

# COMPLEXITY OF RANDOM SMOOTH FUNCTIONS ON COMPACT MANIFOLDS. I

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ABSTRACT. We relate the distribution of eigenvalues of a random symmetric matrix in the Gaussian Orthogonal Ensemble to the distribution of critical values of a random linear combination of eigenfunctions of the Laplacian on a compact Riemann manifold. We then prove a central limit theorem describing what happens when the dimension of the manifold is very large.

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## NOTATIONS

- (i) For any set  $S$  we denote by  $|S| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$  its cardinality. For any subset  $A$  of a set  $S$  we denote by  $I_A$  its characteristic function

$$I_A : S \rightarrow \{0, 1\}, \quad I_A(s) = \begin{cases} 1, & s \in A \\ 0, & s \in S \setminus A. \end{cases}$$

- (ii) For any point  $x$  in a smooth manifold  $X$  we denote by  $\delta_x$  the Dirac measure on  $X$  concentrated at  $x$ .
- (iii) For any random variable  $\xi$  we denote by  $E(\xi)$  and respectively  $\mathit{var}(\xi)$  its expectation and respectively its variance.
- (iv) For any Euclidean space  $V$  we denote by  $\mathcal{S}(V)$  the space of symmetric linear operators  $V \rightarrow V$ . When  $V$  is the Euclidean space  $\mathbb{R}^m$  we set  $\mathcal{S}_m := \mathcal{S}(\mathbb{R}^m)$ .

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(v) For  $v > 0$  we denote by  $d\gamma_v$  the centered Gaussian measure on  $\mathbb{R}$  with variance  $v$ ,

$$d\gamma_v(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{x^2}{2v}} |dx|.$$

Since  $\lim_{v \searrow 0} \gamma_v = \delta_0$ , we set  $\gamma_0 := \delta_0$ .

(vi) If  $\mu$  and  $\nu$  are two finite measures on a common space  $X$ , then the notation  $\mu \propto \nu$  means that

$$\frac{1}{\mu(X)} \mu = \frac{1}{\nu(X)} \nu.$$

## 1. OVERVIEW

**1.1. The setup.** Suppose that  $(M, g)$  is a smooth, compact, connected Riemann manifold of dimension  $m > 1$ . We denote by  $|dV_g|$  the volume density on  $M$  induced by  $g$ . We assume that the metric is normalized so that

$$\text{vol}_g(M) = 1. \quad (*)$$

For any  $\mathbf{u}, \mathbf{v} \in C^\infty(M)$  we denote by  $(\mathbf{u}, \mathbf{v})_g$  their  $L^2$  inner product defined by the metric  $g$ . The  $L^2$ -norm of a smooth function  $\mathbf{u}$  is denoted by  $\|\mathbf{u}\|$ .

Let  $\Delta_g : C^\infty(M) \rightarrow C^\infty(M)$  denote the scalar Laplacian defined by the metric  $g$ . For  $L > 0$  we set

$$U^L = U^L(M, g) := \bigoplus_{\lambda \in [0, L^2]} \ker(\lambda - \Delta_g), \quad d(L) := \dim U^L.$$

We equip  $U^L$  with the Gaussian probability measure.

$$d\gamma^L(\mathbf{u}) := (2\pi)^{-\frac{d(L)}{2}} e^{-\frac{\|\mathbf{u}\|^2}{2}} |d\mathbf{u}|.$$

Fix an orthonormal Hilbert basis  $(\Psi_k)_{k \geq 0}$  of  $L^2(M)$  consisting of eigenfunctions of  $\Delta_g$ ,

$$\Delta \Psi_k = \lambda_k \Psi_k, \quad k_0 \leq k_1 \Rightarrow \lambda_{k_0} \leq \lambda_{k_1}.$$

Then

$$U^L = \text{span}\{\Psi_k; \lambda_k \leq L^2\}.$$

A random (with respect to  $d\gamma^L$ ) function  $\mathbf{u} \in U^L$  can be viewed as a linear combination

$$\mathbf{u} = \sum_{\lambda_k \leq L^2} u_k \Psi_k,$$

where  $u_n$  are i.i.d. Gaussian random variables with mean 0 and variance  $\sigma^2 = 1$ .

For any  $\mathbf{u} \in C^1(M)$  we denote by  $\mathbf{Cr}(\mathbf{u}) \subset M$  the set of critical points of  $\mathbf{u}$  and by  $D(\mathbf{u})$  the set of critical values<sup>1</sup> of  $\mathbf{u}$ . If  $L$  is sufficiently large we can apply [15, Cor. 1.26] to conclude that a random function  $\mathbf{u} \in U^L$  is almost surely (a.s.) Morse, so that the random set  $\mathbf{Cr}(\mathbf{u})$  is a.s. finite.

To a Morse function  $\mathbf{u}$  we associate a Borel measure  $\mu_{\mathbf{u}}$  on  $M$  and a Borel measure  $\sigma_{\mathbf{u}}$  on  $\mathbb{R}$  defined by the equalities

$$\mu_{\mathbf{u}} := \sum_{p \in \mathbf{Cr}(\mathbf{u})} \delta_p, \quad \sigma_{\mathbf{u}} = \mathbf{u}_*(\mu_{\mathbf{u}}) = \sum_{t \in \mathbb{R}} |f^{-1}(t) \cap \mathbf{Cr}(\mathbf{u})| \delta_t.$$

Observe that

$$\text{supp } \mu_{\mathbf{u}} = \mathbf{Cr}(\mathbf{u}), \quad \text{supp } \sigma_{\mathbf{u}} = D(\mathbf{u}).$$

When  $\mathbf{u} \in U^L$  is not a Morse function we set

$$\mu_{\mathbf{u}} := |dV_g|, \quad \sigma_{\mathbf{u}} = \delta_0 = \text{the Dirac measure on } \mathbb{R} \text{ concentrated at the origin.}$$

<sup>1</sup>The set  $D(\mathbf{u})$  is sometime referred to as the *discriminant set* of  $\mathbf{u}$ .

Observe that for any Morse function  $\mathbf{u} \in \mathbf{U}^L$  and any Borel subset  $B \subset \mathbb{R}$  the number  $\sigma_{\mathbf{u}}(B)$  is equal to the number of critical values of  $\mathbf{u}$  in  $B$  counted with multiplicity. We will refer to  $\sigma_{\mathbf{u}}$  as the *variational complexity* of  $\mathbf{u}$ .

We set

$$s_m := \frac{(4\pi)^{-\frac{m}{2}}}{\Gamma(1 + \frac{m}{2})}, \quad d_m := \frac{(4\pi)^{-\frac{m}{2}}}{2\Gamma(2 + \frac{m}{2})}, \quad h_m := \frac{(4\pi)^{-\frac{m}{2}}}{4\Gamma(3 + \frac{m}{2})}. \quad (1.1)$$

Let us observe that

$$s_m = h_m(m+2)(m+4), \quad d_m = (m+4)h_m. \quad (1.2)$$

The statistical significance of these numbers is described in Subsection 2.2. We only want to mention here that the Hörmander-Weyl spectral estimates state that

$$\dim \mathbf{U}^L = s_m L^m + O(L^{m-1}) \quad \text{as } L \rightarrow \infty. \quad (1.3)$$

For  $L \gg 0$ , the correspondence

$$\mathbf{U}^L \ni \mathbf{u} \mapsto \mu_{\mathbf{u}}$$

is a random measure on  $M$  called the *empirical measure* determined by the critical points of a random function. Its expectation is the measure  $\mu^L$  on  $M$  defined by

$$\int_M f d\mu^L = \int_{\mathbf{U}^L} \left( \int_M f d\mu_{\mathbf{u}} \right) d\gamma^L(\mathbf{u}),$$

for any continuous function  $f : M \rightarrow \mathbb{R}$ . Note that the number

$$\mathbf{N}^L := \int_M d\mu^L = \int_{\mathbf{U}^L} |\mathbf{Cr}(\mathbf{u})| d\gamma^L(\mathbf{u})$$

is the expected number of critical points of a random function in  $\mathbf{U}^L$ .

In [16] we showed that there exists a universal constant  $C_m$  that depends only on the dimension  $m$  such that

$$\mathbf{N}^L \sim C_m \dim \mathbf{U}^L \sim C_m s_m L^m \quad \text{as } L \rightarrow \infty, \quad (1.4)$$

and the normalized measures

$$d\bar{\mu}^L := \frac{1}{\mathbf{N}^L} d\mu^L$$

converges weakly to the metric volume measure  $|dV_g|$  as  $L \rightarrow \infty$ . This means that for  $L$  very large we expect the critical set of a random  $\mathbf{u} \in \mathbf{U}^L$  to be close to uniformly distributed on  $M$ . Additionally we showed that

$$\log C_m \sim \frac{1}{2} m \log m \quad \text{as } m \rightarrow \infty.$$

Similarly, the random measure  $\mathbf{U}^L \ni \mathbf{u} \mapsto \sigma_{\mathbf{u}}$  has an expectation

$$\sigma^L := \mathbf{E}_{\mathbf{U}^L}(\sigma_{\mathbf{u}})$$

which is a probability measure on  $\mathbb{R}$  defined by

$$\int_{\mathbb{R}} f(\lambda) d\sigma^L(\lambda) = \int_{\mathbf{U}^L} \left( \int_{\mathbb{R}} f(\lambda) d\sigma_{\mathbf{u}}^L(\lambda) \right) d\gamma^L(\mathbf{u}),$$

for any continuous and bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Results of Adler-Taylor [1] (see Subection 2.1) show that  $\sigma^L$  exists.

**1.2. Statements of the main results.** In this paper we investigate the statistical properties of the measure  $\sigma^L$  as  $L \rightarrow \infty$  and then as  $m \rightarrow \infty$ . First let us point a small annoyance which we will turn to our advantage.

Observe that if  $\mathbf{u} : M \rightarrow \mathbb{R}$  is a fixed Morse function and  $c$  is a constant, then

$$\mathbf{Cr}(c + \mathbf{u}) = \mathbf{Cr}(\mathbf{u}), \quad \mu_{c+\mathbf{u}} = \mu_{\mathbf{u}},$$

but

$$\mathbf{D}(\mathbf{u} + c) = c + \mathbf{D}(\mathbf{u}), \quad \sigma_{\mathbf{u}+c} = \delta_c * \sigma_{\mathbf{u}},$$

where  $*$  denotes the convolution of two finite measures on  $\mathbb{R}$ . More generally, if  $X$  is a scalar random variable with probability distribution  $\nu_X$ , then the expected variational complexity of the random function  $X + \mathbf{u}$  is the measure

$$\mathbf{E}(\sigma_{X+\mathbf{u}}) = \nu_X * \sigma_{\mathbf{u}}.$$

In particular, if the distribution  $\nu_X$  is a Gaussian, then the measure  $\sigma_{\mathbf{u}}$  is uniquely determined by the measure  $\mathbf{E}(\sigma_{X+\mathbf{u}})$  since the convolution with a Gaussian is an injective operation.

Note that the lowest eigenfunction  $\Psi_0$  is the constant function 1. We consider random functions of the form

$$\mathbf{u}_{\omega} = X_{\omega} \Psi_0 + \sum_{\lambda_k \leq L^2} u_k \Psi_k = X_{\omega} + \mathbf{u},$$

where the Fourier coefficients  $u_k$  are i.i.d. standard Gaussians, and  $X_{\omega}$  is a scalar, centered Gaussian random variable with variance  $\omega$  independent of the  $u_k$ 's. In applications  $\omega$  will depend on  $m$  and  $L$ . Equivalently, this means that we replace the Gaussian measure  $d\gamma^L$  on  $U^L$  with a Gaussian measure  $d\gamma_{\omega}^L$  of the form

$$d\gamma_{\omega}^L = \frac{1}{(2\pi)^{\frac{d(L)}{2}} \sqrt{1+\omega}} e^{-\frac{|u_0|^2}{2(1+\omega)} - \frac{\|\mathbf{u}^{\perp}\|^2}{2}} |d\mathbf{u}|,$$

where

$$u_0 = (\mathbf{u}_0, \Psi_0)_g, \quad \mathbf{u}^{\perp} := \mathbf{u} - u_0 \Psi_0.$$

Since  $X_{\omega}$  is independent of  $\mathbf{u}$  we deduce that the expected variational complexity of  $X_{\omega} + \mathbf{u}$  is the measure  $\sigma_{\omega}^L$  on  $\mathbb{R}$  given by

$$\sigma_{\omega}^L = \mathbf{E}(\sigma_{X_{\omega}+\mathbf{u}}) = \gamma_{\omega} * \sigma^L. \quad (1.5)$$

Note that

$$N^L = \int_{\mathbb{R}} d\sigma_{\omega}^L(t) = \int_{\mathbb{R}} d\sigma^L(t).$$

The first goal of this paper is to investigate the behavior of the probability measures  $\frac{1}{N^L} \sigma_{\omega}^L$  as  $L \rightarrow \infty$  for certain very special  $\omega$ 's. For reasons that will become clear during the proof we choose  $\omega$  of the form

$$\omega = \omega(m, L, r) = \bar{\omega}_{m,r} L^m \quad (1.6)$$

where  $r > 0$  and the quantity  $\bar{\omega}_{m,r}$  are uniquely determined by the equality

$$s_m + \bar{\omega}_{m,r} = r \frac{d_m^2}{h_m} =: s_m^{\omega}. \quad (1.7)$$

From (1.2) we deduce that

$$s_m^{\omega} = r \frac{m+4}{m+2} s_m, \quad \bar{\omega}_{m,r} = \left( \frac{r(m+4)}{m+2} - 1 \right) s_m. \quad (1.8)$$

The inequality  $s_m^{\omega} \geq s_m$  shows that the parameter  $r$  must satisfy the  $m$ -dependent constraint

$$r \geq \frac{m+2}{m+4}. \quad (C_m)$$

To formulate our main results we need to briefly recall some terminology from random matrix theory.

For  $v \in (0, \infty)$  and  $N$  a positive integer we denote by  $\text{GOE}_N^v$  the space  $\mathcal{S}_N$  of real, symmetric  $N \times N$  matrices  $A$  equipped with a Gaussian measure such that the entries  $a_{ij}$  are independent, zero-mean, normal random variables with variances

$$\mathbf{var}(a_{ii}) = 2v, \quad \mathbf{var}(a_{ij}) = v, \quad \forall 1 \leq i < j \leq N.$$

We denote by  $\rho_{N,v}(\lambda)$  the *normalized correlation* function of  $\text{GOE}_N^v$ . It is uniquely determined by the equality

$$\int_{\mathbb{R}} f(\lambda) \rho_{N,v}(\lambda) d\lambda = \frac{1}{N} \mathbf{E}_{\text{GOE}_N^v}(\text{tr } f(A)),$$

for any continuous bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . The function  $\rho_{N,v}(\lambda)$  also has a probabilistic interpretation. For any Borel set  $B \subset \mathbb{R}$  the probability that a random  $A \in \text{GOE}_N^v$  has at least one eigenvalue in  $B$  is equal to

$$\int_B \rho_{N,v}(\lambda) d\lambda.$$

For any  $t > 0$  we denote by  $\mathcal{R}_t : \mathbb{R} \rightarrow \mathbb{R}$  the rescaling map  $\mathbb{R} \ni x \mapsto tx \in \mathbb{R}$ . If  $\mu$  is a Borel measure on  $\mathbb{R}$  we denote by  $(\mathcal{R}_t)_* \mu$  its pushforward via the rescaling map  $\mathcal{R}_t$ . The celebrated Wigner semicircle theorem, [2, 13], states that as  $N \rightarrow \infty$  the rescaled probability measures

$$\left(\mathcal{R}_{\frac{1}{\sqrt{N}}}\right)_* \left(\rho_{N,v}(\lambda) d\lambda\right)$$

converge weakly to the semicircle measure given by the density

$$\rho_{\infty,v}(\lambda) := \frac{1}{2\pi v} \times \begin{cases} \sqrt{4v - \lambda^2}, & |\lambda| \leq \sqrt{4v} \\ 0, & |\lambda| > \sqrt{4v}. \end{cases}$$

We can now state the main technical result of this paper.

**Theorem 1.1.** *Fix a positive real number satisfying  $r \geq 1$ . Let  $\omega = \omega(m, L, r)$  be defined by the equalities (1.6) and (1.7). Then as  $L \rightarrow \infty$  the rescaled measures*

$$\frac{1}{N^L} \left(\mathcal{R}_{\frac{1}{\sqrt{s_m^\omega L^m}}}\right)_* \sigma_\omega^L$$

converge weakly to a probability measure  $\sigma_{m,r}$  on  $\mathbb{R}$  satisfying the equality

$$\sigma_{m,r} \propto \gamma_{\frac{(r-1)}{r}} * \left(e^{-\frac{r\lambda^2}{4}} \rho_{m+1,r-1}(\lambda) d\lambda\right), \quad (1.9)$$

where the symbol  $*$  denotes the convolution of two (finite) measures on  $\mathbb{R}$ .

The above result has several interesting consequences.

**Corollary 1.2.** *As  $L \rightarrow \infty$  the rescaled measures*

$$\frac{1}{N^L} \left(\mathcal{R}_{\frac{1}{\sqrt{\dim U^L}}}\right)_* \sigma^L$$

converge weakly to a probability measure  $\sigma_m$  on  $\mathbb{R}$  uniquely determined by the convolution equation

$$\gamma_{\frac{2}{m+2}} * \sigma_m = \left(\mathcal{R}_{\sqrt{\frac{m+4}{m+2}}}\right)_* \sigma_{m,1},$$

where  $\sigma_{m,1} \propto e^{-\frac{\lambda^2}{4}} \rho_{m+1,1}(\lambda) d\lambda$ .

The above result shows that the large  $L$  behavior of the average complexity  $\sigma^L$  is independent of the background manifold  $M$ .

**Corollary 1.3.** *As  $m \rightarrow \infty$ , the measures  $\sigma_m$  converge weakly to the Gaussian measure  $\gamma_2$ .*

Let us briefly describe the principles hiding behind the above results. Theorem 1.1 follows from a Kac-Rice type formula of Adler-Taylor [1] aided by refined spectral estimates due to L. Hörmander, [11], and X. Bin, [5]. Corollary 1.2 is a rather immediate consequence of Theorem 1.1 while Corollary 1.3 follows from Corollary 1.2 via a refined version of Wigner’s semicircle theorem.

The basic facts covering the Kac-Rice formula are presented in Subsection 2.1 while the proofs of the above results are presented in Subsections 2.2, 2.3, 2.4. We have included two probabilistic appendices. In Appendix A we have collected a few basic facts about Gaussian measures used throughout the paper. In the more exotic Appendix B we discuss a family of symmetric random matrices and some of their properties needed in the main body of the paper.

We want to comment on the similarities and differences between this paper and A. Auffinger’s dissertation [3] which was a catalyst for the present research.

Auffinger considers random fields on the round sphere  $S^N$  with covariance kernel given by the function

$$\mathcal{R}_{N,p} : S^N \times S^N \rightarrow \mathbb{R}, \quad \mathcal{R}_{N,p}(\mathbf{q}, \mathbf{q}') = |\mathbf{q} \bullet \mathbf{q}'|^p,$$

where  $\bullet$  denotes the canonical inner product in  $\mathbb{R}^{N+1} \supset S^N$  and  $p$  is a fixed real number  $p \in \{2\} \cup [3, \infty)$ . Among many other things, Auffinger studies the behavior of the variational complexity of such a random function as  $N \rightarrow \infty$ .

The eigenvalues of the Laplacian on the round sphere  $S^N$  of radius 1 are

$$\lambda_k(N) = k(k + N - 1), \quad k \geq 0$$

with multiplicity

$$M_k(N) = \frac{2k + N - 1}{k + N - 1} \binom{k + N - 1}{N - 1} \sim 2 \frac{k^{N-1}}{(N - 1)!} \quad \text{as } k \rightarrow \infty.$$

If  $L_k := \sqrt{\lambda_k(N)}$ , then we can identify  $\mathbf{U}^{L_k}$  with the vector space consisting of the restrictions to  $S^N \subset \mathbb{R}^{N+1}$  of the polynomials in  $(N + 1)$  variables and of degree  $\leq k$ . We have

$$\dim \mathbf{U}^{L_k} \sim \frac{2k^N}{N!} \quad \text{as } k \rightarrow \infty.$$

Using the classical addition theorem for harmonic polynomials, [14, §1.2, Thm.2], we deduce that the covariance kernel of the random function defined by  $\mathbf{U}^{L_k}$  is

$$\mathcal{E}_{N,k}(\mathbf{q}, \mathbf{q}') = \frac{1}{\sigma_N} \sum_{j=0}^k M_j(N) P_{j,N+1}(\mathbf{q} \bullet \mathbf{q}'),$$

where  $\sigma_N$  denotes the “area” of  $S^N$  and  $P_{n,\ell}(t)$  denotes the Legendre polynomial of degree  $n$  and order  $\ell$ , i.e.,

$$P_{n,\ell}(t) = (-1)^n R_n(\ell) = 2^{-n} \frac{\Gamma(\frac{\ell-1}{2})}{\Gamma(n + \frac{\ell-1}{2})} (1 - t^2)^{-\frac{\ell-3}{2}} \left( \frac{d}{dt} \right)^n (1 - t^2)^{n + \frac{\ell-3}{2}}. \quad (1.10)$$

In our paper we first let  $k$  go to infinity and then we let  $N \rightarrow \infty$ . In this case, our Corollary 1.2 is a statement concerning the distribution of critical values of the restriction to  $S^N$  of a polynomial of large degree in  $(N + 1)$ -variables.

Let us point out that the limit  $k \rightarrow \infty$  leads to rather singular phenomena. The random function (field) defined by  $\mathbf{U}^{L_k}$  converges to a generalized random function à la Gelfand-Vilenkin, [10], whose covariance kernel is the Dirac delta-distribution concentrated along the diagonal of  $S^N \times S^N$ . The

sample functions of this process are a.s. nondifferentiable so in the limit the notion of critical point loses its meaning.

## 2. PROOFS

**2.1. A Kac-Rice type formula.** As we have already mentioned, the key result behind Theorem 1.1 is a Kac-Rice type result which we intend to discuss in some detail in this section. This result gives an explicit, yet quite complicated description of the measure  $\sigma_\omega^L$ . More precisely, for any Borel subset  $B \subset \mathbb{R}$  the Kac-Rice formula provides an integral representation of  $\sigma_\omega^L(B)$  of the form

$$\sigma_\omega^L(B) = \int_M f_{L,\omega,B}(\mathbf{p}) |dV_g(\mathbf{p})|,$$

for some integrable function  $f_{L,\omega,B} : M \rightarrow \mathbb{R}$ . The core of the Kac-Rice formula is an explicit probabilistic description of the density  $f_{\omega,L,B}$ .

Fix a point  $\mathbf{p} \in M$ . This determines three Gaussian random variables.

$$\begin{aligned} (U^L, \gamma_\omega^L) \ni \mathbf{u}_\omega &\mapsto \mathbf{u}_\omega(\mathbf{p}) \in \mathbb{R}, \\ (U^L, \gamma_\omega^L) \ni \mathbf{u}_\omega &\mapsto d\mathbf{u}_\omega(\mathbf{p}) \in T_{\mathbf{p}}^*M, \\ (U^L, \gamma_\omega^L) \ni \mathbf{u}_\omega &\mapsto \text{Hess}_{\mathbf{p}}(\mathbf{u}_\omega) \in \mathcal{S}(T_{\mathbf{p}}M), \end{aligned} \tag{RV_\omega}$$

where  $\text{Hess}_{\mathbf{p}}(\mathbf{u}_\omega) : T_{\mathbf{p}}M \times T_{\mathbf{p}}M \rightarrow \mathbb{R}$  is the Hessian of  $\mathbf{u}_\omega$  at  $\mathbf{p}$  defined in terms of the Levi-Civita connection of  $g$  and then identified with a symmetric endomorphism of  $T_{\mathbf{p}}M$  using again the metric  $g$ . More concretely, if  $(x^i)_{1 \leq i \leq m}$  are  $g$ -normal coordinates at  $\mathbf{p}$ , then

$$\text{Hess}_{\mathbf{p}}(\mathbf{u}_\omega) \partial_{x^j} = \sum_{i=1}^m \partial_{x^i x^j}^2 \mathbf{u}_\omega(\mathbf{p}) \partial_{x^i}.$$

For  $L$  very large the map  $U^L \ni \mathbf{u} \mapsto d\mathbf{u}(\mathbf{p}) \in T_{\mathbf{p}}^*M$  is surjective which implies that the covariance form of the Gaussian random vector  $d\mathbf{u}_\omega(\mathbf{p})$  is positive definite. We can identify it with a symmetric, positive definite linear operator

$$S(d\mathbf{u}_\omega(\mathbf{p})) : T_{\mathbf{p}}M \rightarrow T_{\mathbf{p}}M.$$

More concretely, if  $(x^i)_{1 \leq i \leq m}$  are  $g$ -normal coordinates at  $\mathbf{p}$ , then we can identify  $S(d\mathbf{u}_\omega(\mathbf{p}))$  with a  $m \times m$  real symmetric matrix whose  $(i, j)$ -entry is given by

$$S_{ij}(d\mathbf{u}_\omega(\mathbf{p})) = \mathbf{E}(\partial_{x^i} \mathbf{u}_\omega(\mathbf{p}) \cdot \partial_{x^j} \mathbf{u}_\omega(\mathbf{p})).$$

**Theorem 2.1.** Fix a Borel subset  $B \subset \mathbb{R}$ . For any  $\mathbf{p} \in M$  define

$$f_{L,\omega,B}(\mathbf{p}) := (\det(2\pi S(\mathbf{u}_\omega(\mathbf{p})))^{-\frac{1}{2}} \mathbf{E}(|\det \text{Hess}_{\mathbf{p}}(\mathbf{u}_\omega)| \cdot \mathbf{I}_B(\mathbf{u}_\omega(\mathbf{p})) \mid d\mathbf{u}_\omega(\mathbf{p}) = 0),$$

where  $\mathbf{E}(\text{var} \mid \text{cons})$  stands for the conditional expectation of the variable **var** given the constraint **cons**. Then

$$\sigma_\omega^L(B) = \int_M f_{L,\omega,B}(\mathbf{p}) |dV_g(\mathbf{p})|. \tag{2.1}$$

□

This theorem is a special case of a general result of Adler-Taylor, [1, Thm. 11.2.1]. The many technical assumptions in Adler-Taylor Theorem are trivially satisfied in this case. In [16] we proved this theorem in the case  $B = \mathbb{R}$  and  $\omega = 0$ . The strategy used there can be modified to yield the more general Theorem 2.1.

For the above theorem to be of any use we need to have some concrete information about the Gaussian random variables  $(RV_\omega)$ . All the relevant statistical invariants of these variables can be extracted from the covariance kernel of the random function  $\mathbf{u}_\omega$ . This is the function  $\mathcal{E}_\omega^L : M \times M \rightarrow \mathbb{R}$  defined by the equality

$$\begin{aligned} \mathcal{E}_\omega^L(\mathbf{p}, \mathbf{q}) &= \mathbf{E}(\mathbf{u}_\omega(\mathbf{p})\mathbf{u}_\omega(\mathbf{q})) = \mathbf{E}((X + \mathbf{u}(\mathbf{p})) \cdot (X + \mathbf{u}(\mathbf{q}))) \\ &= \omega + \sum_{\lambda_k \leq L^2} \Psi_k(\mathbf{p})\Psi_k(\mathbf{q}) =: \omega + \mathcal{E}^L(\mathbf{p}, \mathbf{q}). \end{aligned}$$

The function  $\mathcal{E}^L$  is the spectral function of the Laplacian, i.e., the Schwartz kernel of the orthogonal projection onto  $U^L$ . Fortunately, a lot is known about the behavior of  $\mathcal{E}^L$  as  $L \rightarrow \infty$ , [5, 8, 11, 18].

**2.2. Proof of Theorem 1.1.** Fix  $L \gg 0$ . For any  $\mathbf{p} \in M$  we have the centered Gaussian vector  $(RV_\omega)$ ,  $\omega = 0$ ,

$$(U^L, \gamma^L) \ni u \mapsto (\mathbf{u}(\mathbf{p}), d\mathbf{u}(\mathbf{p}), \nabla^2 \mathbf{u}(\mathbf{p})) \in \mathbb{R} \oplus T_{\mathbf{p}}^*M \oplus \mathcal{S}(T_{\mathbf{p}}M).$$

We fix normal coordinates  $(x^i)_{1 \leq i \leq m}$  at  $\mathbf{p}$  and we can identify the above Gaussian vector with the centered Gaussian vector

$$(\mathbf{u}(\mathbf{p}), (\partial_{x^i} \mathbf{u}(\mathbf{p}))_{1 \leq i \leq m}, \partial_{x^i x^j}^2(\mathbf{u}(\mathbf{p}))_{1 \leq i, j \leq m}) \in \mathbb{R} \oplus \mathbb{R}^m \oplus \mathcal{S}_m.$$

In [16, §3] we showed that the spectral estimates of Bin-Hörmander [5, 11] imply the following asymptotic estimates.

**Lemma 2.2.** *For any  $1 \leq i, j, k, \ell \leq m$  we have the uniform in  $\mathbf{p}$  asymptotic estimates as  $L \rightarrow \infty$*

$$\mathbf{E}(\mathbf{u}(\mathbf{p})^2) = s_m L^m (1 + O(L^{-1})), \quad (2.2a)$$

$$\mathbf{E}(\partial_{x^i} \mathbf{u}(\mathbf{p}) \partial_{x^j} \mathbf{u}(\mathbf{p})) = d_m L^{m+2} \delta_{ij} (1 + O(L^{-1})), \quad (2.2b)$$

$$\mathbf{E}(\partial_{x^i x^j}^2 \mathbf{u}(\mathbf{p}) \partial_{x^k x^\ell}^2 \mathbf{u}(\mathbf{p})) = h_m L^{m+4} (\delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) (1 + O(L^{-1})), \quad (2.2c)$$

$$\mathbf{E}(\mathbf{u}(\mathbf{p}) \partial_{x^i x^j}^2 \mathbf{u}(\mathbf{p})) = -d_m L^{m+2} \delta_{ij} (1 + O(L^{-1})), \quad (2.2d)$$

$$\mathbf{E}(\mathbf{u}(\mathbf{p}) \partial_{x^i} \mathbf{u}(\mathbf{p})) = O(L^m), \quad \mathbf{E}(\partial_{x^i} \mathbf{u}(\mathbf{p}) \partial_{x^j x^j}^2 \mathbf{u}(\mathbf{p})) = O(L^{m+2}), \quad (2.2e)$$

where the constants  $s_m, d_m, h_m$  are defined by (1.1).  $\square$

Now let  $\omega = \omega(m, L, r)$  be defined as in (1.6), (1.7). Using the notation (1.8) we deduce from the above that in the case of the random function  $\mathbf{u}_\omega$  we have the estimates

$$\mathbf{E}(\mathbf{u}_\omega(\mathbf{p})^2) = s_m^\omega L^m (1 + O(L^{-1})), \quad (2.3a)$$

$$\mathbf{E}(\partial_{x^i} \mathbf{u}_\omega(\mathbf{p}) \partial_{x^j} \mathbf{u}_\omega(\mathbf{p})) = d_m L^{m+2} \delta_{ij} (1 + O(L^{-1})), \quad (2.3b)$$

$$\mathbf{E}(\partial_{x^i x^j}^2 \mathbf{u}_\omega(\mathbf{p}) \partial_{x^k x^\ell}^2 \mathbf{u}_\omega(\mathbf{p})) = h_m L^{m+4} (\delta_{ij} \delta_{k\ell} + \delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) (1 + O(L^{-1})), \quad (2.3c)$$

$$\mathbf{E}(\mathbf{u}_\omega(\mathbf{p}) \partial_{x^i x^j}^2 \mathbf{u}_\omega(\mathbf{p})) = -d_m L^{m+2} \delta_{ij} (1 + O(L^{-1})), \quad (2.3d)$$

$$\mathbf{E}(\mathbf{u}_\omega(\mathbf{p}) \partial_{x^i} \mathbf{u}_\omega(\mathbf{p})) = O(L^m), \quad \mathbf{E}(\partial_{x^i} \mathbf{u}_\omega(\mathbf{p}) \partial_{x^j x^j}^2 \mathbf{u}_\omega(\mathbf{p})) = O(L^{m+2}). \quad (2.3e)$$

From the estimate (2.3b) we deduce that

$$\mathcal{S}(d\mathbf{u}_\omega(\mathbf{p})) = d_m L^{m+2} (\mathbb{1}_m + O(L^{-1})),$$

so that

$$\sqrt{|\det \mathcal{S}(\mathbf{u}_\omega(\mathbf{p}))|} = (d_m)^{\frac{m}{2}} L^{\frac{m(m+2)}{2}} (1 + O(L^{-1})) \quad \text{as } L \rightarrow \infty. \quad (2.4)$$

Consider the rescaled random vector

$$(s^L, v^L, H^L) = (s^{L, \omega, \mathbf{p}}, v^{L, \omega, \mathbf{p}}, H^{L, \omega, \mathbf{p}}) := (L^{-\frac{m}{2}} \mathbf{u}_\omega(\mathbf{p}), L^{-\frac{m+2}{2}} d\mathbf{u}_\omega(\mathbf{p}), L^{-\frac{m+4}{2}} \nabla^2 \mathbf{u}_\omega(\mathbf{p})).$$

Form the above we deduce the following uniform in  $\mathbf{p}$  estimates as  $L \rightarrow \infty$ .

$$\mathbf{E}((s^L)^2) = s_m^\omega(1 + O(L^{-1})), \quad (2.5a)$$

$$\mathbf{E}(v_i^L v_j^L) = d_m \delta_{ij}(1 + O(L^{-1})), \quad (2.5b)$$

$$\mathbf{E}(H_{ij}^L H_{kl}^L) = h_m(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})(1 + O(L^{-1})), \quad (2.5c)$$

$$\mathbf{E}(s^L H_{ij}^L) = -d_m L^{m+2} \delta_{ij}(1 + O(L^{-1})), \quad (2.5d)$$

$$\mathbf{E}(s^L v_i^L) = O(L^{-1}), \quad \mathbf{E}(v_i^L H_{jk}^L) = O(L^{-1}). \quad (2.5e)$$

The probability distribution of the variable  $s^L$  is

$$d\gamma_{s^L}(x) = \frac{1}{\sqrt{2\pi s_m(L)}} e^{-\frac{x^2}{2s_m(L)}} |dx|,$$

where

$$s_m(L) = s_m^\omega + O(L^{-1}).$$

Fix a Borel set  $B \subset \mathbb{R}$ . We have

$$\begin{aligned} \mathbf{E}(|\det \nabla^2 \mathbf{u}_\omega(\mathbf{p})| \mathbf{I}_B(\mathbf{u}_\omega(\mathbf{p})) \mid d\mathbf{u}_\omega(\mathbf{p}) = 0) &= L^{\frac{m(m+4)}{2}} \mathbf{E}(|\det H^L| \mathbf{I}_{B'}(s^L) \mid v^L = 0) \\ &= L^{\frac{m(m+4)}{2}} \underbrace{\int_{L^{-\frac{m}{2}} B} \mathbf{E}(|\det H^L| \mid s^L = x, v^L = 0) \frac{e^{-\frac{x^2}{2s_m(L)}}}{\sqrt{2\pi s_m(L)}} |dx|}_{=: q_{L,\mathbf{p}}(L^{-\frac{m}{2}} B)}. \end{aligned} \quad (2.6)$$

Using (2.4) and (2.6) we deduce from Theorem 2.1 that

$$\sigma_\omega^L(B) = L^m \left( \frac{1}{2\pi d_m} \right)^{\frac{m}{2}} \int_M q_{L,\mathbf{p}}(L^{-\frac{m}{2}} B) \rho_L(\mathbf{p}) |dV_g(\mathbf{p})|,$$

where  $\rho_L : M \rightarrow \mathbb{R}$  is a function that satisfies the uniform in  $\mathbf{p}$  estimate

$$\rho_L(\mathbf{p}) = 1 + O(L^{-1}) \quad \text{as } L \rightarrow \infty. \quad (2.7)$$

Hence

$$\frac{1}{L^m} \left( \mathcal{R}_{L^{-\frac{m}{2}}} \right)_* \sigma_\omega^L(B) = \left( \frac{1}{2\pi d_m} \right)^{\frac{m}{2}} \int_M q_{L,\mathbf{p}}(B) \rho_L(\mathbf{p}) |dV_g(\mathbf{p})|. \quad (2.8)$$

To continue the computation we need to investigate the behavior of  $q_{L,\mathbf{p}}(B)$  as  $L \rightarrow \infty$ . More concretely, we need to elucidate the nature of the Gaussian vector

$$(H^L \mid s^L = x, v^L = 0).$$

We will achieve this via the regression formula (A.3). For simplicity we set

$$Y^L := (s^L, v^L) \in \mathbb{R} \oplus \mathbb{R}^m.$$

The components of  $Y$  are

$$Y_0^L = s^L, \quad Y_i^L = v_i^L, \quad 1 \leq i \leq m.$$

Using (2.5a), (2.5b) and (2.5e) we deduce that for any  $1 \leq i, j \leq m$  we have

$$\mathbf{E}(Y_0^L Y_i^L) = s_m^\omega \delta_{0i} + O(L^{-1}), \quad \mathbf{E}(Y_i^L Y_j^L) = d_m \delta_{ij} + O(L^{-1}).$$

If  $\mathbf{S}_Y$  denotes the covariance operator of  $Y$ , then we deduce that

$$\mathbf{S}(Y^L)_{0,i}^{-1} = \frac{1}{s_m^\omega} \delta_{0i} + O(L^{-1}), \quad \mathbf{S}(Y^L)_{ij}^{-1} = \frac{1}{d_m} \delta_{ij} + O(L^{-1}). \quad (2.9)$$

We now need to compute the covariance operator  $\mathbf{Cov}(H, Y)$ . To do so we equip  $\mathcal{S}_m$  with the inner product

$$(A, B) = \text{tr}(AB), \quad A, B \in \mathcal{S}_m$$

The space  $\mathcal{S}_m$  has a canonical orthonormal basis

$$\widehat{\mathbf{E}}_{ij}, \quad 1 \leq i \leq j \leq m,$$

where

$$\widehat{\mathbf{E}}_{ij} = \begin{cases} \mathbf{E}_{ij}, & i = j \\ \frac{1}{\sqrt{2}}\mathbf{E}_{ij}, & i < j \end{cases}$$

and  $\mathbf{E}_{ij}$  denotes the symmetric matrix nonzero entries only at locations  $(i, j)$  and  $(j, i)$  and these entries are equal to 1. Thus a matrix  $H \in \mathcal{S}_m$  can be written as

$$A = \sum_{i \leq j} a_{ij} \mathbf{E}_{ij} = \sum_{i \leq j} \widehat{a}_{ij} \widehat{\mathbf{E}}_{ij},$$

where

$$\widehat{a}_{ij} = \begin{cases} a_{ij}, & i = j, \\ \sqrt{2}a_{ij}, & i < j. \end{cases}$$

The covariance operator  $\mathbf{Cov}(H, Y)$  is a linear map

$$\mathbf{Cov}(H^L, Y^L) : \mathbb{R} \oplus \mathbb{R}^m \rightarrow \mathcal{S}_m$$

given by

$$\mathbf{Cov}(H^L, Y^L) \left( \sum_{\alpha=0}^m y_\alpha e_\alpha \right) = \sum_{i < j, \alpha} \mathbf{E}(\widehat{H}_{ij}^L Y_\alpha^L) y_\alpha \widehat{\mathbf{E}}_{ij} = \sum_{i < j, \alpha} \mathbf{E}(H_{ij}^L Y_\alpha^L) y_\alpha \mathbf{E}_{ij},$$

where  $e_0, e_1, \dots, e_m$  denotes the canonical orthonormal basis in  $\mathbb{R} \oplus \mathbb{R}^m$ . Using (2.5d) and (2.5e) we deduce that

$$\mathbf{Cov}(H^L, Y^L) \left( \sum_{\alpha=0}^m y_\alpha e_\alpha \right) = -y_0 d_m \mathbb{1}_m + O(L^{-1}). \quad (2.10)$$

We deduce that the transpose  $\mathbf{Cov}(H, Y)^\vee$  satisfies

$$\mathbf{Cov}(H^L, Y^L)^\vee \left( \sum_{i \leq j} \widehat{a}_{ij} \widehat{\mathbf{E}}_{ij} \right) = -d_m \text{tr}(A) e_0 + O(L^{-1}). \quad (2.11)$$

The covariance operator of the random symmetric matrix

$$Z^L = Z^{L,x} := (H^L |_{s^L = x, v^L = 0})$$

is then

$$\mathbf{S}(Z^L) = \mathbf{S}(H^L) - \mathbf{Cov}(H^L, Y^L) \mathbf{S}(Y^L)^{-1} \mathbf{Cov}(H^L, Y^L)^\vee.$$

This means that

$$\mathbf{E}(\widehat{z}_{ij}^L \widehat{z}_{kl}^L) = (\widehat{\mathbf{E}}_{ij}, \mathbf{S}(Z^L) \widehat{\mathbf{E}}_{kl}).$$

Using (2.9), (2.10) and (2.11) we deduce that

$$\mathbf{Cov}(H^L, Y^L) \mathbf{S}(Y^L)^{-1} \mathbf{Cov}(H^L, Y^L)^\vee \left( \sum_{i \leq j} \widehat{a}_{ij} \widehat{\mathbf{E}}_{ij} \right) = \frac{d_m^2}{s_m^\omega} \text{tr}(A) \mathbb{1}_m + O(L^{-1})$$

$$\mathbf{E}((z_{ij}^L)^2) = h_m + O(L^{-1}), \quad \mathbf{E}(z_{ii}^L z_{jj}^L) = h_m - \frac{d_m^2}{s_m^\omega} + O(L^{-1}), \quad \forall i < j,$$

$$\mathbf{E}((z_{ii}^L)^2) = 3h_m - \frac{d_m^2}{s_m^\omega} + O(L^{-1}), \quad \forall i$$

and

$$\mathbf{E}(z_{ij}^L z_{kl}^L) = O(L^{-1}), \quad \forall i < j, \quad k \leq \ell, \quad (i, j) \neq (k, \ell).$$

We can rewrite these equalities in the compact form

$$\mathbf{E}(z_{ij}^L z_{kl}^L) = \left( h_m - \frac{d_m^2}{s_m} \right) \delta_{ij} \delta_{kl} + h_m (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) + O(L^{-1}).$$

Note that

$$h_m - \frac{d_m^2}{s_m} \stackrel{(1.2)}{=} \frac{r-1}{r} h_m.$$

We set

$$\kappa = \kappa(r) := \frac{(r-1)}{2r}$$

so that

$$\mathbf{E}(z_{ij}^L z_{kl}^L) = 2\kappa h_m \delta_{ij} \delta_{kl} + h_m (\delta_{ik} \delta_{j\ell} + \delta_{i\ell} \delta_{jk}) + O(L^{-1}).$$

Using (A.4) we deduce that the expectation of  $Z^L$  is

$$\mathbf{E}(Z^L) = \mathbf{Cov}(H^L, Y^L) \mathbf{S}(Y^L)^{-1} (x e_0) = -\frac{x}{r(m+4)} \mathbb{1}_m + O(L^{-1}). \quad (2.12)$$

We deduce that the Gaussian random matrix  $Z^{L,x}$  converges uniformly in  $\mathbf{p}$  as  $L \rightarrow \infty$  to the random matrix  $A - \frac{x}{r(m+4)} \mathbb{1}_m$ , where  $A$  belongs to the Gaussian ensemble  $\mathfrak{S}_m^{2\kappa h_m, h_m}$  described in Appendix B. Thus

$$\begin{aligned} \lim_{L \rightarrow \infty} q_{L, \mathbf{p}}(B) &= q_\infty(B) := \int_B \mathbf{E}_{\mathfrak{S}_m^{2\kappa h_m, h_m}} \left( \left| \det \left( A - \frac{x}{r(m+4)} \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{x^2}{2s_m^\omega}}}{\sqrt{2\pi s_m^\omega}} dx \\ &= (h_m)^{\frac{m}{2}} \int_B \mathbf{E}_{\mathfrak{S}_m^{2\kappa, 1}} \left( \left| \det \left( A - \frac{x}{r(m+4)\sqrt{h_m}} \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{x^2}{2s_m^\omega}}}{\sqrt{2\pi s_m^\omega}} dx \\ &= (h_m)^{\frac{m}{2}} \int_{(s_m^\omega)^{-\frac{1}{2}} B} \mathbf{E}_{\mathfrak{S}_m^{2\kappa, 1}} \left( \left| \det \left( A - \alpha_m y \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy, \end{aligned}$$

where

$$\alpha_m = \frac{\sqrt{s_m^\omega}}{r(m+4)\sqrt{h_m}} \stackrel{(1.2)}{=} \frac{1}{\sqrt{r}}.$$

This proves that

$$\lim_{L \rightarrow \infty} \mathcal{R}_{(s_m^\omega)^{-\frac{1}{2}}} q_{L, \mathbf{p}}(B) = (h_m)^{\frac{m}{2}} \underbrace{\int_B \mathbf{E}_{\mathfrak{S}_m^{2\kappa, 1}} \left( \left| \det \left( A - \frac{y}{\sqrt{r}} \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy}_{=: \mu_m(B)}.$$

Using the last equality, the normalization (\*) and the estimate (2.7) in (2.8) we conclude

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{1}{s_m L^m} \left( \mathcal{R}_{(s_m^\omega L^m)^{-\frac{1}{2}}} \right)_* \sigma_\omega^L(B) &= \frac{1}{s_m} \left( \frac{h_m}{2\pi d_m} \right)^{\frac{m}{2}} \mu_m(B) \\ &\stackrel{(1.2)}{=} \left( \frac{2}{m+4} \right)^{\frac{m}{2}} \Gamma \left( 1 + \frac{m}{2} \right) \mu_m(B). \end{aligned} \quad (2.13)$$

Observe that the probability density of  $\mu_m$  is

$$\frac{d\mu_m}{dy} = \mathbf{E}_{\mathcal{S}_m^{2\kappa,1}} \left( \left| \det \left( A - \frac{y}{\sqrt{r}} \mathbb{1}_m \right) \right| \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}. \quad (2.14)$$

We now distinguish two cases.

**Case 1.**  $r > 1$  From Lemma B.2 we deduce that

$$\begin{aligned} & \mathbf{E}_{\mathcal{S}_m^{2\kappa,1}} \left( \left| \det \left( A - \frac{y}{\sqrt{r}} \mathbb{1}_m \right) \right| \right) \\ &= 2^{\frac{m+3}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\sqrt{2\pi\kappa}} \int_{\mathbb{R}} \rho_{m+1,1}(\lambda) e^{-\frac{1}{4\tau^2}(\lambda - (\tau^2+1)\frac{y}{\sqrt{r}})^2 + \frac{(\tau^2+1)y^2}{4r}} d\lambda, \end{aligned} \quad (2.15)$$

where

$$\tau^2 := \frac{\kappa}{\kappa-1} = \frac{r-1}{r+1}.$$

Thus

$$\begin{aligned} \frac{d\mu_m}{dy} &= 2^{\frac{m+3}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{2\pi\sqrt{\kappa}} e^{\frac{(\tau^2+1-2r)y^2}{4r}} \int_{\mathbb{R}} \rho_{m+1,1}(\lambda) e^{-\frac{1}{4\tau^2}(\lambda - (\tau^2+1)\frac{y}{\sqrt{r}})^2} d\lambda \\ &= 2^{\frac{m+3}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{2\pi\sqrt{\kappa}} \int_{\mathbb{R}} \rho_{m+1,1}(\lambda) e^{-\frac{1}{4\tau^2}(\lambda - (\tau^2+1)\frac{y}{\sqrt{r}})^2 - \frac{ry^2}{2(r+1)}} d\lambda. \end{aligned}$$

An elementary computation yields a pleasant surprise

$$-\frac{1}{4\tau^2} \left( \lambda - (\tau^2+1)\frac{y}{\sqrt{r}} \right)^2 - \frac{ry^2}{2(r+1)} = -\frac{1}{4}\lambda^2 - \left( \sqrt{\frac{1}{2(r-1)}}\lambda - y\sqrt{\frac{r}{2(r-1)}} \right)^2.$$

Now set

$$\beta = \beta(r) := \frac{1}{(r-1)}.$$

We deduce

$$\begin{aligned} \frac{d\mu_m}{dy} &= 2^{\frac{m+3}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{2\pi\sqrt{\kappa}} \int_{\mathbb{R}} \rho_{m+1,1}(\lambda) e^{-\frac{1}{4}\lambda^2} e^{-\frac{\beta}{2}(\lambda - \sqrt{r}y)^2} d\lambda \\ (\lambda &:= \sqrt{r}\lambda) \\ &= 2^{\frac{m+3}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{2\pi\sqrt{\kappa}} \int_{\mathbb{R}} \sqrt{r}\rho_{m+1,1}(\sqrt{r}\lambda) e^{-\frac{r}{4}\lambda^2} e^{-\frac{r\beta}{2}(\lambda - y)^2} d\lambda \\ &\stackrel{(B.6)}{=} 2^{\frac{m+3}{2}} \Gamma \left( \frac{m+3}{2} \right) \frac{1}{\sqrt{2\pi\kappa r\beta}} \int_{\mathbb{R}} \rho_{m+1,1/r}(\lambda) e^{-\frac{r}{4}\lambda^2} d\gamma_{\frac{1}{\beta r}}(y - \lambda) d\lambda. \end{aligned}$$

Using the last equality in (2.13) and then invoking the estimate (1.4) we obtain the case  $r > 1$  of Theorem 1.1.

**Case 2.**  $r = 1$ . The proof of Theorem 1.1 in this case follows a similar pattern. Note first that in this case  $\kappa = 0$  so invoking Lemma B.1 we obtain the following counterpart of (2.15)

$$\mathbf{E}_{\text{GOE}_m^1} \left( \left| \det \left( A - y\mathbb{1}_m \right) \right| \right) = 2^{\frac{m+4}{2}} \Gamma \left( \frac{m+3}{2} \right) e^{\frac{y^2}{4}} \rho_{m+1,1}(y).$$

Using this in (2.14) we deduce immediately (1.9) in the case  $r = 1$ . This completes the proof of Theorem 1.1.  $\square$

**2.3. Proof of Corollary 1.2.** We use Theorem 1.1 with  $r = 1$ . Using (1.5), (1.6) and (1.8) we deduce that in this case

$$\sigma_{\omega}^L = \gamma_{\frac{2s_m L^m}{m+2}} * \sigma^L. \quad (2.16)$$

Using the equality

$$s_m L^m = s_m^{\omega} L^m \cdot \frac{m+2}{m+4}$$

we deduce

$$\frac{1}{N^L} \left( \mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}} \right)_* \sigma_{\omega}^L = \left( \mathcal{R}_{\sqrt{\frac{m+4}{m+2}}} \right)_* \left( \frac{1}{N^L} \left( \mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}} \right)_* \sigma^L \right). \quad (2.17)$$

Using (2.16) we deduce that

$$\left( \mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}} \right)_* \sigma_{\omega}^L = \gamma_{\frac{2}{m+2}} * \left( \frac{1}{N^L} \left( \mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}} \right)_* \sigma^L \right).$$

Using the spectral estimates (1.3), the equality (2.17) and Theorem 1.1 we deduce

$$\lim_{L \rightarrow \infty} \gamma_{\frac{2}{m+2}} * \left( \frac{1}{N^L} \left( \mathcal{R}_{\frac{1}{\sqrt{\dim UL}}} \right)_* \sigma^L \right) = \lim_{L \rightarrow \infty} \frac{1}{N^L} \left( \mathcal{R}_{\frac{1}{\sqrt{s_m L^m}}} \right)_* \sigma_{\omega}^L = \left( \mathcal{R}_{\sqrt{\frac{m+4}{m+2}}} \right)_* \sigma_{m+1,1}.$$

We can now conclude by invoking Lévy's continuity theorem [12, Thm. 15.23(ii)].  $\square$

**2.4. Proof of Corollary 1.3.** By invoking Levy's continuity theorem and Corollary 1.2 we see that it suffices to show that the probability measures  $\sigma_{m,1}$  converge weakly to the Gaussian measure  $\gamma_2$ .

Set

$$\begin{aligned} \bar{R}_m(x) &:= \sqrt{m} \rho_{m+1,1}(\sqrt{m}x) = \rho_{m+1, \frac{1}{m}}(x), \\ R_{\infty}(x) &= \frac{1}{2\pi} \mathbf{I}_{\{|x| \leq 2\}} \sqrt{4-x^2}. \end{aligned}$$

Fix  $c \in (0, 2)$ . In [16, §4.2]. we proved that

$$\lim_{m \rightarrow \infty} \sup_{|x| \leq c} |\bar{R}_m(x) - R_{\infty}(x)| = 0, \quad (2.18a)$$

$$\sup_{|x| \geq c} |\bar{R}_m(x) - R_{\infty}(x)| = O(1) \text{ as } m \rightarrow \infty. \quad (2.18b)$$

We deduce that

$$\rho_{m+1,1}(\lambda) e^{-\frac{\lambda^2}{4}} = \sqrt{\frac{4\pi}{m}} \bar{R}_m \left( \frac{\lambda}{\sqrt{m}} \right) \frac{1}{\sqrt{4\pi}} e^{-\frac{\lambda^2}{4}}, \quad (2.19)$$

and

$$I_m := \int_{\mathbb{R}} \rho_{m+1,1}(\lambda) e^{-\frac{\lambda^2}{4}} d\lambda = \sqrt{\frac{4\pi}{m}} \int_{\mathbb{R}} \bar{R}_m(x) \sqrt{\frac{m}{4\pi}} e^{-\frac{mx^2}{4}} dx = \sqrt{\frac{4\pi}{m}} \int_{\mathbb{R}} \bar{R}_m(x) d\gamma_{\frac{2}{m}}(x).$$

The estimates (2.18a), (2.18b) imply that

$$I_m \sim \sqrt{4\pi} R_{\infty}(0) m^{-\frac{1}{2}} \text{ as } m \rightarrow \infty.$$

To prove that the probability measures

$$\frac{1}{I_m} \rho_{m+1,1}(\lambda) e^{-\frac{\lambda^2}{4}} d\lambda$$

converges weakly to  $\gamma_2$  it suffices to show that the finite measures

$$\nu_m := m^{\frac{1}{2}} \rho_{m+1,1}(\lambda) e^{-\frac{\lambda^2}{4}} d\lambda$$

converge weakly to the finite measure

$$\nu_\infty := R_\infty(0)e^{-\frac{\lambda^2}{4}}d\lambda.$$

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a bounded continuous function. Using (2.19) we deduce that

$$\int_{\mathbb{R}} f(\lambda)d\nu_m(\lambda) = \int_{\mathbb{R}} f(\lambda)\bar{R}_m(m^{-\frac{1}{2}}\lambda)e^{-\frac{\lambda^2}{4}}d\lambda.$$

We deduce that

$$\begin{aligned} \int_{\mathbb{R}} f(\lambda)d\nu_m(\lambda) - \int_{\mathbb{R}} f(\lambda)d\nu_\infty(\lambda) &= \int_{\mathbb{R}} f(\lambda) \left( \bar{R}_m(m^{-\frac{1}{2}}x) - R_\infty(0) \right) e^{-\frac{\lambda^2}{4}} d\lambda \\ &= \underbrace{\int_{\mathbb{R}} f(\lambda)\mathbf{I}_{\{|\lambda| \leq c\sqrt{m}\}} \left( \bar{R}_m(m^{-\frac{1}{2}}x) - R_\infty(0) \right) e^{-\frac{\lambda^2}{4}} d\lambda}_{A_m} \\ &\quad + \underbrace{\int_{\mathbb{R}} f(\lambda)\mathbf{I}_{\{|\lambda| \geq c\sqrt{m}\}} \left( \bar{R}_m(m^{-\frac{1}{2}}x) - R_\infty(0) \right) e^{-\frac{\lambda^2}{4}} d\lambda}_{B_m}. \end{aligned}$$

The estimate (2.18a) coupled with the dominated convergence theorem imply that  $A_m \rightarrow 0$  as  $m \rightarrow \infty$ . The estimate (2.18b) and the dominated convergence theorem imply that  $B_m \rightarrow 0$  as  $m \rightarrow \infty$ .  $\square$

#### APPENDIX A. GAUSSIAN MEASURES AND GAUSSIAN VECTORS

For the reader's convenience we survey here a few basic facts about Gaussian measures. For more details we refer to [6]. A *Gaussian measure* on  $\mathbb{R}$  is a Borel measure  $\gamma_{\mu,v}$ ,  $v \geq 0$ ,  $m \in \mathbb{R}$ , of the form

$$\gamma_{\mu,v}(x) = \frac{1}{\sqrt{2\pi v}} e^{-\frac{(x-\mu)^2}{2v}} dx.$$

The scalar  $\mu$  is called the *mean*, while  $v$  is called the *variance*. We allow  $v$  to be zero in which case

$$\gamma_{\mu,0} = \delta_\mu = \text{the Dirac measure on } \mathbb{R} \text{ concentrated at } \mu.$$

For a real valued random variable  $X$  we write

$$X \in \mathbf{N}(\mu, v) \tag{A.1}$$

if the probability measure of  $X$  is  $\gamma_{\mu,v}$ .

Suppose that  $\mathbf{V}$  is a finite dimensional vector space. A *Gaussian measure* on  $\mathbf{V}$  is a Borel measure  $\gamma$  on  $\mathbf{V}$  such that, for any  $\xi \in \mathbf{V}^\vee$ , the pushforward  $\xi_*(\gamma)$  is a Gaussian measure on  $\mathbb{R}$ ,

$$\xi_*(\gamma) = \gamma_{\mu(\xi), \sigma(\xi)}.$$

One can show that the map  $\mathbf{V}^\vee \ni \xi \mapsto \mu(\xi) \in \mathbb{R}$  is linear, and thus can be identified with a vector  $\boldsymbol{\mu}_\gamma \in \mathbf{V}$  called the *barycenter* or *expectation* of  $\gamma$  that can be alternatively defined by the equality

$$\boldsymbol{\mu}_\gamma = \int_{\mathbf{V}} \mathbf{v} d\gamma(\mathbf{v}).$$

Moreover, there exists a nonnegative definite, symmetric bilinear map

$$\boldsymbol{\Sigma} : \mathbf{V}^\vee \times \mathbf{V}^\vee \rightarrow \mathbb{R} \text{ such that } \sigma(\xi)^2 = \boldsymbol{\Sigma}(\xi, \xi), \quad \forall \xi \in \mathbf{V}^\vee.$$

The form  $\boldsymbol{\Sigma}$  is called the *covariance form* and can be identified with a linear operator  $\mathbf{S} : \mathbf{V}^\vee \rightarrow \mathbf{V}$  such that

$$\boldsymbol{\Sigma}(\xi, \eta) = \langle \xi, \mathbf{S}\eta \rangle, \quad \forall \xi, \eta \in \mathbf{V}^\vee,$$

where  $\langle -, - \rangle : \mathbf{V}^\vee \times \mathbf{V} \rightarrow \mathbb{R}$  denotes the natural bilinear pairing between a vector space and its dual. The operator  $\mathbf{S}$  is called the *covariance operator* and it is explicitly described by the integral formula

$$\langle \xi, \mathbf{S}\eta \rangle = \Lambda(\xi, \eta) = \int_{\mathbf{V}} \langle \xi, \mathbf{v} - \boldsymbol{\mu}_\gamma \rangle \langle \eta, \mathbf{v} - \boldsymbol{\mu}_\gamma \rangle d\gamma(\mathbf{v}).$$

The Gaussian measure is said to be *nondegenerate* if  $\boldsymbol{\Sigma}$  is nondegenerate, and it is called *centered* if  $\boldsymbol{\mu} = 0$ . A nondegenerate Gaussian measure on  $\mathbf{V}$  is uniquely determined by its covariance form and its barycenter.

**Example A.1.** Suppose that  $\mathbf{U}$  is an  $n$ -dimensional Euclidean space with inner product  $(-, -)$ . We use the inner product to identify  $\mathbf{U}$  with its dual  $\mathbf{U}^\vee$ . If  $A : \mathbf{U} \rightarrow \mathbf{U}$  is a symmetric, positive definite operator, then

$$d\gamma_A(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det A}} e^{-\frac{1}{2}(\mathbf{A}^{-1}\mathbf{u}, \mathbf{u})} |d\mathbf{u}| \quad (\text{A.2})$$

is a centered Gaussian measure on  $\mathbf{U}$  with covariance form described by the operator  $A$ .  $\square$

If  $\mathbf{V}$  is a finite dimensional vector space equipped with a Gaussian measure  $\gamma$  and  $\mathbf{L} : \mathbf{V} \rightarrow \mathbf{U}$  is a linear map then the pushforward  $\mathbf{L}_*\gamma$  is a Gaussian measure on  $\mathbf{U}$  with barycenter

$$\boldsymbol{\mu}_{\mathbf{L}_*\gamma} = \mathbf{L}(\boldsymbol{\mu}_\gamma)$$

and covariance form

$$\boldsymbol{\Sigma}_{\mathbf{L}_*\gamma} : \mathbf{U}^\vee \times \mathbf{U}^\vee \rightarrow \mathbb{R}, \quad \boldsymbol{\Sigma}_{\mathbf{L}_*\gamma}(\eta, \eta) = \boldsymbol{\Sigma}_\gamma(\mathbf{L}^\vee\eta, \mathbf{L}^\vee\eta), \quad \forall \eta \in \mathbf{U}^\vee,$$

where  $\mathbf{L}^\vee : \mathbf{U}^\vee \rightarrow \mathbf{V}^\vee$  is the dual (transpose) of the linear map  $\mathbf{L}$ . Observe that if  $\gamma$  is nondegenerate and  $\mathbf{L}$  is surjective, then  $\mathbf{L}_*\gamma$  is also nondegenerate.

Suppose  $(\mathcal{S}, \mu)$  is a probability space. A *Gaussian* random vector on  $(\mathcal{S}, \mu)$  is a (Borel) measurable map

$$X : \mathcal{S} \rightarrow \mathbf{V}, \quad \mathbf{V} \text{ finite dimensional vector space}$$

such that  $X_*\mu$  is a Gaussian measure on  $\mathbf{V}$ . We will refer to this measure as the *associated Gaussian measure*, we denote it by  $\gamma_X$  and we denote by  $\boldsymbol{\Sigma}_X$  (respectively  $\mathbf{S}(X)$ ) its covariance form (respectively operator),

$$\boldsymbol{\Sigma}_X(\xi_1, \xi_2) = \mathbf{E}(\langle \xi_1, X - \mathbf{E}(X) \rangle \langle \xi_2, X - \mathbf{E}(X) \rangle).$$

Note that the expectation of  $\gamma_X$  is precisely the expectation of  $X$ . The random vector is called *nondegenerate*, respectively *centered*, if the Gaussian measure  $\gamma_X$  is such.

Let us point out that if  $X : \mathcal{S} \rightarrow \mathbf{U}$  is a Gaussian random vector and  $\mathbf{L} : \mathbf{U} \rightarrow \mathbf{V}$  is a linear map, then the random vector  $\mathbf{L}X : \mathcal{S} \rightarrow \mathbf{V}$  is also Gaussian. Moreover

$$\mathbf{E}(\mathbf{L}X) = \mathbf{L}\mathbf{E}(X), \quad \boldsymbol{\Sigma}_{\mathbf{L}X}(\xi, \xi) = \boldsymbol{\Sigma}_X(\mathbf{L}^\vee\xi, \mathbf{L}^\vee\xi), \quad \forall \xi \in \mathbf{V}^\vee,$$

where  $\mathbf{L}^\vee : \mathbf{V}^\vee \rightarrow \mathbf{U}^\vee$  is the linear map dual to  $\mathbf{L}$ . Equivalently,  $\mathbf{S}(\mathbf{L}X) = \mathbf{L}\mathbf{S}(X)\mathbf{L}^\vee$ .

Suppose that  $X_j : \mathcal{S} \rightarrow \mathbf{V}_j$ ,  $j = 1, 2$ , are two *centered* Gaussian random vectors such that the direct sum  $X_1 \oplus X_2 : \mathcal{S} \rightarrow \mathbf{V}_1 \oplus \mathbf{V}_2$  is also a centered Gaussian random vector with associated Gaussian measure

$$\gamma_{X_1 \oplus X_2} = p_{X_1 \oplus X_2}(\mathbf{x}_1, \mathbf{x}_2) |d\mathbf{x}_1 d\mathbf{x}_2|.$$

We obtain a bilinear form

$$\mathbf{cov}(X_1, X_2) : \mathbf{V}_1^\vee \times \mathbf{V}_2^\vee \rightarrow \mathbb{R}, \quad \mathbf{cov}(X_1, X_2)(\xi_1, \xi_2) = \boldsymbol{\Sigma}(\xi_1, \xi_2),$$

called the *covariance form*. The random vectors  $X_1$  and  $X_2$  are independent if and only if they are uncorrelated, i.e.,

$$\mathbf{cov}(X_1, X_2) = 0.$$

We can then identify  $\mathbf{cov}(X_1, X_2)$  with a linear operator  $\mathbf{Cov}(X_1, X_2) : \mathbf{V}_2 \rightarrow \mathbf{V}_1$ , via the equality

$$\begin{aligned} \mathbf{E}(\langle \xi_1, X_1 \rangle \langle \xi_2, X_2 \rangle) &= \mathbf{cov}(X_1, X_2)(\xi_1, \xi_2) \\ &= \langle \xi_1, \mathbf{Cov}(X_1, X_2) \xi_2^\dagger \rangle, \quad \forall \xi_1 \in \mathbf{V}_1^\vee, \quad \xi_2 \in \mathbf{V}_2^\vee, \end{aligned}$$

where  $\xi_2^\dagger \in \mathbf{V}_2$  denotes the vector metric dual to  $\xi_2$ . The operator  $\mathbf{Cov}(X_1, X_2)$  is called the *covariance operator* of  $X_1, X_2$ .

The conditional random variable  $(X_1|X_2 = x_2)$  has probability density

$$p_{(X_1|X_2=x_2)}(\mathbf{x}_1) = \frac{p_{X_1 \oplus X_2}(\mathbf{x}_1, \mathbf{x}_2)}{\int_{\mathbf{V}_1} p_{X_1 \oplus X_2}(\mathbf{x}_1, \mathbf{x}_2) |d\mathbf{x}_1|}.$$

For a measurable function  $f : \mathbf{V}_1 \rightarrow \mathbb{R}$  the conditional expectation  $\mathbf{E}(f(X_1)|X_2 = x_2)$  is the (deterministic) scalar

$$\mathbf{E}(f(X_1)|X_2 = x_2) = \int_{\mathbf{V}_1} f(\mathbf{x}_1) p_{(X_1|X_2=x_2)}(\mathbf{x}_1) |d\mathbf{x}_1|.$$

If  $X_2$  is nondegenerate, the *regression formula*, [4], implies that the random vector  $(X_1|X_2 = x_2)$  is a Gaussian vector with covariance operator

$$\mathbf{S}(X_1|X_2 = x_2) = \mathbf{S}(X_1) - \mathbf{Cov}(X_1, X_2) \mathbf{S}(X_2)^{-1} \mathbf{Cov}(X_2, X_1), \quad (\text{A.3})$$

and expectation

$$\mathbf{E}(X_1|x_2 = x_2) = Cx_2, \quad (\text{A.4})$$

where  $C$  is given by

$$C = \mathbf{Cov}(X_1, X_2) \mathbf{S}(X_2)^{-1}. \quad (\text{A.5})$$

## APPENDIX B. A CLASS OF RANDOM SYMMETRIC MATRICES

We denote by  $\mathcal{S}_m$  the space of real symmetric  $m \times m$  matrices. This is an Euclidean space with respect to the inner product

$$(A, B) := \text{tr}(AB).$$

This inner product is invariant with respect to the action of  $\text{SO}(m)$  on  $\mathcal{S}_m$ . We set

$$\widehat{\mathbf{E}}_{ij} := \begin{cases} \mathbf{E}_{ij}, & i = j \\ \frac{1}{\sqrt{2}} \mathbf{E}_{ij}, & i < j. \end{cases}$$

The collection  $(\widehat{\mathbf{E}}_{ij})_{i \leq j}$  is a basis of  $\mathcal{S}_m$  orthonormal with respect to the above inner product. We set

$$\hat{a}_{ij} := \begin{cases} a_{ij}, & i = j \\ \sqrt{2} a_{ij}, & i < j. \end{cases}$$

The collection  $(\hat{a}_{ij})_{i \leq j}$  the orthonormal basis of  $\mathcal{S}_m^\vee$  dual to  $(\widehat{\mathbf{E}}_{ij})$ . The volume density induced by this metric is

$$|dA| := \prod_{i \leq j} d\hat{a}_{ij} = 2^{\frac{1}{2} \binom{m}{2}} \prod_{i \leq j} da_{ij}.$$

Throughout the paper we encountered a 2-parameter family of Gaussian probability measures on  $\mathcal{S}_m$ . More precisely for any real numbers  $u, v$  such that

$$v > 0, \quad mu + 2v > 0,$$

we denote by  $\mathfrak{S}_m^{u,v}$  the space  $\mathfrak{S}_m$  equipped with the centered Gaussian measure  $d\Gamma_{u,v}(A)$  uniquely determined by the covariance equalities

$$\mathbf{E}(a_{ij}a_{k\ell}) = u\delta_{ij}\delta_{k\ell} + v(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \quad \forall 1 \leq i, j, k, \ell \leq m.$$

In particular we have

$$\mathbf{E}(a_{ii}^2) = u + 2v, \quad \mathbf{E}(a_{ii}a_{jj}) = u, \quad \mathbf{E}(a_{ij}^2) = v, \quad \forall 1 \leq i \neq j \leq m.$$

while all other covariances are trivial. The ensemble  $\mathfrak{S}_m^{0,v}$  is a rescaled version of the Gaussian Orthogonal Ensemble (GOE) and we will refer to it as  $\text{GOE}_m^v$ .

For  $u > 0$  the ensemble  $\mathfrak{S}_m^{u,v}$  can be given an alternate description. More precisely a random  $A \in \mathfrak{S}_m^{u,v}$  can be described as a sum

$$A = B + X\mathbb{1}_m, \quad B \in \text{GOE}_m^v, \quad X \in \mathbf{N}(0, u), \quad B \text{ and } X \text{ independent.}$$

We write this

$$\mathfrak{S}_m^{u,v} = \text{GOE}_m^v \hat{+} \mathbf{N}(0, u)\mathbb{1}_m, \quad (\text{B.1})$$

where  $\hat{+}$  indicates a sum of *independent* variables.

The Gaussian measure  $d\Gamma_{u,v}$  coincides with the Gaussian measure  $d\Gamma_{u+2v, u, v}$  defined in [16, App. B]. We recall a few facts from [16, App. B].

The probability density  $d\Gamma_{u,v}$  has the explicit description

$$d\Gamma_{u,v}(A) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}} \sqrt{D(u,v)}} e^{-\frac{1}{4v} \text{tr} A^2 - \frac{u'}{2} (\text{tr} A)^2} |dA|,$$

where

$$D(u, v) = (2v)^{(m-1)+\binom{m}{2}} (mu + 2v),$$

and

$$u' = \frac{1}{m} \left( \frac{1}{mu + 2v} - \frac{1}{2v} \right) = -\frac{u}{2v(mu + 2v)}.$$

In the special case  $\text{GOE}_m^v$  we have  $u = u' = 0$  and

$$d\Gamma_{0,v}(A) = \frac{1}{(2\pi v)^{\frac{m(m+1)}{4}}} e^{-\frac{1}{4v} \text{tr} A^2} |dA|. \quad (\text{B.2})$$

We have a *Weyl integration formula* [2] which states that if  $f : \mathfrak{S}_m \rightarrow \mathbb{R}$  is a measurable function which is invariant under conjugation, then the value  $f(A)$  at  $A \in \mathfrak{S}_m$  depends only on the eigenvalues  $\lambda_1(A) \leq \dots \leq \lambda_n(A)$  of  $A$  and we have

$$\mathbf{E}_{\text{GOE}_m^v}(f(X)) = \frac{1}{\mathbf{Z}_m(v)} \int_{\mathbb{R}^m} f(\lambda_1, \dots, \lambda_m) \underbrace{\left( \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| \right) \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}}}_{=: Q_{m,v}(\lambda)} |d\lambda_1 \dots d\lambda_m|, \quad (\text{B.3})$$

where the normalization constant  $\mathbf{Z}_m(v)$  is defined by

$$\begin{aligned} \mathbf{Z}_m(v) &= \int_{\mathbb{R}^m} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}} |d\lambda_1 \dots d\lambda_m| \\ &= (2v)^{\frac{m(m+1)}{4}} \underbrace{\int_{\mathbb{R}^m} \prod_{1 \leq i < j \leq m} |\lambda_i - \lambda_j| \prod_{i=1}^m e^{-\frac{\lambda_i^2}{2}} |d\lambda_1 \dots d\lambda_m|}_{=: \mathbf{Z}_m}. \end{aligned}$$

The precise value of  $\mathbf{Z}_m$  can be computed via Selberg integrals, [2, Eq. (2.5.11)], and we have

$$\mathbf{Z}_m = (2\pi)^{\frac{m}{2}} m! \prod_{j=1}^m \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{1}{2})} = 2^{\frac{m}{2}} m! \prod_{j=1}^m \Gamma\left(\frac{j}{2}\right). \quad (\text{B.4})$$

For any positive integer  $n$  we define the *normalized* 1-point correlation function  $\rho_{n,v}(x)$  of  $\text{GOE}_n^v$  to be

$$\rho_{n,v}(x) = \frac{1}{\mathbf{Z}_n(v)} \int_{\mathbb{R}^{n-1}} Q_{n,v}(x, \lambda_2, \dots, \lambda_n) d\lambda_1 \cdots d\lambda_n.$$

For any Borel measurable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  we have [7, §4.4]

$$\frac{1}{n} \mathbf{E}_{\text{GOE}_n^v}(\text{tr } f(X)) = \int_{\mathbb{R}} f(\lambda) \rho_{n,v}(\lambda) d\lambda. \quad (\text{B.5})$$

The equality (B.5) characterizes  $\rho_{n,v}$ . Let us observe that for any constant  $c > 0$ , if

$$A \in \text{GOE}_n^v \iff cA \in \text{GOE}_n^{c^2v}.$$

Hence for any Borel set  $B \subset \mathbb{R}$  we have

$$\int_{cB} \rho_{n,c^2v}(x) dx = \int_B \rho_{n,v}(y) dy.$$

We conclude that

$$c\rho_{n,c^2v}(cy) = \rho_{n,v}(y), \quad \forall n, c, y. \quad (\text{B.6})$$

The behavior of the 1-point correlation function  $\rho_{n,v}(x)$  for  $n$  large is described by *Wigner semicircle law* which states that for any  $v > 0$  the sequence of measure on  $\mathbb{R}$

$$\rho_{n,vn^{-1}}(x) dx = n^{\frac{1}{2}} \rho_{n,v}(n^{\frac{1}{2}}x) dx$$

converges weakly as  $n \rightarrow \infty$  to the semicircle distribution

$$\rho_{\infty,v}(x) |dx| = \mathbf{I}_{\{|x| \leq 2\sqrt{v}\}} \frac{1}{2\pi v} \sqrt{4v - x^2} |dx|.$$

The expected value of the absolute value of the determinant of a random  $A \in \text{GOE}_m^v$  can be expressed neatly in terms of the correlation function  $\rho_{m+1,v}$ . More precisely we have the following result first observed by Y.V. Fyodorov [9] in a context related to ours.

**Lemma B.1.** *Suppose  $v > 0$  Let  $c \in \mathbb{R}$ . Then*

$$\mathbf{E}_{\text{GOE}_m^v}(|\det(A - c\mathbb{1}_m)|) = 2^{\frac{3}{2}} (2v)^{\frac{m+1}{2}} \Gamma\left(\frac{m+3}{2}\right) e^{\frac{c^2}{4v}} \rho_{m+1,v}(c).$$

*Proof.* Using the Weyl integration formula we deduce

$$\begin{aligned} \mathbf{E}_{\text{GOE}_m^v}(|\det(A - c\mathbb{1}_m)|) &= \frac{1}{\mathbf{Z}_m(v)} \int_{\mathbb{R}^m} \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}} |c - \lambda_i| \prod_{i \leq j} |\lambda_i - \lambda_j| d\lambda_1 \cdots d\lambda_m \\ &= \frac{e^{\frac{c^2}{4v}}}{\mathbf{Z}_m(v)} \int_{\mathbb{R}^m} e^{-\frac{c^2}{4v}} \prod_{i=1}^m e^{-\frac{\lambda_i^2}{4v}} |c - \lambda_i| \prod_{i \leq j} |\lambda_i - \lambda_j| d\lambda_1 \cdots d\lambda_m \\ &= \frac{e^{\frac{c^2}{4v}} \mathbf{Z}_{m+1}(v)}{\mathbf{Z}_m(v)} \frac{1}{\mathbf{Z}_{m+1}(v)} \int_{\mathbb{R}^m} Q_{m+1,v}(c, \lambda_1, \dots, \lambda_m) d\lambda_1 \cdots d\lambda_m \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{\frac{c^2}{4v}} \mathbf{Z}_{m+1}(v)}{\mathbf{Z}_m(v)} \rho_{m+1,v}(c) = v^{\frac{m+1}{2}} \frac{e^{\frac{c^2}{4v}} \mathbf{Z}_{m+1}}{\mathbf{Z}_m} \rho_{m+1,v}(c) \\
&= (m+1)\sqrt{2}(2v)^{\frac{m+1}{2}} e^{\frac{c^2}{4v}} \Gamma\left(\frac{m+1}{2}\right) \rho_{m+1,v}(c) = 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}} \Gamma\left(\frac{m+3}{2}\right) e^{\frac{c^2}{4v}} \rho_{m+1,v}(c).
\end{aligned}$$

□

The above result admits the following generalization, [3, Lemma 3.2.3].

**Lemma B.2.** *Let  $u > 0$ . Then*

$$\mathbf{E}_{\mathcal{S}_m^{u,v}}(|\det(A - c\mathbb{1}_m)|) = 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}} \Gamma\left(\frac{m+3}{2}\right) \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{\frac{(c-x)^2}{4v} - \frac{x^2}{2u}} dx.$$

In particular, if  $u = 2kv$ ,  $k < 1$  we have

$$\mathbf{E}_{\mathcal{S}_m^{2kv,v}}(|\det(A - c\mathbb{1}_m)|) = 2^{\frac{3}{2}}(2v)^{\frac{m}{2}} \Gamma\left(\frac{m+3}{2}\right) \frac{1}{\sqrt{2\pi k}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{-\frac{1}{4vt_k^2}(x+t_k^2c)^2 + \frac{(t_k^2+1)c^2}{4v}} dx,$$

( $\lambda := c - x$ )

$$= 2^{\frac{3}{2}}(2v)^{\frac{m}{2}} \Gamma\left(\frac{m+3}{2}\right) \frac{1}{\sqrt{2\pi k}} \int_{\mathbb{R}} \rho_{m+1,v}(\lambda) e^{-\frac{1}{4vt_k^2}(\lambda - (t_k^2+1)c)^2 + \frac{(t_k^2-1)c^2}{4v}} d\lambda,$$

where

$$t_k^2 := \frac{1}{\frac{1}{k} - 1} = \frac{k}{1-k}.$$

*Proof.* Recall the equality (B.1)

$$\mathcal{S}_m^{u,v} = \text{GOE}_m^v \hat{+} \mathbf{N}(0, u) \mathbb{1}_m.$$

We deduce that

$$\begin{aligned}
&\mathbf{E}_{\mathcal{S}_m^{u,v}}(|\det(A - c\mathbb{1}_m)|) = \mathbf{E}(\det(B + (X - c)\mathbb{1})) \\
&= \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \mathbf{E}_{\text{GOE}_m^v}(|\det(B - (c - X)\mathbb{1}_m)| \mid X = x) e^{-\frac{x^2}{2u}} dx \\
&= \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \mathbf{E}_{\text{GOE}_m^v}(|\det(B - (c - x)\mathbb{1}_m)|) e^{-\frac{x^2}{2u}} dx \\
&= 2^{\frac{3}{2}}(2v)^{\frac{m+1}{2}} \Gamma\left(\frac{m+3}{2}\right) \frac{1}{\sqrt{2\pi u}} \int_{\mathbb{R}} \rho_{m+1,v}(c-x) e^{\frac{(c-x)^2}{4v} - \frac{x^2}{2u}} dx.
\end{aligned}$$

Now observe that if  $u = 2kv$  then

$$\begin{aligned}
&\frac{(c-x)^2}{4v} - \frac{x^2}{2u} = -\frac{x^2}{4kv} + \frac{1}{4v}(x^2 - 2cx + c^2) \\
&= \frac{1}{4v} \left( -\frac{1}{t_k^2} x^2 - 2cx - c^2 t_k^2 \right) + \frac{c^2(1+t_k^2)}{4v} = -\frac{1}{4vt_k^2}(x + t_k^2c)^2 + \frac{c^2(1+t_k^2)}{4v}.
\end{aligned}$$

□

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