THE ANATOMY OF A SINGULARITY

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Contents

1. Some basic facts

Denote by $\mathcal{O} = \mathcal{O}_{N+1}$ the ring of germs of holomorphic functions $f = f(z_0, \cdots, z_N)$ defined in a neighborhood of $\vec{0} \in \mathbb{C}^{N+1}$. We denote by $\mathfrak{m} \subset \mathcal{O}$ the maximal ideal of $\mathcal{O}$,

$$f \in \mathfrak{m} \Longleftrightarrow f(\vec{0}) = 0.$$

Let $f \in \mathfrak{m}$. Assume $\vec{0}$ is an isolated critical point of f, i.e. $\vec{0}$ is an isolated point of the variety

$$\partial_{z_i} f = 0, \ \ \forall i = 0, \cdots, N.$$

We define the Jacobian ideal of f to be the ideal $J_f \subset \mathcal{O}$ generated by $\partial_{z_i} f$, $i = 0, \cdots, N$. From the analytical Nullstellensatz we deduce

$$\sqrt{J_f} = \mathfrak{m} \Longleftrightarrow \exists k > 0 : \mathfrak{m}^k \subset J_f \Longleftrightarrow A_f := \dim_{\mathbb{C}} \mathcal{O}/J_f < \infty.$$

The finite dimensional commutative $\mathbb{C}$ -algebra A_f is called the local algebra of the critical point $\vec{0}$ of f. Its dimension is called the *Milnor number* of f at $\vec{0}$ and it is denoted by $\mu = \mu(f, \vec{0})$. It has a natural structure of $\mathbb{C}\{t\}$-algebra

$$t \cdot (g \bmod J_f) = (fg) \bmod J_F, \ \ \forall g \in \mathcal{O}.$$

For every positive integer N we denote by $j_N(f)$ the N-th jet of f. It can be identified with a polynomial of degree N in $n+1$ complex variables.

Two germs $f, g \in \mathfrak{m}$ are called *right-equivalent* and we write this $f \sim_r g$ if g is obtained from g by a change in variables.

Theorem 1.1 (Finite determinacy). *(a) (Mather-Tougeron) Let $f \in \mathfrak{m}$ have an isolated singularity at 0. Then*

$$f \sim_r j_{\mu+1}(f).$$

(b) (Mather-Yau) Let $f, g \in \mathfrak{m}$ have isolated singularities at 0. Then

$$f \sim_r g \Longleftrightarrow A_f \cong A_g \ \text{ as } \ \mathbb{C}\{t\} - \text{algebras}.$$

□

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Notes for a Felix Klein Seminar.

Example 1.2 (Brieskorn singularities)**.** Consider three integers $p, q, r \geq 2$ and consider the function

$$f = f_{p,q,r}(x, y, z) = az^p + by^q + cz^r.$$

Then $\mu = (p-1)(q-1)(r-1)$. The local algebra $A_{p,q,r}$ is generated by the monomials $e_{ijk} = x^i y^j z^k$ where $0 \leq i < p$, $0 \leq j < q$, $0 \leq k < r$. We see that this algebra is isomorphic to the $\mathbb{C}$-group algebra of the Abelian group $\mathbb{Z}/p \times \mathbb{Z}/q \times \mathbb{Z}/r$. The singularity described by $f_{2,2,n+1}$ is called the A_n singularity. It has Milnor number n.

□

Example 1.3. Consider the polynomial

$$D_4 = D_4(x, y, z) = x^2 y - y^3 + z^2.$$

$\vec{0}$ is an isolated critical point of D_4,the local algebra has dimension 4, and we can explicitly determine a basis

$$e_0 = 1, \;\; e_1 = x, \;\; e_2 = y, \;\; e_3 = y^2.$$

It is easy to compute the multiplication table of the local algebra $\mathcal{A}_{D_4} = \mathcal{O}_3/J_{D_4}$.

	$e_1 = x$	$e_2 = y$	$e_3 = y^2$
$e_1 = x$	$3e_3$	0	0
$e_2 = y$	0	e_3	0
$e_3 = y^2$	0	0	0

Note that the D_4-singularity is *weighted homogeneous.* We recall that a function $f = f(z_1, \cdots, z_N)$ is called weighted homogeneous if there exist integers, i.e. there exists integers $m_1, \cdots, m_N, m$ such that

$$f(t^{m_1} z_1, t^{m_N} z_N) = t^m D_4(z_1, \cdots, z_N), \;\; \forall t \in \mathbb{C}^*.$$

The rational numbers $w_i = m_i/m$ are called the *weights.* The weights of the D_4 singularity are

$$w_1 = w_2 = \frac{1}{3}, \;\; w_3 = \frac{1}{2}.$$

A weighted homogeneous polynomial satisfies the *Euler identity*

$$f = \sum_i w_i \frac{\partial f}{\partial z_i}.$$

Note that for such a function we have $f \in J_f$ so the $\mathbb{C}\{t\}$ module of A_f is very simple: t acts trivially.

□

2. The Milnor fibration and the Gauss-Manin connection

Let $f \in \mathfrak{m}$ have an isolated singularity at 0. Set $\mu = \mu(f, 0)$. According to Milnor, for $\varepsilon > 0$ sufficiently small we can find an open neighborhood $X = X_\varepsilon$ of $0 \in \mathbb{C}^{N+1}$ so that $f(X_\varepsilon) = \mathbb{D}_\varepsilon = \{|z| < \varepsilon\} \subset \mathbb{C}$ such that the induced map

$$f : X^* := X \setminus f^{-1}(0) \to \mathbb{D}^*_\varepsilon$$

is a local trivial fibration called the Milnor fibration. Its typical fiber X_f is smooth $2N$-dimensional manifold with boundary called the Milnor fiber. Its boundary is a $(2N-1)$-manifold called the *link of the singularity.* The Milnor fiber which has the homotopy type of a wedge of μ spheres of dimension N,

$$X_f \simeq \underbrace{S^N \vee \cdots \vee S^N}_{\mu}$$

The Milnor fibration defines a monodromy map

$$\mathcal{M}_f : \pi_1(\mathbb{D}^*_\varepsilon) \to \mathbf{Aut}_{\mathbb{Z}}(\tilde{H}_N(X_f, \mathbb{Z})),$$

where $\tilde{H}_\bullet$ denotes *reduced homology.* We denote by $[\mathcal{M}_f]_{\mathbb{Z}}$ its $\mathbb{Z}$-conjugacy class and by $[\mathcal{M}_f]_{\mathbb{C}}$ its $\mathbb{C}$-conjugacy class. The complex conjugacy class is completely determined by the complex Jordan normal form of $\mathcal{M}_f$.

Theorem 2.1 (Monodromy Theorem, Griffith-Deligne)**.** *All the eigenvalues of* $\mathcal{M}_f$ *are roots of* 1 *and its Jordan cells have dimension* $\leq (N+1)$.

Example 2.2. (a) Consider the germ $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, $f(z) = z^n$. Then the Milnor fiber f^{-1} can be identified with the group $\mathfrak{R}_n$ of n-th roots of 1,

$$\mathfrak{R}_n = \{1, \rho, \cdots, \rho^{n-1};\;\; \rho = e^{\frac{2\pi \mathbf{i}}{n}}\}.$$

The Milnor number is $(n-1)$. This is equal to the rank of the reduced homology $\tilde{H}_0(f^{-1}(0), \mathbb{Z})$ which can be identified with the subgroup of the group algebra $\mathbb{Z}[\mathfrak{R}_n]$

$$\tilde{H}_0(f^{-1}(1), \mathbb{Z}) \cong \Big\{ \sum_{k=0}^{n-1} a_k \rho^k \in \mathbb{Z}[\mathfrak{R}_n];\;\; \sum_{k=0}^{n-1} a_k = 0 \Big\}.$$

As basis in this group we can choose the "polynomials"

$$e_k := \rho^k - \rho^{k-1}, ;\;\; k = 1, \cdots, n-1.$$

Then

$$\mathcal{M}_f(e_k) = \begin{cases} e_{k+1} & \text{if} \quad k < n-1 \\ -(e_1 + \cdots + e_{n-1}) & \text{if} \quad k = n-1 \end{cases}$$

We deduce $\mathcal{M}^n_{A_{n-1}} = \mathbb{I}$, i.e. all the eigenvalues of the monodromy are n-th roots of 1.

(b) (Thom-Sebastiani) If $f = f(x_1, \cdots, x_p) \in \mathcal{O}_p$ and $g = g(y_1, \cdots, y_q) \in \mathcal{O}_q$ have isolated singularities at the origin, then so does $f * g \in \mathcal{O}_{p+q}$

$$f * g(x, y) = f(x_1, \cdots, x_p) + g(y_1, \cdots, y_q)$$

and

$$X_{f*g} \simeq X_f * X_g := \text{the join of the Milnor fibers } X_f \text{ and } X_g$$

("$\simeq$" denotes homotopy equivalence)

$$\mu(f * g, 0) = \mu(f, 0) \cdot \mu(g, 0),\;\; [\mathcal{M}_{f*g}]_{\mathbb{C}} = [\mathcal{M}]_f \otimes [\mathcal{M}]_g$$

Note that if $q = 1$ and $g(y) = y^2$ then

$$X_{f*y^2} \simeq \Sigma X_f,$$

where Σ denotes the suspension operation. The operation $f \mapsto f * y^2$ is called *stabilization* and two singularities are called *stably equivalent* if they become right-equivalent after a finite number of stabilizations. Note that the singularity presented $\{z^n = 0\}$ discussed in part (a) is stably equivalent to the A_{n-1}-singularity.

A theorem of J. Mather states that two hypersurface singularities $\{f(x_1, \cdots, x_p) = 0\}$ and $\{g(y_1, \cdots, y_q) = 0\}$ are stably equivalent if and only if their local algebras A_f and A_g are isomorphic as $\mathbb{C}$-algebras.

(c) $\mathcal{M}_{D_4}$ was computed by Arnold. It is related to the Coxeter group with the same name. The Milnor fiber X_{D_4} is a 4-manifold with boundary and the intersection form q on $\Lambda = H_2(X_{D_4}, \mathbb{Z})$ has a particularly nice form described in the Dynkin diagram below.

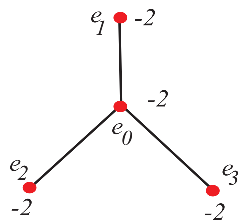

FIGURE 1. *The Dynkin diagram D_4.*

This means that Λ has a canonical integral basis consisting of vanishing spheres, i.e. embedded 2-spheres e_0, e_1, e_2, e_3 with self intersection -2, $q(e_\alpha, e_\alpha) = -2$, $\forall \alpha = 0, 1, 2, 3$. Moreover

$$q(e_0, e_i) = 1,\quad q(e_i, e_j) = 0,\quad \forall i, j = 1, 2, 3.$$

A vanishing sphere e_α determines an involution R_α of Λ, the so called *Picard-Lefschetz transformations* associated to e_α. More explicitly, it is the q-orthogonal reflection in the hyperplane q-orthogonal to e_α, i.e.

$$R_\alpha(v) = v - 2\frac{q(v, e_\alpha)}{q(e_\alpha, e_\alpha)} = v + q(v, e_\alpha).$$

Then $\mathcal{M}_{D_4}$ is conjugate (over $\mathbb{Z}$) with the Coxeter transformation

$$T_{D_4} = R_0R_1R_2R_3 \in \mathrm{GL}(\Lambda).$$

From the equality $T_{D_4}^6 = \mathbb{I}$ (the Coxeter number of D_4 is 6) we deduce that all the eigenvalues of $\mathcal{M}_{D_4}$ are 6-th order roots of 1.

□

Using local trivializations in the Milnor fibration $f : X^* \to \mathbb{D}^*$ we can parallel transport[1] cycles in a fiber $X_t := f^{-1}(t) \cap X$ to nearby fibers and we obtain in this fashion the locally constant sheaf H_f whose stalk at $t \in \mathbb{D}^*$ is $\tilde{H}_N(X_t, \mathbb{Z})$. It is called the sheaf of *vanishing cycles*. Its sections are families of vanishing cycles varying continuously from fiber to fiber. We will refer to these as *locally constant* vanishing cycles. We denote by $\underline{\mathbb{Z}}$ the constant sheaf on $\mathbb{D}^*$ and we set

$$H^f := \underline{Hom}_{\mathbb{Z}}(H_f, \underline{\mathbb{Z}}),$$

where $\underline{Hom}(\mathcal{F}, \mathcal{G})$ denotes the sheaf of morphisms between two sheaves $\mathcal{F}, \mathcal{G}$. Consider the sheaf $\mathcal{E}$ of smooth complex valued functions on $\mathbb{D}^*$. The sheaf

$$\mathcal{H}^f := \underline{Hom}_{\mathbb{Z}}(H_f, \mathcal{E}) \cong H^f \otimes_{\mathbb{Z}} \mathcal{E}$$

is a locally free sheaf of $\mathcal{E}$ modules and thus can be interpreted as the sheaf of sections of rank μ-complex vector bundle over $\mathbb{D}^*$ which we also denote by $\mathcal{H}^f$. It is called the *cohomological Milnor bundle.*

This bundle is equipped with a canonical holomorphic structure and a canonical flat connection ∇ constructed as follows.

[1]This a C^∞ *but not a holomorphic* construction, as one may think. That is why the fact that the Gauss-Manin connection ends up having a *holomorphic* (even *algebraic*!) nature is somewhat surprising.

Given $t_0 \in \mathbb{D}^*$, a small contractible neighborhood U of $t_0 \in \mathbb{D}_*$ and a $\mathbb{Z}$-basis $\{e_1, \cdots, e_\mu\}$ of vanishing cycles in X_t, we obtain by parallel transport a trivialization of H_f over U and then by duality a local frame (e^i) of $\mathcal{H}^f \mid_U$. Any $s \in \Gamma(U, \mathcal{H}^f)$ can be written as $s = \sum_k s_k e^k$, $s_k = \langle s, e_k \rangle \in \mathcal{E}(U)$. s is declared holomorphic if all the components s_k are holomorphic functions. We set

$$\nabla s := \sum_k (ds_k) \otimes e_k \in \Gamma(U, T^*U \otimes \mathcal{H}^f).$$

These notions are independent of the various choices. ∇ is called the *topological Gauss-Manin connection.* We denote by $\mathcal{H}^f_{hol}$ the sheaf of holomorphic sections of $\mathcal{H}^f$.

Brieskorn has constructed free, coherent sheaves of $\mathcal{O}_\mathbb{D}$-modules $\mathcal{L}_0, \mathcal{L}_1 \to \mathbb{D}$, together with an *injective* morphisms of $\mathcal{O}_\mathbb{D}$-modules $\varphi : \mathcal{L}_1 \hookrightarrow \mathcal{L}_0$ and *isomorphisms* $\beta_i : \mathcal{H}^f \to \mathcal{L}_i \mid_{\mathbb{D}^*}$, $i = 0, 1$ such that over $\mathbb{D}^*$ the diagram below is commutative

$$\begin{array}{ccc} \mathcal{L}_1 & \xrightarrow{\ \varphi\ } & \mathcal{L}_0 \\ {\scriptstyle \beta_1} \nwarrow & & \nearrow {\scriptstyle \beta_0} \\ & \mathcal{H}^f_{hol} & \end{array} .$$

Moreover, if we denote by t the local coordinate on $\mathbb{D}$ such that $\mathcal{O}_{\mathbb{D},0} \cong \mathbb{C}\{t\}$ then there exists a natural isomorphisms of $\mathbb{C}\{t\}$-modules

$$\rho : \big(\mathcal{L}_0/\varphi(\mathcal{L}_1)\big)_0 \to A_f.$$

The sheaves $\mathcal{L}_i$ are also known as the *Brieskorn lattices.* Each is an extension to $\mathbb{D}$ of the coherent sheaf $\mathcal{H}^f$. Note also that the quotient $\mathcal{L}_0/\varphi(\mathcal{L}_1)$ is a coherent sheaf supported at the center of $\mathbb{D}$.

We describe the restrictions to $\mathbb{D}^*$ of the sheaves $\mathcal{L}_i$ and the morphisms φ, β_i^{-1}. Denote by Ω^k sheaf of holomorphic k-forms on X, i.e. differential forms ω locally described as

$$\omega = \sum_\alpha \omega_\alpha dz_{\alpha_1} \wedge \cdots dz_{\alpha_k}.$$

Given a small open disk $U \subset \mathbb{D}^*$ we set ${}^fU = f^{-1}(U)$ and

$$\mathcal{L}_1(U) \approx \Omega^N({}^fU) \bmod \Big(d\Omega^{N-1}({}^fU) + df \wedge \Omega^{N-1}({}^fU) \Big).$$

We use the symbol "$\approx$" instead of "$=$" since the above definition is only "morally correct".

The restriction of a holomorphic form $\omega \in \Omega^N(U_f)$ to fiber X_t, $t \in U$ is a closed form ω_t and we denote by $[\omega_t] \in H^N(X_t, \mathbb{C})$ the class it defines. This cohomology class depends only on the image of $\omega \in \mathcal{L}_1(U)$.

Given $\omega \in \mathcal{L}_1(U)$ we obtain a *holomorphic* section[2] $[\omega] \in \Gamma(U, \mathcal{H}^f)$ determined by the following rule: for every locally constant vanishing cycle $U \ni t \mapsto c_t \in H_n(X_t, \mathbb{Z})$

$$\langle [\omega], c \rangle (t) = \int_{c_t} [\omega \mid_{X_t}].$$

The resulting map

$$\mathcal{L}_1 \mid_{\mathbb{D}^*} \ni \omega \longmapsto [\omega] \ni \mathcal{H}^f_{hol}$$

is an isomorphism whose inverse is β_1.

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[2]The holomorphic nature of this section is by no means obvious since the cycle c_t only varies smoothly with t.

The sheaf $\mathcal{L}_0$ is intimately related to the notion of Poincaré residue. Given $U \subset \mathbb{D}^*$ as above and $\omega \in \Omega^{N+1}({}^fU)$, we deduce from the fact that $df \neq 0$ on X^* that ω can be written as

$$\omega = df \wedge \eta, \;\; \eta \in \Omega^N_X({}^fU).$$

η is uniquely determined only modulo $df \wedge \Omega^{N-1}({}^fU)$ and we denote by $\frac{\omega}{df}$ the image of η in $\Omega^N \bmod df \wedge \Omega^{N-1}$. Note that

$$\omega = df \wedge \eta = df \wedge \eta' \Longrightarrow \eta\,|_{X_t} = \eta'\,|_{X_t}, \;\; \forall t \in U.$$

Hence $\frac{\omega}{df}$ defines on each fiber X_t a closed form $\frac{\omega}{df}\,|_{X_t}$. Its cohomology class does not change if we add to ω forms of the type $df \wedge d\eta$, $\eta \in \Omega^{N-1}_X$ since $\frac{df\wedge d\eta}{df} = d\eta$. We get a map

$$\Omega^{N+1}({}^fU) \bmod df \wedge d\Omega^{N-1}({}^fU) \to H^N(X_t, \mathbb{C}), \;\; \omega \longmapsto \big[\frac{\omega}{df}\,|_{X_t}\big].$$

The cohomology class $\big[\frac{\omega}{df}\,|_{X_t}\big]$ is called the *Poincaré residue of* ω along X_t. We will denote it by $\mathbf{Res}_f(\omega, X_t)$. Now set

$$\mathcal{L}_0({}^fU) \approx \Omega^{N+1}({}^fU) \bmod df \wedge d\Omega^{N-1}({}^fU).$$

For $\omega \in \mathcal{L}_0({}^fU)$ we can integrate $\mathbf{Res}_f(\omega, X_t)$ over locally constant vanishing cycles and obtain a *holomorphic* section $\mathbf{Res}_f(\omega) \in \Gamma(U, \mathcal{H}^f)$. Arnold refers to this section as the *geometric section* determined by the top dimensional form ω. The resulting morphism of sheaves

$$\mathbf{Res}_f : \mathcal{L}_0\,|_{\mathbb{D}^*} \to \mathcal{H}^f_{hol}, \;\; \omega \mapsto \mathbf{Res}_f(\omega),$$

is an isomorphism whose inverse is β_0. The map

$$\Omega^N \ni \omega \longmapsto df \wedge \omega \in \Omega^{N+1}$$

induces a morphism

$$\mathcal{L}_1 \approx \Omega^N \bmod\big(df \wedge \Omega^{N-1} + d\Omega^{N-1}\big) \longrightarrow \Omega^{N+1} \bmod\big(df \wedge d\Omega^{N-1}\big) = \mathcal{L}_0.$$

This is precisely the morphism φ.

The exterior differentiation $d : \Omega^N \to \Omega^{N+1}$ induces a morphism of sheaves

$$d : \mathcal{L}_1\,|_{\mathbb{D}^*} \to \mathcal{L}_0\,|_{\mathbb{D}^*}\,.$$

This is intimately related to the (topological) Gauss-Manin connection.

Theorem 2.3 (Gelfand-Leray formula)**.** *The following diagram of sheaves and morphisms of sheaves is commutative*

$$\begin{array}{ccc} \mathcal{L}_1\,|_{\mathbb{D}^*} & \xrightarrow{\;d\;} & \mathcal{L}_0\,|_{\mathbb{D}^*} \\ \uparrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \mathbf{Res}_f} \\ \mathcal{H}^f & \xrightarrow[\nabla_t]{} & \mathcal{H}^f \end{array}$$

Hence if we start with $\omega \in \Omega^n({}^fU)$ *we obtain a section* $[\omega] \in \Gamma(U, \mathcal{H}^f_{hol})$ *and for every locally constant vanishing cycle* $t \mapsto c_t$ *we have*

$$\frac{d}{dt}\int_{c_t} [\omega] = \int_{c_t} \big[\frac{d\omega}{df}\big].$$

Suppose $S = S_\theta \subset \mathbb{D}^*$ is an angular sector

$$S = \{ t \in \mathbb{D}^* \ \ |\arg t| \leq \theta \}, \ \ \theta \in (0,\pi).$$

We fix a branch of $\log t$ on U such that $\log 1 = 0$ and for every real number α we set $t^\alpha = e^{\alpha \log t}$. Define

$$\Lambda^f := \{ r \in \mathbb{R}; \ \exp(2\pi \mathbf{i} r) \ \text{ is an eigenvalue of the monodromy } \mathcal{M}_f \},$$

and $\Lambda^f_\nu = \Lambda^f \cap (\nu, \infty)$, $\forall \nu \in \mathbb{R}$. From the monodromy theorem we deduce that Λ^f consists of finitely many arithmetic progression of rational numbers. We have the following fundamental result.

Theorem 2.4 (Regularity Theorem, Deligne-Griffiths)**.** *Denote by $j = j_f$ the largest dimension of the Jordan cells of $\mathcal{M}_f$. Suppose $\omega \in \Omega^{N+1}(X)$ and $S_\theta \ni t \stackrel{c}{\longmapsto} c_t$ is a parallel vanishing cycle. Then there exists a real number ν and for every $\alpha \in \Lambda^f_\nu$ a polynomial $P_\alpha = P_{\alpha,\omega,c} \in \mathbb{C}[s]$ of degree $< j$ such as $t \to 0$ in S we have the asymptotic expansion*

$$\int_{c_t} [\mathbf{Res}_f\, \omega] \sim \sum_{r \in \Lambda^f_\nu} t^\alpha P_\alpha(\log t).$$

Remark 2.5. Let $\omega \in \Omega^{N+1}(X)$. We can write $\omega = g d\vec{z}$, where $d\vec{z} = dz_0 \wedge \cdots \wedge dz_N$ and g is a holomorphic function on X. Since 0 is an isolated critical point of f we deduce from the analytical Nullstellensatz that there exists an integer $\ell > 0$ such that

$$f^\ell \in \mathfrak{m}^\ell \subset J_f.$$

In other words, there exist an open neighborhood V of 0 in X and holomorphic functions $a^0, \cdots, a^n$ on V such that

$$f^\ell = \sum_k a^k \partial_{z_k} f \quad \text{on } V.$$

If we denote by A the vector field $A = \sum_k a^k \partial_{z_k}$ and we denote by ι_A the contraction by A then we can rewrite the above equality as

$$f^\ell d\vec{z} = df \wedge \iota_A d\vec{z}.$$

In particular, we deduce that on $V^* - V \setminus f^{-1}(0)$ we have the equality

$$g dV = f^{-\ell} g df \wedge \iota_A d\vec{z} \Longleftrightarrow \frac{\omega}{df} = f^{-\ell} \iota_A \omega.$$

We can assume V has the form $V = f^{-1}(\mathbb{D}_\varepsilon) \cap X$. Now observe that $\iota_A\omega$ defines a section

$$[g\iota_A\omega] \in \Gamma(\mathbb{D}^*_\varepsilon, \mathcal{L}_1) \ \text{ and } \ [f^{-\ell}\iota_A\omega] = t^{-\ell}[\iota_A\omega] \in \Gamma(\mathbb{D}^*_\varepsilon, \mathcal{L}_1).$$

We have

$$\int_{c_t} \mathbf{Res}_f(\omega) = t^{-\ell} \int_{c_t} [\iota_A\omega], \ \ \forall 0 < |t| \ll 1.$$

This shows that we can expect these integrals will "explode" as $t \to 0$ so we can expect that the real number ν in the regularity theorem is < 0.

On the other hand, according to Malgrange, the polynomial $P_\alpha(s) \equiv 0$ if $\alpha \leq -1$ so that in the above theorem we can assume $\nu = -1$. Thus these integrals explode but slower than t^{-1}.

□

3. The spectrum of a singularity

Suppose $(e_1, \cdots, e_\mu)$ is a basis of vanishing cycles in X_{t_0} for some t_0. We can extend them by parallel transport over U to a trivialization $H_f\,|_U$. For every holomorphic function g on X we obtain μ asymptotic expansions

$$I_{\omega_g,e_k}(t) := \int_{e_k(t)} \mathrm{Res}_f(\omega_g) \sim \sum_{\alpha\in\Lambda^f_{-1}} t^\alpha P_{\alpha,\omega,k}(\log t),\ \ \omega_g = g dz^0\wedge\cdots\wedge dz^n.$$

We set

$$\nu_k(\omega) = \min\{\alpha\in\Lambda^f_{-1};\ \ P_{\alpha,\omega,k}\neq 0\}$$

and we define the *order* of the geometric section $s_g = \mathrm{Res}(\omega_g)$ to be

$$\nu = \nu(\omega_g) = \min\{\nu_k(\omega);\ \ k = 1,\cdots,\mu\}.$$

If we denote by (e^k) the basis if $H^f\,|_U$ dual to (e_i) then we set

$$s_{max}(\omega_g) = \sum_{k=1}^{\mu} t^\nu P_{\nu,\omega_g,k}(\log t)e^k \in \Gamma(U, \mathcal{H}^f_{hol}).$$

This section is independent of the basis (e_i) and moreover, it extends to a section of $\mathcal{H}^f_{hol}$ over the entire punctured disk $\mathbb{D}^*$. It is called the *principal part* of the geometric section $\mathbf{Res}_f(\omega)$.

Example 3.1. Consider the function $f : X = \mathbb{C}\to\mathbb{C}$, $z\mapsto t = z^n$. Let $\zeta := e^{\frac{2\pi\mathbf{i}}{n}}$. For every $t = \rho e^{\mathbf{i}\theta}$ in the sector $S = S_{\pi/2} = \{\mathbf{Re}\, z > 0\}$ we set

$$t^{1/n} = \rho^{1/n}e^{\frac{\mathbf{i}\theta}{n}},\ \ e_k(t) = t^{1/n}\big(\zeta^k - \zeta^{(k-1)}\big) \in \tilde{H}_0(f^{-1}(t),\mathbb{Z}),\ \ k = 1,\cdots,n-1.$$

For $1\leq m < n$ we set $\omega_m = z^{m-1}dz = \frac{1}{m}d(z^m)\in\Omega^1(X)$. Then

$$\frac{\omega_m}{df} = \frac{z^{m-1}dz}{nz^{n-1}dz} = \frac{1}{n}z^{m-n}\in\Omega^0(X^*).$$

For $t\in S$ we have

$$\int_{e_k(t)}\frac{\omega_m}{df} = \frac{1}{n}\big((t^{1/n}\zeta^k)^{(m-n)} - (t^{1/n}\zeta^{k-1})^{(m-n)}\big) = \frac{1}{n}(\zeta^{km})t^{\frac{(m-n)}{n}}\big(1-\zeta^{-m}\big).$$

We conclude that

$$\nu(\omega_m) = \frac{m}{n} - 1 < 0,\ \ 1\leq m < n.$$

□

Returning to the general case, let us make the change in variables $t = e^s$, $\mathbf{Re}\, s < 0$ and we (ambiguously) set $e_k(s) = e_k(e^s)$. Fix $t_0\in\mathbb{D}^*$, $\mathbf{Im}\, t_0 = 0$ and $s_0 = \log t_0\in\mathbb{R}$. Set

$$\underline{\mathbf{e}}(s) = \big[\,e_1(s),\cdots,e_\mu(s)\,\big],\ \ \bar{\mathbf{e}}(s) = \begin{bmatrix} e^1(s)\\ \vdots \\ e^\mu(s)\end{bmatrix}.$$

In the basis $(\underline{\mathbf{e}}(s_0))$ the monodromy $\mathcal{M}_f$ is represented by a $\mu\times\mu$ matrix $M = \big(m^i_j\big)_{1\leq i,j\leq\mu}$ and we have the equalities

$$\underline{\mathbf{e}}(s_0+2\pi\mathbf{i}) = \underline{\mathbf{e}}(s_0)\cdot M \Longleftrightarrow e_i(s_0+2\pi\mathbf{i}) = \sum_j m^j_i e_j(s_0).$$

$$\bar{\mathbf{e}}(s_0) = M \cdot \bar{\mathbf{e}}(s_0 + 2\pi\mathbf{i}) \Longleftrightarrow e^i(s_0) = \sum_j m^i_j e^j(s_0 + 2\pi\mathbf{i}).$$

Given $\omega \in \Omega^{N+1}(X)$ we define the row vector

$$\vec{I}_\omega = [I_{\omega,1}(s), \cdots, I_{\omega,\mu}(s)], \;\; I_{\omega,k} = \int_{e_k(s)} \mathbf{Res}_f\,\omega.$$

Note that

$$\vec{I}_\omega(s + 2\pi\mathbf{i}) = \vec{I}_\omega(s) \cdot M.$$

If we pick μ-forms $\omega_1, \cdots, \omega_\mu \in \Omega^{n+1}(X)$ we can form the $\mu \times \mu$ *period matrix*

$$P(s) = \begin{bmatrix} \vec{I}_{\omega_1}(s) \\ \vdots \\ \vec{I}_{\omega_\mu}(s) \end{bmatrix}.$$

It satisfies

$$P(s + 2\pi\mathbf{i}) = P(s) \cdot M.$$

Since $M \in \mathrm{GL}_\mu(\mathbb{Z})$ we deduce that

$$\det P(s + 2\pi\mathbf{i}) = \pm \det P(s).$$

Thus $\det P(s)^2$ is a well defined meromorphic function $t \mapsto \delta(t; \omega_1, \cdots, \omega_\mu)$ on $\mathbb{D}$ with a possible pole at $t = 0$. We denote by $\nu(\omega_1, \cdots, \omega_\mu) \in \frac{1}{2}\mathbb{Z}$ its order at $t = 0$ divided by 2.

Theorem 3.2 (Varchenko)**.**

$$\nu(\omega_1, \cdots, \omega_\mu) \geq \max\Big\{ \frac{N-1}{2}\mu, \; \sum_{j=1}^{\mu} \nu(\omega_j) \Big\}$$

with equality for a generic choice of $\{\omega_1, \cdots, \omega_\mu\}$*. In such a generic case we also have the equality*

$$\nu(\omega_1, \cdots, \omega_\mu) = \frac{N-1}{2}\mu = \sum_{j=1}^{\mu} \nu(\omega_j).$$

We will refer to such a generic choice as a $\mathbb{C}\{t\}$*-basis and we will use the notation* $\underline{\omega}$ *to denote an ordered* $\mathbb{C}\{t\}$*-basis.* □

We define a *rational divisor* on $\mathbb{R}$ to be a finite formal linear combination of the form

$$\sum_{q\in\mathbb{Q}} n_q \cdot (q), \quad n_q \in \mathbb{Z}, \quad n_q = 0 \text{ for all but finitely may } q\text{'s}.$$

In other words, a rational divisor is an element

$$\mathbb{Z}^{(\mathbb{Q})} = \text{functions } f : \mathbb{Q} \to \mathbb{Z} \text{ with finite support}.$$

For a rational number q we denote by $(q) \in \mathbb{Z}^{(\mathbb{Q})}$ the Dirac function supported at q. For any function $f : \mathbb{Q} \to \mathbb{Q}$ with finite fibers and any divisor $D \in \mathbb{Z}^{(\mathbb{Q})}$ we define

$$f^*D = \sum_{r\in\mathbb{Q}} n_{f(r)}(f(r)) = \sum_{q\in\mathbb{Q}} \sum_{f(r)=q} n_q(q).$$

A divisor will be called invariant with respect to f if $D = f^*D$.

Given a $\mathbb{C}\{t\}$-basis $\underline{\omega} = (\omega_1, \cdots, \omega_1)$ we set

$$(\underline{\omega}) = \sum_{i=1}^{\mu} \big(\nu(\omega_i) \big).$$

Following Steenbrink and Varchenko, we define for every $\alpha \in \Lambda^f$ the subsheaf $\mathcal{S}_\alpha$ of $\mathcal{H}^f_{hol}$ spanned over $\mathcal{O}_{\mathbb{D}^*}$ by the principal parts of the geometric sections of order α. One can show that each of them is a locally free sheaf and defines a sub-bundle of $\mathcal{H}^f$. The multiplication by t defines an inclusion

$$\mathcal{S}_{\alpha-1} \hookrightarrow \mathcal{S}_\alpha.$$

Note that $\mathcal{S}_\alpha = 0$ for all $\alpha \leq -1$. It is a highly nontrivial fact that $\mathcal{S}_N = \mathcal{H}^f_{hol}$.

The *spectrum* of f is the divisor $\mathrm{sp}\,(f) \in \mathbb{Z}^{(\mathbb{Q})}$ defined by

$$\mathrm{sp}\,(f) = \sum_{\alpha \in \Lambda^f_{-1}} \big(\dim_{\mathbb{C}} \mathcal{S}_\alpha / t \cdot \mathcal{S}_{\alpha-1} \big) \cdot (\alpha).$$

If we write

$$\mathrm{sp}\,(f) = \sum_{\alpha \in \Lambda^f_{-1}} n_\alpha \cdot (\alpha)$$

then the numbers α such that $n_\alpha \neq 0$ are called the *spectral numbers of* f. The integer n_α is called the multiplicity of α (in the spectrum of f). Since $\mathcal{S}_N = \mathcal{H}^f_{hol}$ we deduce

$$n_\alpha = 0, \;\; \forall \alpha \geq N.$$

Theorem 3.3 (Varchenko)**.** *Suppose* $f = f(z_0, \cdots, z_N) \in \mathcal{O}_{N+1}$*. Then the spectrum* $\mathrm{sp}\,(f)$ *is well defined, i.e. it is indeed a rational divisor supported inside the interval* $(-1, N)$*. Moreover, for any* $\mathbb{C}\{t\}$*-basis* $\underline{\omega}$ *of* f *we have the equality*

$$\mathrm{sp}\,(f) = (\underline{\omega}).$$

and $\mathrm{sp}\,(f)$ *is invariant with respect to the reflection in the midpoint of* $[-1, N]$*.* □

To every divisor $D = \sum_q n_q(q) \in \mathbb{Z}^{(\mathbb{Q})}$ we associate the Laurent-Puiseux polynomial

$$S_D(T) = \sum_q n_q T^q.$$

Note that the polynomial S_D completely determines the divisor D. When $D = \mathrm{sp}\,(f)$ we set

$$S_f(T) := S_{\mathrm{sp}\,(f)}(T).$$

We will refer to $S_f(T)$ as the *spectral polynomial of* f.

Theorem 3.4 (Varchenko)**.**

$$S_{f*g}(T) = T \cdot S_f(T) \cdot S_f(T).$$

□

Remark 3.5. If, following Saito, we define

$$\tilde{S}_f(T) = T S_f(T)$$

then the last equality has the more natural form

$$\tilde{S}_{f*g}(T) = \tilde{S}_f(T) \cdot \tilde{S}_g(T).$$

□

Example 3.6. Consider again the function $f(z) = z^n$ discussed in Example 3.1 so that

$$N = 0,\;\; \mu = n-1,\;\; \frac{N-1}{2}\mu = -\frac{n-1}{2}.$$

Then the period matrix is given by

$$P_k^m(t) = \int_{e_k(t)} \omega_m = \frac{1}{n}(\zeta^{km})t^{\frac{(m-n)}{n}}\big(1-\zeta^{-m}\big).$$

and we have

$$\det P(t) = \frac{1}{n^{n-1}}\Big(\prod_{m=1}^{n-1} t^{\frac{(m-n)}{n}}\big(1-\zeta^{-m}\big)\Big)\cdot \det[\zeta^{km}]_{1\le k,m\le n-1}.$$

The last determinant is a Vandermonde determinant and it is non zero. Hence the order of $\det P(t)$ at zero is

$$\sum_{m=1}^{n-1}\Big(\frac{m}{n}-1\Big) = -\frac{n-1}{2} = \frac{N-1}{2}\mu.$$

Thus the collection $\{z^m dz\}_{1\le m\le n-1}$ is a basis and we deduce

$$S_{z^n}(T) = \sum_{m=1}^{n-1} T^{\frac{m}{n}-1} = T^{-1}\sum_{m=1}^{n-1} T^{m/n} = T^{-1}\frac{T^{\frac{1}{n}}-T}{1-T^{\frac{1}{n}}}$$

Using Theorem 3.4 we deduce that for a Brieskorn singularity $f_{a_0,\cdots,a_N} = z_0^{a_0}+\cdots+z_N^{a_N}$ we have

$$S_{f_{a_0,\cdots,a_N}}(T) = T^{-1}\prod_{j=0}^{N}\frac{T^{1/a_j}-T}{1-T^{1/a_j}}.$$

More generally, if f is a quasihomogeneous function with weights $w_0,\cdots,w_N$ then

$$S_f = T^{-1}\prod_{j=0}^{N}\frac{T^{w_j}-T}{1-T^{w_j}}.$$

In particular, the D_4 singularity is quasihomogeneous with weights $(1/3,/1/3,1/2)$ and we have

$$S_{D_4}(T) = T^{-1}\left(\frac{T^{1/3}-T}{1-T^{1/3}}\right)^2\frac{T^{1/2}-T}{1-T^{1/2}} = T^{1/6}(1+T^{1/3})^2 = T^{1/6}+2T^{1/2}+T^{5/6}.$$

□

The *geometric genus* of the isolated singularity defined by $f \in \mathcal{O}_{N+1}$ is the number of nonpositive spectral numbers of f counted with their multiplicities. In terms of a $\mathbb{C}\{t\}$-basis $\underline{\omega} = \{\omega_1,\cdots,\omega_\mu\}$ of f, the geometric genus is the number of ω_j's with the property that there exists a locally constant vanishing cycle c_t such that the integral of ω_j along c_t does not converge to zero as $t \to 0$ inside an angular sector. We denote the geometric genus by $p_g(f,0)$. For example $p_g(z^n,0) = n-1$, $p_g(D_4,0) = 0$.

For generic f's the geometric genus can be given a combinatorial description, similar in spirit to the above description of $p_g(z^n,0)$.

Let $f = f(z_0, \cdots, z_N)$. Set $L := \mathbb{Z}^{N+1}$, $L^+ := \mathbb{Z}^{n+1}_{\geq 0}$, $L_{\mathbb{R}} = L \otimes \mathbb{R}$. For $\alpha \in L$ we set $\vec{z}^{\alpha} := z_0^{\alpha_0} \cdots z_N^{\alpha_N}$. We can write

$$f = \sum_{\alpha \in L^+} f_{\alpha} \vec{z}^{\alpha}.$$

We set

$$\operatorname{supp} f = \{ \alpha \in L_+;\ \ f_{\alpha} \neq 0 \}.$$

The *(local) Newton polyhedron* of f, denoted by $\Gamma_+(f)$ is the convex hull of $\operatorname{supp}(f) + L^+$. The germ f is called *convenient* if its Newton polyhedron intersects all the coordinate axes of $L_{\mathbb{R}}$. Equivalently, this means that for every $j = 0, \cdots, N$, there exists $n_j \in \mathbb{N}$ such that the monomial $z_n^{n_j}$ enters into the Taylor expansion of f. We can assume without a loss of generality that f is a convenient *polynomial*. Indeed, according to Mather-Tougeron theorem, the analytic type of the singularity described by f does not change if we modify arbitrarily the terms in the Taylor expression of degree $> \mu + 1$. In particular, we can replace f by $j_{\mu+1}(f) + \sum_{j=0}^{N} z_j^{\mu+2}$ and not change the analytic type of the singularity.

The Newton polyhedron is the intersection of finitely many half-spaces. Its boundary has compact and noncompact faces. The *Newton diagram* of f, denoted by $\Delta(f)$, is the union of all the compact faces. These are compact polyhedra of dimensions $\leq N$. For each face γ of the Newton diagram we set

$$f_{\gamma} = \sum_{\alpha \in \gamma} f_{\alpha} \vec{z}^{\alpha}.$$

The polynomial f is called *Newton nondegenerate* if for every face γ of $\Delta(f)$ the polynomials

$$\frac{\partial f_{\gamma}}{\partial z_j},\ \ j = 0, 1, \cdots, N$$

have no common zero on $(\mathbb{C}^*)^{N+1}$. This condition is generic in the space of convenient polynomials with a fixed Newton polyhedron.

Let $\vec{w}_0 = (1, \cdots, 1)$. A monomial $\vec{z}^{\alpha}$ is called *subdiagramatic* if $\alpha + \vec{w}_0$ does not lie in the interior of the Newton polyhedron.

Theorem 3.7 (Khovanski-Varchenko-Saito)**.** *Suppose $f \in \mathcal{O}_{N+1}$ is a Newton nondegenerate convenient polynomial. Then $p_g(f, 0)$ is equal to the number of subdiagramatic monomials.* □

Example 3.8. Consider the singularity D_4. The defining polynomial $x^2y - y^3 + z^2$ is not convenient, but near 0 it is right equivalent to $cx^6 + x^2y - y^3 + z^2$, where c is a complex number. The Newton diagram of this polynomial is depicted in Figure 2. It consists of 0- dimensional, 1-dimensional and 2-dimensional faces. The 2-dimensional faces are the triangles ACD and BCD. The 1-dimensional faces are the edges of these triangles and the 0-dimensional faces are the vertices of these triangles. We have

$$f_{ACD} = cx^6 + x^2y + z^2,\ \ \frac{\partial f_{ACD}}{\partial x} = 6cx^5 + xy,\ \ \frac{\partial f_{ACD}}{\partial y} = x^2,\ \ \frac{\partial f_{ACD}}{\partial z} = 2z.$$

$$f_{BCD} = y^3 + x^2y + z^2,\ \ \frac{\partial f_{BCD}}{\partial x} = 2xy,\ \ \frac{\partial f_{BCD}}{\partial y} = 3y^2 + x^2,\ \ \frac{\partial f_{BCD}}{\partial z} = 2z.$$

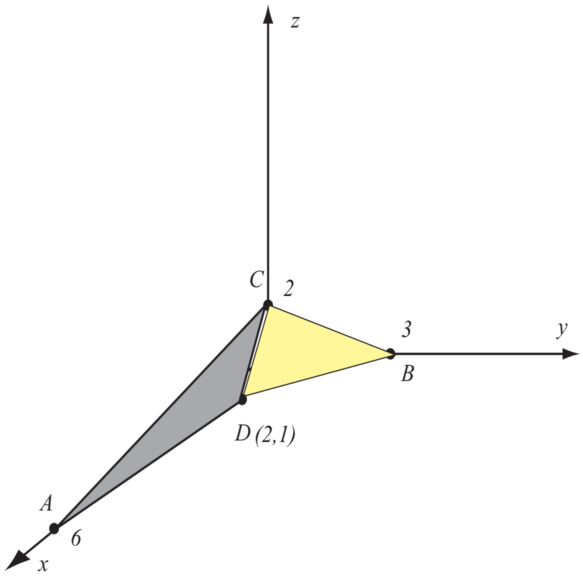

FIGURE 2. *The Newton diagram of* $cx^6 + x^2y - y^3 + z^2$.

etc. One can check that for $c \neq 0$ this is Newton nondegenerate. The two top dimensional faces of the Newton diagram are contained in the planes

$$ACD \subset \{ \underbrace{\frac{1}{6}x + \frac{2}{3}y + \frac{1}{2}z}_{:=\ell_1(x,y,z)=1} \}, \;\; BCD \subset \{ \underbrace{\frac{1}{3}x + \frac{1}{3}y + \frac{1}{2}z}_{:=\ell_2(x,y,z)} = 1 \}$$

the Newton polyhedron is defined by

$$\ell_1(x,y,z) \geq 1 \;\text{ and }\; \ell_2(x,y,z) \geq 1.$$

A subdiagramatic monomial $x^m y^n z^p$ satisfies

$$\ell_1(m,n,p) + \ell_1(1,1,1) \leq 1 \;\text{ or }\; \ell_2(m,n,p) + \ell_2(1,1,1) \leq 1.$$

Equivalently this means

$$\frac{m}{6} + \frac{2n}{3} + \frac{p}{2} + \frac{4}{3} \leq 1 \;\text{ or }\; \frac{m}{3} + \frac{n}{3} + \frac{p}{2} + \frac{7}{6} \leq 1, \;\; m,n,p \geq 0.$$

Clearly there are no such monomials so that $p_g(D_4, 0) = 0$ as expected.

□

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