

AN INVITATION TO TAME GEOMETRY

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1. GOALS

- Describe large categories of nice spaces and maps.
- Describe some of the nice properties of these nice spaces and maps.
- Describe nice applications of these nice spaces and maps.

2. REASONABLE CATEGORIES OF SPACES

The spaces belonging to reasonable category \mathcal{S} should be subsets of some Euclidean space so that

$$\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}^n, \quad \mathcal{S}^n = \text{the collection of spaces in } \mathcal{S} \text{ which are subsets of } \mathbb{R}^n.$$

Via the inclusions $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ we can regard \mathcal{S}^n as a subcollection of \mathcal{S}^{n+1} . A reasonable category should satisfy the following requirements.

E_1 . The collection \mathcal{S}^n contains all the real algebraic subsets of \mathbb{R}^n , i.e., the subsets described by finitely many polynomial equations.

E_2 . The collection \mathcal{S}^n contains all the closed affine halfspaces.

P_1 . The collection \mathcal{S}^n is closed under all the boolean operations \cup, \cap, \setminus , i.e.

$$A, B \in \mathcal{S}^n \implies A \cup B, A \cap B, A \setminus B \in \mathcal{S}^n.$$

P_2 . If $A \in \mathcal{S}^m$ and $B \in \mathcal{S}^n$ then $A \times B \in \mathcal{S}^{m+n}$.

P_3 . If $A \in \mathcal{S}^m$ and $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is an affine map, then $T(A) \in \mathcal{S}^n$.

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M. If $A, B \in \mathcal{S}$ then an \mathcal{S} -morphism $A \rightarrow B$ is a map $f : A \rightarrow B$ such that its graph $\Gamma_f \subset A \times B$ belongs to \mathcal{S} .

We will refer to the sets in \mathcal{S} as \mathcal{S} -*definable* or *constructible sets*, and to the morphisms as *definable* or *constructible maps*.

3. CONNECTION WITH HUMAN LANGUAGE

The boolean operations \cup, \cap, \setminus correspond to the logical operators AND, OR, NOT = \wedge, \vee, \neg . The projection $\pi : A \times B \rightarrow B$ corresponds to the existential quantifier \exists . For example, if $Z \subset A \times B$, then $\pi(Z)$ can be described as

$$Z = \{a \in A; \exists b \in B; (a, b) \in Z\}.$$

Note that the universal quantifier can be expressed as a composition $\neg\exists\neg$. We obtain the following *metaprinciple*

If \mathcal{S} is a reasonable category and a set A is defined by a statement involving only the basic logic operators and \mathcal{S} -definable sets, then A is also \mathcal{S} definable.

Example 3.1. Suppose that \mathcal{S} is a reasonable category, $f : A \rightarrow B$ is \mathcal{S} -definable, and $S \subset B$ is also definable. Then

$$f^{-1}(S) := \{a \in A; \exists s \in S; (a, s) \in \Gamma_f\}$$

is definable. Note that if $a \in \mathbb{R}^n$, and ε_0 then the map

$$F : \mathbb{R}^n \rightarrow \mathbb{R}, \quad F(x) = |x - a|^2 - \varepsilon^2$$

is definable since it is a polynomial. The set $S = (0, \infty)$ is definable because it is the complement of a closed halfspace and thus $F^{-1}(S)$ is definable. Note that this set is precisely the open Euclidean ball $B_a(\varepsilon)$ of center a and radius ε . \square

Example 3.2. Suppose $A \in \mathcal{S}^m, B \in \mathcal{S}^n$. Consider the set

$$A_0 := \{a \in A; (a, b) \in A \times B, \forall b \in B\}.$$

Then

$$A \setminus A_0 = \{a \in A; \exists b \in B; (a, b) \notin A \times B\} = \pi(\mathbb{R}^{m+n} \setminus (A \times B)),$$

where $\pi : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ denotes the natural projection. This shows that A_0 is \mathcal{S} -definable. \square

Example 3.3. Suppose \mathcal{S} is a reasonable category, and $A \in \mathcal{S}^n$. We denote by $\mathbf{cl}(A)$ its closure in \mathbb{R}^n . Observe that

$$\mathbf{cl}(A) = \{x \in \mathbb{R}^n; \forall \varepsilon \in (0, \infty), \exists a \in A \cap B_a(\varepsilon)\}.$$

We can rewrite the above description by saying that $\mathbf{cl}(A)$ consists of all $x \in \mathbb{R}^n$ such that the following statement is true

$$\forall \varepsilon \left(\varepsilon > 0 \Rightarrow \exists a (a \in A) \wedge (|x - a| < \varepsilon) \right).$$

This shows that $\mathbf{cl}(A)$ is \mathcal{S} -definable because the logical operator \Rightarrow can also be rewritten as $\forall \neg$. \square

4. EXAMPLES OF REASONABLE CATEGORIES

Example 4.1 (Semialgebraic sets). Consider the collection \mathcal{S}_{alg} consisting of *semialgebraic* subsets. More precisely

$$A \in \mathcal{S}_{\text{alg}}^n \iff A = \bigcup_{k=1}^N A_k,$$

where for every $k = 1, \dots, N$ the set A_k described by finitely many polynomial inequalities.

A theorem of Tarski-Seidenberg (1950's) states that \mathcal{S}_{alg} is a reasonable category, and in fact, it is the *smallest* reasonable category. Observe that every set $A \in \mathcal{S}_{\text{alg}}^1$ is a finite union of intervals (possibly of infinite or zero length). \square

Example 4.2. (a) Suppose that \mathcal{S} is a reasonable category, and \mathcal{A}^k is a collection of subsets of \mathbb{R}^k . We set $\mathcal{A} := \cup_k \mathcal{A}^k$, and we denote by $\mathcal{S}[\mathcal{A}]$ the smallest reasonable category containing \mathcal{S} and all the collections \mathcal{A}^k . We say that $\mathcal{S}[\mathcal{A}]$ is the category obtained from \mathcal{S} by adjoining the collection \mathcal{A} .

(b) We denote by \mathcal{S}_{an} the category obtained from \mathcal{S}_{alg} by adjoining the graphs of real analytic functions

$$f : [0, 1]^n \rightarrow \mathbb{R}.$$

The sets obtained in this fashion are called *subanalytic sets* and first appeared in the works of A. Gabrielov, R. Hardt and H. Hironaka in late 60s and early 70s.

(c) We denote by $\widehat{\mathcal{S}}_{\text{an}}$ the smallest reasonable category \mathcal{S} containing \mathcal{S}_{an} , and satisfying the property:

If $f : (0, 1) \rightarrow \mathbb{R}$ is C^1 and \mathcal{S} -definable then so are its antiderivatives.

Note that $\mathcal{S}_{\text{exp}} \subset \widehat{\mathcal{S}}_{\text{an}}$ because $\log t$ is an antiderivative of $1/t$ so $\log t$ is $\widehat{\mathcal{S}}_{\text{an}}$ -definable, and e^t is the inverse of $\log t$ and thus it is also $\widehat{\mathcal{S}}_{\text{an}}$ -definable. \square

5. TAME CATEGORIES

A reasonable category \mathcal{S} is called *tame* or *o-minimal* (order minimal) if it satisfies the condition

T. Any set $A \subset \mathcal{S}^1$ is a finite union of intervals.

Example 5.1. (a) The Tarski-Seidenberg theorem in the 50s implies that the category \mathcal{S}_{alg} is tame.

(b) Work of Gabrielov, Hardt, Hironaka in the 70s implies that \mathcal{S}_{an} is tame.

(c) Work of Khovanski and Wilkie in the 90s implies that $\widehat{\mathcal{S}}_{\text{an}}$ is tame. \square

In the sequel we will refer interchangeably to the spaces in $\widehat{\mathcal{S}}_{\text{an}}$ as tame, or definable or, constructible. Let us point out that any compact real analytic manifold is a tame set.

6. PROPERTIES OF TAME SETS AND MAPS

We list some nice properties of tame sets and maps. For proofs we refer to [1, 2].

(1) If $f : (0, 1) \rightarrow \mathbb{R}$ is a tame map (not necessarily continuous, then for any positive integer p there exists a partition

$$0 = a_0 < a_1 < \dots < a_N = 1, \quad N = N(p)$$

such that the restriction of f to every subinterval (a_k, a_{k+1}) is of class C^p and weakly monotone.

- (2) Suppose $A, B \subset \mathbb{R}^N$ are compact tame sets and $f : A \rightarrow B$ is a tame map. Then f is continuous if and only if its graph is a closed subset of $A \times B$.
- (3) Any tame set A is a disjoint union of finitely many real analytic subsets

$$A = \bigsqcup_{k=1}^N S_k. \quad (6.1)$$

If we define $\dim A = \max \dim S_k$ then the dimension of A is independent of the choice of stratification (6.1). Moreover

$$\dim A > \dim(\mathbf{cl}(A) \setminus A).$$

- (4) If $f : A \rightarrow B$ is a continuous map then there exists a finite partition of B into finitely many real analytic manifolds $B = \bigsqcup_{k=1}^N B_k$ such that each of the maps $f : f^{-1}(B_k) \rightarrow B_k$ is a tamely trivial fiber bundle. In particular, there can be only finitely many topological types amongst the fibers of f . (See Fig 1.)

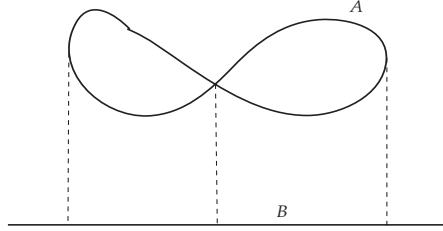


FIGURE 1. A piecewise fibration.

- (5) Any tame set A can be triangulated which means that there exists pair $(F, \{\Delta_i\}_{i \in I})$ where F is a tame homeomorphism from A to a tame subset M of some Euclidean space \mathbb{R}^N and $\{\Delta_i\}_{i \in I}$ is a finite family of mutually disjoint affine open simplices in \mathbb{R}^N , of various dimensions, such that

$$M = \bigcup_{i \in I} \Delta_i$$

and for every $i, j \in I$ the intersection $\mathbf{cl}(\Delta_i) \cap \mathbf{cl}(\Delta_j)$ is either empty, or it is a common face of $\mathbf{cl}(\Delta_i)$ and $\mathbf{cl}(\Delta_j)$.

7. EULER CHARACTERISTIC

Suppose A is a tame set. For any triangulation $\mathcal{T} = (F, \{\Delta_i\}_{i \in I})$. we set

$$\chi(\mathcal{T}) = \sum_{i \in I} (-1)^{\dim \Delta_i}$$

The integer $\chi(\mathcal{T})$ is independent of the triangulation \mathcal{T} , and it is called the *Euler characteristic* of A . If A is a tame locally closed subset of \mathbb{R}^n then

$$\chi(A) = \sum_{k \geq 0} H_c^k(A, \mathbb{R}),$$

where H_c^\bullet denotes the cohomology with compact supports. Equivalently, $\chi(A)$ is the Euler characteristic of the Borel-Moore homology of A .

We say that two tame sets A and B are *scissor equivalent* if there exists a tame, but not necessarily continuous, bijection $f : A \rightarrow B$. We have the following fundamental result of Lou van der Dries.

Scissor Principle. *Two tame sets A and B are scissor equivalent if and only if they have the same dimension and Euler characteristic.*

Note that the scissor principle implies that if two tame sets are tamely homeomorphic then they have the same Euler characteristic.

8. “MOTIVIC” INTEGRATION

For every tame set X we denote by \mathcal{T}_X the collection of tame subsets of X . Suppose G is an Abelian group. A G -valuation on X is a map $\mu : \mathcal{T}_X \rightarrow G$ such that if A, B are two tame subsets of X then

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B), \quad \forall A, B \in \mathcal{T}_X.$$

Example 8.1. The Euler characteristic is a \mathbb{Z} -valuation. □

Example 8.2. For every $A \in \mathcal{T}_X$ we denote by $\mathbb{I}_A : X \rightarrow \mathbb{Z}$ the characteristic function of A

$$\mathbb{I}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

We denote by \mathcal{C}_X the Abelian subgroup of the additive group $\text{Map}(X, \mathbb{Z})$ generated by the characteristic functions of tame subsets. The functions in \mathcal{C}_X are called *constructible*. Note that $f : X \rightarrow \mathbb{Z}$ is constructible if and only if its range is finite, and for every $n \in \mathbb{Z}$ the level set $f^{-1}(n)$ is tame. We have

$$f = \sum_{n \in \mathbb{Z}} n \mathbb{I}_{f^{-1}(n)}.$$

From the equality

$$\mathbb{I}_{A \cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_{A \cap B}$$

we deduce that the map

$$\mathbb{I} : \mathcal{T}_X \rightarrow \mathcal{C}_X, \quad A \mapsto \mathbb{I}_A$$

is a \mathcal{C}_X -valuation on X called the *universal valuation* on \mathcal{T}_X . Note that any morphism of groups $\Phi : \mathcal{C}_X \rightarrow G$ defines a G -valuation φ on X given by

$$\varphi(A) = \Phi(\mathbb{I}_A), \quad \forall A \in \mathcal{T}_X. \quad \square$$

We have the following fundamental theorem,[4].

Groemer Extension Theorem. For every G -valuation φ on X there exists a unique morphism of Abelian groups $\Phi : \mathcal{C}_X \rightarrow G$ such that

$$\varphi(A) = \Phi(\mathbb{I}_A).$$

The morphism Φ extending the valuation φ is called the *integral with respect to the valuation* φ , and for every $f \in \mathcal{C}_X$ we set

$$\int f d\varphi = \int_X f d\varphi := \Phi(f).$$

We deduce that the Euler characteristic defines a linear map $\mathcal{C}_X \rightarrow \mathbb{Z}$ called the *integral with respect to the Euler characteristic*.

Let us point out that the construction $X \mapsto \mathcal{C}_X$ is bi-functorial.

Any tame map $\pi : X \rightarrow Y$ induces a pullback morphism

$$\pi^* : \mathcal{C}_Y \rightarrow \mathcal{C}_X, \quad \mathcal{C}_Y \ni f \mapsto \pi^* f := f \circ \pi \in \mathcal{C}_X.$$

Suppose now that $\pi : X \rightarrow Y$ is a tame continuous map. For every $y \in Y$ we set $A_y := A \cap \pi^{-1}(y)$. We define a map

$$\pi_* : \mathcal{T}_X \rightarrow \mathcal{C}_Y, \quad \mathcal{T}_X \ni A \mapsto \pi_*(A),$$

$$\pi_*(A) \in \mathcal{C}_Y, \quad \pi_*(A)(y) = \chi(A_y) = \int_X \mathbb{I}_{\pi^{-1}(y)} \mathbb{I}_A d\chi = \int_{\pi^{-1}(y)} \mathbb{I}_A.$$

Since $\pi|_A$ is a piecewise fibration we deduce that the function $\pi_*(A)$ is indeed constructible. Note also that if A, B are two disjoint tame subsets of X then the sets A_y and B_y are disjoint for every y so that

$$\chi(A_y \cup B_y) = \chi(A_y) + \chi(B_y)$$

which shows that the map $\mathcal{T}_X \rightarrow \mathcal{C}_Y$ is a \mathcal{C}_Y -valuation. We obtain in this fashion a morphism of Abelian groups $\pi_* : \mathcal{C}_Y \rightarrow \mathcal{C}_X$ called the *integration along fibers*. For $f \in \mathcal{C}_X$ we have

$$\pi_* f(y) = \int_X \mathbb{I}_{\pi^{-1}(y)} f d\chi = \int_{\pi^{-1}(y)} f d\chi.$$

The operations π^* and π_* satisfy several desirable properties first formulated by Grothendieck while working with coherent sheaves.

Functoriality. If $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ are tame continuous maps then

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*, \quad (\beta \circ \alpha)^* = \alpha^* \circ \beta^*.$$

Projection formula. If $\alpha : X \rightarrow Y$ is a tame continuous map $f \in \mathcal{C}_X$ and $g \in \mathcal{C}_Y$ then

$$\alpha_*(f \cdot \alpha^*(g)) = \alpha_*(f) \cdot g.$$

Base change formula. If $X \xrightarrow{\rho} S$ and $T \xrightarrow{\beta} S$ are tame continuous maps and we define

$$T \times_S X := \{ (t, x) \in T \times X; \beta(t) = \rho(x) \},$$

then we have a commutative (cartesian) diagram

$$\begin{array}{ccc} T \times_S X & \xrightarrow{\pi_X} & X \\ \pi_T \downarrow & & \downarrow \rho \\ T & \xrightarrow{\beta} & S \end{array}$$

and

$$\beta^* \circ \rho_* = (\pi_T)_* \circ \pi_X^*.$$

9. INTEGRAL KERNELS AND TRANSFORMS

Suppose X and Y are tame sets. An *integral kernel* from X to Y is a function $\mathbf{K} \in \mathcal{C}_{Y \times X}$. Given such a kernel we define a linear map

$$\mathcal{J}_{\mathbf{K}} : \mathcal{C}_X \rightarrow \mathcal{C}_Y, \quad \mathcal{C}_X \ni f \mapsto (\pi_Y)_*(\pi_X^*(f) \cdot \mathbf{K}) \in \mathcal{C}_Y$$

where $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are the natural projections. The linear map $\mathcal{J}_{\mathbf{K}}$ is called the *integral transform defined by the kernel \mathbf{K}* . We will represent it as a “roof”

$$\begin{array}{ccc} & (Y \times X, \mathbf{K}) & \\ \pi_Y \swarrow & & \searrow \pi_X \\ Y & & X \end{array}$$

More intuitively, for any $f \in \mathcal{C}_X$, the integral transform $\mathcal{J}_{\mathbf{K}}f$ is a constructible function on Y such that

$$\mathcal{J}_{\mathbf{K}}f(y) = \int_X \mathbf{K}(y, x)f(x)d\chi(x).$$

The following result follows rather easily from the properties of the pushforward and the pullback.

Composition formula. If S_0, S_1, S_2 are tame sets, $\mathbf{K}_{10} \in \mathcal{C}_{S_1 \times S_0}$, $\mathbf{K}_{21} \in \mathcal{C}_{S_2 \times S_1}$, then

$$\mathcal{J}_{\mathbf{K}_{21}} \circ \mathcal{J}_{\mathbf{K}_{10}} = \mathcal{J}_{\mathbf{K}_{21} * \mathbf{K}_{10}}$$

where

$$\mathbf{K}_{21} * \mathbf{K}_{10}(s_2, s_0) = \int_{S_1} \mathbf{K}_{21}(s_2, s_1)\mathbf{K}_{10}(s_1, s_0)d\chi(s_1).$$

More rigorously $\mathbf{K}_{20} = \mathbf{K}_{21} * \mathbf{K}_{10}$ is given by the equality

$$\mathbf{K}_{20} = \pi_*(\ell_{21}^* \mathbf{K}_{21} \cdot r_{10}^* \mathbf{K}_{10}),$$

where

$$\begin{array}{ccccc} & (S_2 \times S_1) \times_{S_1} (S_1 \times S_0) & & & \\ & \ell_{21} \swarrow & \pi \downarrow & \searrow r_{10} & \\ S_2 \times S_1, \mathbf{K}_{21} & & S_2 \times S_0, \mathbf{K}_{20} & & S_1 \times S_0, \mathbf{K}_{10} \\ \ell_2 \downarrow & \swarrow r_1 & & \swarrow \ell_1 & \downarrow r_0 \\ & S_2 & & S_1 & & S_0 \end{array}$$

10. TOPOLOGICAL TOMOGRAPHY

Denote by $\mathbf{Graf}^1(\mathbb{R}^n)$ the Grassmannian of affine hyperplanes in \mathbb{R}^n . This is a constructible set. Note that the lines in \mathbb{R}^2 are hyperplanes in \mathbb{R}^2 .

The classical Radon transform associates to a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ a function $\mathcal{R}f$ on $\mathbf{Graf}^1(\mathbb{R}^1)$ such that the value of $\mathcal{R}f$ on the line L is equal to the integral of f along the line L . The classical Radon inversion formula allows the reconstruction of f from its Radon transform. In particular, if f is the characteristic function of a bounded open set $\Omega \subset \mathbb{R}^2$, then we can completely reconstruct Ω if we know the length of the intersection of any line L with the region Ω . We want to show that if Ω is a tame open set, then we can completely

reconstruct Ω if we know only the number of connected components of the intersection of Ω with any line.

We have a natural constructible set

$$A = \{ (H, x) \in \mathbf{Graff}^1(\mathbb{R}^n) \times \mathbb{R}^n; H \ni x \}$$

We regard the characteristic function of A as a kernel from \mathbb{R}^n to $\mathbf{Graff}^1(\mathbb{R}^n)$, and we denote by \mathcal{R}_n the associated integral transform. Let us compute $\mathcal{R}_n(\mathbb{I}_S)$, where $S \subset \mathbb{R}^n$ is a tame set.

We look at the integral transform given by the roof

$$\begin{array}{ccc} & (\mathbf{Graff}^1(\mathbb{R}^n) \times \mathbb{R}^n, \mathbb{I}_A) & \\ \lambda \swarrow & & \searrow \rho \\ \mathbf{Graff}^1(\mathbb{R}^n) & & \mathbb{R}^n \end{array}$$

Note that $\rho^*(\mathbb{I}_S)$ is the characteristic function of the set

$$A_S = \{ (H, x) \in \mathbf{Graff}^1(\mathbb{R}^n) \times \mathbb{R}^n; x \in S \cap H \}$$

Then, for any $H_0 \in \mathbf{Graff}^1(\mathbb{R}^n)$ we have a homeomorphism

$$\lambda^{-1}(H_0) \cap A_S \ni (H_0, x) \mapsto x \in H_0 \cap S.$$

We deduce that

$$\lambda_*(\mathbb{I}_{A_S})(H_0) = \chi(H_0 \cap S).$$

Hence

$$\mathcal{R}_n(\mathbb{I}_S) : \mathbf{Graff}^1(\mathbb{R}^n) \rightarrow \mathbb{Z}$$

is given by

$$\mathcal{R}_n(\mathbb{I}_A) = \chi(H \cap S) = \int_{\mathbb{R}^n} \mathbb{I}_{H \cap S} d\chi,$$

so that \mathcal{R}_n is a topological version of the Radon transform. More generally, for every hyperplane H we have

$$\mathcal{R}_n f(H) = \int_{\mathbb{R}^n} \mathbb{I}_H f d\chi.$$

Consider now the dual set

$$A^\dagger := \{ (x, H) \in \mathbb{R}^n \times \mathbf{Graff}^1(\mathbb{R}^n); x \in H, \},$$

and denote by $\mathcal{R}_n^\dagger : \mathcal{C}_{\mathbf{Graff}^1(\mathbb{R}^n)} \rightarrow \mathcal{C}_{\mathbb{R}^n}$ the integral transform defined by the kernel \mathbb{I}_{A^\dagger} . We have the following result due to A. Khovaskii and P. Schapira.

Inversion Formula. For any $f \in \mathcal{C}_{\mathbb{R}^n}$ We have

$$\mathcal{R}_n^\dagger \circ \mathcal{R}_n(f) = (-1)^{n+1} f + \frac{1 + (-1)^n}{2} \left(\int_{\mathbb{R}^n} f d\chi \right) \mathbb{I}_{\mathbb{R}^n}$$

We can express $\int f d\chi$ in terms of the Radon transform $\mathcal{R}_n f$ as follows. Consider the linear map

$$\pi : \mathbb{R}^n \rightarrow \mathbb{R}, \quad (x^1, \dots, x^n) \mapsto x^1.$$

The fiber of π over $t \in \mathbb{R}$ is the hyperplane H_t given by the equation $x^1 = t$.

Denote by c_k the constant map $\mathbb{R}^k \rightarrow \{*\}$. Note that $\mathcal{C}_* = \mathbb{Z}$

$$(c_k)_*(f) = \int_{\mathbb{R}^k} f d\chi, \quad \forall f \in \mathcal{C}_{\mathbb{R}^k}.$$

On the other hand,

$$\int_{\mathbb{R}^n} f d\chi = (c_n)_*(f) = (c_1)_* \circ \pi_*(f) = \int_{\mathbb{R}} (\pi_* f)(t) d\chi(t).$$

Now observe that

$$\pi_* f(t) = \int_{\mathbb{R}^n} \mathbb{I}_{\pi^{-1}(t)} f d\chi = \int_{\mathbb{R}^n} \mathbb{I}_{H_t} f = \mathcal{R}_n(H_t).$$

Hence

$$\int f d\chi = \int_{\mathbb{R}} \mathcal{R}_n f(H_t) d\chi(t).$$

so that

$$f = (-1)^{n+1} \mathcal{R}_n^\dagger \circ \mathcal{R}_n(f) + \frac{1 + (-1)^n}{2} \left(\int_{\mathbb{R}} \mathcal{R}_n f(H_t) d\chi(t) \right).$$

If we define

$$B = \{(x, H) \in \mathbb{R}^n \times \mathbf{Graff}^1(\mathbb{R}^n); H = \pi^{-1}(\pi(x))\}$$

then we deduce that

$$\int_{\mathbb{R}} \mathcal{R}_n f(H_t) d\chi(t) = \mathcal{J}_{\mathbb{I}_B}(\mathcal{R}_n f).$$

Finally if we set

$$K = (-1)^{n+1} \mathbb{I}_{A^\dagger} + \frac{1 + (-1)^n}{2} \mathbb{I}_B \in \mathcal{C}_{\mathbb{R}^n \times \mathbf{Graff}^1(\mathbb{R}^n)}$$

then we deduce that

$$\mathcal{J}_K \circ \mathcal{R}_n(f) = f, \quad \forall f \in \mathcal{C}_{\mathbb{R}^n}.$$

This proves that the Radon transform is injective.

REFERENCES

- [1] M. Coste: *An Introduction to o-minimal geometry*, Real Algebraic and Analytic Geometry Network, <http://www.ihp-raag.org/publications.php>
- [2] L.van der Dries: *Tame Topology and o-minimal Structures*, London Math. Soc. Series, vol. 248, Cambridge University Press, 1998.
- [3] A. Khovanskii: *Fewnomials*, Transl. Math. Monogr., **88**, Amer.Math. Soc., 1991.
- [4] D.A. Klain, G.-C. Rota: *Introduction to Geometric Probability*, Cambridge University Press, 1997.

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