sigma)$ *and will refer to it as the* Seiberg-Witten invariant of (M,σ).

Proposition 2.6 implies that $\mathbf{sw}_M(\sigma)$ is zero for all but finitely many *spin*^{c} structures σ. From the identity (2.6) we deduce that when $b_1(M) > 1$ we have

$$\mathbf{sw}_M(\sigma) = \mathbf{sw}_M(\bar{\sigma}), \ \ \forall \sigma \in Spin^c(M). \tag{2.7}$$

We set $H = H^2(M,\mathbb{Z}) \cong H_1(M,\mathbb{Z})$. We denote the group operation on H multiplicatively and for every ring R we denote by $R[H]$ the group R-algebra associated to H and R. The elements P of $R[H]$ have the description

$$P = \sum_{h\in H} r_h h,$$

where the coefficients r_h are in R and all but finitely many of them are 0. As we know the group H acts freely and transitively on $Spin^c(M)$. Define

$$\mathbf{SW}_{M,\bullet} : Spin^c(M) \to \mathbb{Z}[H], \ \ \sigma \mapsto \mathbf{SW}_{M,\sigma} = \sum_{\mathcal{H}\in H} \mathbf{sw}_M(h^{-1}\sigma)h.$$

Observe that this map is H-equivariant, i.e. for any $g \in H$ we have

$$\mathbf{SW}_{M,g\sigma} = g \sum_{\mathcal{H}\in H} \mathbf{sw}_M(h^{-1}g\sigma)g^{-1}h = g \cdot \mathbf{SW}_{M,\sigma}.$$

Observe that the involution $h \mapsto h^{-1}$ induces an involution $\mathbb{Z}[H] \ni P \mapsto \bar{P} \in \mathbb{Z}[H]$ The symmetry equality (2.7) takes the form

$$\mathbf{SW}_{M,\bar{\sigma}} = c(\sigma)\overline{\mathbf{SW}}_{M,\sigma}. \tag{2.8}$$

The cases $b_1(M) = 0, 1$ require special care. We will discuss them separately.

§2.3.1 The case $b_1(M) = 1$.

In this case the wall $\mathcal{W}_\sigma$ is a codimension one submanifold of the space of parameters. We will show that the complement consists of two connected components called *chambers*. An orientation of the one dimensional vector space $H^1(M,\mathbb{R})$ will then produce a transversal orientation of the wall so we can define a *positive* chamber and a *negative* chamber. The integer $\mathbf{sw}_M(\sigma, u)$, $u \in \mathcal{U}_\sigma \setminus \mathcal{W}_\sigma$ then depends only on the chamber to which u belongs. We thus get two invariants $\mathbf{sw}_M^\pm(M)$ corresponding to the two chambers. We will prove a *wall crossing formula* relating these two invariants.

To begin with, we fix an orientation of $H^1(M,\mathbb{R})$ by choosing a generator $\mathfrak{o}$ of $H^1(M,\mathbb{Z})$. We get a morphism

$$\deg_{\mathfrak{o}} : H^2(M,\mathbb{Z}) \to \mathbb{Z}, \;\; \deg_{\mathfrak{o}}(\omega) = \int_M \mathfrak{o} \wedge \omega.$$

In particular we obtain a map

$$\deg_{\mathfrak{o}} : Spin^c(M) \to \mathbb{Z}, \;\; \deg_{\mathfrak{o}}(\sigma) := \deg_{\mathfrak{o}}(c(\sigma)).$$

Since the tangent bundle of M is trivializable we deduce

$$\deg_{\mathfrak{o}}(\sigma) \in 2\mathbb{Z}, \;\; \forall \sigma \in Spin^c(M).$$

Observe now that since $H^1(M,\mathbb{R})$ is one dimensional we have the equivalences

$$u = (g,h,\eta,\mu) \in \mathcal{W}_\sigma \iff [*\eta]_g = 2\pi[c(\sigma)] \iff \int_M \mathfrak{o} \wedge *\eta = 2\pi \deg_{\mathfrak{o}}(\sigma).$$

Thus if we define

$$w_\sigma : \mathcal{U}_\sigma \to \mathbb{R}, \;\; w_\sigma(g,h,\eta,\mu) = \int_M \mathfrak{o} \wedge *\eta - 2\pi \deg_{\mathfrak{o}}(\sigma)$$

we deduce that

$$\mathcal{W}_\sigma = w_\sigma^{-1}(0).$$

Define the $\pm$-chamber $\mathcal{U}_\sigma^\pm$ by

$$\mathcal{U}_\sigma^\pm = \Big\{ u \in \mathcal{U}_\sigma; \;\; \pm w_\sigma(u) > 0 \Big\}.$$

We get two integers

$$\mathbf{sw}_M^\pm(\sigma) := \mathbf{sw}_M(\sigma, u), \;\; u \in \mathcal{U}_\sigma^\pm \cap \mathcal{U}_\sigma^\dagger.$$

Theorem 2.13 (Wall Crossing Formula).

$$\mathbf{sw}_M^+(\sigma) - \mathbf{sw}_M^-(\sigma) = \frac{1}{2}\deg_{\mathfrak{o}}(\sigma).$$

Idea of proof. The proof uses a refinement of the cobordism argument employed in the proof of Theorem 2.11. We sketch the main lines of this proof and we refer for details to [6].

Suppose $u_\pm \in \mathcal{U}_\sigma^\pm \cap \mathcal{U}_\sigma^\dagger$. For any smooth path $u : [-1,1] \to \mathcal{U}_\sigma$ such that $u(\pm 1) = u_\pm$ form again the moduli space $\widehat{\mathfrak{M}}_{\sigma,u} \subset \mathcal{B}_\sigma \times [0,1]$ as in the proof of Theorem 2.11. Set $w_\sigma(t) := w_\sigma(u(t))$. We define the *resonance locus* of the path to consist only the pairs (S, t) where $\mathsf{S} = (0, A)$ is a reducible $(\sigma, u(t))$-monopole such that $\ker(\mathfrak{D}_A + \mu(t)) \neq 0$. We can generically choose the path $u(t)$ to satisfy the following properties.

• $\frac{d}{dt}w_\sigma(t) > 0$, $\forall t \in [-1,1]$, and $w(0) = 0$. In other words, the path $u(t)$ crosses the wall $\mathcal{W}_\sigma$ transversally at time $t = 0$. Set $u_0 = u(0) = (g_0, h_0, \eta_0, \mu_0)$. Note that the resonance locus is contained in the time slice $t = 0$ since there are no reducible $(\sigma, u(t))$ monopoles for $t \neq 0$.
• The resonance locus is as good as possible, i.e. it is a finite set $\mathcal{R} \in \mathfrak{M}_{\sigma,u_0}$ and every monopole $(0, A) \in \mathcal{R}$ we have

$$\dim_{\mathbb{C}} \ker(\mathfrak{D}_A + \mu_0) = 1.$$

Moreover if $\psi_0 \in \ker(\mathfrak{D}_A + \mu_0) \setminus 0$ then

$$\xi_A := \int_M \langle \mathbf{c}(\mathbf{i}[\mathfrak{o}]_{g_0})\psi_0, \psi_0 \rangle dV_{g_0} \neq 0. \tag{2.9}$$

• $C := \widehat{\mathfrak{M}}_{\sigma,u} \setminus \mathfrak{M}_{\sigma,u_0}$ is smooth, naturally oriented and 1-dimensional.

One can then show that the closure of $\widehat{\mathfrak{M}}_{\sigma,u} \setminus \mathfrak{M}_{\sigma,u_0}$ in $\mathcal{B}_\sigma \times [0,1]$ is a manifold with boundary (see Figure 5)

$$\partial \bar{C} = \mathfrak{M}_{\sigma,u_-} \cup \mathcal{R} \cup \mathfrak{M}_{\sigma,u_+}.$$

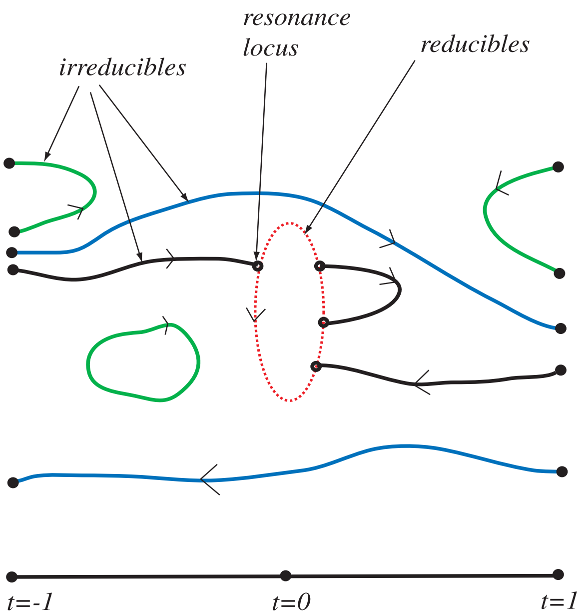

Figure 5: *A singular cobordism.*

The reducible component $\mathfrak{M}^0_{\sigma,0}$ is diffeomorphic to the circle $H^1(M,\mathbb{R})/4\pi H^1(M,\mathbb{Z})$ which is naturally oriented by $\mathfrak{o}$. We can describe this component as a path

$$[0,1] \ni \to (0, A(s)) \in \mathcal{C}_\sigma.$$

We obtain a real analytic family of Fredholm selfadjoint operators with compact resolvent

$$s \mapsto T_s = \mathfrak{D}_{A(s)} + \mu_0.$$

Observe that

$$\mathbf{sf}_s(T_\bullet) \neq 0 \Longleftrightarrow (0, A(s)) \in \mathcal{R}.$$

Moreover

$$\mathbf{sf}_s(T_\bullet) = \mathrm{sign}\,(\xi_{A_s}) = \pm 1, \quad \forall (0, A_s) \in \mathcal{R}.$$

The map

$$[0,1] \ni s \mapsto \mathbf{sf}_s(T_\bullet) \in \mathbb{Z}$$

thus induces a map

$$\nu : \mathcal{R} \to \{\pm 1\}.$$

One can then show that we have the following equality of 0-cycles

$$\partial \bar{C} = \sum_{[\mathsf{S}_+] \in \mathfrak{M}_{\sigma, u_+}} \epsilon([\mathsf{S}_+]) \cdot [\mathsf{S}_+] - \sum_{[\mathsf{S}_-] \in \mathfrak{M}_{\sigma, u_-}} \epsilon([\mathsf{S}_-]) \cdot [\mathsf{S}_-] - \sum_{[\mathsf{R}] \in \mathcal{R}} \nu(\mathsf{R}) \cdot [\mathsf{R}].$$

We deduce that

$$\sum_{[\mathsf{S}_+] \in \mathfrak{M}_{\sigma, u_+}} \epsilon([\mathsf{S}_+]) - \sum_{[\mathsf{S}_-] \in \mathfrak{M}_{\sigma, u_-}} \epsilon([\mathsf{S}_-]) = \sum_{[\mathsf{R}] \in \mathcal{R}} \nu(\mathsf{R})$$

or equivalently

$$\mathbf{sw}^+_M(\sigma) - \mathbf{sw}^-_M(\sigma) = \mathbf{sf}\,(T_\bullet).$$

Using the results in [1] we deduce after an elementary computation that

$$\mathbf{sf}\,(T_\bullet) = \frac{1}{2} \deg_{\mathfrak{o}} \sigma.$$

$\square$

The wall crossing formula can be conveniently encoded as follows. Set

$$\Theta = \Theta_H = \sum_{h \in \mathrm{Tors}\,(H)} h \in \mathbb{Z}[H],$$

and define

$$\mathbf{SW}^{\pm}_{M,\sigma,\mathfrak{o}} = \sum_{h \in H} \mathbf{sw}^{\pm}_M(h^{-1}\sigma) h.$$

Then

$$\mathbf{SW}^{+}_{M,\sigma,\mathfrak{o}} - \mathbf{SW}^{-}_{M,\sigma,\mathfrak{o}} = \frac{1}{2} \sum_{h \in H} \big(\deg_{\mathfrak{o}}(c(\sigma) - 2\deg_{\mathfrak{o}}(h)\big) h.$$

Let $\deg^{\pm}_{\mathfrak{o}} = \max(\pm \deg_{\mathfrak{o}}, 0)$. Then

$$\mathbf{SW}^{+}_{M,\sigma,\mathfrak{o}} - \frac{1}{2} \sum_h \deg^+_{\mathfrak{o}} c(h^{-1}\sigma) h = \mathbf{SW}^{-}_{M,\sigma,\mathfrak{o}} - \frac{1}{2} \sum_h \deg^-_{\mathfrak{o}} c(h^{-1}\sigma) h.$$

Observe that the left-hand-side (and a fortiori the right-hand-side) defines a H-equivariant map

$$\mathbf{SW}^0_{M,\bullet} : Spin^c(M) \to \mathbb{Z}[[H]].$$

The H equivariance of

$$Spin^c(M) \ni \sigma \mapsto \mathbf{SW}^0_{M,\sigma} \in \mathbb{Z}[[H]]$$

implies that there exists a function

$$\mathbf{sw}^0_M : Spin^c(M) \to \mathbb{Z}$$

such that

$$\mathbf{SW}^0_{M,\sigma} = \sum_{h\in H} \mathbf{sw}^0_M(h^{-1}\sigma)h.$$

More precisely

$$\mathbf{sw}^0_M(\sigma) = \mathbf{sw}^+_{M,\sigma} - \frac{1}{2}\deg^+_{\mathfrak{o}} c(\sigma).$$

We say that $\mathbf{sw}^0_M$ is the *modified Seiberg-Witten invariant.* We want to present a different interpretation of $\mathbf{sw}^0$. Define a smaller space of parameters

$$\mathcal{U}_{\sigma,\mathfrak{o}} := \Big\{u = (g,h,\eta,\mu) \in \mathcal{U}_\sigma;\ 0 < \Big|\int_M \mathfrak{o}\wedge *_g\eta\Big| < \hbar\Big\}$$

where $\hbar$ is a small constant significantly smaller that 2π. We deduce that $\mathcal{U}_{\sigma,go}$ is an open subset of $\mathcal{U}_\sigma \setminus \mathcal{W}_\sigma$. We set $\mathcal{U}^\dagger_{\sigma,\mathfrak{o}} := \mathcal{U}_{\sigma,\mathfrak{o}} \cap \mathcal{U}^\dagger_\sigma$. Observe that if $\deg_{\mathfrak{o}} c(\sigma) = 0$ then

$$\mathbf{sw}^0_M = \mathbf{sw}^+_M(\sigma) = \mathbf{sw}^-_M(\sigma) = \mathbf{sw}_M(\sigma,u).$$

The choice $\hbar < 2\pi$ implies

$$\mathcal{U}^\dagger_{\sigma,\mathfrak{o}} \subset \mathcal{U}^+_\sigma \quad \text{if } \deg_{\mathfrak{o}} c(\sigma) < 0,\ \ \mathcal{U}^\dagger_{\sigma,\mathfrak{o}} \subset \mathcal{U}^-_\sigma \quad \text{if } \deg_{\mathfrak{o}} c(\sigma) > 0$$

Hence if $\deg_{\mathfrak{o}} c(\sigma) < 0$ we have

$$\mathbf{sw}^0_M(\sigma) = \mathbf{sw}^+_\sigma = \mathbf{sw}_M(\sigma,u),\ \ \forall u \in \mathcal{U}^\dagger_{\sigma,\mathfrak{o}}.$$

On the other hand, if $\deg_{\mathfrak{o}} c(\sigma) > 0$ then for every $u \in \mathcal{U}^\dagger_{\sigma,\mathfrak{o}}$ we have

$$\mathbf{sw}_M(\sigma,u) = \mathbf{sw}^-_M(\sigma) = \mathbf{sw}^+_{M,\sigma} - \frac{1}{2}\deg_{\mathfrak{o}} c(\sigma) = \mathbf{sw}^0_M(\sigma)$$

We conclude

$$\mathbf{sw}^0_{M,\sigma} = \mathbf{sw}_M(\sigma,u),\ \ \forall u \in \mathcal{U}^\dagger_{\sigma,\mathfrak{o}}. \tag{2.10}$$

Using Proposition 2.6 we deduce that $\mathbf{sw}^0_M(\sigma)$ is zero for all but finitely many σ's so that

$$\mathbf{SW}^0_{M,\sigma} \in \mathbb{Z}[H],\ \ \forall \sigma \in Spin^c(M).$$

Note also that

$$\mathbf{sw}^0_{M,\sigma} = \mathbf{sw}^0_M(\bar\sigma),\ \ \forall \sigma \in Spin^c(M).$$

We want to present yet another, less canonical description of $\mathbf{SW}^0_{M,\sigma}$ which is useful in concrete computations. Fix $T \in H$ such that $\deg_{\mathfrak{o}} T = 1$ and set $d(\sigma) := \frac{1}{2}\deg_{\mathfrak{o}} c(\sigma)$. Then

$$\frac{1}{2}\sum_h \deg^+_{\mathfrak{o}} c(h^{-1}\sigma)h = \Theta \sum_{n<d(\sigma)} \underbrace{(d(\sigma)-n)}_{m} T^n = \Theta\sum_{m>0} mT^{d(\sigma)-m} = \Theta\cdot T^{d(\sigma)}\sum_{m>0} T^{-m}.$$

Similarly

$$\frac{1}{2}\sum_h \deg^-_{\mathfrak{o}} c(h^{-1}\sigma)h = \Theta \sum_{n>d(\sigma)} \underbrace{(n-d(\sigma))}_{k} T^n = \Theta T^{d(\sigma)}\sum_{k>0} kT^k$$

Hence

$$\mathbf{SW}^0_{M,\sigma} = \mathbf{SW}^+_{M,\sigma} - \Theta\cdot T^{d(\sigma)}\sum_{m>0} T^{-m} = \mathbf{SW}^-_{M,\sigma} - \Theta T^{d(\sigma)}\sum_{k>0} kT^k.$$

Example 2.14. Suppose $M = S^1 \times S^2$. We orient $H^1(M, \mathbb{Z})$ via the cohomology class $\mathfrak{o} = \frac{1}{2\pi} d\theta$, where $d\theta$ denotes the angular form on S^1. Denote by $T \in H_1(M, \mathbb{Z})$ the homology class corresponding to the fiber S^1 of this trivial S^1-bundle. Since M admits a metric a positive scalar curvature we deduce that

$$\mathbf{sw}_M^0(\sigma) = 0, \ \ \forall \sigma \in Spin^c(M)$$

so that

$$\mathbf{SW}_{M,\sigma}^+ = T^{\frac{1}{2}\deg_{\mathfrak{o}} c(\sigma)} \sum_{m<0} mT^{-m} \approx T^{\frac{1}{2}\deg_{\mathfrak{o}} c(\sigma)} \frac{T}{(1-T)^2}.$$

□

§2.3.2 The case $b_1(M) = 0$.

For every $u \in \mathcal{U}_\sigma$ consider a fixed smooth configuration $\mathsf{S}_0 = (0, B_u)$ and $\sigma \in Spin^c(M)$. Since $c(\sigma)$ is a torsion class we can choose B_u to be a smooth flat connection. *In the remainder of this subsection we will assume $F_{B_u} = 0$.* For every $u = (g, h, \eta, \mu)$ there exists a unique $\mathcal{G}$-orbit of reducible monopoles $[\mathsf{S}_u] = [(0, A_u)]$. This is determined by solving the equation

$$F_{A_u} + \mathbf{i} *_g \eta = 0.$$

If we write $A_u = B_u + \mathbf{i} a_u$ then the above equality becomes

$$da_u = - *_g \eta.$$

Since $*_g\eta$ is a closed 2-form and $H^1(M, \mathbb{R}) \cong H^2(M, \mathbb{R}) \cong 0$ it follows from Hodge theory that *there exists a unique co-closed 1-form a_u satisfying the above equality.* we will denote it by $-d^{-1}(*_g\eta)$ so that

$$A_u := B_u - \mathbf{i} d^{-1}(*_g\eta). \tag{2.11}$$

Observe that the real number

$$\Xi(u) := \int_M d^{-1}(*_g\eta) \wedge *_g\eta$$

is independent of the choice of the flat connection B_u and thus *depends only on the parameter u.*

For every $u = (g, h, \eta, \mu) \in \mathcal{U}_\sigma^*$, the moduli space $\mathfrak{M}_{\sigma,u}$ consists of finitely many gauge orbits of irreducible (σ, u)-monopoles and the unique orbit $[\mathsf{S}_u] = [(0, A_u)]$ of reducible monopoles. We get a *non-constant* function

$$\mathcal{U}_\sigma^* \ni u \mapsto \mathbf{sw}_M(\sigma, u) = \sum_{[\mathsf{S}] \in \mathfrak{M}_{\sigma,u}^*} \epsilon([\mathsf{S}]).$$

For every $u_0, u_1 \in \mathcal{U}_\sigma$ we denote by $\Phi(u_1, u_2)$ the spectral flow of the affine path of Dirac operators

$$\Phi(u_1, u_0) = \mathbf{sf}_{\mathbb{C}}(\mathfrak{D}_{A_t} + \mu_t, \ \ t \in [0, 1]),$$

where

$$\mu_t := (1-t)\mu_{u_0} + t\mu_{u_1}, \quad A_t = (1-t)A_{u_0} + tA_{u_1}.$$

Theorem 2.15.

$$\mathbf{sw}_M(\sigma, u_1) - \mathbf{sw}_M(\sigma, u_0) = -\Phi(u_1, u_0), \ \ \forall u_0, u_1 \in \mathcal{U}_\sigma^*.$$

Idea of proof Again we use a cobordism approach. We can generically find a smooth path

$$u : [0,1] \to \mathcal{U}_\sigma, \quad t \mapsto u(t) = (g_t, h_t, \eta_t, \mu_t), \quad u(k) = u_k, \quad k = 0, 1$$

satisfying the following properties.

• The resonance locus

$$\mathcal{R} := \{t \in [0,1]; \;\; \ker(\mathfrak{D}_{A_{u(t)}} + \mu_t) \neq 0\}$$

is finite. Moreover, for every $t_0 \in \mathcal{R}$ we have $\dim_{\mathbb{C}} \ker(\mathfrak{D}_{A_{u(t_0)}} + \mu_{t_0}) = 1$, and if ψ_0 is a generator of this one dimensional space then

$$\xi_{t_0} := \int_M \Big(\langle \boldsymbol{c}(A'_{u(t_0)})\psi_0, \psi_0 \rangle + \mu'_{t_0} |\psi_0|^2 \Big) dV_{g_{t_0}} \neq 0, \tag{2.12}$$

where

$$A'_{u(t_0)} := \frac{d}{dt}|_{t=t_0} A_{u(t)}, \quad \mu'_{t_0} := \frac{d}{dt}|_{t=t_0} \mu_t.$$

• The space

$$\widehat{\mathfrak{M}}^*_{\sigma,u} := \{([\mathsf{S}], t) \in \mathcal{B}^*_\sigma \times [0,1], \;\; \mathcal{SW}_{\sigma,u(t)}(\mathsf{S}) = 0\}$$

is a smooth 1-dimensional manifold.

One can show (see [6] for details) that $\widehat{\mathfrak{M}}^*_{\sigma,u}$ is naturally oriented and its closure in $\mathcal{B}_\sigma \times [0,1]$ is a manifold with boundary. Moreover (see Figure 6)

$$\partial \widehat{\mathfrak{M}}^*_{\sigma,u} = \sum_{[\mathsf{S}''] \in \mathfrak{M}^*_{\sigma,u_1}} \epsilon([\mathsf{S}''])[\mathsf{S}''] - \sum_{[\mathsf{S}'] \in \mathfrak{M}^*_{\sigma,u_0}} \epsilon([\mathsf{S}'])[\mathsf{S}'] + \sum_{t \in \mathcal{R}} \mathrm{sign}\,(\xi_t)[\mathsf{S}_{u(t)}]$$

We deduce that

$$\sum_{[\mathsf{S}''] \in \mathfrak{M}^*_{\sigma,u_1}} \epsilon([\mathsf{S}'']) - \sum_{[\mathsf{S}'] \in \mathfrak{M}^*_{\sigma,u_0}} \epsilon([\mathsf{S}']) + \sum_{t \in \mathcal{R}} \mathrm{sign}\,(\xi_t) = 0.$$

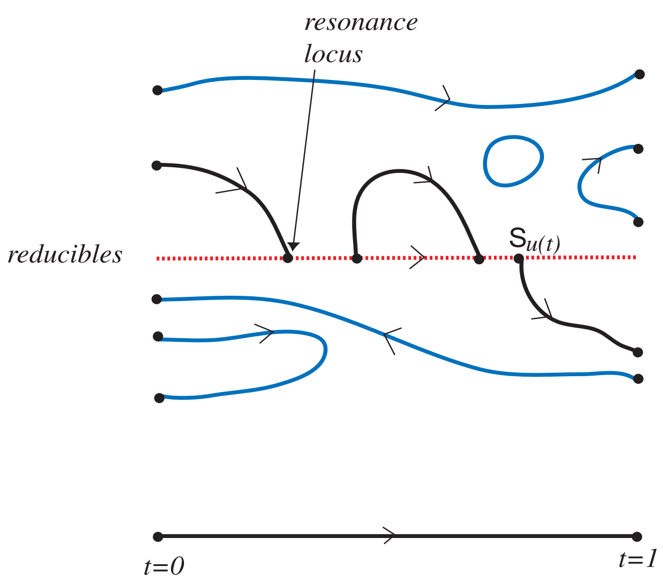

Figure 6: *A singular cobordism.*

A simple computation shows that

$$\sum_{t\in\mathcal{R}} \operatorname{sign}(\xi_t) = \mathbf{sf}_{\mathbb{C}}(\mathfrak{D}_{A_{u(t)}} + \mu_t,\ \ t\in[0,1]),$$

while the homotopy invariance of the spectral flow implies

$$\mathbf{sf}_{\mathbb{C}}(\mathfrak{D}_{A_{u(t)}} + \mu_t,\ \ t\in[0,1]) = \Phi(u_1,u_0).$$

Hence

$$\mathbf{sw}_M(\sigma,u_1) - \mathbf{sw}_M(\sigma,u_0) = \sum_{[\mathsf{S}'']\in\mathfrak{M}^*_{\sigma,u_1}} \epsilon([\mathsf{S}'']) - \sum_{[\mathsf{S}']\in\mathfrak{M}^*_{\sigma,u_0}} \epsilon([\mathsf{S}']) = -\Phi(u_1,u_0).$$

□

To produce a topological invariant we will construct a function

$$F_\sigma : \mathfrak{U}^*_\sigma \to \mathbb{R}$$

such that

$$F(u_1) - F(u_0) = \Phi(u_1,u_0),\ \ \forall u_0,u_1\in\mathfrak{U}^*_\sigma.$$

Then the number

$$\mathbf{sw}^0_M(\sigma,u) = \mathbf{sw}_M(\sigma,u) + F_\sigma(u),\ \ \forall u\in\mathfrak{U}^*_\sigma$$

is independent of u and is thus a topological invariant of the pair (M,σ). To do this we first need the map

$$\mathfrak{U}_\sigma \to \mathfrak{U}_\sigma,\quad u=(g,h,\eta,\mu)\mapsto \underline{u} := (g,h,\eta,0).$$

Observe that

$$C_u = C_{\underline{u}}.$$

Then

$$\Phi(u_1,u_0) = \Phi(u_1,\underline{u}_1) + \Phi(\underline{u}_1,\underline{u}_0) - \Phi(u_0,\underline{u}_0).$$

Hence it suffices to find

$$G_\sigma : \mathfrak{U}_\sigma \to \mathbb{R}$$

such that

$$G_\sigma(u_1) - G_\sigma(u_0) = \Phi(\underline{u}_1,\underline{u}_0)$$

for then the function

$$F_\sigma(u) = \Phi(u,\underline{u}) + G_\sigma(u)$$

will do the trick. To achieve this we need to use the Atiyah-Patodi-Singer index theorem. Given $u_0,u_1\in\mathfrak{U}^*_\sigma$ we a smooth path

$$u:[0,1]\to\mathfrak{U}^*_\sigma,\ \ u(t) = (g_t,h_t,\eta_t,0)$$

such that $u(t)\equiv\underline{u}_i$, for $|t-i|\ll 1$, $i=0,1$. Form the cylinder

$$\hat{M} := [0,1]\times M$$

equipped with the metric $\hat{g} := dt^2 + g_t$. The $spin^c$ structure σ induces a $spin^c$ structure $\hat{\sigma}$ on $\hat{M}$. The associated line bundle $\det\hat{\sigma}$ is then equipped with a metric $\hat{h}$

$$\hat{h}\,|_{t\times M} = h_t.$$

Fix a hermitian connection $\hat{A}$ on $\det \hat{\sigma}$. Set $\mathfrak{D}_i := \mathfrak{D}_{A_{u_i}}$, $d_i := \dim_{\mathbb{C}} \ker \mathfrak{D}_i$ .The Atiyah-Patodi-Singer index theorem implies that

$$\frac{1}{2}(\eta_{\mathfrak{D}_1} + d_1) - \frac{1}{2}(\eta_{\mathfrak{D}_0} + d_0) = \Phi(\underline{u}_1, \underline{u}_0) + \frac{1}{8}\int_{\hat{M}} \Big(-\frac{1}{3} p_1(\nabla^{\hat{g}}) + c_1(\hat{A})^2 \Big)$$

For $u = (g, h, \eta, \lambda)$ we denote by $\eta_{sign}(u)$ the eta invariant of odd signature operator corresponding to the metric g. The Atiyah-Patodi-Singer index theorem implies

$$\eta_{sign}(u_1) - \eta_{sign}(u_0) = \frac{1}{3}\int_{\hat{M}} p_1(\nabla^{\hat{g}}).$$

Define the *Kreck-Stolz* invariant of $u \in \mathcal{U}_\sigma$ to be

$$KS(u) = 4\Big(\eta_{\mathfrak{D}_{A_u}} + \dim_{\mathbb{C}} \ker \mathfrak{D}_{A_u} \Big) + \eta_{sign}(u).$$

We conclude that

$$KS(u_1) - K_S(u_0) = 8\Big(\Phi(\underline{u}_1) - \Phi(\underline{u}_0) \Big) + \int_{\hat{M}} c_1(\hat{A})^2.$$

A simple computations shows that

$$\int_{\hat{M}} c_1(\hat{A})^2 = \frac{1}{4\pi^2}\Big(\Xi(u_1) - \Xi(u_0) \Big).$$

Thus the function

$$G_\sigma : \mathcal{U}_\sigma \to \mathbb{R}, \;\; u \mapsto \frac{1}{8} KS(u) - \frac{1}{32\pi^2}\Xi(u)$$

satisfies

$$G_\sigma(u_1) - G_\sigma(u_0) = \Phi(\underline{u}_1) - \Phi(\underline{u}_0).$$

Definition 2.16. *The* modified Seiberg-Witten invariant *of (M, σ) is the real number*

$$\mathbf{sw}^0_M(\sigma) := \mathbf{sw}_M(\sigma, u) + \frac{1}{8} KS(u) - \frac{1}{32\pi^2}\Xi(u) + \Phi(u, \underline{u}), \;\; u \in \mathcal{U}^\dagger_\sigma.$$

We set

$$\mathbf{SW}^0_{M,\sigma} := \sum_{h \in H} \mathbf{sw}^0_M(h^{-1}\sigma) h \in \mathbb{R}[H].$$

§2.4 A combinatorial description of the Seiberg-Witten invariant Suppose X is a closed, oriented manifold such that $\chi(X) = 0$. Fix a point $x_0 \in X$. A *spider* on X is a pair consisting of a finite *good* open cover $(U_\alpha)_{\alpha \in A}$ of X an a collection of smooth paths $\gamma_\sigma : [0, 1] \to X$, one for each simplex σ of the nerve $\mathcal{N}(U_\bullet)$ of the cover, such that

$$\gamma_\sigma(0) = x_0, \;\; \gamma_\sigma(1) \in U_\sigma, \;\; \forall \sigma.$$

Two spiders $\big\{ (U_\alpha)_\alpha, \gamma^0_\sigma \big\}$ $\big\{ (U_\alpha)_\alpha, \gamma^1_\sigma \big\}$ corresponding to the same good cover are said to be *homologic* if

$$\sum_\sigma (-1)^{\dim \sigma} \big(\bar{\gamma}^0_\sigma - \gamma^1_\sigma \big) = 0 \in H_1(X, \mathbb{Z}),$$

where for every $\sigma \in \mathcal{N}(U_\bullet)$ we denoted by $\bar{\gamma}^0_\sigma$ the path obtained from γ^0_σ by extending it with an arc in U_σ connecting $\gamma^0_\sigma(1)$ to $\gamma^1_\sigma(1)$.

Fix a spider $\mathfrak{s} = \{(U_\alpha)_{\alpha\in A}, \gamma_\sigma\}$ and a *nontrivial* character $\chi : H_1(X, \mathbb{X}) \to \mathbb{C}^*$. Set $x_\sigma := \gamma_\sigma(1)$. The character χ determines a pair (L, A), where L is a complex line bundle, and A is a flat connection on L. Denote by Γ_χ the sheaf of sections of L covariant constant with respect to A. This is a locally constant sheaf and, as the notation suggests, it depends only on the character χ. Denote by $(C^\bullet_\chi(U_\bullet), \delta)$ the Čech complex associated to this cover and this locally constant sheaf.

More precisely

$$C^k_\chi(U_\bullet) := \bigoplus_{\dim\sigma=k+1} \Gamma_\chi(U_\sigma) \cong \bigoplus_{\dim\sigma=k} L_{x_\sigma}.$$

Using the parallel transport T_σ along γ_σ we can identify $L_{x_0} \cong L_{x_\sigma}$ so that

$$C^k_\chi(U_\bullet) \cong \bigoplus_{\dim\sigma=k} L_{x_0}$$

Once we fix a basis $\mathbf{e}_0$ of L_{x_0} we deduce that $C^k_\chi(U_\bullet)$ is also equipped with an unordered basis. For each simplex $\tau = (\alpha_0, \cdots, \alpha_m)$ of $\mathcal{N}(U_\bullet)$ and each $i = 0, \cdots, m$ we set

$$\tau_i = (\alpha_0, \cdots \hat{\alpha}_i, \cdots, \alpha_m).$$

The coboundary operator δ is defined by

$$\delta\Big(\bigoplus_{\dim\sigma=k} z_\sigma\mathbf{e}_0\Big)_\tau = \bigoplus_{\dim\tau=k+1} T_\tau^{-1}\Big(\sum_{i=0}^{k+1}(-1)^i T_{\tau_i}(z_{\tau_i}\mathbf{e}_0)\,|_{U_\tau}\Big).$$

We can now define the torsion $\mathcal{T}_{X,\mathfrak{s}}(\chi)$ of the triple $(X, \mathfrak{s}, \chi)$. Following Turaev, we declare it to be zero if the cohomology of $(C^\bullet_\chi(U^\bullet), \delta)$ is nontrivial. If this complex is acyclic, then the torsion is equal (up to an undetermined sign) to the determinant of this *acyclic, based complex* (see [13, Chap. 1] for more details on the determinants of complexes. The torsion depends only on the homology class of the spider.

Example 2.17. Consider the simplest case $X = S^1$. We identify the circle with the boundary of the triangle $[V_0V_1V_2]$ in Figure 7. We denote by M_i the barycenter of the edge opposite to V_i. The dual cell decomposition of the simplicial complex $\partial[V_0V_1V_2]$ defines a good open cover of the circle whose nerve is isomorphic to the 1-dimensional simplicial complex $\partial[V_0V_1V_2]$. For example the open set U_0 is a small neighborhood of the union of segments $[V_0M_1]$ and $[V_0M_2]$.

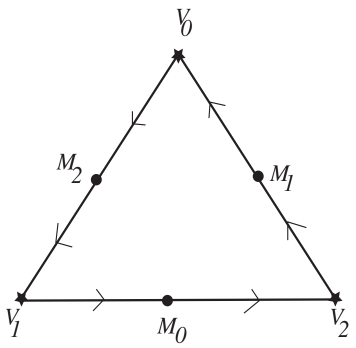

Figure 7: *A simplicial decomposition of the circle.*

The group $H_1(S^1, \mathbb{Z})$ is generated by the 1-cycle $[S^1] := [V_0V_1] + [V_1V_2] + [V_2V_0]$. A character of $H_1(S^1, \mathbb{Z})$ is uniquely determined by its value on this cycle. Fix a character χ such that

$$\chi([S^1]) = t \in \mathbb{C} \setminus \{0, 1\}.$$

We choose a spider $\mathfrak{s}$ as follows. $x_0 = V_0$, x_σ is the barycenter of the simplex σ.

$$x_{(i)} = V_i,\;\; x_{(01)} = M_2,\;\; x_{(12)} = M_0,\;\; x_{(02)} = M_1.$$

γ_σ is the path running counterclockwise from x_0 to x_σ. E.g.

$$\gamma_{(01)} = [V_0M_2],\;\; \gamma_{(1)} = [V_0V_1],\;\; x_{(12)} = [V_0V_1] + [V_1M_0],\;\; \text{etc.}$$

Fix a basis $\mathbf{e}_0$ of L_{V_0}. For every σ we denote by $\mathbf{e}_\sigma$ the basis of L_{x_σ} obtained from $\mathbf{e}_0$ via the parallel transport along γ_σ. If τ is a codimension 1 face of σ, fix a path $\gamma_{\tau\sigma} : [0,1] \to U_\sigma$ from x_τ to x_σ. We get a cycle

$$\gamma_\tau + \gamma_{\tau\sigma} - \gamma_\sigma = n_{\sigma\tau}[S^1] \in H_1(S^1, \mathbb{Z}).$$

We have

$$C^0_\chi = L_{V_0} \oplus L_{V_1} \oplus L_{V_2},\;\; C^1_\chi = L_{M_0} \oplus L_{M_1} \oplus L_{M_2}.$$

We will represent the elements of C^0_χ (respectively C^0_χ) as vectors

$$\begin{bmatrix} z_{V_0} \\ z_{V_1} \\ z_{V_2} \end{bmatrix},\;\; \text{resp.}\;\; \begin{bmatrix} \zeta_{M_0} \\ \zeta_{M_1} \\ \zeta_{M_2} \end{bmatrix}.$$

Then

$$\delta \begin{bmatrix} z_{V_0} \\ z_{V_1} \\ z_{V_2} \end{bmatrix} = \begin{bmatrix} t^{n_{M_0V_1}} z_{V_1} - t^{n_{M_0V_2}} z_{V_2} \\ t^{n_{M_1V_2}} z_{V_2} - t^{n_{M_1V_0}} z_{V_0} \\ t^{n_{M_2V_0}} z_{V_0} - t^{n_{M_2V_1}} z_{V_1} \end{bmatrix} = \begin{bmatrix} 0 & t^{n_{M_0V_1}} & -t^{n_{M_0V_2}} \\ -t^{n_{M_1V_0}} & 0 & t^{n_{M_1V_2}} \\ t^{n_{M_2V_0}} & -t^{n_{M_2V_1}} & 0 \end{bmatrix} \cdot \begin{bmatrix} z_{V_0} \\ z_{V_1} \\ z_{V_2} \end{bmatrix}.$$

The integers $n_{M_iV_j}$ are the winding numbers of the path that $\gamma_{V_j} + [V_jM_i] - \gamma_{M_i}$. Of these integers, only one is nonzero, namely $n_{M_2V_0}$ which is equal to 1. We deduce that

$$\delta \begin{bmatrix} z_{V_0} \\ z_{V_1} \\ z_{V_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ t & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_{V_0} \\ z_{V_1} \\ z_{V_2} \end{bmatrix}.$$

The determinant of this matrix is $(t-1)$ and the torsion is the inverse of this determinant so that

$$\mathcal{T}_{S^1,\mathfrak{s}}(\chi) = \pm \frac{1}{(1-\chi)}$$

$\square$

As explained by Turaev in [14], on a closed oriented 3-manifold M there is a natural bijection between the set of *spin*^{c} structures and the set of homology classes of spiders. Moreover, the sign ambiguity can be removed by fixing an orientation on the determinant line $\det H^*(M, \mathbb{R})$. On a closed, oriented 3-manifold, the Poincaré duality induces a natural orientation on this line. This the one we will consistently use in the future. Set $H := H_1(M, \mathbb{Z})$, $\hat{H} := \mathrm{Hom}(H, \mathbb{C}^*)$. The torsion associates to each *spin*^{c} structure σ on M a holomorphic function $\mathcal{T}_{M,\sigma}$ on $\hat{H} \setminus \{1\}$.

By Hartogs' theorem, this function has a holomorphic extension at $\chi = 1$ if $b_1(M) > 1$. If $b_1(M) = 1$, it has a pole of order two at $\chi = 1$ while if $b_1(M) = 0$ we have

$$\mathcal{T}_{M,\chi}(1) = 0.$$

When $b_1(M) = 1$ we can define a *modified* torsion as follows. Fix an orientation $\mathfrak{o}$ of $H_1(M,\mathbb{R})$. Fix $T \in H$ such that $\deg_{\mathfrak{o}}(T) = 1$. For every $\sigma \in Spin^c(M)$ we set

$$\hat{W}_{\sigma,\mathfrak{o}} : \hat{H} \setminus \{1\} \to \mathbb{C}$$

$$W_\sigma(\chi) = |\mathrm{Tors}\, H| \cdot \begin{cases} \chi(T)^{-d(\sigma)} \cdot \frac{1}{\big(\chi(T)-1\big)\big(\chi(T)^{-1}-1\big)} & \text{if} \quad \chi(T) \neq 1 \\ 0 & \text{if} \quad \chi(T) = 1 \end{cases}, \quad d(\sigma) = \frac{1}{2}\deg_{\mathfrak{o}} c(\sigma). \tag{2.13}$$

We set

$$\mathcal{T}^0_{M,\sigma}(\chi) := \mathcal{T}_{M,\sigma}(\chi) + \hat{W}_\sigma(\chi).$$

Suppose G is a finitely generated Abelian group of rank r. We set $\hat{G} := \mathrm{Hom}(G,\mathbb{C}^*)$. For every function $f : G \to \mathbb{C}$ we define its Fourier transform to be the distribution $\hat{f}$ on $\hat{G}$ defined by

$$\hat{f}(\chi) = \sum_{g\in G} f(g)\chi^{-1}(g).$$

We can recover f from $\hat{f}$ vie the Fourier inversion formula

$$f(g) = \frac{1}{(2\pi)^r}\int_{|\chi|=1} \hat{f}(\chi)\chi(g)d\chi.$$

For example, the function $W_{\sigma,\mathfrak{o}}$ in (2.13) is the Fourier transform of the function

$$W_{\sigma,\mathfrak{o}} : H \to \mathbb{C}$$

defined by the generating series

$$\sum_{h\in H} W_{\sigma,\mathfrak{o}}(h) = -\Theta_H \cdot T^{d(\sigma)} \sum_{k\geq 0} kT^k, \quad \Theta_H := \sum_{h\in \mathrm{Tors}\, H} h \in \mathbb{Z}[H],$$

i.e.

$$W_{\sigma,\mathfrak{o}}(h) = -\max(\deg_{\mathfrak{o}} h - d(\sigma), 0).$$

Theorem 2.18 (Meng-Taubes-Turaev). *Suppose M is a closed oriented 3-manifold. Set $H := H_1(M,\mathbb{Z})$.*
(a) If $b_1(M) > 0$ there exists $\epsilon_M = \pm 1$ such that for every $\sigma \in Spin^c(M)$ we have

$$\mathcal{T}_{M,\sigma} = \epsilon_M \widehat{\mathbf{SW}}_{M,\sigma}(\chi), \ \ \forall \chi \in \hat{H}.$$

(b) If $b_1(M) = 1$ then

$$\mathcal{T}^0_{M,\sigma}(\chi) = \widehat{\mathbf{SW}}^0_{M,\sigma}(\chi), \ \ \forall \chi \in \hat{H}.$$

For a general outline of the proof[1] we refer to [7, 15].

In the case $b_1(M) = 0$ one can check on simple examples that $\widehat{\mathbf{SW}}_{M,\sigma}(1) \neq 0 = \mathcal{T}_{M,\sigma}(1)$ so the Meng-Taubes theorem does not extend to this case in the form above. In this case we defined a modified torsion as follows

$$\mathcal{T}^0_{M,\sigma}(\chi) = \begin{cases} \mathcal{T}_{M,\sigma}(\chi) & \text{if} \quad \chi \neq 1 \\ -CW_M & \text{if} \quad \chi(M) \end{cases},$$

where CW is the Casson-Walker invariant of M normalized as in C. Lescop's book [5]. We have the following result.

[1] As of April 2003, there is no published *complete* proof of this fact.

Theorem 2.19 (Nicolaescu). *If M is a rational homology 3-sphere, $H = H_1(M, \mathbb{Z})$ then*

$$\mathcal{T}^0_{M,\sigma}(\chi) = \widehat{\mathbf{SW}}_{M,\sigma}(\chi), \;\; \forall \sigma \in Spin^c(M), \;\; \chi \in \hat{H}.$$

For a proof we refer to [12]. We want to mention that the equality $\mathcal{T}^0_{M,\sigma}(1) = \widehat{\mathbf{SW}}_{M,\sigma}(1)$ is equivalent to

$$-CW_M = \sum_{\sigma} \mathbf{sw}^0_M(\sigma).$$

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