

ON A MULTI-DIMENSIONAL CHECKERS GAME

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ABSTRACT. We show that certain configurations in an infinite, multi-dimensional checkers game are unattainable. We achieve this with the aid of an entropy like function associated to a configuration of checkers which does not increase during a game.

1. FORMULATION OF THE PROBLEM

Consider the n -dimensional lattice $\Lambda = \mathbb{Z}^n$. A point in Λ is described by an n -uple of integers $x = (x_1, \dots, x_n)$. We denote by e_1, \dots, e_n the standard integral basis of Λ .

We think of Λ as an infinite, n -dimensional checkers board and we will refer to its points as *positions*. Some of the positions are occupied by checkers. We will refer to the quantity x_1 as the *height* of a position. We are allowed to move checkers, one at a time, subject to the following rule.

A checker located in the position x can only move to the positions $x + \delta e_i$, $\delta \in \mathbb{Z}$, $|\delta| \geq 2$, $i = 1, \dots, n$. Such move is *legal* if and only if the all the positions in the interior of the segment connecting x to $x + \delta e_i$ are occupied. After such a move the checkers inside this segment must be removed. The integer δ (which could be positive or negative) is called the *stretch* of the move.

In Figure 1 we illustrate a move of stretch 3 in a 2-dimensional situation, where a \bullet indicates a checker, and \circ indicates a position which was vacated after the move. We denote this move by $T_{x,i}^\delta$, or by T_i^δ when the particular position of x is irrelevant.



FIGURE 1. A *stretch 3* move in the x_1 direction.

We define a *game* to be a finite succession of legal moves starting from the configuration where all positions of height ≤ 0 are occupied by checkers. During a game the set of occupied positions changes. We want to prove that there is a universal height (depending only on the dimension n) such that during any game no checker can move above that height. More precisely we will prove the following result.

Theorem 1.1. *During any game the height of any occupied position satisfies the inequality*

$$x_1 \leq 3n - 2.$$

Date: April 8, 2005.

Remark 1.2. For $n = 1$ the result is obviously optimal. For $n = 2$ one can verify easily that one can move a checker at height 3 whereas the above theorem puts the upper bound at 4. However I believe that the estimate $x_1 \leq 3n - 2$ is not optimal. \square

Acknowledgment This note is a generalization of a *Budapest Semester in Mathematics* problem which I learned from Andras Stipsicz, who in turn attributes it to L. Lovasz. In the Budapest problem the board is two dimensional and the all the moves during a game are required to have stretch ± 2 .

2. PROOF OF THE MAIN RESULT

Let us introduce some notations. We define a *configuration* to be a subset of Λ . One should think of a configuration as the set of occupied positions at some stage during a game. We regard the moves $T_{x,i}^\delta$ as acting on set of *configurations*. Note that not every configuration belongs to the domain of a given $T_{x,i}^\delta$.

For every $x \in \Lambda$ we denote by $|x|$ the *taxi cab norm*

$$|x| := |x_1| + \cdots + |x_n|.$$

We split the coordinates $x = (x_1, \dots, x_n)$ into two groups, x_1 and $x' = (x_2, \dots, x_n)$ so that

$$|x| = |x_1| + |x'|.$$

Denote by C_n the cone

$$C_n := \{x \in \Lambda; x_1 \geq 0, |x'| \leq |x_1|\}.$$

For any integer $r \geq 0$ we denote by $C_n[-r]$ its translate

$$C_n[-r] := -re_1 + C_n,$$

and we set

$$C_n[-r]^- := C_n[-r] \cap \{x_1 \leq 0\}.$$

Note that $C_n[-r]^-$ can be alternatively characterized by

$$C_n[-r]^- = \{x \in \Lambda; x_1 \leq 0, |x| \leq r\}.$$

Now define

$$\begin{aligned} \nu : C_n &\rightarrow \mathbb{Z}, \quad \nu(x) = x_1 - |x'|, \\ \nu_r : C_n[-r] &\rightarrow \mathbb{Z}, \quad \nu_r(x) = \nu(x + re_1). \end{aligned}$$

The values of ν_3 on $C_2[-3]^-$ are indicated in Figure 2.

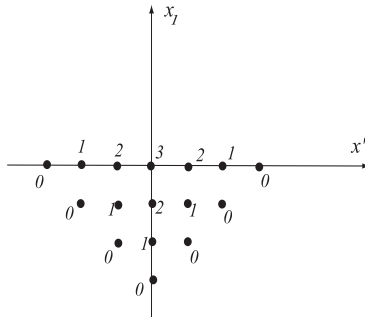


FIGURE 2. $C_n[-r]^-$ for $n = 2$ and $r = 3$.

Denote by w the unique positive root of the quadratic equation

$$t^2 - t - 1 = 0.$$

For later reference note that

$$w - 1 = \frac{1}{w} > 0, \quad \frac{w + 1}{w} = w, \quad (2.1)$$

and

$$w^k \leq w^{k-1} + \dots + w + 1, \quad w^{-k} < w^{-k+1} + \dots + w^{-1} + 1, \quad \forall k \geq 2. \quad (2.2)$$

For every set $S \subset C_n[-r]$ such that x_1 is bounded on S we set

$$\mathcal{E}_{n,r}(S) := \sum_{x \in S} w^{\nu_r(x)}.$$

We will refer to $\mathcal{E}_{n,r}(S)$ as the r -entropy of a configuration.

Suppose that during a game one of the checkers reaches a position of height $x_1 = h > 0$. By a change in coordinates we can assume that position is he_1 . Since during the game we perform only finitely moves there exists $r > 0$ such that none of the positions outside $C_n[-r]$ intervene in the games. We can rephrase this by saying that we have a game confined to the region¹ $C_n[-r]$ such that all positions in $C_n[-r]^-$ are initially occupied by checkers. The first crucial observation is that the moves $T_{x,i}^\delta$ do not increase the r -entropy of a configuration in $C_n[-r]$.

Lemma 2.1.

$$\mathcal{E}_{n,r}(T_{x,i}^\delta S) \leq \mathcal{E}_{n,r}(S), \quad \text{for all feasible } n, i, \delta, x,$$

and any configuration S such that $S, T_{x,i}^\delta S \subset C_n[-r]$.

Proof To prove this inequality we have to distinguish three cases.

(a) $i = 1$. Set $k = |\delta|$. This checker move changes the entropy of a configuration by an amount equal to $w^p(w^k - w^{k-1} - \dots - w - 1)$ or $w^p(w^{-k} - w^{-k+1} - \dots - w^{-1} - 1)$, for some $p \in \mathbb{Z}$, and by (2.2) both these quantities are ≤ 0 .

(b) $i > 1$ and $x_i(x_i + \delta) \geq 0$. Again, this checker move changes the entropy of a configuration by an amount equal to $w^p(w^k - w^{k-1} - \dots - w - 1)$ or $w^p(w^{-k} - w^{-k+1} - \dots - w^{-1} - 1)$, for some $p \in \mathbb{Z}$.

(c) $i > 1$ and $x_i(x_i + \delta) < 0$. In this case the point $\bar{x} = (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ lies in the interior of the segment connecting x and $x + \delta e_i$. Then

$$\mathcal{E}_{n,r}(T_{x,i}^\delta S) - \mathcal{E}_{n,r}(S) < w^{\nu_r(x + \delta e_i)} - w^{\nu_r(\bar{x})} - w^{\nu_r(x)} < 0.$$

The last inequality follows from the fact that the dependence of function $x \mapsto \nu_r(x)$ on the radii $|x_i|$, $i > 1$, is strictly decreasing. □

We deduce that for any configuration of occupied positions during the game we have

$$\mathcal{E}_{n,r}(S) \leq \mathcal{E}_{n,r}(C_n[-r]^-).$$

If S is one of these configuration which contains the occupied position he_1 then

$$\mathcal{E}_{n,r}(S) \geq \nu_r(he_1) = w^{r+h}.$$

¹This means that during the games checkers inside the region are not allowed move outside, and no checker from outside the region is allowed to move in.

Thus, the height that can be reached by a checker during a game confined to the cone $C_n[-r]$ is constrained by the inequality

$$w^{r+h} \leq \mathcal{E}_{n,r}(C_n[-r]^-).$$

Theorem 1.1 now follows from the following arithmetic fact whose proof is presented in the next section.

Lemma 2.2 (Main Estimate).

$$\mathcal{E}_{n,r}(C_n[-r]^-) < \omega^{r+3n-1}, \quad \forall n \geq 1, \quad r \geq 0.$$

3. PROOF OF THE MAIN ESTIMATE

Set

$$E_{n,r} = \mathcal{E}_{n,r}(C_n[-r]^-), \quad E_n = E_n(t) = \sum_{r \geq 0} E_{n,r} t^r,$$

$$G(t) = \frac{1}{1-wt}, \quad S(t) = \frac{1}{1-t}, \quad P(t) = (1+t).$$

Given two formal power series

$$A(t) = \sum_{r \geq 0} a_r t^r, \quad B(t) = \sum_{r \geq 0} b_r t^r$$

with real coefficients we write

$$A \prec B \iff a_r < b_r \quad \forall r \geq 0,$$

and

$$A \preceq B \iff a_r \leq b_r \quad \forall r \geq 0.$$

Note that $0 \prec S$, $0 \preceq P$ and

$$A \prec B, \quad 0 \prec C \implies AC \prec BC.$$

The Main Estimate can now be rephrased as

$$E_n \prec w^{3n-1}G. \tag{3.1}$$

Note that

$$E_{1,r} = \sum_{k=0}^r w^k = \frac{w^{r+1} - 1}{w - 1} \stackrel{(2.1)}{=} w^{r+2} - w$$

so that

$$E_1 = \sum_{r \geq 0} w^{r+2} t^r - w \sum_{r \geq 0} t^r = w^2 G - S \prec w^2 G. \tag{3.2}$$

If we set $\hat{x} := (x_1, \dots, x_{n-1})$ we deduce

$$E_{n,r} = \sum_{x \in C_n[-r]^-} w^{r-|x|} = \sum_{x_1 \leq 0, |x| \leq r} w^{r-|x|} = \sum_{|x_n|=0}^r \left(\sum_{x_1 \leq 0, |\hat{x}| \leq r-|x_n|} w^{r-|x_n|-|\hat{x}|} \right).$$

Since the equation $|x_n| = k$ has two solutions for $k > 0$ and one solution for $k = 0$ we deduce

$$\begin{aligned} E_{n,r} &= 2 \sum_{k=0}^n \left(\sum_{x_1 \leq 0, |\hat{x}| \leq r-k} w^{r-k-|\hat{x}|} \right) - \sum_{x_1 \leq 0, |\hat{x}| \leq r} w^{r-|\hat{x}|} \\ &= 2 \sum_{k=0}^r E_{n-1,r-k} - E_{n-1,r}. \end{aligned}$$

In terms of generating functions the last equality can be rewritten as

$$E_n = \frac{2}{1-t}E_{n-1} - E_{n-1} = \frac{(1+t)}{(1-t)}E_{n-1} = PSE_{n-1}. \quad (3.3)$$

Using the equality (3.2) we deduce

$$E_n = (PS)^{n-1}E_1. \quad (3.4)$$

At this point observe that

$$PG = \sum_{r \geq 0} (w^r + w^{r-1})t^r - 1 = \sum_{r \geq 0} w^{r+1}t^r - 1 = wG - 1 \preccurlyeq wG.$$

Similarly

$$\begin{aligned} SG &= \sum_{r \geq 0} \left(\sum_{k=0}^r w^k \right) t^r = \sum_{r \geq 0} \frac{w^{r+1} - 1}{w - 1} t^r \stackrel{(2.1)}{=} \sum_{r \geq 0} (w^{r+2} - w) t^r \\ &= w^2G - wS \prec w^2G. \end{aligned}$$

Putting these together we deduce

$$PSG \prec w^3G$$

and thus

$$E_n = (PS)^{n-1}E_1 \prec (PS)^{(n-1)}(w^2G) \prec w^{3n-1}G.$$

This concludes the proof of the Main Estimate. □

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