Solutions to Homework # 1

Hatcher, Chap. 0, Problem 4. Denote by i_A the inclusion map $A \hookrightarrow X$. Consider a homotopy $F : X \times I \to X$ such that

$$F_0 := \mathbb{1}_X, \;\; F_1(X) \subset A, \;\; F_t(A) \subset A.$$

We claim that $g := F_1$ is a homotopy inverse of i_A, i.e.

$$g \circ i_A \simeq \mathbb{1}_A, \;\; i_A \circ g \simeq \mathbb{1}_X.$$

To prove the first part consider the homotopy $g_t = F_{1-t}|_A$. Observe that

$$g_0 = g \circ i_A, \;\; g_1 = F_0 \circ i_A = \mathbb{1}_A.$$

To prove the second part we consider the homotopy $H_t = F_{1-t} : X \to X$. Observe that $F_1 = i_A \circ F_1$ since $F_1(X) \subset A$. On the other hand, $F_0 = \mathbb{1}_X$.

□

Hatcher, Chap. 0, Problem 5. Suppose $F : X \times I \to X$ is a deformation retraction of X onto a point x_0. This means

$$F_t(x_0) = x_0, \;\; \forall t, \;\; F_0 = \mathbb{1}_X, \;\; F_1(X) = \{x_0\}.$$

We want to prove a slightly stronger statement, namely, that for any neighborhood U of x_0 there exists a smaller neighborhood $V \subset U$ of x_0 such that $F_t(V) \subset U$, $\forall t \in I$.

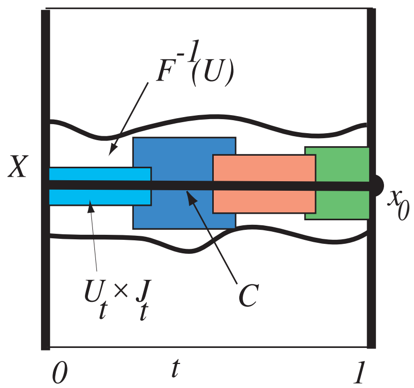


Figure 1: *Constructing contractible neighborhoods of* x_0.

Consider the pre-image of U via F,

$$F^{-1}(U) = \Big\{(x,t) \in X \times I; \;\; F_t(x) \in U\Big\}.$$

Note that $C := \{x_0\} \times I \subset F^{-1}(U)$ (see Figure 1).

For every $t \in I$ we can find a neighborhood U_t of $x_0 \in X$, and a neighborhood J_t of $t \in I$ such that (see Figure 1)

$$U_t \times J_t \subset F^{-1}(U).$$

The set C is covered by the family of open sets $\{U_t \times J_t\}_{t \in I}$, and since C is compact, we can find $t_1, \dots, t_n \in I$ such that

$$C \subset \bigcup_k U_{t_k} \times J_{t_k}.$$

In particular, the set

$$V := \bigcap_k U_{t_k}$$

is an open neighborhood of x_0, and $V \times I \subset F^{-1}(U)$. This means $F_t(V) \subset U$, $\forall t$, i.e. we can regard F_t as a map from V to U, for any t.

If we denote by i_V the inclusion $V \hookrightarrow U$ we deduce that the composition $F_t \circ i_V$ defines a homotopy

$$F : V \times I \to U$$

between $F_0 = i_V$ and $F_1 =$ the constant map. In other words i_V is null-homotopic. □

Hatcher, Chap. 0, Problem 9. Suppose X is contractible and $A \hookrightarrow X$ is a retract of X. Choose a retraction $r : X \to A$, and a contraction of X to a point which we can assume lies in A

$$F : X \times I \to X, \;\; F_0 = \mathbb{1}_X, \;\; F_1(x) = a_0, \;\; \forall x.$$

Consider the composition

$$G : A \times I \stackrel{i_A \times \mathbb{1}_I}{\longrightarrow} X \times I \stackrel{F}{\to} X \stackrel{r}{\to} A.$$

This is a homotopy between the identity map $\mathbb{1}_A$ and the constant map $A \to \{a_0\}$.

□

Hatcher, Chap. 0, Problem 14. We denote by c_i the number of i-cells. In Figure 2 we have depicted three cell decompositions of the 2-sphere. The first one has

$$c_0 = 1 = c_2, \;\; c_1 = 0.$$

The second one has

$$c_0 = n + 1, \;\; c_1 = n, \;\; c_2 = 1, \;\; n > 0.$$

The last one has

$$c_0 = n + 1, \;\; c_1 = n + k, \;\; c_2 = k + 1, \;\; k \geq 0.$$

Any combination of nonnegative integers c_0, c_1, c_2 such that

$$c_0 - c_1 + c_2 = 2, \;\; c_0, c_2 > 0$$

belongs to one of the three cases depicted in Figure 2. □

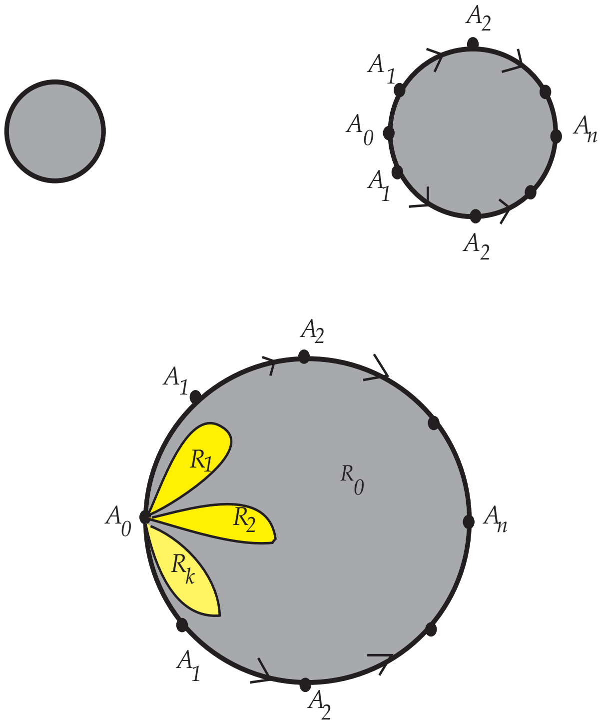


Figure 2: *Cell decompositions of the 2-sphere.*

Solutions to Homework # 2

Hatcher, Chap. 0, Problem 16.[1] Let

$$\mathbb{R}^\infty := \bigoplus_{n\geq 1}\mathbb{R} = \Big\{\vec{x} = (x_k)_{k\geq 1};\ \ \exists N:\ \ x_n = 0,\ \ \forall n\geq N\Big\}.$$

We define a topology on $\mathbb{R}^\infty$ by declaring a set $S\subset\mathbb{R}^\infty$ closed if and only if, $\forall n\geq 0$, the intersection S of with the finite dimensional subspace

$$\mathbb{R}^n = \big\{(x_k)_{k\geq 1};\ \ x_k = 0,\ \ \forall k > n\,\big\},$$

is closed in the Euclidean topology of $\mathbb{R}^n$. For each $\vec{x}\in\mathbb{R}^\infty$ set

$$|\vec{x}| := \Big(\sum_{k=0}^{\infty} x_k^2\Big)^{1/2}.$$

S^∞ is homeomorphic to the "unit sphere" in $\mathbb{R}^\infty$, $S^\infty \cong \big\{\vec{x}\in\mathbb{R}^\infty;\ \ |\vec{x}| = 1\big\}$.

Observe that S^∞ is a deformation retract of $\mathbb{R}^\infty\setminus\{0\}$ so it suffices to show that $\mathbb{R}^\infty\setminus\{0\}$ is contractible. Define $F:\mathbb{R}^\infty\times[0,1]\to\mathbb{R}^\infty$ by

$$(\vec{x},t)\mapsto F_t(\vec{x}) = \Big((1-t)x_0, tx_0 + (1-t)x_1, tx_1 + (1-t)x_2,\dots\Big)$$

Observe that $F_t(\mathbb{R}^\infty\setminus\{0\})\subset\mathbb{R}^\infty\setminus\{0\}$, $\forall t\in[0,1]$.

Indeed, this is obviously the case for F_0 and F_1. Suppose $t\in(0,1)$, and $F_t(\vec{x}) = 0$. This means

$$x_0 = 0,\ \ x_{k+1} = \frac{t}{t-1}x_k,\ \ \forall k = 0,1,2,\dots,$$

so that $\vec{x} = 0$.

We have thus constructed a homotopy $F:\mathbb{R}^\infty\setminus\{0\}\times I\to\mathbb{R}^\infty\setminus\{0\}$ between $F_0 = \mathbb{1}$ and $F_1 = S$, the shift map, $(x_0,x_1,x_2,\cdots)\stackrel{S}{\longmapsto}(0,x_0,x_1,x_2,\dots)$. It is convenient to write this map as $\vec{x}\mapsto(0,\vec{x})$.

Consider now the homotopy $G:(0\oplus\mathbb{R}^\infty\setminus\{0\})\times I\to\mathbb{R}^\infty\setminus\{0\}$ given by

$$G_t(0,\vec{x}) = (t,(1-t)\cdot\vec{x}).$$

If we first deform $\mathbb{R}^\infty\setminus\{0\}$ to $0\oplus\mathbb{R}^\infty\setminus\{0\}$ following F_t, and then to $(1,0)\in\mathbb{R}^\infty$ following G_t, we obtain the desired contraction of $\mathbb{R}^\infty\setminus\{0\}$ to a point. □

FIGURE 1. *This CW-complex deformation retracts to both the cylinder (yellow) and the Möbius band (grey).*

Hatcher, Chap. 0, Problem 17. (b) Such a *CW* complex is depicted in Figure 1. For part (a) consider a continuous map $f:S^1\to S^1$. Fix a point a in S^1. A cell decomposition

[1]See Example 1.B.3 in Hatcher's book.

is depicted in Figure 2. It consists of two vertices $a, f(a)$, three 1-cells e_0, e_1, t, and a single 2-cell C. The attaching map of C maps the right vertical side of C onto $S^1 = e_1/\partial e_1$ via f.

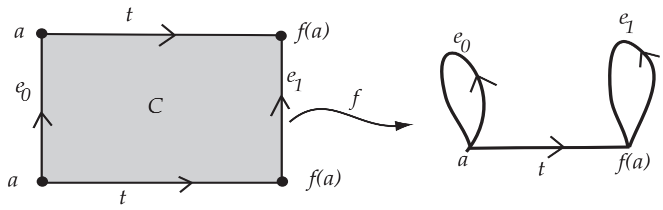


FIGURE 2. *A cell decomposition of a map $f : S^1 \to S^1$.*

□

Hatcher, Chap. 0, Problem 22. We investigate each connected component of the graph separately so we may as well assume that the graph is connected. We distinguish two cases.

Case 1. *The graph has vertices on the boundary of the half plane.* We can deform the graph inside the half-plane so that all its vertices lie on the boundary of the half-plane (see Figure 3). More precisely, we achieve this by collapsing the edges which connect *two* different vertices, and one of them is in the interior of the half-plane.

Rotating this collapsed graph we obtain a closed subset X of $\mathbb{R}^3$ which is a finite union of sets of the type R or S as illustrated in Figure 3. More precisely, when an edge connecting different vertices is rotated, we obtain a region of type S which is a 2-sphere. When a loop is rotated, we obtain a region of type R, which is a 2-sphere with a pair of points identified.

Two regions obtained by rotating two different edges will intersect in as many points as the two edges. Thus, two regions of X can intersect in $0, 1$ or 2 points. Using the arguments in Example 0.8 and 0.9 in **Hatcher** we deduce that X is a wedge of S^1's and S^2's.

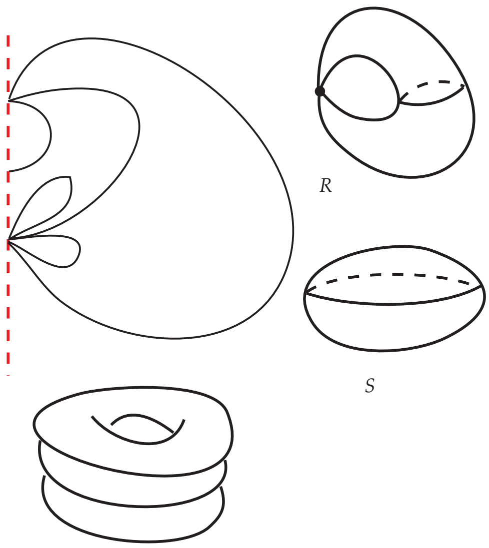


FIGURE 3. *Rotating a planar graph.*

Case 2. *There are no vertices on the boundary.* In this case the graph can be deformed inside the half plane to a wedge of circles. By rotating this wedge we obtain a space homotopic to collection of tori piled one on top another (see Figure 3). □

Hatcher, Chap. 0, Problem 23. Suppose A, B are contractible subcomplexes of X such that $X = A \cup B$, and $A \cap B$ is also contractible. Since B is contractible we deduce $X/B \simeq X$. The inclusion $A \hookrightarrow X$ maps $A \cap B$ into B, and thus defines an injective continuous map

$$j : A/A \cap B \hookrightarrow X/B \simeq X.$$

Since $X = A \cup B$, the above map is a *bijection.* Note also that j maps closed sets to closed sets. From the properties of quotient topology we deduce that j is a *homeomorphism.*

Now observe that since $A \cap B$ is contractible we deduce

$$A \simeq A/A \cap B$$

so that $A/A \cap B$ is contractible.

□

Sec. 1.1, Problem 5. (a) $\Longrightarrow$ (b) Suppose we are given a map $f : S^1 \to X$. We want to prove that it extends to a map $\tilde{f} : D^2 \to X$, given that f is homotopic to a constant. Consider a homotopy

$$F : S^1 \times I \to X, \;\; F(e^{\mathbf{i}\theta}, 0) = x_0 \in X, \;\; F(e^{\mathbf{i}\theta}, 1) = f(e^{\mathbf{i}\theta}), \;\; \forall \theta \in [0, 2\pi].$$

Identify D^2 with the set of complex numbers of norm ≤ 1 and set

$$\tilde{f}(re^{\mathbf{i}\theta}) = F(e^{\mathbf{i}\theta}, r).$$

(b)$\Longrightarrow$ (c)Suppose $f : (S^1, 1) \to (X, x_0)$ is a loop at $x_0 \in X$ we want to show that $[f] = 1 \in \pi_1(X, x_0)$. From (b) we deduce that there exists $\tilde{f} : (D^2, 1) \to (X, x_0)$ such that the diagram below is commutative.

$$\begin{array}{ccc} (S^1, 1) & \overset{i}{\hookrightarrow} & (D^2, 1) \\ & \overset{f}{\searrow} & \Big\downarrow \tilde{f} \\ & & (X, x_0) \end{array}.$$

We obtain the following commutative diagram of group morphisms.

$$\begin{array}{ccc} \pi_1(S^1, 1) & \overset{i_*}{\longrightarrow} & \pi_1(D^2, 1) \\ & \overset{f_*}{\searrow} & \Big\downarrow \tilde{f}_* \\ & & \pi_1(X, x_0) \end{array}.$$

Since $\pi_1(D^2, 1) = \{1\}$ we deduce that i_* is the trivial morphism so that $f_* = \tilde{f}_* \circ i_*$ must be the trivial morphism as well.

The identity map $\mathbf{1}_{S^1} : (S^1, 1) \to (S^1, 1)$ defines a loop on S^1 whose homotopy class is a generator of $\pi_1(S^1, 1)$, and we have $f_*([\mathbf{1}_{S^1}])$ is trivial in $\pi_1(X, x_0)$. This homotopy class is precisely the homotopy class represented by the loop f.

(c) $\Longrightarrow$ (a). Obvious.

□

Sec. 1.1, Problem 9. Assume the sets A_i are *open, bounded and connected.*

Set

$$A := A_1 \cup A_2 \cup A_3, \;\; V_i := \operatorname{vol}(A_i).$$

For every unit vector $\vec{n} \in S^2$ and every t we denote by $H^+_{\vec{n},t}$ the half space determined by the plane through $t\vec{n}$, of normal vector $\vec{n}$, and situated on the same side of this plane as $\vec{n}$. More precisely, if $(\bullet, \bullet)$ denotes the Euclidean inner product in $\mathbb{R}^3$, then

$$H^+_{\vec{n},t} := \Big\{ \vec{x} \in \mathbb{R}^3; \;\; (\vec{x}, \vec{n}) \geq t \Big\}.$$

Set

$$V_3^+(\vec{n},t) := \mathrm{vol}\,(A_3 \cap H^+_{\vec{n},t}).$$

Observe that $t \mapsto V_3^+(\vec{n},t)$ is a continuous, non-increasing function such that

$$\lim_{t\to\infty} V_3^+(\vec{n},t) = 0, \quad \lim_{t\to-\infty} V_3^+(\vec{n},t) = V_3.$$

The intermediate value theorem implies that the level set

$$S_{\vec{n}} = \Big\{ t \in \mathbb{R};\;\; V_3^+(\vec{n},t) = \frac{1}{2}V_3 \Big\}$$

is closed and bounded so it must be compact. $t \mapsto V^+(\vec{n},t)$ is non-increasing we deduce that $S_{\vec{n}}$ must be a closed, bounded interval of the real line. Set

$$t_{min}(\vec{n}) := \min S_{\vec{n}}, \;\; T_{max}(\vec{n}) := \max S_{\vec{n}}, \;\; s(\vec{n}) = \frac{1}{2}(t_{min}(\vec{n}) + T_{max}(\vec{n})).$$

The numbers $t(\vec{n})$, and $T(\vec{n})$ have very intuitive meanings. Think of the family of hyperplanes

$$H_t := \{\vec{x} \in \mathbb{R}^3;\;\; (\vec{x},\vec{n}) = t\}$$

as a hyperplane depending on time t, which moves while staying perpendicular to $\vec{n}$. For $t \ll 0$ the entire region A_3 will be on the side of H_t determined by $\vec{n}$, while for very large t the region A_3 will be on the other side of H_t, determined by $-\vec{n}$. Thus there must exist moments of time when H_t divides A into regions of equal volume. $t_{min}(\vec{n})$ is the first such moment, and $T_{max}(\vec{n})$ is the last such moment. Observe that

$$T_{max}(-\vec{n}) = -t_{min}(\vec{n}), \;\; t_{min}(-\vec{n}) = -T_{max}(\vec{n}), \;\; s(-\vec{n}) = -s(\vec{n}).$$

Set

$$H^+_{\vec{n}} := H^+_{\vec{n},s(\vec{n})}.$$

Observe that $H^+_{\vec{n}}$ and $H^+_{-\vec{n}}$ are complementary half-spaces.

Lemma 1. *$S_{\vec{n}}$ consists of a single point so that $t_{min}(\vec{n}) = T_{max}(\vec{n}) = s(\vec{\eta})$.*

Lemma 2. *The map $S^2 \ni \vec{n} \to s(\vec{n}) \in \mathbb{R}$ is continuous*

We will present the proofs of these lemmata after we have completed the proof of the claim in problem 9.

Set

$$V_i^+(\vec{n}) = \mathrm{vol}\,(\,A_i \cap H^+_{\vec{n}}\,), \;\; i = 1,2,3.$$

We need to prove that there exists $\vec{n} \in S^2$ such that

$$V_i^+(\vec{n}) = \frac{1}{2}V_i, \;\; i = 1,2,3.$$

Note that $V_3^+(\vec{n}) = \frac{1}{2}V_3$ so we only need to find $\vec{n}$ such that

$$V_i^+(\vec{n}) = \frac{1}{2}V_i, \quad i = 1, 2.$$

Define

$$f : S^2 \to \mathbb{R}^2, \quad f(\vec{n}) := \Big(V_1^+(\vec{n}) + V_2^+(\vec{n}), V_1^+(\vec{n}) \Big).$$

$H_{\vec{n}}^+$ and $H_{-\vec{n}}^+ = \mathbb{R}^3$ are complementary half spaces so that

$$V_i^+(\vec{n}) + V_i^+(-\vec{n}) = \mathrm{vol}\,(A_i), \quad i = 1, 2, 3. \tag{1}$$

Lemma 2 implies that f is continuous, and using the Borsuk-Ulam theorem we deduce that there exists $\vec{n}_0$ such that

$$f(\vec{n}_0) = f(-\vec{n}_0).$$

The equality (1) now implies that

$$V_1^+(\vec{n}_0) + V_2^+(\vec{n}_0) = \frac{1}{2}\Big(\mathrm{vol}\,(A_1) + \mathrm{vol}\,(A_2)\Big),$$

and

$$V_1^+(\vec{n}_0) = \frac{1}{2}\mathrm{vol}\,(A_1).$$

These equalities imply that $V_2^+(\vec{n}_0) = \frac{1}{2}\mathrm{vol}\,(A_2)$.

□

Proof of Lemma 1. Observe that since the set A_3 is compact we can find a sufficiently large $R > 0$ such that

$$A_3 \subset B_R(0).$$

Set for brevity

$$G_{\vec{n}}(t) = V_3^+(\vec{n}, t).$$

Observe that for each $\vec{n}$ we have

$$G_{\vec{n}}(t) = 0, \quad \forall t \geq R, \quad G_{\vec{n}}(t) = V_3, \quad \forall t \leq -R.$$

We claim that for every $t \in S_{\vec{n}}$ there exists $\varepsilon_t > 0$ such that $\forall h \in (0, \varepsilon_t)$ we have

$$G_{\vec{n}}(t-h) > G_{\vec{n}}(t) > G_{\vec{n}}(t+h),$$

which shows that if $S_{\vec{n}}$ were an interval then $G_{\vec{n}}$ could not have a constant value $(V_3/2)$ along it.

Now observe that

$$G_{\vec{n}}(t-h) - G_{\vec{n}}(t) = \mathrm{vol}\,\Big(A_3 \cap \{\vec{x};\ t-h < (\vec{x}, \vec{n}) < t\}\Big)$$

Now observe that the region $A_3 \cap \{\vec{x};\ t-h < (\vec{x},\vec{n}) < t\}$ is open. Since A_3 *is connected* we deduce that for every h sufficiently small it must be *nonempty* and thus it has *positive* volume. The inequality $G_{\vec{n}}(t) > G_{\vec{n}}(t+h)$ is proved in a similar fashion.

□

Proof of Lemma 2. We continue to use the same notations as above.

Suppose $\vec{n}_k \to \vec{n}_0$ as $k \to \infty$. Set $G_k := G_{\vec{n}_k}$, $G_0 := G_{\vec{n}_0}$. Note that

$$\lim_{k\to\infty} G_k(t) = G_0(t), \quad \forall t \in [-R, R] \tag{2}$$

On the other hand

$$\begin{aligned} |G_k(t+h) - G_k(t)| &= \mathrm{vol}\Big(A_3 \cap \big\{\vec{x};\ \ t \le (\vec{x},\vec{n}_k) \le t+h\big\}\Big) \\ &\le \mathrm{vol}\Big(B_R(0) \cap \big\{\vec{x};\ \ t \le (\vec{x},\vec{n}_k) \le t+h\big\}\Big) \le \pi R^2 h \end{aligned} \tag{3}$$

so that the family of functions (G_k) is equicontinuous. Using (2) we deduce from the Arzela-Ascoli theorem that the sequence of function G_k converges *uniformly* to G_0 on $[-R, R]$.

Observe that the sequence $t_{min}(\vec{n}_k)$ lies $[-R, R]$ so it has a convergent subsequence. Choose such a subsequence $\tau_j := t_{min}(\vec{n}_{k_j}) \to t_0 \in [-R, R]$. Since the sequence G_{k_j} converges uniformly to G_0 and

$$G_{k_j}(\tau_j) = V_3/2$$

we deduce[1]

$$G_0(t_0) = V_3/2,$$

so that $t_0 \in S_{\vec{n}_0}$. Since $S_{\vec{n}}$ consists of a single point we deduce that for every convergent subsequence of $t_{min}(\vec{n}_k)$ we have

$$\lim_{j\to\infty} t_{min}(\vec{n}_{k_j}) = t_{min}(\vec{n}_0).$$

This proves the continuity of $\vec{n} \mapsto s(\vec{n}) = t_{min}(\vec{n})$.

□

Sec. 1.1, Problem 16. We argue by contradiction in each of the situations (a)-(f). Suppose there exists a retraction $r : X \to A$.

(a) In this case r_* would induce a surjection from the trivial group $\pi_1(\mathbb{R}^3, p)$ to the integers $\pi_1(S^1, p)$.

(b) In this case r_* would induce a surjection from the infinite cyclic group $\pi_1(S^1 \times D^2)$ to the direct product of infinite cyclic groups $\pi_1(S^1 \times S^1)$. This is not possible since

$$\mathrm{rank}\,\pi_1(S^1 \times S^1) = 2 > 1 = \mathrm{rank}\,\pi_1(S^1 \times D^2).$$

[1]This also follows directly form (3) without invoking the Arzela-Ascoli theorem.

(c) The inclusion $i : A \hookrightarrow X$ induces the trivial morphism $i_* : \pi_1(A) \to \pi_1(X)$. Hence $\mathbf{1}_{\pi_1(A)} = r_* \circ i_*$ is trivial. This is a contradiction since $\pi_1(A)$ is not trivial.
(d) Observe first that S^1 is a retract of $S^1 \vee S^1$ so that there exist surjections

$$\pi_1(S^1 \vee S^1) \twoheadrightarrow \pi_1(S^1).$$

In particular $\pi_1(S^1 \vee S^1)$ is nontrivial so that there cannot exist surjections $\pi_1(D^2 \vee D^2) \twoheadrightarrow \pi_1(S^1 \vee S^1)$.
(e) Let p, q be two distinct points on ∂D^2, and $X = D^2/\{p, q\}$. Denote by x_0 the point in X obtained by identifying p and q. The chord $\tilde{C}$ connecting p and q defines a circle C on X. C is a deformation retract of X so that

$$\pi_1(X) \cong \pi_1(C) \cong \mathbb{Z}.$$

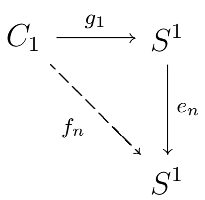
To prove that A is not a retract of X it suffices to show that $\pi_1(S^1 \vee S^1)$ is not a quotient of $\mathbb{Z}$. We argue[2] by contradiction.

Suppose $\pi_1(S^1 \vee S^1)$ is a quotient of $\mathbb{Z}$. Since there are surjections $\pi_1(S^1 \wedge S^1) \to \mathbb{Z}$ we deduce that $\pi_1(S^1 \vee S^1)$ must be isomorphic to $\mathbb{Z}$. In particular there exists *exactly two* surjections

$$\pi_1(S^1 \wedge S^1) \twoheadrightarrow \mathbb{Z}.$$

We now show that in fact there are infinitely many thus yielding a contraction. We denote the two circles entering into $S^1 \vee S^1$ by C_1 and C_2. Since C_i is a deformation retract of $C_1 \wedge C_2$ we deduce that $[C_i]$ is an element of infinite order in $\pi_1(C_1 \vee C_2)$.

Denote by $e_n : S^1 \to S^1$ the map $\theta \mapsto e^{\mathbf{i}\theta}$. Fix homeomorphisms $g_i : C_i \to S^1$ and define $f_n : C_1 \to C_2$ by the composition



$f_n := e_n \circ g_1.$

Define $r_n : C_1 \vee C_2 \to S_1$ by

$$r_n \,|_{C_1} = f_n, \;\; r_n \,|_{C_2} = g_2 \tag{4}$$

Observe that

$$r_{n*}([C_1]) = [e_n] \in \pi_1(S^1), \;\; r_{n*}([C_2]) = [e_1] \in \pi_1(S^1)$$

Using the isomorphism $\mathbb{Z} \to \pi_1(S^1)$, $n \mapsto [e_n]$ we deduce that $r_{n*} \neq r_{m*}$ if $n \neq m$.

[2]We can achieve this much faster invoking Seifert-vanKampen theorem.

(f) Observe first that $\pi_1(X) \cong \mathbb{Z}$, where the generator is the core circle C of the Móbus band. A is a circle so that $\pi_1(A) \cong \mathbb{Z}$. In terms of these isomorphisms the morphism $i_* : \pi_1(A) \to \pi_1(X)$ induced by $i : A \hookrightarrow X$ has the description

$$i_*(n[A]) = 2n[C].$$

Clearly there cannot exist any surjection $f : \pi_1(X) \twoheadrightarrow \pi_1(A)$ such that

$$[A] = f \circ i_*([A]) = 2k[A], \;\; k[A] := f_*([C]).$$

□

Sec. 1.1, Problem 17. We have already constructed these retraction in (4). Using the notations there we define

$$R_n : C_1 \vee C_2 \to C_2$$

by $R_n := g_2^{-1} \circ r_n$. Since

$$R_{n*} \neq R_{m*}, \;\; \forall m \neq m$$

we deduce that these retractions are pairwise non-homotopic.

□

Sec. 1.1, Problem 20. Fix a homotopy

$$F : X \times I \to X, \;\; f_s(\bullet) = F(\bullet, s)$$

such that $f_0 = f_1 = \mathbf{1}_X$. Denote by $g : I \to X$ the loop

$$g(t) = f_t(x_0).$$

Consider another loop at x_0, $h : (I, \partial I) \to (X, x_0)$ and form the map (see Figure 1).

$$H : I_t \times I_s \to X, \;\; H(s,t) = F(h(s), t).$$

Set $u_0 = g \cdot h$, $u_1 = h \cdot g$. A homotopy (u_t) rel x_0 connecting u_0 to u_1 is depicted at the bottom of Figure 1.

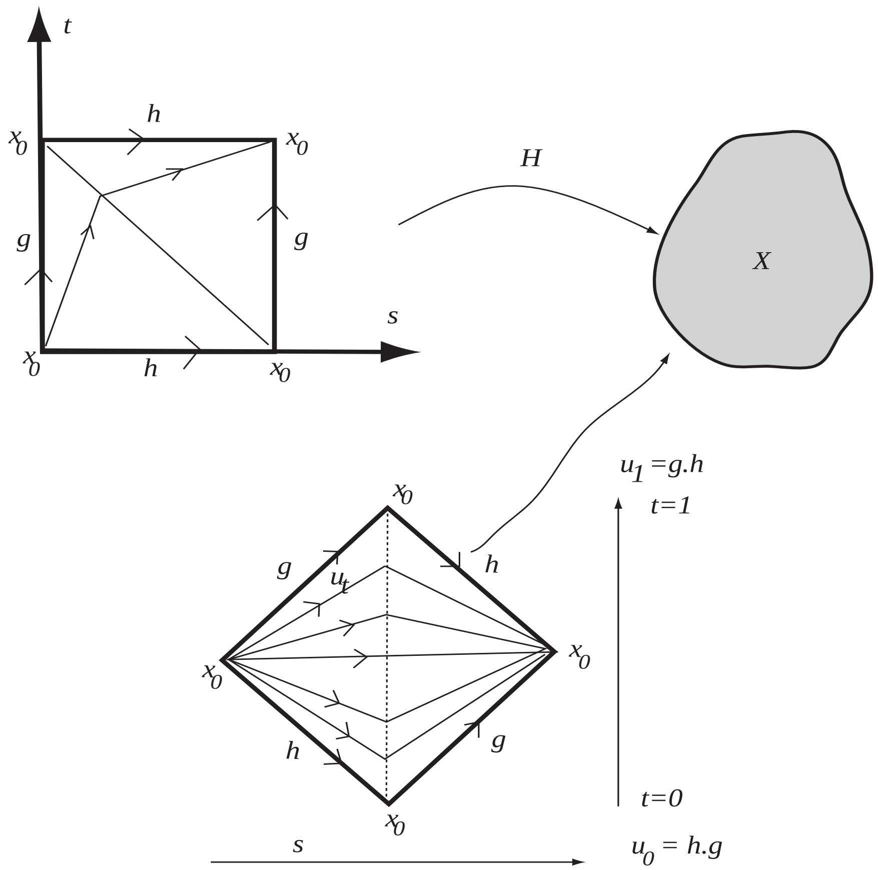


Figure 1: $g \cdot h \simeq h \cdot g$.

Sec. 1.2, Problem 8.

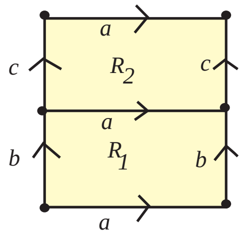


Figure 1: *A cell decomposition*

The space in question has the cell decomposition depicted in Figure 1. It consists of one 0-cell $\bullet$, three 1-cells a, b, c and two 2-cells, R_1 and R_2. We deduce that the fundamental group has the presentation

generators: a, b, c

relations $R_1 = aba^{-1}b^{-1} = 1$, $R_2 = aca^{-1}c^{-1} = 1$.

□

Sec. 1.2, Problem 10. We will first compute the fundamental group of the complement of $a \cup b$ in the cylinder $D^2 \times I$ (see Figure 2), and then show that the loop defined by c defines a nontrivial element in this group.

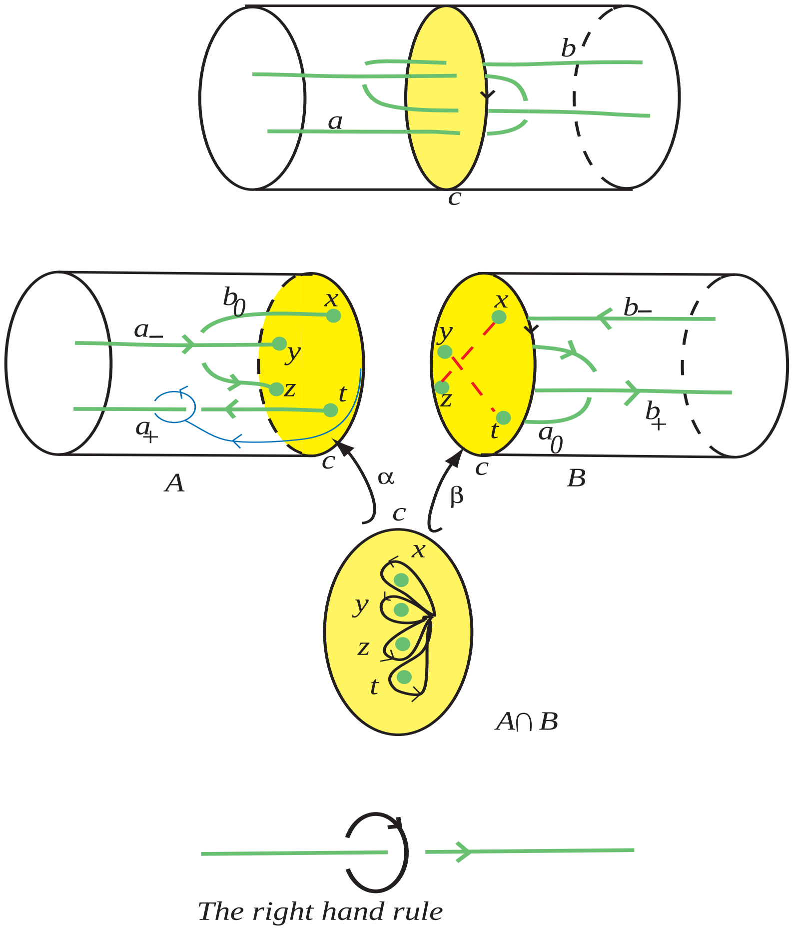


Figure 2: *If you cannot untie it, cut it.*

Cut the solid torus along the "slice" $D^2 \times \{1/2\}$ into two parts A and B as in Figure 2. We will use the Seifert-vanKampen theorem for this decomposition of $D^2 \times I$. We compute the fundamental groups $\pi_1(A, pt)$, $\pi_1(B, pt)$, $\pi_1(A \cap B, pt)$, where pt is a point situated on the boundary c of the slice.

- $A \cap B$ is a homotopically equivalent to the wedge of four circles (see Figure 2), and thus $\pi_1(A \cap B, pt)$ is a free group with four generators x, y, z, t depicted[1] in Figure 2.

[1] **Warning:** The order in which the elements x, y, z, t are depicted is rather subtle. You should keep in mind that since the two arcs a and b link then the segment which connects the entrance and exit points of b (x and z) must intersect the segment which connects the entrance and exit points of a (y and t); see Figure 2.

The intersection of $a \cup b$ with A consists of three oriented arcs $a_{\pm}$, b_0. Suppose g is one of these arcs. We will denote by ℓ_g the loop oriented by the right hand rule going once around the arc g. (The loop ℓ_{a_+} is depicted in Figure 2.)

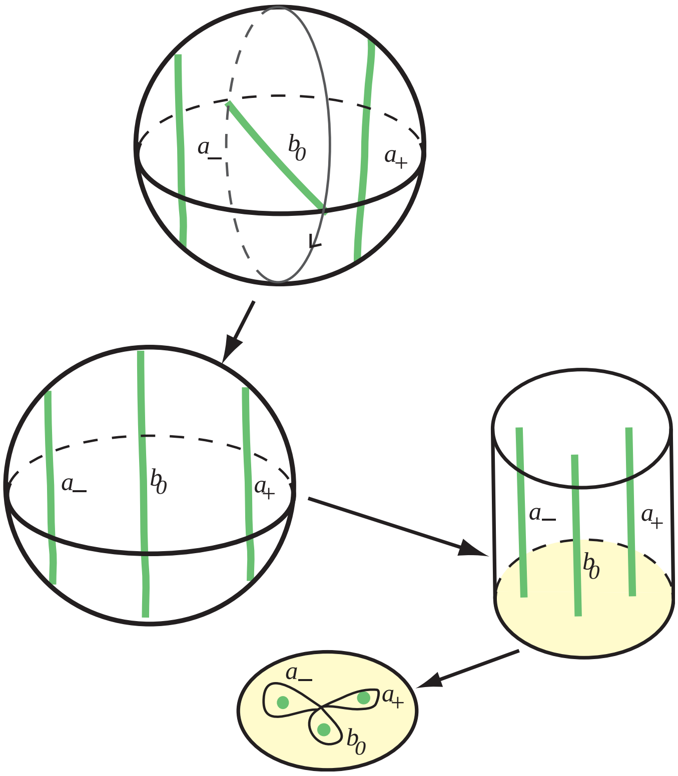


Figure 3: *Pancaking a sphere with three solid tori deleted*

As shown in Figure 3 the complement of these arcs in A is homotopically equivalent to a disk with three holes bounding the loops $\ell_{a_{\pm}}$ and ℓ_{a_0}. This three-hole disk is homotopically equivalent to a wedge of three circles and we deduce that $\pi_1(A, pt)$ is the free group with generators $\ell_{a_{\pm}}$, ℓ_{b_0}. We deduce similarly that $\pi_1(B, pt)$ is the free group with generators $\ell_{b_{\pm}}$ and ℓ_{a_0}.

Denote by α the natural inclusion $A \cap B \hookrightarrow A$ and by β the natural inclusion $A \cap B \hookrightarrow B$ (see Figure 2). We want to compute the induced morphisms α_* and β_*. Upon inspecting

Figure 2 we deduce[2] the following equalities.

$$\begin{cases} \alpha_*(x) = \ell_{b_0} \\ \alpha_*(y) = \ell_{a_-}^{-1} \\ \alpha_*(z) = \ell_{b_0}^{-1} \\ \alpha_*(t) = \ell_{a_+} \end{cases}, \quad \begin{cases} \beta_*(x) = \ell_{b_-} \\ \beta_*(y) = \ell_{a_0}^{-1} \\ \beta_*(z) = \ell_{b_+}^{-1} \\ \beta_*(t) = \ell_{a_0} \end{cases}. \tag{$\dagger$}$$

Thus the fundamental group of the complement of $a \cup b$ in $D^2 \times I$ is the group G defined by
generators: $\ell_{a_\pm}$, ℓ_{a_0}, $\ell_{b_\pm}$, ℓ_{b_0},
relations: $\ell_{b_0} = \ell_{b_-}$, $\ell_{a_-}^{-1} = \ell_{a_0}^{-1}$, $\ell_{b_0}^{-1} = \ell_{b_+}^{-1}$, $\ell_{a_+} = \ell_{a_0}$.

It follows that G is the free group with two generators ℓ_b $(= \ell_{b_\pm} = \ell_{b_0})$ and ℓ_a $(= \ell_{a_\pm} = \ell_{a_0})$. Inspecting Figure 2 we deduce that the loop c defines the element

$$\alpha_*(xyzt)^{-1} = \left(\ell_{b_0}\ell_{a_-}^{-1}\ell_{b_0}^{-1}\ell_{a_+}\right)^{-1}$$
$$= \left(\ell_b\ell_a^{-1}\ell_b^{-1}\ell_a^{-1}\right) = [\ell_b, \ell_a^{-1}]^{-1} \neq 1.$$

□

[2] Be very cautions with the right hand rule.

Sec. 1.2, Problem 11. Consider the wedge of two circles

$$(X, x_0) = (C_1, x_1) \vee (C_2, x_2), \quad x_i \in C_i,$$

and a continuous map $f : (X, x_0) \to (X, x_0)$. Consider the mapping torus of f

$$T_f := X \times I / \{(x, 0) \sim (f(x), 1)\},$$

and the loop $\gamma : (I, \partial I) \to (T_f, (x_0, 0)), \quad \gamma(s) = (x_0, s)$. We denote by C its image in T_f. Observe that C is homeomorphic to a circle and the closed set $A = X \times \{0\} \cup C \subset T_f$ is homeomorphic to $X \vee C \cong X \vee S^1$. The complement $T_f \setminus A$ is homeomorphic to

$$X \setminus \{x_0\} \times (0, 1) \cong \underbrace{(C_1 \setminus x_1) \times (0, 1)}_{R_1} \cup \underbrace{(C_2 \setminus x_2) \times (0, 1)}_{R_2}.$$

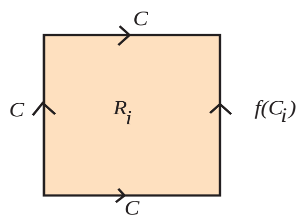


Figure 4: *Attaching maps*

In other words, the complement is the union of two open 2-cells R_1, R_2, and thus T_f is obtained from A by attaching two 2-cells. The attaching maps are depicted in Figure 4. Thus the fundamental group of T_f has the presentation

generators: C_1, C_2, C

relations $R_i = Cf_*(C_i)C^{-1}C_i^{-1} = 1$, $i = 1, 2$.

□

Sec. 1.2, Problem 14. We define a counterclockwise on each face using the outer normal convention as in Milnor's little book. For each face R of the cube we denote by R_* the opposite face, and by $R^{\circlearrowleft}$ the counterclockwise rotation by 90° of the face R. We denote by F, T, S the front, top, and respectively side face of the cube as in Figure 5.

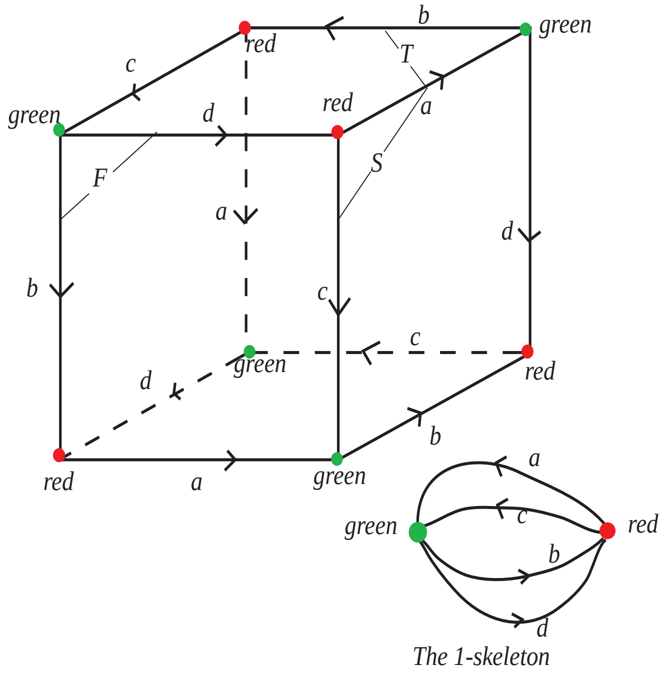


Figure 5: *A 3-dimensional CW-complex*

We make the identifications

$$F \longleftrightarrow F_*^{\circlearrowleft}, \;\; T \longleftrightarrow T_*^{\circlearrowleft}, \;\; S \longleftrightarrow S_*^{\circlearrowleft}.$$

In Figure 5 we labelled the objects to be identified by identical symbols or colors. We get a CW complex with two 0-0cellls (the green and red points), four 1-cells, a, b, c, d, three 2-cells, F, T, S, and one 3-cell, the cube itself. For fundamental group computations the 3-cell is irrelevant.

The 1-skeleton is depicted in Figure 5 and by collapsing the contractible subcomplex d to a point we deduce that it is homotopically equivalent to a wedge of three circles. In other words the fundamental group of the 1-skeleton (with base point the red 0-cell) is the free group with three generators

$$\alpha = a \cdot d, \;\; \beta = b^{-1} \cdot d, \;\; \gamma = c \cdot d.$$

Attaching the three 2-cells has the effect of adding three relations

$$F = ac^{-1}d^{-1}b = \alpha\gamma^{-1}\beta^{-1} = 1, \;\; T = abcd = \alpha\beta^{-1}\gamma = 1, \;\; S = adb^{-1}c^{-1} = \alpha\beta\gamma^{-1} = 1. \quad (1)$$

Thus the fundamental group is isomorphic to the group G with generators α, β, γ and relations (1).

We deduce from the first relation

$$\beta = \alpha\gamma^{-1} \Longrightarrow \alpha(\alpha\gamma^{-1})\gamma^{-1} = 1 \Longrightarrow \alpha^2 = \gamma^2.$$

Using the third relation we deduce

$$\gamma = \alpha\beta \Longrightarrow \alpha^2 = \gamma^2 = \alpha\beta\gamma.$$

Using the second and third relation we deduce that

$$\alpha = \gamma^{-1}\beta = \gamma\beta^{-1} \Longrightarrow \gamma^2 = \beta^2.$$

Hence

$$\alpha^2 = \beta^2 = \gamma^2 = \alpha\beta\gamma \tag{2}$$

Observe that

$$\alpha^2\beta = \beta^2\beta = \beta\beta^2 = \beta\alpha^2, \text{ and similarly } \alpha^2\gamma = \gamma\alpha^2$$

so that the α^2 lies in the center of G. α^2 is an element of order 2, and the cyclic subgroup $\langle\alpha^2\rangle$ it generates is a normal subgroup. Consider the quotient $H := G/\langle\alpha^2\rangle$. We deduce that H has the presentation

$$H = \Big\langle \alpha, \beta, \gamma \Big| \alpha^2 = \beta^2 = \gamma^2 = \alpha\beta\gamma = 1 \Big\rangle,$$

which shows that $H \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. It follows that $\operatorname{ord} G = 8$.

Denote by Q the subgroup of nonzero quaternions generated by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. We have a surjective morphism $G \to Q$ given by

$$\alpha \mapsto \boldsymbol{i}, \quad \beta \mapsto \boldsymbol{j}, \quad \gamma \mapsto \boldsymbol{k}.$$

Since $\operatorname{ord}(G) = \operatorname{ord}(Q)$ we deduce that this must be an isomorphism.

□

Sec. 1.3, Problem 9. Suppose $f : X \to S^1$ is a continuous map, and $x_0 \in X$. Then $f_*\pi_1(X, x_0)$ is a finite subgroup of $\pi_1(S^1, f(x_0) \cong \mathbb{Z}$ and thus it must be the trivial subgroup. It follows that f has a lift $\tilde{f}$ to the universal cover

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle \exp} \\ X & \xrightarrow{f} & S^1 \end{array}$$

Since $\mathbb{R}$ is contractible we deduce that $\tilde{f}$ is nullhomotopic. Thus $f = \exp \circ \tilde{f}$ must be nullhomotopic as well.

□

Sec. 1.3, Problem 18. Every normal cover of X has the form

$$Y := \tilde{X}/G \stackrel{p}{\to} X$$

where $G \trianglelefteq \pi_1(X)$. In this case $\operatorname{Aut}(Y/X) \cong \pi_1(X)/G$. We deduce that the cover $X/G \to X$ is Abelian iff G contains all the commutators in $\pi_1(X)$, i.e.

$$G_0 := [\pi_1(X), \pi_1(X)] \leq G.$$

Consider the cover.

$$X_{ab} := X/G_0 \stackrel{p_{ab}}{\to} X.$$

Note that $\operatorname{Aut}(X_{ab}/X) \cong \operatorname{Ab}(\pi_1(X))$ acts freely and transitively on X_{ab}. We deduce that for any Abelian cover of the form X/G we have an isomorphism of covers

$$X/G \cong X_{ab}/(G/G_0)$$

so that X_{ab} is a normal covering of X/G.

For example, when $X = S^1 \vee S^1$ we have $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$, $\operatorname{Ab}(\mathbb{Z} * \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$. The universal Abelian cover of $S^1 \vee S^1$ is homomorphic to the closed set in $\mathbb{R}^2$

$$X_{ab} \cong \Big\{ (x, y) \in \mathbb{R}^2 \;\; x \in \mathbb{Z} \;\text{ or }\; y \in \mathbb{Z} \Big\}.$$

The group $\mathbb{Z}^2$ acts on this set by

$$(x, y) \cdot (m, n) := (x + m, y + n)$$

This action is even and the quotient is X. The case $S^1 \vee S^1 \vee S^1$ can be analyzed in a similar fashion.

□

Sec. 1.3, Problem 24. Suppose we are given a based G-covering

$$(X_0, x_0) \overset{p^0}{\twoheadrightarrow} (X_1, x_1) := (X_0, x_0)/G.$$

We want to classify the coverings $(X, x) \overset{p^0}{\twoheadrightarrow} (X_1, x_1)$ which interpolate between X_0 and X_1, i.e. there exists a covering map $(X_0, x_0) \overset{q}{\twoheadrightarrow} (X, x)$ such that the diagram below is commutative.

$$\begin{array}{ccc}
(X_0, x_1) & \overset{q}{\twoheadrightarrow} & (X, x) \\
 & \searrow^{p^0} & \downarrow p \\
 & & (X_1, x_1)
\end{array}$$

We will denote such coverings by $(X, x; q, p)$ A morphism between two such covers $(X', x'; q', p')$ and $(X, x; q, p)$ is a pair of continuous maps $f : (X, x) \to (X, x')$, such that the diagram below is commutative

$$\begin{array}{ccc}
(X_0, x_1) & \overset{q}{\twoheadrightarrow} & (X, x) \\
\downarrow q' & \overset{f}{\swarrow} & \downarrow p \\
(X', x') & \overset{p'}{\twoheadrightarrow} & (X_1, x_1)
\end{array}$$

Suppose $(X, x; q, p)$ is such an intermediate cover. Set $F_i := \pi_1(X_i, x_i)$, $F := \pi_1(X, x)$. Since $X_0 \overset{p^0}{\twoheadrightarrow} X_1$ is a G-covering we obtain a short exact sequence

$$1 \hookrightarrow F_0 \overset{p^0_*}{\hookrightarrow} F_1 \overset{\mu}{\twoheadrightarrow} G \to 1$$

Note that we also have a commutative diagram

$$\begin{array}{ccc}
 & & F_1 \\
 & \nearrow^{p^0_*} & \uparrow p_* \\
F_0 & \overset{q_*}{\hookrightarrow} & F
\end{array}$$

which can be completed to a commutative diagram

$$\begin{array}{ccccccccc}
1 & \hookrightarrow & F_0 & \overset{p^0_*}{\hookrightarrow} & F_1 & \overset{\mu}{\twoheadrightarrow} & G & \twoheadrightarrow & 1 \\
 & & \updownarrow \mathbf{1}_{F_0} & & \uparrow p_* & & \uparrow \mu\circ p_* & & \\
1 & \hookrightarrow & F_0 & \overset{q_*}{\hookrightarrow} & F & \twoheadrightarrow & H := F_0/q_*F & \twoheadrightarrow & 1
\end{array} \qquad (F; q_*, p_*)$$

Consider another such commutative diagram,

$$\begin{array}{ccccccccc} 1 & \hookrightarrow & F_0 & \overset{p^0_*}{\hookrightarrow} & F_1 & \overset{\mu}{\twoheadrightarrow} & G & \twoheadrightarrow & 1 \\ & & {\scriptstyle \mathbf{1}_{F_0}}\updownarrow & & \uparrow{\scriptstyle p'_*} & & \uparrow{\scriptstyle \mu\circ p'_*} & & \\ 1 & \hookrightarrow & F_0 & \overset{q'_*}{\hookrightarrow} & F' & \twoheadrightarrow & H' := F_0/q_*F' & \twoheadrightarrow & 1 \end{array} \qquad (F'; q'_*, p'_*)$$

We define a morphism $(F; q_*, p_*) \to (F'; q'_*, p'_*)$ to be a group morphism $\varphi : F \to F'$ such that the diagrams below are commutative

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & F' \\ & {\scriptstyle p_*}\searrow \quad \swarrow{\scriptstyle p'*} & \\ & F_1 & \end{array} \quad , \quad \begin{array}{ccc} F & \xrightarrow{\varphi} & F' \\ & {\scriptstyle q_*}\searrow \quad \swarrow{\scriptstyle q'*} & \\ & F_1 & \end{array}$$

We denote by $\mathcal{I}$ the collection of intermediate coverings $(X_0, x_0) \overset{q}{\twoheadrightarrow} (X, x) \overset{p}{\twoheadrightarrow} (X_1, x_1)$, and by $\mathcal{D}$ the collection of the diagrams of the type $(F; q_*, p_*)$.

We have constructed a map $\Xi : \mathcal{I} \to \mathcal{D}$ which associates to a covering $(X, x; q, p)$ the diagram $\Xi(X, x; q, p) :=(F; q_*, p_*)\in \mathcal{D}$. Moreover if $(X', x'; q', p') \in \mathcal{I}$, with associated diagram $(F'; q'_*, p'_*)$, and $f : (X, x; , q, p) \to (X', x'; q', p')$ is morphism of intermediate coverings, then the group morphism $f_* : F \to F'$ induces a morphism of diagrams

$$\Xi(f) : \Xi(X, x; q, p) \to \Xi(X', x'; q', p').$$

Note that for every coverings $C, C', C'' \in \mathcal{I}$, and every morphisms $C \overset{g}{\to} C' \overset{f}{\to} C''$ we have

$$\Xi(\mathbf{1}_C) = \mathbf{1}_{\Xi(C)}, \ \ \Xi(f \circ g) = \Xi(f) \circ \Xi(g).$$

Thus two coverings $C, C' \in I$ are isomorphic iff the corresponding diagrams are isomorphic, $\Xi(C) \cong \Xi(C')$.

This shows that we have an injective correspondence $[\Xi]$ between the collection $[\mathcal{I}]$ of isomorphisms classes of intermediate coverings and the collection $[\mathcal{D}]$ of isomorphism classes of diagrams.

Conversely, given a diagram $D \in \mathcal{D}$

$$\begin{array}{ccccccccc} 1 & \hookrightarrow & F_0 & \overset{p^0_*}{\hookrightarrow} & F_1 & \overset{\mu}{\twoheadrightarrow} & G & \twoheadrightarrow & 1 \\ & & {\scriptstyle \mathbf{1}_{F_0}}\updownarrow & & \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \mu\circ\beta} & & \\ 1 & \hookrightarrow & F_0 & \overset{\alpha}{\hookrightarrow} & F & \twoheadrightarrow & H := F_0/\alpha F & \twoheadrightarrow & 1 \end{array} \qquad (F; \alpha, \beta)$$

we can form $(Y, y; a, b) \in \mathcal{I}$ where

$$(Y, y) := (X_0, x_0)/\mu \circ \beta(H),$$

$a : (X_0, x_0) \to (Y, y)$ is the natural projection, and $b : (Y, y) \in (X_1, x_1)$ is the map

$$(Y, y) \ni z \cdot H \mapsto z \cdot G \in (X_1, x_1),$$

where for $z \in X_0$ we have denoted by $z \cdot H$ (resp. $z \cdot G$) the H-orbit (resp the G-orbit) of z. Observe that the diagram $\Xi(Y, y; a, b)$ associated to $(Y, y; a, b)$ is isomorphic to the initial diagram $(F; \alpha, \beta)$. We thus have a bijection[1]

$$[\Xi] : [\mathfrak{I}] \to [\mathfrak{D}].$$

To complete the solution of the problem it suffices to notice that the isomorphism class of the diagram $(F; \alpha, \beta)$ is uniquely determined by the subgroup $\mu \circ \beta(F) \leq G$. Conversely, to every subgroup $H \hookrightarrow G$ we can associate the diagram

$$\begin{array}{ccccccccc}
1 & \hookrightarrow & F_0 & \overset{p^0_*}{\hookrightarrow} & F_1 & \overset{\mu}{\twoheadrightarrow} & G & \twoheadrightarrow & 1 \\
 & & {\scriptstyle \mathbf{1}_{F_0}} \updownarrow & & \uparrow & & \uparrow & & \\
1 & \hookrightarrow & F_0 & \overset{p^0_*}{\hookrightarrow} & \mu^{-1}(H) & \overset{\mu}{\twoheadrightarrow} & H & \twoheadrightarrow & 1
\end{array} \qquad (\mu^{-1}(H); p^0_*, \text{inclusion})$$

□

[1] In more modern language, we have constructed two categories $\mathfrak{I}$ and $\mathfrak{D}$, and an equivalence of categories $\Xi : \mathfrak{I} \to \mathfrak{D}$.

Solutions to Homework # 3

Sec. 2.1, Problem 1. It is The Möbius band; see Figure 1.

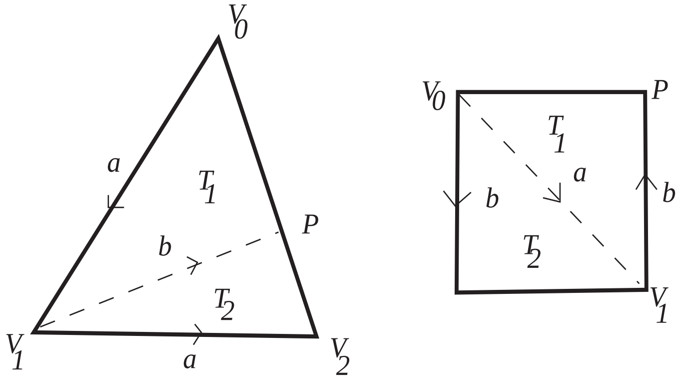


FIGURE 1. *The Möbius band*

□

Sec. 2.1, Problem 2. For the problem with the Klein bottle the proof is contained in Figure 2, where we view the tetrahedron as the upper half-ball in $\mathbb{R}^3$ by rotating the face $[V_0V_1V_2]$ about $[V_1V_2]$ so that the angle between the two faces with common edge $[V_1V_2]$ increases until it becomes 180°. We now see the Klein bottle sitting at the bottom of this upper half-ball. All the other situations (the torus and $\mathbb{RP}^2$) are dealt with similarly.

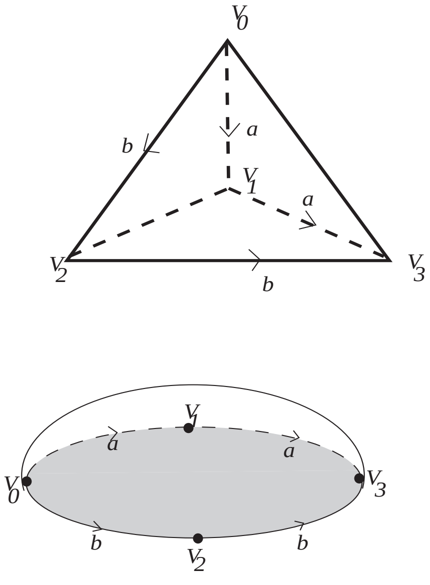


FIGURE 2. *A 3-dimensional Δ-complex which deformation retracts to the Klein bottle.*

Sec. 2.1, Problem 4.

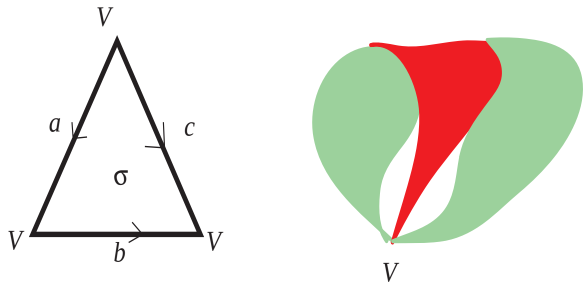


FIGURE 3. *The homology of a parachute.*

In this case we have
$C_n(K) = 0$ if $n \geq 3$ or $n \leq 0$, and

$$C_2(K) = \mathbb{Z}\langle\sigma\rangle,\ \ C_1(K) = \mathbb{Z}\langle a, b, c\rangle,\ \ C_0(K) = \mathbb{Z}\langle V\rangle$$

and the boundary operator is determined by the equalities

$$\partial\sigma = a + b - c,\ \ \partial a = \partial b = \partial c = \partial V = 0.$$

Then $Z_2(C_*(K)) = 0$, $Z_1(C_*(K)) = C_1(K) = \mathbb{Z}\langle a, b, c\rangle$, $Z_0(C_*(K)) = C_0(K)$. Hence $H_2^\Delta(|K|) = (0)$. Moreover

$$B_1(C_*(K)) = \text{span}_\mathbb{Z}(a + b - c) \subset \mathbb{Z}\langle a, b, c\rangle$$

so that

$$H_1^\Delta(|K|) \cong \mathbb{Z}\langle a, b, c\rangle/\text{span}_\mathbb{Z}(a + b - c).$$

The images of a and b in $H_1^\Delta(|K|)$ define a basis of $H_1^\Delta(|K|)$. It is clear that $H_0^\Delta(|K|) \cong \mathbb{Z}$. □

Sec. 2.1, Problem 5.

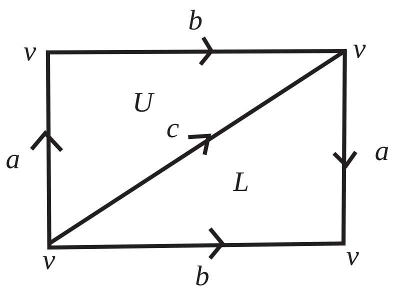


FIGURE 4. *The homology of the Klein bottle.*

We have

$$C_2 = \mathbb{Z}\langle U, L\rangle,\ \ C_1 = \mathbb{Z}\langle a, b, c\rangle,\ \ C_0 = \mathbb{Z}\langle v\rangle.$$

and

$$\partial U = a + b - c,\ \ \partial L = c + a - b,\ \ \partial a = \partial b = \partial c = \partial v = 0.$$

If follows that $Z_2 = 0 \cong H_2^\Delta(|K|) = 0$, and $H_0^\Delta \cong \mathbb{Z}$. The first homology group has the presentation

$$\mathbb{Z}\langle U, L\rangle \xrightarrow{P} \mathbb{Z}\langle a, b, c\rangle \twoheadrightarrow H_1^\Delta \to 0$$

where P is the 3×2 matrix

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Using the Maple procedure ismith we can diagonalize P over the integers

$$D_0 := \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} = APB,$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

This means that by choosing the $\mathbb{Z}$-basis $\mu_1 := A^{-1}a$, $\mu_2 := A^{-1}b$, $\mu_3 = A^{-1}c$ in $\mathbb{Z}\langle a, b, c\rangle$, and the $\mathbb{Z}$-basis $e := BU$, $f := BL$ in $\mathbb{Z}\langle U, L\rangle$ we can represent the linear operator P as the diagonal matrix D_0. We deduce that H_1^Δ has an equivalent presentation with three generators μ_1, μ_2, μ_3 and two relations

$$\mu_1 = 0, \ \ 2\mu_2 = 0.$$

Thus

$$H_1^\Delta \cong \mathbb{Z}_2\langle \mu_2\rangle \oplus \mathbb{Z}\langle \mu_3\rangle.$$

Using the MAPLE procedure inverse we find that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

so that μ_2 is given by the 2nd column of A^{-1} and μ_3 is given by the third column of A^{-1}

$$\mu_2 = c - b, \ \ \mu_3 = c.$$

□

Solutions to Homework # 4

Problem 6, §2.1 We begin by describing the equivalence classes of k-faces, $k = 0, 1, 2$. Let $\Delta_i[v_0^i v_1^i v_2^i]$.

• *The 0-faces.* We have

$$[v_0^0 v_1^0] \sim [v_1^0 v_2^0] \sim [v_0^0 v_2^0]$$

so that

$$v_0^0 \sim v_1^0 \sim v_2^0.$$

Denote by v^0 the equivalence class containing these vertices. Note that

$$[v_0^1 v_2^1] \sim [v_0^0 v_1^0] \Longrightarrow v_0^1 \sim v^0, \;\; v_2^1 \sim v^0$$

$$[v_0^1 v_1^1] \sim [v_1^1 v_2^1] \Longrightarrow v_1^1 \sim v^0.$$

Iterating this procedure we deduce that there exists a single equivalence class of vertices.

• *The 1-faces.* Denote by e_0 the equivalence class containing the edges of Δ_0. Then all the edges $[v_1^i v_2^i]$ belong to this equivalence class. We also have another n-equivalence classes e_i containing the pair $[v_0^i v_1^i]$, $[v_1^i v_2^i]$. Observe that

$$[v_0^i v_2^i] \sim e_{i-1}, \;\; i = 1, \cdots, n.$$

• *The 2-faces.* We have $n + 1$ equivalence classes of 2-faces, $\Delta_0, \Delta_1, \cdots, \Delta_n$.

• *$\partial : C_2 \to C_1$.* We have

$$C_2 = \mathbb{Z}\langle \Delta_0, \cdots, \Delta_1 \rangle, \quad C_1 = \mathbb{Z}\langle e_0, e_1, \cdots, e_n \rangle$$

$$\partial \Delta_0 = e^0, \;\; \partial \Delta_i = [v_0^i v_1^i] + [v_1^i v_2^i] - [v_0^i v_2^i] = 2e_i - e_{i-1}.$$

• *$\partial : C_1 \to C_0$.* We have

$$C_0 = \mathbb{Z}\langle v^0 \rangle$$

and

$$\partial e_i = 0, \;\; \forall i = 0, 1, \cdots, n.$$

• *Z_2 and H_2.* We have $B_2 = 0$ and

$$Z_2 = \Big\{ \sum_{i=0}^{n} x_i \Delta_i; \;\; \sum_{i=0}^{n} x_i \partial \Delta_i = 0 \Big\}$$

Thus

$$\sum_{i=0}^{n} x_i \Delta_i \in Z_2 \Longleftrightarrow \begin{cases} x_n & = & 0 \\ -x_n + 2x_{n-1} & = & 0 \\ \vdots & \vdots & \vdots \\ -x_2 + 2x_1 & = & 0 \\ -x_1 + x_0 & = & 0 \end{cases}$$

We deduce $Z_2 = 0$ so that $H_2 = 0$.

• *Z_1 and H_1.* We have $Z_1 = C_1$ and H_1 has the presentation

$$\Big\langle e_0, e_1, \cdots, e_n \big| \;\; 0 = 2e_n - e_{n-1} = \cdots 2e_1 - e_0 = e_0 \Big\rangle.$$

Hence

$$e_{n-1} = 2e_n, \;\; e_{n-2} = 2e_{n-1}, \cdots, e_0 = 2e_1 = 0$$

so that H_1 is the cyclic group of order 2^n generated by e_n. By general arguments we have $H_0 = \mathbb{Z}$.

□

Sec. 2.1, Problem 7. Consider a regular tetrahedron $\Delta_3 = [P_0P_1P_2P_3]$, and fix two opposite edges $a = [P_0P_1]$, $b = [P_2P_3]$. Now glue the faces of this tetrahedron according to the prescriptions

- Type (a) gluing: $[P_0P_1P_2] \sim [P_0P_1P_3]$.
- Type (b) gluing: $[P_0P_2P_3] \sim [P_1P_2P_3]$.

To see that the space obtained by these identifications is homeomorphic to S^3 we cut the tetrahedron with the plane passing through the midpoints of the edges of Δ_3 different from a and b (see Figure 2).

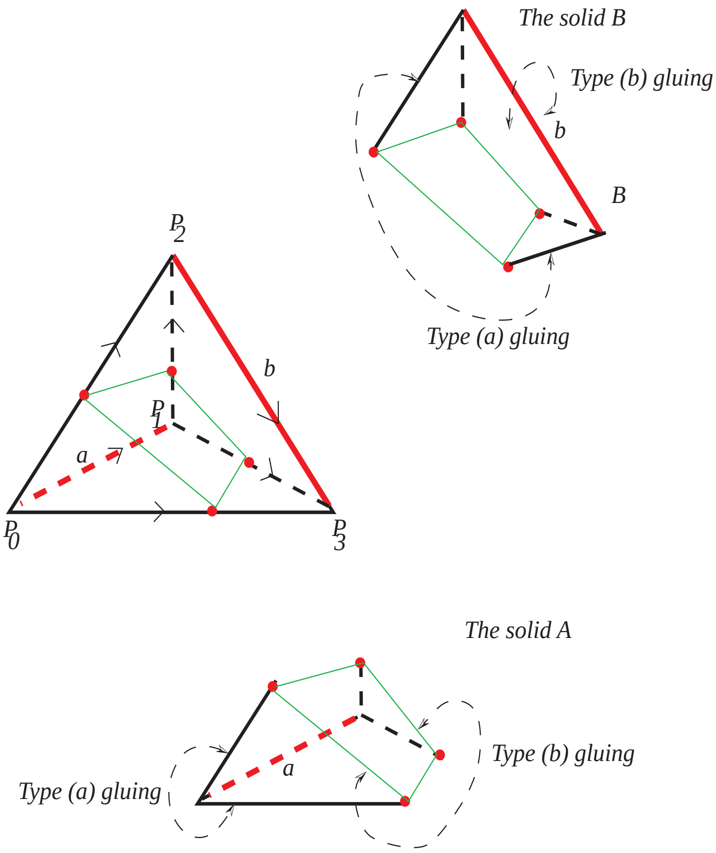


FIGURE 1. *Gluing the faces of a tetrahedron to get a 3-sphere.*

We get a solid A containing the edge a and a solid B containing the edge B. By performing first the type (b) gluing and then the type (a) gluing on the solid B we obtain a solid torus. Then performing first the type (a) gluing and next the type (b) gluing on the solid A we obtain another solid torus. We obtain in this fashion the standard decomposition of S^3 as an union of two solid tori

$$S^3 \cong \partial D^4 \cong \partial(D^2 \times D^2) = (\partial D^2 \times D^2) \cup (D^2 \times \partial D^2).$$

□

Problem 8, §2.1 Hatcher. Denote by $[V_0^iV_1^iV_2^iV_3^i]$ the i-th 3-simplex.

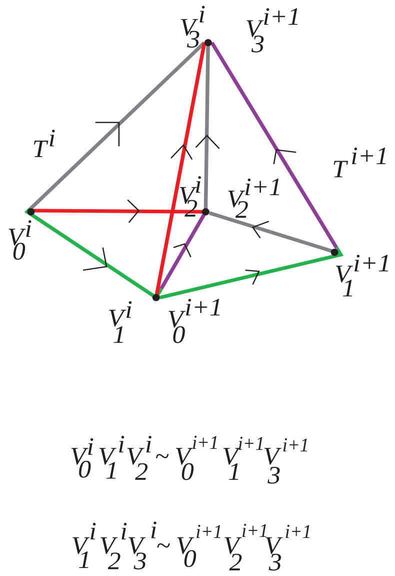


Figure 2. *Cyclic identifications of simplices*

To describe the associated chain complex we need to understand the equivalence classes of k-faces, $k = 0, 1, 2, 3$.

• *0-faces.* We deduce $V_0^i \sim V_0^{i+1}$ $\forall i$ mod n and we denote by U_0 the equivalence class containing V_0^i.

Similarly $V_1^i \sim V_1^{i+1}$ and we denote by U_1 the corresponding equivalence class. Since $V_1^i \sim V_0^{i+1}$ we deduce $U_0 = U_1$.

Now observe that $V_2^i \sim V_2^{i+1}$ and we denote by U_2 the corresponding equivalence class. Similarly the vertices V_3^i determine a homology class U_3 and we deduce from $V_2^i \sim V_3 i + 1$ that $U_2 = U_3$. Thus we have only two equivalence classes of vertices, U_0 and U_2. The vertices V_0^i, V_1^j belong to U_0 while the vertices V_2^i, V_3^j belong to U_2.

• *1-faces.* The simplex T^i has six 1-faces (edges) (see Figure 2).

A vertical edge $v_i = [V_2^iV_3^i]$.
A horizontal edge $h_i = [V_0^iV_1^i]$.
Two bottom edges: bottom-right $br_i = [V_1^iV_2^i]$ and bottom-left $bl_i = [V_0^iV_2^i]$.
Two top edges: top-right $tr_i = [V_1^iV_3^i]$ and top-left $tl_i = [V_0^iV_3^i]$.

Inspecting Figure 2 we deduce the following equivalence relations.

$$br_i \sim bl_{i+1}, \;\; tr_i \sim tl_{i+1}, \;\; v_i \sim v_{i+1}, \tag{0.1}$$

$$h_i \sim h_{i+1}, \;\; bl_i \sim tl_{i+1}, \;\; br_i \sim tr_{i+1}. \tag{0.2}$$

We denote by v the equivalence class containing the vertical edges and by h the equivalence class containing the horizontal edges.

Observe next that

$$bl_i \sim tl_{i+1} \sim tr_i, \;\; \forall i$$

so that $bl_i \sim tr_i$ for all i. Denote by e_i the equivalence class containing bl_i. Observe that

$$bl_i \sim tr_i \sim e_i, \;\; tl_i \sim e_{i-1}, \;\; br_i \sim e_{i+1}.$$

We thus have $(n+2)$ equivalence classes of edges v, h and e_i, $i = 1, \cdots, n$.

• *2-faces.* Each simplex T^i has four 2-faces
A bottom face $B_i = [V_0^i V_1^i V_2^i]$.
A top face $\tau_i = [V_0^i V_1^i V_3^i]$.
A left face $L_i = [V_0^i V_2^i V_3^i]$.
A right face $R_i = [V_1^i V_2^i V_3^i]$.

We have the identifications

$$R_i \sim L_{i+1}, \;\; B_i \sim \tau_{i+1}.$$

We denote by B_i the equivalence class of B_i, by L_i the equivalence class of L_i and by R_i the equivalence class of R_i. Observe that

$$R_i = L_{i+1}, \;\; \forall i \mod n.$$

There are exactly $2n$ equivalence classes of 2-faces.

• *3-faces.* There are exactly n three dimensional simplices $T^1, \cdots, T^n$.

• *The associated chain complex.*

$$C_0 = \mathbb{Z}\langle U_0, U_2\rangle, \;\; C_1 = \mathbb{Z}\langle v, h, e_i; \;\; 1 \leq i \leq n\rangle$$

$$C_2 = \mathbb{Z}\langle B_i, R_j; \;\; 1 \leq i, j, k \leq n\rangle, \;\; C_3 = \mathbb{Z}\langle T^i; \;\; 1 \leq i \leq n\rangle.$$

The boundary operators are defined as follows.

• $\partial : C_3 \to C_2$

$$\partial T^i = R_i - L_i + \tau_i - B_i = R_i - R_{i-1} + B_{i-1} - B_i.$$

• $\partial : C_2 \to C_1$

$$\partial B_i = h + br_i - bl_i = h + e_{i+1} - e_i, \;\; \partial R_i = v - tr_i + br_i = v + e_{i+1} - e_i,$$

• $\partial : C_1 \to C_0$

$$\partial e_i = U_2 - U_0, \;\; \partial h = 0, \;\; \partial v = 0.$$

For every sequence of elements $x = (x_i)_{i\in\mathbb{Z}}$ we define its "derivative" to be the sequence

$$\Delta_i x = (x_{i+1} - x_i), \;\; i \in \mathbb{Z}.$$

Using this notation we can rewrite

$$\partial T^i = \Delta_{i-1}R - \Delta_{i-1}B, \;\; \partial B_i = h + \Delta_i e, \;\; \partial R_i = v + \Delta_i e.$$

• *The groups of cycles.*

$$Z_0 = C_0,$$

$$Z_1 = \Big\{ah + bv + \sum_i k_i e_i \in C_1; \;\; a, b, k_i \in \mathbb{Z}, \;\; \sum_i k_i = 0\Big\}$$

$$= \mathrm{span}_{\mathbb{Z}}\Big\{v, h, \Delta_i e; \;\; 1 \leq i \leq n\Big\}^1.$$

[1] Here we use the elementary fact that the subgroup of $\mathbb{Z}^n$ described by the condition $x_1 + \cdots + x_n = 0$ is a free Abelian group with basis $e_2 - e_1, e_3 - e_2, \cdots, e_n - e_{n-1}$, where (e_i) is the canonical basis of $\mathbb{Z}^n$

Suppose

$$c = \sum_i x_i B_i + \sum_j y_j R_j \in Z_2.$$

Then

$$0 = \partial C = \Big(\sum_i x_i\Big) h + \big(\sum_j y_j\Big) v + \sum_i (x_i + y_i) \Delta_i e$$

(use Abel's trick[2])

$$= \Big(\sum_i x_i\Big) h + \big(\sum_j y_j\Big) v - \sum_i \Delta_i (x+y) e_{i+1}.$$

We deduce

$$\sum_i x_i = \sum_j y_j = 0, \;\; \Delta_i(x+y) = \Delta_i x + \Delta_i y = 0, \;\; \forall y.$$

The last condition implies that $(x_i + y_i)$ is a constant α independent of i. Using the first two conditions we deduce

$$0 = \sum_i (x_i + y_i) = n\alpha$$

so that $x_i = -y_i$, for all i. This shows

$$Z_2 = \Big\{ \sum_i x_i (B_i - R_i); \;\; x_i \in \mathbb{Z}, \;\; \sum_i x_i = 0 \Big\}.$$

To find Z_3 we proceed similarly. Suppose

$$c = \sum_i x_i T^i \in Z_3.$$

Then

$$0 = \partial c = \sum_i x_i \Delta_{i-1}(R - B) = -\sum_i (R_i - B_i) \Delta_i x = -\sum (\Delta_i x) R_i + \sum_i (\Delta_i x) B_i.$$

We deduce $\Delta_i x = 0$ for all i, i.e. x_i is independent of i. We conclude that

$$Z_3 = \Big\{ xT; \;\; x \in \mathbb{Z}; \;\; T = \sum_i T^i \Big\}$$

In particular we conclude $H_3 \cong \mathbb{Z}$.

- *The groups of boundaries and the homology.* We have

$$B_0 = \mathrm{span}_{\mathbb{Z}}(U_2 - U_0) \subset \mathbb{Z}\langle U_0, U_2 \rangle.$$

We deduce

$$H_0 = Z_0/B_0 = C_0/B_0 = \mathbb{Z}\langle U_0, U_2 \rangle / \mathrm{span}_{\mathbb{Z}}(U_2 - U_0) \cong \mathbb{Z}.$$

$$B_1 = \mathrm{span}_{\mathbb{Z}}(\partial B_i, \partial R_j; \;\; 1 \le i, j \le n) \subset \mathbb{Z}\langle h, v, e_i; \;\; 1 \le i \le n \rangle.$$

Thus H_1 admits the presentation

$$H_1 = Z_1/B_1 = \Big\langle h, v, \Delta_i e; \;\; h = v = -\Delta_i e, \sum_i \Delta_i e = 0 \;\; 1 \le i \le n \Big\rangle$$

---

[2] Abel's trick is a discrete version of the integration-by-parts formula. More precisely if R is a commutative ring, M is an R-module, $(x_i)_{i\in\mathbb{Z}}$ is a sequence in R, $(y_i)_{i\in\mathbb{Z}}$ is a sequence in M then we have

$$\sum_{i=1}^n (\Delta_i x) \cdot y_i = x_{n+1} y_n - x_1 y_0 - \sum_{j=1}^n x_j \cdot (\Delta_{j-1} y).$$

Using the equality

$$\sum_{i=1}^{n} \Delta_i e = 0$$

we deduce $nh = nv = 0$. This shows $H_1 \cong \mathbb{Z}/n\mathbb{Z}$.

Using the fact that for every sequence $x_i \in \mathbb{Z}$ $i \in \mathbb{Z}/n\mathbb{Z}$ such that $\sum_i x_i = 0$ there exists a sequence $y_i \in \mathbb{Z}$, $i \in \mathbb{Z}/n\mathbb{Z}$ such that

$$x_i = \Delta_i y, \ \ \forall i.$$

Any element $c \in Z_2$ has the form

$$c = \sum_i x_i(R_i - B_i),$$

where $\sum_i x_i = 0$. Choose y_i as above such that $x_i = -\Delta_i y$, $\forall i \bmod n$. Then

$$c = \partial \sum_i y_i T^i$$

so that $Z_2 = B_2$, i.e. $H_2 = 0$.

□

Problem 11, §2.1, Hatcher. Denote by i the canonical map $A \hookrightarrow X$. Suppose $r : A \to X$ is a retraction, i.e. $r \circ i = \mathbb{1}_A$. Then the morphisms induced in homology satisfy

$$r_* \circ i_* = \mathbb{1}_{H_n(A)}.$$

This shows that i_* is one-to-one since $i_*(u) = i_*(v)$ implies

$$u = r_* \circ i_*(u) = r_*(i_*(u)) = r_*(i_*(v)) = r_* \circ i_*(v) = v.$$

□

Solutions to Homework # 5

Problem 17, §2.1, Hatcher. Denote by A_n a set consisting of n distinct points in X. The long exact sequence of the triple (X, A_n, A_{n-1}) is

$$\cdots \to H_k(A_n, A_{n-1}) \to H_k(X, A_{n-1}) \to H_k(X, A_n) \to H_{k-1}(A_n, A_{n-1}) \to \cdots$$

We deduce that for $k \geq 2$ we have isomorphisms

$$H_k(X, A_{n-1}) \to H_k(X, A_n).$$

Thus for every $k \geq 2$ and every $n \geq 1$ we have an isomorphism

$$H_k(X) \cong \tilde{H}_k(X) \cong H_k(X, A_1) \to H_k(X, A_n). \tag{5.1}$$

For $k = 1$ we have an exact sequence

$$0 \to H_1(X, A_{n-1}) \to H_1(X, A_n) \to H_0(A_n, A_{n-1}) \xrightarrow{j_n} H_0(X, A_{n-1})$$

Since $H_0(A_n, A_{n-1})$ is a *free* Abelian group $\ker j_n$ is free Abelian and we have

$$H_1(X, A_n) \cong H_1(X, A_{n-1}) \oplus \ker j_n.$$

Assume X is a path connected CW-complex. Then X/A_{n-1} is path connected so that $H_0(X, A_{n-1}) \cong 0$. Hence

$$H_1(X, A_n) \cong H_1(X, A_{n-1}) \oplus H_0(A_n, A_{n-1})$$
$$\cong H_1(X, A_{n-1}) \oplus \tilde{H}_0(A_n/A_{n-1}) \cong H_1(X, A_{n-1}) \oplus \mathbb{Z}.$$

Hence[1]

$$H_1(X, A_n) \cong H_1(X, A_1) \oplus \mathbb{Z}^{n-1} \cong H_1(X) \oplus \mathbb{Z}^{n-1}. \tag{5.2}$$

Finally assuming the path connectivity of X as above we deduce

$$H_0(X, A_n) \cong \tilde{H}_0(X/A_n) \cong 0. \tag{5.3}$$

Now apply (5.1)-(5.3) using the information

$$H_0(S^2) \cong H_0(S^1 \times S^1) \cong \mathbb{Z}, \;\; H_1(S^2) = 0,$$
$$H_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}, \;\; H_2(S^2) \cong H_2(S^1 \times S^1) \cong \mathbb{Z}.$$

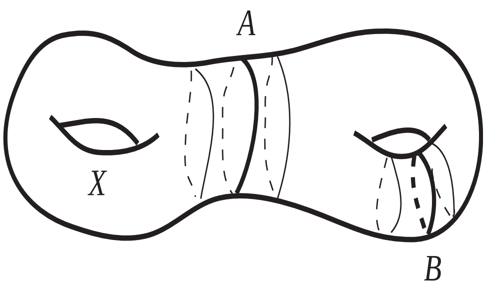


FIGURE 1. *The cycle A is separating while B is non-separating*

(b) Denote by $\tilde{A}$ a collar around A and by $\tilde{B}$ a collar around B. Then $\tilde{A}$ deformation retracts onto A while $\tilde{B}$ deformation retracts onto B. Then

$$H_*(X, A) \cong H_*(X, \tilde{A}) \overset{\text{excision}}{\cong} H_*(X - A, \tilde{A} - A).$$

[1]Can you visualize the isomorphisms in (5.2)?

The space $X - A$ has two connected components Y_1, Y_2 both homeomorphic to a torus with a disk removed. Then $\tilde{A} - A$ consists of two collars around the boundaries of Y_j so that

$$H_*(X - A, \tilde{A} - A) \cong H_*(Y_1, \partial Y_1) \oplus H_*(Y_2, \partial Y_2).$$

We now use the following simple observation. Suppose Σ is a surface, S is a finite set of points in Σ, and DS is a set of disjoint disks centered at the points in S. By homotopy invariance we have

$$H_*(\Sigma, S) \cong H_*(\Sigma, D_S).$$

Denote by Σ_S the manifold with boundary obtained by removing the disks D_S. Using excision again we deduce

$$H_*(\Sigma, D_S) \cong H_*(\Sigma_S, \partial\Sigma_S)$$

so that

$$H_*(\Sigma_S, \partial\Sigma_S) \cong H_*(\Sigma, S) \tag{5.4}$$

Note that the groups on the right hand side were computed in part (a).

We deduce that

$$H_*(X, A) \cong H_*(\text{torus, pt}) \oplus H_*(\text{torus, pt}).$$

Observe that $X - B$ is a torus with two disks removed so that

$$H_*(X, B) \cong H_*(\text{torus}, \{\text{ pt}_1, \text{ pt}_2\}).$$

□

Problem 20, §2.1 (a) Consider the cone over X

$$CX = I \times X / \{0\} \times X.$$

We will regard X as a subspace of CX via the inclusion

$$X \cong \{1\} \times X \hookrightarrow CX.$$

Then CX is contractible and we deduce

$$\tilde{H}_*(CX) = 0.$$

(CX, X) is a good pair, and $SX = CX/X$so that

$$\tilde{H}_*(SX) \cong H_*(CX, X).$$

From the long exact sequence of the pair (CX, X) we deduce

$$\cdots \to H_{k+1}(CX) \to H_{k+1}(CX, X) \to H_k(X) \to H_k(CX) \to \cdots \tag{5.5}$$

Thus for $k \geq 1$ we have

$$H_k(CX) = H_{k+1}(CX) = 0$$

so that

$$H_{k+1}(SX) \cong H_{k+1}(CX, X) \cong H_k(X).$$

Using $k = 0$ in (5.5) we deduce

$$0 \to H_1(CX, X) \to H_0(X) \to H_0(CX)$$

The inclusion induced morphism $H_0(X) \to H_0(CX)$ is onto so that

$$\tilde{H}_1(SX) \cong H_1(CX, X) \cong \ker(H_0(X) \to H_0(CX)) \cong \tilde{H}_0(X).$$

(b) Denote by $S_n X$ the space obtained by attaching n-cones over X along their bases using the tautological maps (see Figure 2).

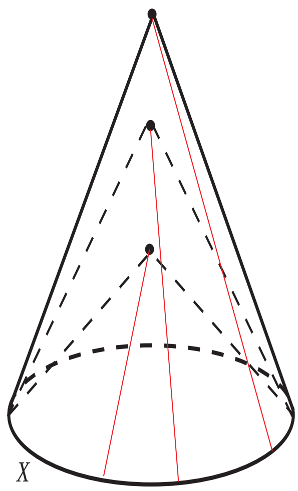


FIGURE 2. *Stacking-up several cones*

We see a copy of X inside S_nX. It has an open neighborhood U which deformation retracts onto this copy of X and such that its complement is homeomorphic to a disjoint union of n cones on X. The Mayer-Vietoris sequence of the decomposition

$$S_nX = S_{n-1}X \cup_X CX$$

is

$$\cdots \to H_k(X) \to H_k(S_{n-1}X) \oplus H_k(CX) \to H_k(S_nX) \to H_{k-1}(X) \to \cdots .$$

For $k > 0$ we have $H_k(CX) = 0$. Moreover, the inclusion induced morphism $H_k(X) \to H_k(S_{n-1}X)$ is trivial since any cycle in X bounds inside[2] $S_{n-1}X$. Hence we get a short exact sequence

$$0 \to H_k(S_{n-1}X) \to H_k(S_nX) \to H_{k-1}(X) \to H_{k-1}(S_{n-1}X).$$

For $k > 1$ we have

$$H_{k-1}(X) \cong \ker\Big(H_{k-1}(X) \to H_{k-1}(S_{n-1}X) \Big)$$

while for $k = 1$ we have

$$\tilde{H}_{k-1}(X) \cong \ker\Big(H_{k-1}(X) \to H_{k-1}(S_{n-1}X) \Big).$$

Thus, for every $k \geq 1$ we have the short exact sequence

$$0 \to H_k(S_{n-1}X) \to H_k(S_nX) \to \tilde{H}_{k-1}(X) \to 0. \tag{5.6}$$

Now observe that there exists a natural retraction

$$r : S_nX \to S_{n-1}X.$$

To describe it consider first the obvious retraction from the disjoint union of n cones to the disjoint union of $(n-1)$ cones

$$\tilde{r} : \{1, \cdots, n\} \times CX \to \{1, \cdots, n-1\} \times CX, \;\; \tilde{r}(j,p) = \begin{cases} (j,p) & \text{if} \quad j < n \\ (1,p) & \text{if} \quad j = n \end{cases}$$

Now observe that

$$\tilde{r}(\{1, \cdots, n\} \times X) = \{1, \cdots, n-1\} \times X$$

[2]The cone on z bounds z.

and

$S_nX = \{1, \cdots, n\} \times CX/\{1, \cdots, n\} \times X, \;\; S_{n-1}X = \{1, \cdots, n-1\} \times CX/\{1, \cdots, n-1\} \times X$

so that $\tilde{r}$ descends to a retraction

$$r : S_nX \to S_{n-1}X.$$

This shows that the sequence (5.6) splits so that

$$H_k(S_nX) \cong H_k(S_{n-1}X) \oplus \tilde{H}_{k-1}(X) \cong^{\text{inductively}} \cdots \cong \oplus_{j=1}^{n-1} \tilde{H}_{k-1}(X).$$

□

Problem 27, §2.1 (a) We have the following commutative diagram

$$\begin{array}{ccccccccc}
H_{n+1}(A) & \longrightarrow & H_{n+1}(X) & \longrightarrow & H_{n+1}(X,A) & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
H_{n+1}(B) & \longrightarrow & H_{n+1}(Y) & \longrightarrow & H_{n+1}(Y,B) & \longrightarrow & H_n(B) & \longrightarrow & H_n(Y)
\end{array}$$

The rows are exact. The morphisms induced on absolute homology are isomorphisms so the five lemma implies that the middle vertical morphism between relative homology groups is an isomorphism as well.

(b) We argue by contradiction. Suppose there exists a map $g : (D^n, D^n \setminus 0) \to (D^n, \partial D^n)$ such that $g \circ f$ is homotopic as maps of pairs with $\mathbb{1}_{(D^n, \partial D^n)}$. If $x \in D^n \setminus 0$ then, $g(tx) \in \partial D^n$, $\forall t \in (0, 1]$. We deduce that

$$g(0) = \lim_{t \searrow 0} g(tx) \in \partial D^n.$$

Hence $g(D^n) \subset \partial D^n$ so we can regard g as a map $D^n \to \partial D^n$. Note that $g\,|_{\partial D^n} \simeq \mathbb{1}_{\partial D^n}$. Equivalently, if we denote by i the natural inclusion $\partial D^n \hookrightarrow D^n$ then we have

$$g \circ i \simeq \mathbb{1}_{\partial D^n},$$

so that for every $k \geq 0$ we get a commutative diagram

$$\begin{array}{ccc}
\tilde{H}_k(\partial D^n) & \xrightarrow{i_*} & \tilde{H}_k(D^n) = 0 \\
 & \searrow^{\mathbb{1}_{H_k(\partial D^n)}} & \downarrow g_* = 0 \\
 & & \tilde{H}_k(\partial D^n)
\end{array}$$

In particular for $k = n - 1$ we have $\tilde{H}_{n-1}(\partial D^n) \cong \mathbb{Z}$ and we reached a contradiction.

□

Problem 28, §2.1 The cone on the 1-skeleton of Δ_3 is depicted in Figure 3.

Before we proceed with the proof let us introduce a bit of terminology. The cone X is linearly embedded in $\mathbb{R}^3$ so that it is equipped with a metric induced by the Euclidean metric. For every point $x_0 \in X$ we set

$$B_r(x_0) := \{x \in X; \;\; |x - x_0| \leq r\}.$$

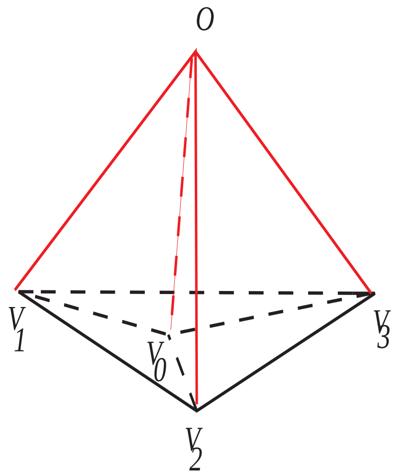


FIGURE 3. *A cone over the* 1*-skeleton of a tetrahedron.*

By excising $X - B_r(x_0)$, $0 < r \ll 1$ we deduce

$$H_*(X, X - x_0) \cong H_*(B_r(x_0), B_r(x_0) - x_0).$$

Now observe that $B_r(x_0)$ deformation retracts onto $L_r(x_0)$, the *link* of x_0 in X,

$$L_r(x_0) = \{x \in X;\ \ |x - x_0| = r.$$

Hence

$$H_*(X, X - x_0) \cong H_*(B_r(x_0), L_r(x_0)) \cong \tilde{H}_*(B_r(x_0)/L_r(x_0)).$$

We now discuss separately various cases (see Figure 4).

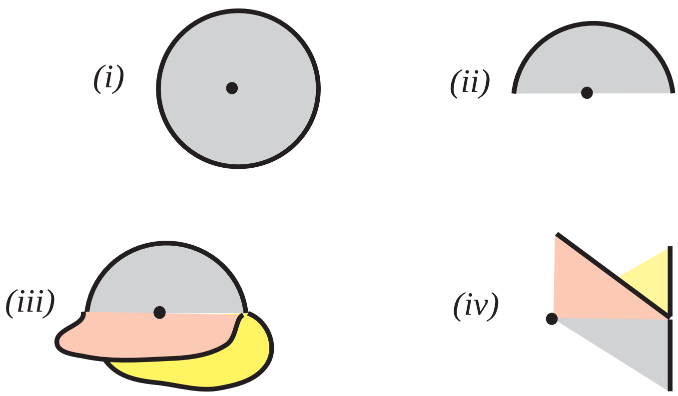


FIGURE 4. *The links of various points on* X*.*

(i) *x_0 is in the interior of a 2-face.* In this case $B_r(x_0)/L_r(x_0) \cong S^2$ for all $r \ll 1$ so that

$$H_*(X, X - x_0) \cong \tilde{H}_*(S^2).$$

(ii)*x_0 is inside one of the edges $[V_iV_j]$.* In this case $B_r(x_0)$ is the upper half-disk, and the link is the upper half-circle.

$$H_*(X, X - x_0) \cong 0.$$

(iii) *x_0 is inside one of the edges $[OV_i]$.* In this case B_r consists of three half-disks glued along their diameters. The link consists of three arcs with identical initial points and final points. Then $B_r(x_0)/L_r(x_0) \simeq S^2 \vee S^2$ so that

$$H_*(X, X - x_0) \cong \tilde{H}_*(S^2 \vee S^2) \cong \tilde{H}_*(S^2) \oplus \tilde{H}_*(S^2).$$

(iv) *x_0 is one of the vertices V_i.* In this case B_r consists of three circular sectors with a common edge. The link is the wedge of three arcs. In this case B_r/L_r is contractible so that

$$H_*(X, X - x_0) \cong 0$$

(v) *$x_0 = O$.* In this case $B_r \cong X$ and the link coincides with the 1-skeleton of Δ_3. We denote this 1-skeleton by Y. Using the long exact sequence of the pair (X, Y) and the contractibility of X we obtain isomorphisms

$$H_n(X, Y) \cong \tilde{H}_{n-1}(Y) \cong \begin{cases} 0 & \text{if} \quad n \neq 2 \\ \mathbb{Z}^3 & \text{if} \quad n = 2 \end{cases}.$$

We deduce that the boundary points are the points in (ii) and (iv). These are precisely the points situated on Y.

To understand the invariant sets of a homeomorphism f of X note first that

$$H_*(X, X - x) \cong H_*(X, X - f(x)).$$

In particular any homeomorphism of X induces by restriction a homeomorphism of Y. By analyzing in a similar fashion the various local homology groups $H_*(Y, Y - y)$ we deduce that any homeomorphism of Y maps vertices to vertices so it must permute them.

Any homeomorphism f of X maps the vertex O to itself. Also, it maps any point on one of the edges $[OV_i]$ to a point on an edge $[OV_j]$. Thus any homeomorphism permutes the edges $[OV_i]$. We deduce that the nonempty subsets of X left invariant by all the homeomorphisms of X are obtained from the following sets

$$\{O\}, \ \ \{V_0, V_1, V_2, V_3\}, \ \ Y, \ \ [OV_0] \cup \cdots \cup [OV_3], \ \ X.$$

via the basic set theoretic operations $\cup$, $\cap$, $\setminus$.

□

Homework

1. We denote by $\mathbb{Z}[t]$ the ring of polynomials with integer coefficients in one variable t. If $A, B \in \mathbb{Z}[t]$, we say that A *dominates* B, and we write this $A \succeq B$, if there exists a polynomial $Q \in \mathbb{Z}[t]$, with *nonnegative coefficients* such that

$$A(t) = B(t) + (1+t)Q(t).$$

(a) Show that if $A_0 \succeq B_0$, $A_1 \succeq B_1$ and $C \succeq 0$ then

$$A_0 + A_1 \succeq B_0 + B_1 \ \text{ and } \ CA_0 \succeq CB_0.$$

(b) Suppose $A(t) = a_0 + a_1 t + \cdots a_n t^n \in \mathbb{Z}[t]$, $B = b_0 + b_1 t + \cdots + b_m t^m$. Show that $A \succeq B$ if and only if, for every $k \geq 0$ we have

$$\sum_{i+j=k} (-1)^i a_j \geq \sum_{i+j=k} (-1)^i b_j, \tag{$M_\geq$}$$

$$\sum_{j\geq 0} (-1)^j a_j = \sum_{k\geq 0} (-1)^j b_j. \tag{$M_=$}$$

(c) We define a graded Abelian group to be a sequence of Abelian groups $C_\bullet := (C_n)_{n\geq 0}$. We say that $C_\bullet$ is of *finite type* if

$$\sum_{n\geq 0} \operatorname{rank} C_n < \infty.$$

The *Poincaré polynomial* of a graded group $C_\bullet$ of finite type is defined as

$$P_C(t) = \sum_{n\geq 0} (\operatorname{rank} C_n) t^n.$$

The *Euler characteristic* of $C_\bullet$ is the integer

$$\chi(C_\bullet) = P_C(-1) = \sum_{n\geq 0} (-1)^n \operatorname{rank} C_n.$$

A short exact sequence of graded groups $(A_\bullet)$, $(B_\bullet)$, $(C_\bullet)$ is a sequence of short exact sequences

$$0 \to A_n \to B_n \to C_n \to 0, \ \ n \geq 0.$$

Prove that if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of graded Abelian groups of finite type, then

$$P_B(t) = P_A(t) + P_C(t). \tag{2}$$

(d)(*Morse inequalities. Part 1*) Suppose

$$\cdots \to C_n \stackrel{\partial}{\to} C_{n-1} \stackrel{\partial}{\to} \cdots \stackrel{\partial}{\to} C_1 \stackrel{\partial}{\to} C_0 \to 0$$

is a chain complex such that the grade group $C_\bullet$ is of finite type. We denote by H_n the n-th homology group of this complex and we form the corresponding graded group $H_\bullet = (H_n)_{n\geq 0}$. Show that $H_\bullet$ is of finite type and

$$P_C(t) \succeq P_H(t) \ \text{ and } \ \chi(C_\bullet) = \chi(H_\bullet).$$

(e) (*Morse inequalities. Part 2*) Suppose we are given three finite type graded groups $A_\bullet$, $B_\bullet$ and $C_\bullet$ which are part of a long exact sequence

$$\cdots \to A_k \stackrel{i_k}{\to} B_k \stackrel{j_k}{\to} C_k \stackrel{\partial_k}{\to} A_{k-1} \to \cdots \to A_0 \to B_0 \to C_0 \to 0.$$

Show that

$$P_A(t) + P_C(t) \succeq P_B(t),$$

and

$$\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet).$$

Proof. (a) We have

$$A_0(t) = B_0(t) + (1+t)Q_0(t),\ \ A_1(t) = B_1(t) + (1+t)Q_1(t)$$

so that

$$A_0(t) + A_1(t) = B_0(t) + B_1(t) + (1+t)\big(\,Q_0(t) + Q_1(t)\,\big).$$

Note that if Q_0 and Q_1 have nonnegative integral coefficients, so does Q_0+Q_1. Next observe that

$$CA_0 = CB_0 + (1+t)CQ.$$

If C and Q have nonnegative integral coefficients, so does CQ.
(b) Use the identity

$$(1+t)^{-1} = \sum_{k\geq 0}(-1)^k t^k.$$

Then

$$A - B = (1+t)Q \Longleftrightarrow Q(t) = (1+t)^{-1}\big(\,A(t) - B(t)\,\big)$$
$$\Longleftrightarrow q_n = \sum_{i+j=n}(-1)^i(a_j - b_j),\ \text{where}\ \ Q = \sum_n q_n t^n.$$

Hence

$$q_n \geq 0,\ \ \forall n \Longleftrightarrow \sum_{i+j=n}(-1)^i a_j \geq \sum_{i+j=n}(-1)^i b_j.$$

This proves $(M_{\geq})$. The equality $(M_{=})$ is another way of writing the equality

$$A(-1) = B(-1).$$

(c) Set $a_n = \operatorname{rank} A_n$, $b_n = \operatorname{rank} B_n$, $c_n = \operatorname{rank} C_n$. If

$$0 \to A_n \to B_n \to C_n \to 0,\ \ n \geq 0.$$

is a short exact sequence then

$$b_n = a_n + c_n \Longrightarrow \sum_{n\geq 0} b_n t^n = \sum_{n\geq 0} a_n t^n + \sum_{n\geq 0} c_n t^n.,$$

which is exactly (2).
(d) Observe that we have short exact sequences

$$0 \to Z_n(C) \to C_n \stackrel{\partial}{\to} B_{n-1}(C) \to 0, \tag{3}$$
$$0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0. \tag{4}$$

We set

$$z_n := \operatorname{rank} Z_n(C),\ \ b_n = \operatorname{rank} B_n(C),\ \ h_n = \operatorname{rank} H_n(C),\ \ c_n = \operatorname{rank} C_n.$$

From (3) we deduce

$$c_n = z_n + b_{n-1},\ \ \forall n \geq 0,$$

where we have $B_{-1}(C) = 0$. Hence

$$P_C(t) = P_Z(t) + tP_B(t).$$

On the other hand, the sequence (4) implies

$$P_Z = P_B + P_H.$$

Hence
$$P_C = P_H + (1+t)P_B \Longrightarrow P_C \succeq P_H.$$
The equality $\chi(C) = \chi(H)$ follows from $(M_=)$.
(e) Set
$$a_k := \operatorname{rank} A_k,\ \ b_k := \operatorname{rank} B_k,\ \ c_k = \operatorname{rank} C_k,$$
$$\alpha_k = \operatorname{rank}\ker i_k,\ \ \beta_k = \operatorname{rank}\ker j_k,\ \ \gamma_k = \operatorname{rank}\ker \partial_k.$$
Then
$$\begin{cases} a_k &= \alpha_k + \beta_k \\ b_k &= \beta_k + \gamma_k \\ c_k &= \gamma_k + \alpha_{k-1} \end{cases} \Longrightarrow a_k - b_k + c_k = \alpha_k + \alpha_{k-1}$$
$$\Longrightarrow \sum_k (a_k - b_k + c_k)t^k = \sum_k t^k(\alpha_k + \alpha_{k-1})$$
$$\Longrightarrow P_{A_\bullet}(t) - P_{B_\bullet}(t) + P_{C_\bullet}(t) = (1+t)Q(t),\ \ Q(t) = \sum_k \alpha_k t^{k-1}.$$

□

Hatcher, §2.1, Problem 14. We will use the identification
$$\mathbb{Z}_n = \Big\{ i/n \in \mathbb{Q}/\mathbb{Z};\, i \in \mathbb{Z} \Big\}.$$
(a) Consider the injection
$$j : \mathbb{Z}_4 \hookrightarrow \mathbb{Z}_8 \oplus \mathbb{Z}_2,\ \ 1/4 \mapsto (1/4, 1/2).$$
Then $(1/8, 0)$ is an element of order 4 in $(\mathbb{Z}_8 \oplus \mathbb{Z}_2)/j(\mathbb{Z}_4)$ so that we have a short exact sequence
$$0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0.$$
(b) Suppose we have a short exact sequence
$$0 \to \mathbb{Z}_{p^m} \xrightarrow{j} A \xrightarrow{\pi} \mathbb{Z}_{p^n} \to 0. \tag{5}$$
Then A is an Abelian group of order p^{m+n} so that it has a direct sum decomposition
$$A \cong \bigoplus_{i=1}^k \mathbb{Z}_{p^{\nu_i}},\ \ \nu_1 \geq \nu_2 \geq \cdots \geq \nu_k,\ \ \sum_i \nu_i = m+n. \tag{6}$$
On the other hand A must have an element of order p^m, and an element of order $\geq p^n$ so that $\nu_1 \geq \max(m,n)$.

Fix an element $a_1 \in A$ which projects onto a generator of $\mathbb{Z}_{p^n}$, and denote by $a_0 \in A$ the image of a generator in $\mathbb{Z}_{p^m}$. Then A is generated by a_0 and a_1 so the number k of summands in (6) is at most 2. Hence
$$A \cong A_{\alpha,\beta} := \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta},\ \ \alpha \geq \max(m,n,\beta),\ \ \alpha + \beta = m+n. \tag{7}$$
We claim that any group $A_{\alpha,\beta}$ as in (7) fits in an exact sequence of the type (5). To prove this we need to find an inclusion $j : \mathbb{Z}_{p^n} \hookrightarrow A_{\alpha,\beta}$ such that the group $A_{\alpha,\beta}/j(\mathbb{Z}_{p^m})$ has an element of order p^n.

Observe first that $\beta \leq \min(m,n)$ because
$$\beta = (m+n) - \alpha = \min(m,n) + \underbrace{(\max(m,n) - \alpha)}_{\leq 0} \leq \min(m,n).$$

Consider the inclusion

$$\mathbb{Z}_{p^m} \to A_{\alpha,\beta} = \mathbb{Z}_{p^\alpha} \oplus \mathbb{Z}_{p^\beta}, \;\; 1/p^m \mapsto (1/p^m, 1/p^\beta).$$

Then the element $g = (1/p^\alpha, 0)$ has order p^n in the quotient $A_{\alpha,\beta}/j(\mathbb{Z}_{p^m})$.

To prove this observe first that the order of g is a power p^ν of p, $\nu \leq n$. Since $p^\nu g \in j(\mathbb{Z}_{p^m})$, there exists $x \in \mathbb{Z}$, $0 < x < p^m$, such that

$$p^\nu g = (1/p^{\alpha-\nu}, 0) = x \cdot (1/p^m, 1/p^\beta) \mod \mathbb{Z}.$$

Hence

$$p^\beta | x, \;\; p^{\alpha+m} | (p^{m+\nu} - x p^\alpha).$$

We can now write $x = x_1 p^\beta$, so that

$$p^{n+m} | (x_1 p^{\alpha+\beta} - p^{m+\nu}).$$

Since $\alpha + \beta = m + n$ we deduce $p^{n+m} | p^{m+\nu}$ so that $n \leq \nu$.

(c)* Consider a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}_n \to 0.$$

We will construct a group morphism $\chi : \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}$ as follows.[1]

For every $x \in \mathbb{Z}_n$ there exists $\hat{x} \in A$ such that $g(\hat{x}) = x$. Then $g(n \cdot \hat{x}) = nx = 0$ so that

$$n \cdot \hat{x} \in \ker g = f(\mathbb{Z}).$$

Hence there exists $k \in \mathbb{Z}$ such that

$$f(k) = n \cdot \hat{x}.$$

Set[2]

$$\chi(x) := \frac{k}{n} \mod \mathbb{Z}.$$

The definition of $\chi(x)$ is independent of the choice $\hat{x}$. Indeed if $\hat{x}' \in A$ is a different element of A such that $g(\hat{x}') = x$ then $\hat{x} - \hat{x}' \in \ker g$ so there exists $s \in \mathbb{Z}$ such that

$$\hat{x} - \hat{x}' = f(s).$$

Then

$$n\hat{x}' = n\hat{x} - f(ns) = f(k - ns)$$

so that $\frac{k}{n} = \frac{k-ns}{n} \mod \mathbb{Z}$.

Now define a map

$$h : A \to \mathbb{Q} \oplus \mathbb{Z}_n, \;\; a \mapsto \Big(\frac{f^{-1}(na)}{n}, g(a) \Big).$$

Observe that h is injective. Its image consists of pairs $(q, x) \in \mathbb{Q} \oplus \mathbb{Z}_n$ such that

$$q = \chi(x) \mod \mathbb{Z}.$$

We deduce that A is isomorphic to $\mathbb{Z} \oplus \operatorname{Im}(\chi)$. The image of χ is a cyclic group whose order is a divisor of n.

Conversely,given a group morphism $\lambda : \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}$, we denote by $C_\lambda \subset \mathbb{Q}/\mathbb{Z}$ its image, and we form the group

$$A_\lambda := \big\{ (q, c) \in \mathbb{Q} \times \mathbb{Z}_n; \;\; q = \lambda(c) \mod \mathbb{Z} \big\}.$$

Observe that $A \cong \mathbb{Z} \oplus C_\lambda$, and C_λ is a finite cyclic group whose order is a divisor of n.

---

[1] A group morphism $G \to \mathbb{Q}/\mathbb{Z}$ is called a *character* of the group.

[2] Less rigorously $\chi(x) = \frac{f^{-1}(ng^{-1}(x))}{n} \mod \mathbb{Z}$.

We have a natural injection

$$f : \mathbb{Z} \hookrightarrow \mathbb{Q} \oplus 0 \hookrightarrow A_\lambda,$$

a natural surjection

$$A_\lambda \hookrightarrow \mathbb{Q} \times \mathbb{Z}_n \twoheadrightarrow \mathbb{Z}_n,$$

and the sequence

$$0 \to \mathbb{Z} \to A_\lambda \to \mathbb{Z}_n \to 0$$

is exact.

Given any divisor m of n, we consider

$$\lambda_m : \mathbb{Z}_n \to \mathbb{Q}/\mathbb{Z}, \ \ \frac{k}{n} \mod \mathbb{Z} \mapsto \frac{k}{m} \mod \mathbb{Z}.$$

Its image is a cyclic group of order m. We have thus shown that there exists a short exact sequences

$$0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$$

if and only if $A \cong \mathbb{Z} \oplus \mathbb{Z}_m$, $m|n$.

□

Homework # 7

Definition 7.1. A space X is said to be of *finite type* if it satisfies the following conditions.
(a) $\exists N > 0$ such that $H_n(X) = 0$, $\forall n > N$.
(b) $\operatorname{rank} H_k(X) < \infty$, $\forall k \geq 0$. □

1. (a) Suppose A, B are open subsets of the space X such that $X = A \cup B$. Assume A, B and $A \cap B$ are of finite type. Prove that X is of finite type and

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

(b) Suppose X is a space of finite type. Prove that

$$\chi(S^1 \times X) = 0.$$

(c) Suppose we are given a structure of finite Δ-complex on a space X. We denote by c_k the number of equivalence classes of k-faces. Prove that

$$\chi(X) = c_0 - c_1 + c_2 - \dots .$$

(d) Let us define a graph to be a connected, 1-dimensional, finite Δ-complex. (A graph is allowed to have loops, i.e., edges originating and ending at the same vertex, see Figure 1.)

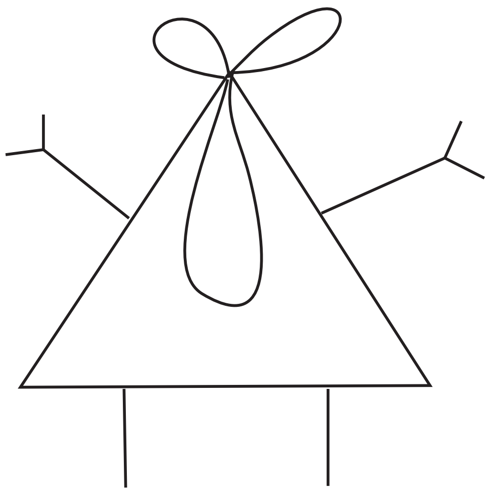


FIGURE 1. *A graph with loops.*

Suppose G is a graph with vertex set V. For simplicity, we assume that it is embedded in the Euclidean space $\mathbb{R}^3$. We denote by $c_0(G)$ the number of vertices, and by $c_1(G)$ the number of edges, and by $\chi(G)$ the Euler characteristic of G. We set

$$\ell(v) := \operatorname{rank} H_1(G, G \setminus \{v\}), \quad d(v) = 1 + \ell(v).$$

Prove that

$$c_1(G) = \frac{1}{2} \sum_{v \in V} d(v), \quad \chi(G) = \frac{1}{2} \sum_{v \in V} \big(1 - \ell(v)\big).$$

Proof. (a) From the Mayer-Vietoris sequence

$$\dots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

that X is of finite type. Using part (e) of Problem 1 in Homework # 6 for the above long exact sequence we deduce

$$\chi(A)+\chi(B)=\chi(A\cap B)+\chi(X).$$

(b) View S^1 as the round circle in the plane

$$S^1=\{\,(x,y)\in\mathbb{R}^2;\;\;x^2+y^2=1\,\}.$$

Denote by p_+ the North pole $p_+=(0,1)$, and by p_- the South pole, $p_-=(0,-1)$. We set

$$A_\pm=(S^1\setminus\{p_\pm\})\times X.$$

Then $A_\pm$ are open subsets of $S^1\times X$ and $S^1\times A_+\cup A_-$. Each of them is homeomorphic to $(0,1)\times X$, and thus homotopic with X and therefore

$$\chi(A_\pm)=\chi(X).$$

The overlap

$$A_0=A_+\cap A_-=(S^1\setminus\{p_+,p_-\})\times X,$$

has two connected components, each homeomorphic to $(0,1)\times X$, and thus homotopic with X so that

$$\chi(A_0)=2\chi(X).$$

From part (a) we deduce that

$$\chi(X)=\chi(A_+)+\chi(A_-)-\chi(A_0)=0.$$

(c) The homology of X can be computed using the Δ-complex structure. Thus, the homology groups $H_k(X)$ are the homology groups of a chain complex

$$\cdots\to\Delta_n(X)\stackrel{\partial}{\to}\Delta_{n-1}(X)\stackrel{\partial}{\to}\cdots,$$

where $\operatorname{rank}\Delta_n(X)=c_n$. The desired conclusion now follows from part (d) of Problem 1 in Homework # 6.

(d) For every $v\in V$ we denote by $B_r(v)$ the closed ball of radius r centered at x, and we set

$$G_r(v):=B_r(v)\cap G.$$

For r sufficiently small $G_r(x)$ is contractible. We assume r is such. Using excision, we deduce

$$H_\bullet(G,G\setminus\{v\})\cong H_\bullet(G_r(v),G_r(x)\setminus\{v\}).$$

We set $G'_r(x):=G_r(v)\setminus\{x\}$. Using the long exact sequence of the pair $(G_r(v),G'_r(v))$ we obtain the exact sequence

$$0=H_1(G_r(x))\to H_1(G_r(v),G'_r(x))\to H_0(\,G'_r(v)\,)\stackrel{i_0}{\to}H_0(\,G_r(v)\,)\cong\mathbb{Z}.$$

Hence

$$\ell(x)=\operatorname{rank}\ker i_0=\operatorname{rank}H_0(G'_r(v))-1\Longrightarrow d(v)=\operatorname{rank}H_0(G'_r(x)).$$

In other words, $d(v)$ is the number of components of $G'_r(v)$, when r is very small. Equivalently, $d(v)$ is the number of edges originating /and/or ending at v, where each loop is to be counted twice. This is called the *degree* of the vertex x. For example, the degree of the top vertex of the graph depicted in Figure 1 is 8, because there are 3 loops and 2 regular edges at that vertex. The equality

$$\sum_{v\in V}d(v)=2c_1(G),$$

is now clear, because in the above sum each edge is counted twice. From part (c) we deduce

$$\chi(G) = c_0(G) - c_1(G)$$

so that

$$\chi(G) = \sum_{v\in V} 1 - \frac{1}{2}\sum_{v\in V} d(v) = \sum_{v\in V} 1 - \frac{1}{2}\sum_{v\in V}(1-\ell(v))$$
$$= \frac{1}{2}\sum_{v\in V}\big(1+\ell(v)\big).$$

□

2. Consider a connected planar graph G situated in a half plane H, such that the boundary of the half plane intersects G in a nonempty set of vertices. Denote by ν the number of such vertices, and by χ_G the Euler characteristic of G. Let S be the space obtained by rotating G about the y axis.
(a) Compute the Betti numbers of S.
(b) Determine these Betti numbers in the special case when G is the graph depicted in Figure 2, where the red dotted line is the boundary of the half plane.

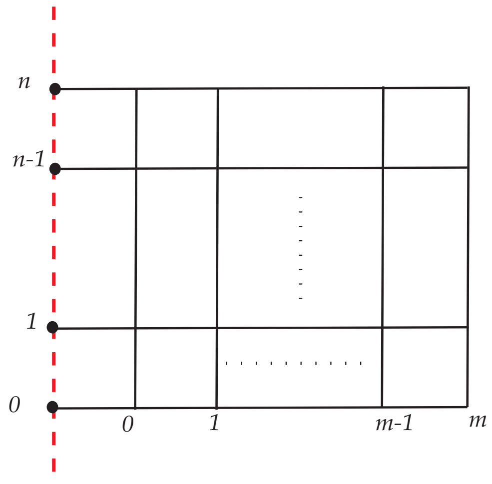


FIGURE 2. *Rotating a planar graph.*

Proof. For every graph Γ, we denote by $c_0(\Gamma)$ (respectively $(c_1(\Gamma))$ the number of vertices (respectively edges) of Γ.

As in Homework # 2, we can deform the graph G inside the halfplane, by collapsing one by one the edges which have at least one vertex not situated on the y-axis. We obtain a new planar graph G_0, that is homotopic to G, and has exactly ν vertices, all situated on the axis of rotation. From the equality

$$\chi_G = \chi(G_0),$$

we deduce

$$\chi_G = c_0(G_0) - c_1(G_0) = \nu - c_1(G_0) \Longrightarrow c_1(G_0) = \nu - \chi_G.$$

Denote by S_0 the space obtained by rotating G_0 about the y-axis. Then S_0 is homotopic with S, and the result you proved in Homework 2 shows that S_0 is a wedge of a number n_1 circles, and a number n_2 of spheres. Using Corollary 2.25 of your textbook we deduce

$$\tilde{H}_k(S_0) = \underbrace{\tilde{H}_k(S^1) \oplus \cdots \oplus \tilde{H}_k(S^1)}_{n_1} \oplus \underbrace{\tilde{H}_k(S^2) \oplus \cdots \oplus \tilde{H}_k(S^2)}_{n_2}.$$

so that

$$b_0(S_0) = 1, \;\; b_1(S_0) = n_1, \;\; b_2(S_0) = n_2, \;\; b_k(S_0) = 0, \;\; \forall k > 2,$$

and its Euler characteristic satisfies

$$\chi(S) = \chi(S_0) = 1 - n_1 + n_2.$$

The 2-spheres which appear in the above wedge decomposition of S_0 are in a bijective correspondence with the edges of G_0 so that

$$b_2(S_0) = n_2 = c_1(G_0) = \nu - \chi_G.$$

For every vertex v of G_0 we denote by S_0^v the intersection of S_0 with a tiny *open* ball centered at v. Note that S_0^v is contractible. Define

$$A := \bigcup_{v \in V} S_0^v, \quad B = S_0 \setminus V.$$

Then A, B are open subsets of S_0 and

$$S_0 = A \cup B.$$

From part (a) of Problem 1 we deduce

$$\chi(S_0) = \chi(A) + \chi(B) - \chi(A \cap B),$$

provided that the spaces A, B and $A \cap B$ are of finite type. A is the disjoint union of ν contractible sets so that A is of finite type and $\chi(A) = \nu$. B is the disjoint union of $c_1(G_0)$ cylinders, one cylinder for each edge of G_0. In particular, B is of finite type and $\chi(B) = 0$. The overlap is the disjoint union of punctured disks, and each of them has finite type and trivial Euler characteristic. Hence

$$\chi(S_0) = \nu.$$

We deduce

$$\nu = 1 - n_1 + n_2 = 1 - n_1 + \nu - \chi_G \Longrightarrow b_1(S_0) = n_1 = 1 - \chi_G = b_1(G).$$

(b) Observe that the graph in Figure 1 has $(m+2)(n+1)$ vertices because there are $n+1$ horizontal lines and $m+2$ vertices on each of them.

To count the edges, observe that there are $(m+1)(n+1)$ horizontal edges and $n(m+1)$ vertical ones. Hence

$$\chi_G = (m+2)(n+1) - (m+1)(n+1) - n(m+1) = n + 1 - n(m+1) = 1 - mn.$$

Since $b_0(G) = 1$, we deduce $b_1(G) = mn$. By rotating G about the vertical axis we obtain a space which is a wedge of mn copies of S^1 and $n + mn$ copies of S^2. □

Solutions to Homework # 8

Problem 3, §2.2. Since $\deg f = 0 \neq (-1)^{n+1}$ we deduce that f must have a fixed point, i.e. there exists $x \in S^n$ such that $f(x) = x$.

Let $g = (-\mathbb{1}) \circ f$. Then $\deg g = \deg(-\mathbb{1}) \cdot \deg f = 0$ so that g must have a fixed point y. Thus $f(y) = -y$.

□

Problem 4, §2.2. Consider a continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$f(0) = f(1) = 0, \;\; f(1/2) = 2\pi.$$

The map

$$I := [0, 1] \to S^1, \;\; t \mapsto \exp(\boldsymbol{i} f(t))$$

induces a continuous surjective map $g : I/\partial I = S^1 \to S^1$. The map f is a lift at $0 \in \mathbb{R}$ of g in the universal cover $\mathbb{R} \stackrel{\exp}{\to} S^1$. Since f starts and ends at the same point we deduce that g is homotopically trivial so that $\deg g = 0$. We have thus constructed a surjection $g : S^1 \to S^1$ of degree zero. Suppose inductively that $f : S^n \to S^n$ is a degree 0 surjection. Then the suspension of f is a degree 0 surjection

$$Sf : S^{n+1} \to S^{n+1}.$$

□

Problem 7, §2.2. Assume E is an n-dimensional real Euclidean space with inner product $\langle \bullet, \bullet \rangle$. Suppose $T : E \to E$ is a linear automorphism, and set

$$S := TT^*.$$

S is selfadjoint, and thus we can find an orthonormal basis $(e_1, \cdots, e_n)$ of E which diagonalizes it,

$$S = \operatorname{diag}(\lambda_1, \cdots, \lambda_n), \;\; \lambda_i > 0.$$

Let

$$D(t) = \operatorname{diag}(\lambda_1^{-t/2}, \cdots, \lambda_n^{-t/2}),$$

so that $D(0) = \mathbb{1}$ and $D(1)^2 = S^{-1}$. Now define

$$T_t = D(t)T, \;\; S_t = T_t T_t^* = D_t S D_t.$$

Observe that sign $\det T_t =$ sign $\det T$, $\forall t$, and

$$S_0 = S, \;\; S_1 = 1,$$

so that T_1 is homotopic through automorphisms with an orthogonal operator. Thus, we can assume from the very beginning that T is orthogonal.

For each $\theta \in [0, 2\pi]$ denote by $R_\theta : \mathbb{C} \to \mathbb{C}$ the counterclockwise rotation by θ. Using the Jordan normal form of an orthogonal matrix we can find an orthogonal decomposition

$$E \cong U \oplus V \oplus \mathbb{C}^m,$$

such that T has the form

$$T = \mathbb{1}_U \oplus (-\mathbb{1}_V) \oplus \bigoplus_{i=1}^m R_{\theta_i}.$$

There exists a homotopy

$$T_s = \mathbb{1}_U \oplus (-\mathbb{1}_V) \oplus \bigoplus_{i=1}^{m} R_{s\theta_i},$$

such that

$$T_0 = \mathbb{1}_U \oplus (-\mathbb{1}_V) \oplus \mathbb{1}_{\mathbb{C}^m}, \;\; T_1 = T, \;\; \det T_0 = \det T_1.$$

Thus T is homotopic to a product of reflections and the claim in the problem is true for such automorphisms.

□

Problem 8, §2.2. It is convenient to identify S^2 with $\mathbb{CP}^1$. As such, its covered by two coordinate charts,

$$U_s = S^2 \setminus \{\text{South Pole}\} \cong \mathbb{C}, \;\; U_n = S^2 \setminus \{\text{North Pole}\} \cong \mathbb{C}.$$

We denote by $x : U_s \to \mathbb{C}$ the complex coordinate on U_s and by $y : U_n \to \mathbb{C}$ the complex coordinate on U_n. On the overlap $U_s \cap U_n$ we have the equality $x = \frac{1}{y}$.

We think of a polynomial as a function $f : U_s \to \mathbb{C}$,

$$f(p) = \sum_{j=0}^{d} a_j x^j, \;\; x^j = x(p)^j.$$

Here we think of U_s as a coordinate chart in a copy of of $\mathbb{CP}^1$ which we denote by $\mathbb{C}^1_{source}$.

We think of the target space $\mathbb{C}$ of f as the coordinate chart V_s of another copy of $\mathbb{CP}^1$ which we denote by $\mathbb{CP}^1_{target}$. We denote the local coordinates on $\mathbb{CP}^1_{target}$ by u on V_s, and v on V_n. Thus we regard $f : U_s \to V_s$ as a function

$$u = \sum_j a_j x^j. \tag{0.1}$$

We identify the South Pole on $\mathbb{CP}^1_{source}$ with the point at ∞ on U_s, $x \to \infty$. Using the equality $y = \frac{1}{x}$ we see that the point at ∞ has coordinate $y = 0$. Similarly, the point at infinity on $\mathbb{CP}^1_{target}$ ($u \to \infty$) has coordinate $v = 0$.

Using (0.1) we deduce that $\lim_{x\to\infty} u(x) = \infty$. Now chage the coordinates in both the source and target space, $x = 1/y$, $v = 1/u$. Hence

$$v(y) = \frac{1}{u(x)} = \frac{1}{u(1/y)} = \frac{1}{\sum_{j=0}^{n} a_j y^{-j}} = \frac{y^d}{\sum_{j=0}^{d} a_j y^{d-j}}.$$

This shows that the polynomial f extends as a smooth map $\mathbb{CP}^1_{source} \to \mathbb{CP}^1_{target}$.

Suppose $r_1, \cdots, r_m$ are the roots of f with multiplicities $\mu_1, \cdots, \mu_m$, $\sum_k \mu_k = d$.

Fix a small disk $\Delta = \{|u| < \varepsilon\}$ centered at the point $u = 0 \in V_s \subset \mathbb{CP}^1_{target}$. We can find small pairwise disjoint disks $D_1, \cdots, D_m$ centered at $r_1, \cdots, r_k \in U_s \subset \mathbb{CP}^1_{source}$ such that

$$f(D_k) \subset \Delta, \;\; \forall 1 \le k \le m.$$

More explicitly $D_k := \{|x - r_k| < \delta_k\}$, where δ_k is a very small positive number. On D_k the polynomial f has the description

$$u(x) = (x - r_k)^{\mu_k} Q_k(x), \;\; Q_k(x) \neq 0, \;\; \forall x \in D_k.$$

Since $Q_k \leq 0$ on D_k we can find a holomorphic function $L_k : D_k \to \mathbb{C}$ such that

$$Q_k = \exp(L_k). \ \Big(\text{Explicitely,} \ \ L_k(x) = \log(Q_k(r_k)) + \int_{r_k}^{x} (dQ_k/Q_k)\Big).$$

For $t \in [0,1]$ we set

$$Q_k^t := \exp(tL_k), \ \ f_k^t = (x - r_k)^{\mu_k} Q_k^t.$$

Observe that

$$|Q_k^t| = |Q_k|^t$$

Set

$$M_k := \sup\{|Q_k(x)|; \ \ |x - r_k| \leq \delta_k\}.$$

If we choose δ_k sufficiently small then

$$|(x - r_k)^{\mu_k} Q_k^t(x)| \leq M_k^t |x - r_k|^{\mu_k} \leq M_k^t \delta_k^{\mu_k} < \varepsilon, \ \ \forall |x - r_k| < \delta_k.$$

Equivalently, this means that if δ_k is sufficiently small then

$$f_k^t(D_k, D_k \setminus \{r_k\}) \subset (\Delta, \Delta \setminus \{0\}).$$

This implies that $f = f^1 : (D_k, D_k \setminus r_k) \to (\Delta, \Delta \setminus 0)$ is homotopic to

$$f^0 : (D_k, D_k \setminus r_k) \to (\Delta, \Delta \setminus 0), \ \ f^0(x) = (x - r_k)^{\mu_k},$$

as maps of pairs. The degree of induced map

$$f^0 : \{|x| = \delta_k\} \to \{|u| = \delta_k^{\mu_k}\} \subset \Delta \setminus 0$$

is μ_k so that $\deg(f, r_k) = \mu_k$. We conclude that

$$\deg f = \sum_k \deg(f, r_k) = \sum_k \mu_k = d.$$

$\square$

Solutions to Homework # 9

Problem 10, §2.2 (a) X has a cell structure with a single vertex v, a single 1-cell e, and two 2-cells $D_\pm$ (the upper and lower hemispheres of S^2.) The cellular complex has the form

$$0 \to \mathbb{Z}\langle D_1, D_2\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v\rangle \to 0.$$

Denote by $\alpha_n : S^n \to S^n$ the antipodal map. Then

$$\partial_2 D_\pm = (1 + \deg \alpha_1)e = 2e, \ \ \partial_1 e = 0.$$

We conclude that

$$H_2(X) \cong \mathbb{Z}\big\langle (D_+ - D_-)\big\rangle \cong \mathbb{Z}, \ \ H_1(X) \cong \mathbb{Z}_2, \ \ H_0(X) \cong \mathbb{Z}.$$

(b) For the space Y obtained by identifying the antipodal points of the equator we obtain a cell complex

$$0 \to \mathbb{Z}\langle D_+, D_-\rangle \xrightarrow{\partial_3} \underbrace{\mathbb{Z}\langle e_2\rangle \xrightarrow{\partial_2} \mathbb{Z}\langle e_1\rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v\rangle \to 0}_{\text{cellular chain complex of } \mathbb{RP}^2},$$

$$\partial D_\pm = (1 + \deg \alpha_2)e_2 = 0.$$

Hence

$$H_3(Y) \cong \mathbb{Z} \oplus \mathbb{Z}, \ \ H_2(Y) \cong H_2(\mathbb{RP}^2) \cong 0, \ \ H_1(Y) \cong \mathbb{Z}/2\mathbb{Z}, \ \ H_0(Y) \cong \mathbb{Z}.$$

□

Problem 14, §2.2. Denote by $\alpha_n : S^n \to S^n$ the antipodal map. Then the map f is even if and only if

$$f \circ \alpha_n = f.$$

Hence

$$\deg f = \deg(f)\deg \alpha_n \Longrightarrow \deg f = (\deg f) \cdot \deg \alpha_n = (-1)^{n+1} \deg f.$$

Hence if n is even then $\deg f = 0$. *Assume next that n is odd.*

Since $\mathbb{RP}^n = S^n/(x \sim -x)$ there exists a continuous map $g : \mathbb{RP}^n \to S^n$ such that the diagram below is commutative

$$\begin{array}{ccc} S^n & \xrightarrow{f} & S^n \\ {\scriptstyle \pi}\downarrow & \nearrow{\scriptstyle g} & \\ \mathbb{RP}^n & & \end{array} \qquad (\dagger)$$

Consider the collapse maps

$$q : \mathbb{RP}^n \to \mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n,$$

Arguing as in the proof of the *Cellular Boundary Formula* (page 140 of the textbook) we deduce that the degree of the map

$$q \circ \pi : S^n \cong \partial\mathbb{RP}^n/\mathbb{RP}^{n-1} \cong S^n,$$

is $1 + (-1)^{n+1} = 2$.

From the long exact sequence of the pair $(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ we deduce that the natural map

$$H_n(\mathbb{RP}^n) \xrightarrow{j_n} H_n(\mathbb{RP}^n, \mathbb{RP}^{n-1}) \cong H_n(\mathbb{RP}^n/\mathbb{RP}^{n-1})$$

is an isomorphism.

By consulting the commutative diagram

$$\begin{array}{ccc} H_n(S^n)\cong\mathbb{Z} & & \\ \downarrow{\scriptstyle \pi_*} & \searrow^{(q\circ\pi)_*=\times 2} & \\ H_n(\mathbb{RP}^n)\cong\mathbb{Z} & \xrightarrow[\cong j_n]{} & H_n(\mathbb{RP}^n/\mathbb{RP}^{n-1})\cong\mathbb{Z} \end{array}$$

we deduce that the induced $\pi_* : H_n(S^n) \cong \mathbb{Z} \to H_n(\mathbb{RP}^n) \cong \mathbb{Z}$ is described by multiplication by ± 2. Using this information in the diagram (†) we deduce that $\deg f = \pm \deg g$, so that $\deg f$ must be even.

To show that there exist even maps $S^{2n-1} \to S^{2n-1}$ of arbitrary even degrees we use the identification

$$S^{2n-1} := \{(z_1, \dots, z_n) \in \mathbb{C}^n;\ \sum_k |z_k|^2 = n\}.$$

We write $z_k = r_k \exp(\boldsymbol{i}\theta_k)$. For every vector $\vec{\nu} = (\nu_1, \nu_2, \dots, \nu_n) \in (\mathbb{Z}^*)^n$ define

$$F_{\vec{\nu}} : S^{2n-1} \to S^{2n-1},\ \ F_{\vec{\nu}}(r_1 e^{\boldsymbol{i}\theta_1}, \dots, r_n e^{\boldsymbol{i}\theta_n}) = (r_1 e^{\nu_1 \boldsymbol{i}\theta_1}, \dots, r_n e^{\nu_n \boldsymbol{i}\theta_n}).$$

Observe that

$$F_{\vec{\nu}}(-\vec{z}) = F_{\vec{\nu}}(e^{\boldsymbol{i}\pi} \cdot \vec{z}).$$

Hence, if all the integers ν_i are odd, the map $F_{\vec{\nu}}$ is odd, i.e., $F_{\vec{\nu}}(-\vec{z}) = -F_{\vec{\nu}}(\vec{z})$.

Now observe that $p_0 := (1, 1, \dots, 1) \in S^{2n-1}$ and

$$F_{\vec{\nu}}^{-1}(p_0) = \{\vec{\zeta} := (\zeta_1, \dots, \zeta_n);\ \ \zeta_k^{\nu_k} = 1\}.$$

Near $\vec{\zeta}$ the map $F_{\vec{\nu}}$ is homotopic to its linearization $D_\zeta F_{\vec{\nu}}$ since for $\vec{z}$ close to $\vec{\zeta}$

$$F_{\vec{\nu}}(\vec{z}) \approx F_{\vec{\nu}}(\vec{\zeta}) + D_\zeta F_{\vec{\nu}} \cdot (\vec{z} - \vec{\zeta}) + O(|\vec{z} - \vec{\zeta}|^2).$$

Near $\vec{\zeta}$ and p_0 we can use the same coordinates $(r_1, \dots, r_{n-1}; \theta_1, \dots, \theta_n)$ and the linearization is given by the matrix

$$D_\zeta F_{\vec{\nu}} = \mathbb{1}_{\mathbb{R}^{n-1}} \oplus \operatorname{diag}(\nu_1, \dots, \nu_n).$$

We have

$$\deg(F_{\vec{\nu}}, \vec{\zeta}) = \det D_\zeta F_{\vec{\nu}} = \operatorname{sign}(\nu_1 \cdots \nu_n).$$

We conclude that

$$\deg F_{\vec{\nu}} = \sum_{\vec{\zeta} \in F_{\vec{\nu}}^{-1}(p_0)} \deg(F_{\vec{\nu}}, \vec{\zeta}) = \nu_1 \nu_2 \cdots \nu_n$$

When $\vec{\nu} = (m, 1, \dots, 1)$ we write F_m instead of $F_{(m,\dots,1)}$. Note that F_m is odd if and only if m is odd.

Denote by $G : S^{2n-1} \to S^{2n-1}$ the continuous map defined as the composition

$$S^{2n-1} \to \mathbb{RP}^{2n-1}/\mathbb{RP}^{2n-1} \cong S^{2n-1}.$$

The map G is even and has degree 2.

Suppose N is an even number. We can write $N = 2^k m$, m, odd number. Define

$$G_N := \underbrace{G \circ \cdots \circ G}_{k} \circ F_m.$$

Then G_N is an even map of degree N. □

Problem 29, §2.2 The standard embedding of a genus 2 Riemann surface in $\mathbb{R}^3$ is depicted in Figure 1. Denote by $j : \Sigma_g \to R$ the natural embedding. It induces a morphism

$$j_* : H_1(\Sigma_g) \to H_1(R).$$

whose kernel consists of cycles on Σ_g which bound on R.

More precisely, $\ker j$ is a free Abelian group of rank g with a basis consisting of the cycles $a_1, \dots, a_g$ (see Figure 1). We can complete $a_1, \dots, a_g$ to a $\mathbb{Z}$-basis $a_1, \dots, a_g; b_1, \dots, b_g$ of $H_1(\Sigma_g)$ (see Figure 1). R is homotopic to the wedge of the circles $b_1, \dots, b_g$.

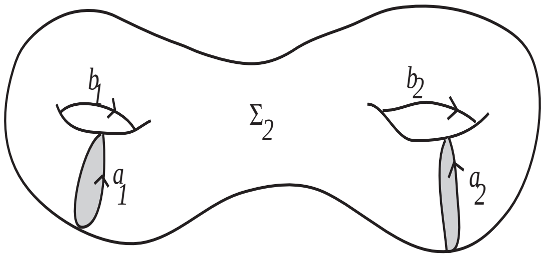


FIGURE 1. Σ_2 *is the "crust" of a double bagel R.*

Consider now two copies R^0, R^1 of the handlebody R. Correspondingly we get two inclusions

$$j^k : \Sigma \hookrightarrow R^k, \quad k = 0, 1.$$

Then $X = R^0 \cup_\Sigma R^1$. Denote by i^k the inclusion $R^k \hookrightarrow X$. The Mayer-Vietoris sequence has the form

$$\cdots \to H_k(R^0) \oplus H_k(R^1) \xrightarrow{s} H_k(X) \xrightarrow{\partial} H_{k-1}(\Sigma) \xrightarrow{\Delta_{k-1}} H_{k-1}(R^0) \oplus H_{k-1}(R^1) \to \cdots.$$

where $\Delta(c) = (\, j^0_*(c), -j^1_*(c)\,)$, and $s(u, v) = i^0_*(u) + i^1_*(v)$. Since R is homotopic to a wedge of circles we deduce $H_k(R) = 0$ for $k > 1$.

Using the portion $k = 3$ in the above sequence we obtain an isomorphism

$$\partial : H_3(X) \to H_2(\Sigma) \cong \mathbb{Z}.$$

For $k = 2$ we obtain an isomorphism

$$\partial : H_2(X) \to \ker \Delta_1 \cong \mathbb{Z}\langle b_1, \dots, b_g\rangle.$$

Since $\ker \Delta_0 = 0$ we obtain an isomorphism

$$H_1(X) \cong \operatorname{coker}(\Delta_1) \cong \frac{\mathbb{Z}^g \oplus \mathbb{Z}^g}{\{\vec{x} \oplus -\vec{x};\ \ \vec{x} \in \mathbb{Z}^g\}} \cong \mathbb{Z}^g.$$

We use the long exact sequence of the pair (R, Σ)

$$\cdots \to H_k(R) \to H_k(R, \Sigma) \xrightarrow{\partial} H_{k-1}(\Sigma) \xrightarrow{j_*} H_{k-1}(R) \to \cdots$$

For $k = 3$ we obtain an isomorphism $\partial : H_3(R, \Sigma) \to H_2(\Sigma)$. For $k = 2$ we obtain an isomorphism

$$\partial : H_2(R, \Sigma) \to \ker j_* \cong \mathbb{Z}\langle a_1, \dots, a_g\rangle$$

(The disks depicted in Figure 1 represent the generators of $H_2(R, \Sigma)$ defined by the above isomorphism.)

For $k = 1$ we have an exact sequence

$$H_1(\Sigma) \xrightarrow{j_*} H_1(R) \to H_1(R, \Sigma) \xrightarrow{\partial} \ker j_* = 0.$$

Since $H_1(\Sigma)\xrightarrow{j_*} H_1(R)$ is onto we deduce $H_1(R,\Sigma)=0$. Finally, $H_0(R,\Sigma)=0$. □

Problem 30, §2.2
(a) Observe that $H_k(T_f)\cong 0$ for $k>3$. Since r is a reflection we deduce $f_*=\deg f\cdot \mathbb{1}=-\mathbb{1}$ on $H_2(S^2)$ and $=\mathbb{1}$ on $H_0(S^2)$. We have the short exact sequence

$$0\to H_3(T_f)\to H_2(S^2)\xrightarrow{2\cdot} H_2(S^2)\to H_2(T_f)\to 0.$$

Hence $H_3(T_f)=0$ and $H_2(T_f)\cong\mathbb{Z}_2$. We also have a short exact sequence

$$0\to H_1(T_f)\to H_0(S^2)\xrightarrow{0} H_0(S^2)$$

so that $H_1(T_f)\cong\mathbb{Z}$.
(b) In this case $1-f_*=-1$ on $H_2(S^2)$, and we deduce as above $H_3(T_f)\cong H_2(T_f)\cong 0$. We conclude similarly that $H_1(T_f)\cong\mathbb{Z}$.

The maps $f:S^1\to S^1$ are described by matrices $A:\mathbb{Z}^2\to\mathbb{Z}^2$. More precisely such a map defines a continuous map $\mathbb{R}^2\to\mathbb{R}^2$ which descends to quotients

$$A:\mathbb{R}^2/\mathbb{Z}^2\to\mathbb{R}^2/\mathbb{Z}^2.$$

Here are the matrices in the remaining three cases.
(c)

$$A:=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.$$

(d)

$$A:=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}.$$

(e)

$$A:=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\cdot\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}.$$

Suppose $f:S^1\times S^1\to S^1\times S^1$ is given by a 2×2 matrix A with integral entries. We need to compute the induced maps $f_*:H_k(T^2)\to H_k(T^2)$. For $k=0$ we always have $f_*=\mathbb{1}$.

For $k=1$ we have $H_1(T^2)\cong H_1(S^1)\oplus H_1(S^1)\cong\mathbb{Z}^2$ and the induced map $f_*:\mathbb{Z}^2\to\mathbb{Z}^2$ coincides with the map induced by the matrix A. For $k=2$ the induced map $f_*:\mathbb{Z}\to\mathbb{Z}$ can be identified with an integer, the degree of f. This can be computed using the computation in Problem 7, §2.2, and local degrees as in Proposition 2.30, page 136. We deduce that

$$\deg f=\det A.$$

The Wang long exact sequence then has the form

$$0\to H_3(T_A)\to H_2(T^2)\stackrel{1-\det A}{\longrightarrow} H_2(T^2)\to H^2(T_A)\to$$
$$\to H_1(T^2)\xrightarrow{1-A} H_1(T^2)\to H_1(T_A)\to H_0(T^2)\xrightarrow{0} H_0(T^2)\to H_0(T_A).$$

In our cases $\det A=\pm 1$ When $\det A=1$ (case (d) and (e)) we have

$$H_3(T_A)\cong H_2(T^2)\cong\mathbb{Z}.$$

In the case (c) we have $1-\det A=2$ and we have

$$H_3(T_A)\cong 0.$$

In the cases (d) and (e) we have short exact sequences

$$0\to H_2(T^2)\to H_2(T_A)\to\ker(1-A)\to 0.$$

In both cases $\ker(1-A)=0$ so that

$$H_2(T_A)\cong\mathbb{Z}.$$

Finally we deduce a short exact sequence

$$0\to\operatorname{coker}(1-A)\to H_1(T_A)\to H_0(T^2)\to 0$$

so that

$$H_1(T_A)\cong\mathbb{Z}\oplus\operatorname{coker}(1-A).$$

In the case (d) we have $1-A=2\cdot\mathbb{1}_{\mathbb{Z}^2}$ so that coker $\cong\mathbb{Z}_2\oplus\mathbb{Z}_2$.

In the case (e) we have

$$1-A=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$$

$\operatorname{coker}(1-A)$ is a group of order $|\det(1-A)|=2$ so it can only be $\mathbb{Z}_2$.

In the case (c) we have $1-\det A=2$ and we get an exact sequence

$$0\to\mathbb{Z}_2\to H_2(T_A)\to\ker(1-A)\to 0\Longrightarrow H_2(T_A)\cong\mathbb{Z}_2\oplus\ker(1-A).$$

Note that

$$1-A=\begin{bmatrix}0&0\\0&2\end{bmatrix}$$

Hence

$$H_2(T_A)\cong\mathbb{Z}_2\oplus\mathbb{Z}.$$

We get again

$$H_1(T_A)\cong\mathbb{Z}\oplus\operatorname{coker}(1-A).$$

so that $\operatorname{coker}(1-A)\cong\mathbb{Z}\oplus\mathbb{Z}_2$. We deduce

$$H_1(T_A)\cong\mathbb{Z}^2\oplus\mathbb{Z}_2.$$

The following table summarizes the above conclusions.

$H_*(T_f)$	H_0	H_1	H_2	H_3
(a)	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
(b)	$\mathbb{Z}$	$\mathbb{Z}$	0	0
(c)	$\mathbb{Z}$	$\mathbb{Z}^2\oplus\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2$	0
(d)	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
(e)	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$

□

Homework # 10: The generalized Mayer-Vietoris principle.

Suppose X is a locally compact topological space, and $\mathfrak{U} = (U_\alpha)_{\alpha\in A}$ is an open cover of X. Assume for simplicity that the set A is finite. Fix a total ordering on A. For each finite subset $S \subset A$ we set

$$U_S := \bigcap_{\alpha\in S} U_\alpha$$

The *nerve* of the cover $\mathfrak{U}$ is the combinatorial simplicial complex $\boldsymbol{N}(\mathfrak{U})$ defined as follows.

- The vertex set of $\boldsymbol{N}(\mathfrak{U})$ is A.
- A finite subset $S \in A$ is a face of $\boldsymbol{N}(\mathfrak{U})$ if and only if $U_S \neq \emptyset$.

For example, this meas that two vertices $\alpha, \beta \in A$ are to be connected by an edge, i.e., $\{\alpha, \beta\}$ is a face of $\boldsymbol{N}(\mathfrak{U})$, if and only if $U_\alpha \cap U_\beta \neq \emptyset$.

In Figure 1 we have depicted two special cases of the above construction

(a) *The nerve of a cover consisting of two open sets U_1, U_2 with nonempty overlap.*

(b) *The nerve of the open cover of the one-dimensional space X depicted in Figure 1.*

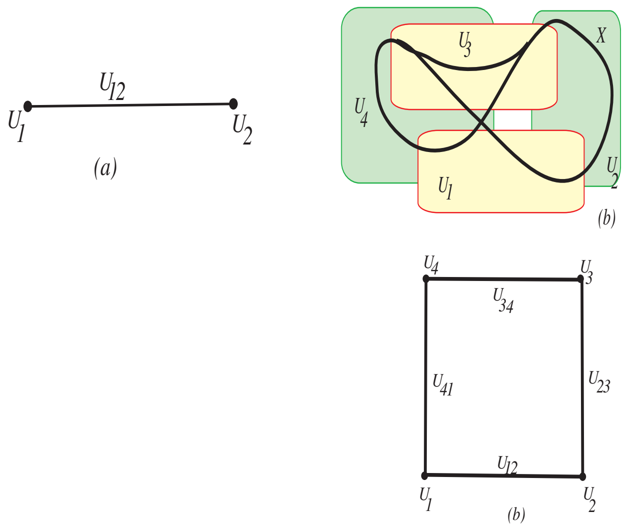


FIGURE 1. *An open cover of a* 1*-dimensional cellular complex X.*

In general, for any X, any open cover $\mathfrak{U}$ as above, and any $p, q \geq 0$ we set

$$K_{p,q}(\mathfrak{U}) := \bigoplus_{S\subset A,\ |S|=q+1} C_p(U_S),$$

where $C_p(U_S)$ denotes the free Abelian group generated by singular simplices $\sigma : \Delta_p \to U_S$. Note that the above direct sum is parameterized by the q-dimensional faces of the nerve $\boldsymbol{N}(\mathfrak{U})$.

The elements of $K_{p,q}$ have the form

$$c = \bigoplus_{|S|=q+1} c_S,\ \ c_S \in C_p(U_S).$$

The chain c assigns to each q-dimensional face S of the nerve $\boldsymbol{N}(\mathfrak{U})$ an element c_S in the group $C_p(U_S)$.

We now form a double complex $(K_{\bullet,\bullet}, \partial_I, \partial_{II})$ as follows.

$$\partial_I : K_{p,q} = \bigoplus_{S\subset A,\, |S|=q+1} C_p(U_S) \longrightarrow \bigoplus_{S\subset A,\, |S|=q+1} C_{p-1}(U_S) = K_{p-1,q}$$

$$\partial_I\big(\oplus_{|S|=q+1} c_S\big) = \oplus_{|S|=q+1}\partial c_S$$

To define ∂_{II}, note that for every inclusion $S' \hookrightarrow S$ we have an inclusion $U_S \hookrightarrow U_{S'}$. In particular, for every

$$S = \{s_0 < s_1 < \cdots < s_q\} \subset A,\ \ U_S \neq \emptyset$$

we have inclusions

$$\varphi_j : U_S \to U_{S\setminus s_j},$$

and thus we have morphisms $\varphi_j : C_p(U_S) \to C_p(U_{S\setminus s_j})$ Given a singular simplex

$$\sigma : \Delta_p \to U_S$$

so that σ determines an element in $K_{p,q}$, we define $\delta\sigma \in K_{p,q-1}$ by

$$\delta\sigma = \sum_{j=0}^{q}(-1)^j\varphi_j(\sigma) \in \bigoplus_{j=0}^{q} C_p(U_{S\setminus s_j}) \subset K_{p,q-1}.$$

The map δ extends by linearity to an morphism $\delta : K_{p,q} \to K_{p,q-1}$ called the *Čech boundary operator*. Note that

$$K_{p,0} = \bigoplus_{\alpha\in A} C_p(U_\alpha).$$

Exercise 10.1. (a) Describe $K_{\bullet,\bullet}$, d_I and δ for the two situations in (a) and (b). Prove that in both these cases $\delta^2 = 0$.
(b) Prove in general that $\delta^2 = 0$, and define

$$d_{II} : K_{p,q} \to K_{p,q-1},\ \ d_{II} = (-1)^p\delta.$$

Show that $d_I d_{II} = -d_{II} d_I$. □

Proof. In both cases we have $U_S = \emptyset$ for $|S| > 2$ so that in both cases we have

$$K_{p,q} = 0,\ \ \forall q \geq 2$$

so that in either case the double complex has the form in Figure 2 where the ∘'s denote the places where $K_{p,q} = 0$.

In case (a) we have

$$K_{p,0} = C_p(U_1) \oplus C_p(U_2),\ \ K_{p,1} = C_p(U_{12}),\ \ U_{12} = U_1 \cap U_2$$

Denote by φ_α the inclusion

$$C_p(U_{12}) \hookrightarrow C_p(U_\alpha).$$

We will identify $\varphi_\alpha(C_p(U_\alpha))$ with $C_p(U_\alpha)$. Then for $(c_1, c_2) \in K_{p,0}$ we have

$$d_I(c_1, c_2) = (\partial c_1, \partial c_2) \in K_{p-1,0}$$

and

$$\delta(c_1, c_2) = 0.$$

For $c \in K_{p,1} = C_p(U_{12})$ we have

$$d_I c = \partial c \in K_{p-1,1},\ \ \delta c = (-\varphi_1(c), \varphi_2(c)) = (-c, c) \in K_{p,0}.$$

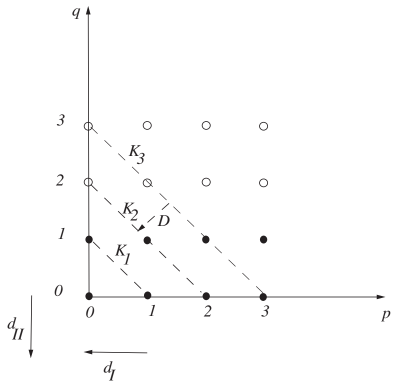


FIGURE 2. *A highly degenerate double complex*

In case (b) we have

$$K_{p,0} = C_p(U_1) \oplus C_p(U_2) \oplus C_p(U_3) \oplus C_p(U_3) \oplus C_p(U_4)$$

We describe the elements of $K_{p,0}$ as quadruples (c_1, c_2, c_3, c_4) and we have

$$\delta(c_1, c_2, c_3, c_4) = 0.$$

$$K_{p,1} = C_p(U_{12}) \oplus C_p(U_{23}) \oplus C_p(U_{34}) \oplus C_p(U_{41}).$$

We describe the elements of $K_{p,1}$ as quadruples $(c_{12}, c_{23}, c_{34}, c_{14})$. Then

$$\delta(c_{12}, c_{23}, c_{34}, c_{14}) = (-c_{14} - c_{12}, c_{12} - c_{23}, c_{23} - c_{34}, c_{34} + c_{14}).$$

The condition $\delta^2 = 0$ is trivially satisfied in both cases.

Consider now the general situation, and let $c \in K_{p,q} = \bigoplus_{|S|=q+1} C_p(U_S)$. We can write

$$c = \bigoplus_{|S|=q+1} c_S$$

We will first show that

$$\delta^2 c_S = 0, \ \ \forall S.$$

Fix one such S. Assume $S = \{0, 1, 2, \cdots q\}$. For every $i, j \in S$ denote by φ_{ij} the inclusion

$$C_p(S) \hookrightarrow C_p(S \setminus \{i, j\}).$$

Then

$$\delta c_S = \sum_{i=0}^{q} (-1)^j \varphi_j(c_S).$$

$$\delta(\delta c_S) = \sum_{i=0}^{q} (-1)^i \delta(\varphi_i c_S) = \sum_{i=0}^{q} (-1)^i \Big(\sum_{j=0}^{i-1} (-1)^j \varphi_j \varphi_i(c_S) + \sum_{j=i+1}^{q} (-1)^{j-1} \varphi_j \varphi_i(c_S) \Big)$$

$$= \sum_{0 \le j < i} (-1)^{i+j} \varphi_{ij}(c_S) + \sum_{0 \le i < j} (-1)^{i+j+1} \varphi_{ij}(c_S) = 0.$$

This proves $\delta^2 = 0$. Form the definition of δ it follows that

$$\delta d_I = d_I \delta.$$

For $c \in K_{p,q}$ we have

$$d_I d_{II} c = (-1)^p d_I \delta c = (-1)^p \delta(d_I c) = (-1)^p \cdot (-1)^{p-1} d_{II} d_I c. \qquad \square$$

Exercise 10.2. Denote by $C_p(X,\mathcal{U})$ the free Abelian group spanned by singular simplices in X whose images lie in some U_α. Note that we have a natural surjection

$$\varepsilon : K_{p,0} \to C_p(X,\mathcal{U}).$$

Prove that for every $p \geq 0$, $q \geq 0$ we have

$$\operatorname{Im}\big(K_{p,q+1} \xrightarrow{\partial_{II}} K_{p,q}\big) = \ker\big(K_{p,q} \xrightarrow{\partial_{II}} K_{p,q-1}\big),$$

and

$$\operatorname{Im}\big(K_{p,1} \xrightarrow{\partial_{II}} K_{p,0}\big) = \ker\big(K_{p,0} \xrightarrow{\varepsilon} C_p\big).$$

(In other words, you have to show that the columns of the expanded double complex

$$(K_{\bullet,\bullet}, \partial_I, \partial_{II}) \xrightarrow{\varepsilon} \big(C_*(X,\mathcal{U}), \partial\big)$$

are exact. *Hint:* Workout some special cases first.

Proof. We have

$$C_p(X,\mathcal{U}) := \sum_\alpha C_p(U_\alpha) \subset C_p(X).$$

The natural map

$$\varepsilon : K_{p,0} = \bigoplus_\alpha C_p(U_\alpha) \to \sum_\alpha C_p(U_\alpha)$$

is given by

$$\bigoplus_\alpha C_p(U_\alpha) \ni \bigoplus_\alpha c_\alpha \mapsto \sum_\alpha c_\alpha$$

For every $\oplus_{|S|=2} c_S \in K_{p,1}$ we have

$$\delta(c_S) = (-c_S) \oplus c_S \in C_p(U_{s_1}) \oplus C_p(U_{s_2}), \quad (S = \{s_1, s_2\}),$$

and clearly $\varepsilon(\delta(c_S)) = 0$. Set

$$K_{p,-1} := C_p(X,\mathcal{U}).$$

We denote by $\boldsymbol{N}(\mathcal{U})_q$ the set of q-faces of the simplicial complex $\boldsymbol{N}(\mathcal{U})$. For $S \in \boldsymbol{N}(\mathcal{U})_q$ we set

$$\mathcal{S}_{p,q}(S) := \big\{\, \sigma : \Delta^p \to X;\ \ \sigma(\Delta^p) \subset U_S \,\big\} = \big\{\, \sigma : \Delta^p \to X;\ \ \sigma(\Delta^p) \in U_s,\ \ \forall s \in S \,\big\}.$$

For each singular simplex $\sigma : \Delta^p \to X$ we set

$$\operatorname{supp}_q(\sigma) := \{S \in \boldsymbol{N}(\mathcal{U})_q;\ \ \sigma(\Delta^p) \subset U_S \Longleftrightarrow \sigma(\Delta^p) \in U_s,\ \ \forall s \in S\}.$$

Denote by $\mathcal{S}_{p,q}$ the set of singular p-simplices $\sigma : \Delta^p \to X$ such that $\operatorname{supp}_q(\sigma) \neq \emptyset$. Then

$$K_{p,q} = \bigoplus_{S \in \boldsymbol{N}(\mathcal{U})_q} \bigoplus_{\sigma \in \mathcal{S}_{p,q}(S)} \mathbb{Z}.$$

We denote by $\{\langle \sigma, S\rangle;\ \ S \in \boldsymbol{N}(\mathcal{U})_q,\ \ \sigma \in \mathcal{S}_{p,q}(S)\}$ the canonical basis of $K_{p,q}$ corresponding to the above direct sum decomposition. We will denote the elements in group by sums

$$c = \sum_{S \in \boldsymbol{N}(\mathcal{U})_q} \sum_{\sigma \in \mathcal{S}_{p,q}(S)} n(\sigma, S)\langle \sigma, S\rangle = \sum_{\sigma \in \mathcal{S}_{p,q}} \sum_{S \in \operatorname{supp}_q(\sigma)} n(\sigma, S)\langle \sigma, S\rangle.$$

Denote by $(C_\bullet(\boldsymbol{N}(\mathcal{U})), \partial)$ the simplicial chain complex associated to the nerve $\boldsymbol{N}(\mathcal{U})$. Then

$$C_q(\boldsymbol{N}(\mathcal{U})) = \bigoplus_{S \in \boldsymbol{N}(\mathcal{U})_q} \mathbb{Z}$$

and we denote by $\{\langle S\rangle;\quad S\in \boldsymbol{N}(\mathcal{U})_q\}$ the canonical basis of $C_q(\boldsymbol{N}(\mathcal{U}))$ determined by the above direct sum decomposition. Observe that for every $\sigma_0\in \mathcal{S}_{p,q}$ we have a canonical projection

$$\pi_q(\sigma_0): K_{p,q}\to C_q(\boldsymbol{N}(\mathcal{U})),$$
$$\sum_{\sigma\in\mathcal{S}_{p,q}}\sum_{S\in\operatorname{supp}_q(\sigma)} n(\sigma,S)\langle\sigma,S\rangle\mapsto \sum_{S\in\operatorname{supp}_q(\sigma_0)} n(\sigma_0,S)\langle S\rangle.$$

We see from the definition of δ that the morphism

$$\pi_*(\sigma_0):(K_{p,\bullet},\delta)\to (C_\bullet(\boldsymbol{N}(\mathcal{U})),\partial)$$

is a chain map. In particular, if

$$c=\sum_{\sigma\in\mathcal{S}_{p,q}}\sum_{S\in\operatorname{supp}_q(\sigma)} n(\sigma,S)\langle\sigma,S\rangle$$

is a δ-cycle, $\delta c=0$, then for every $\tau\in\mathcal{S}_{p,q}$ we get a ∂-cycle in $C_*(\boldsymbol{N}(\mathcal{U}))$,

$$\pi_q(\tau)c=\sum_{S\in\operatorname{supp}_q(\tau)} n(\tau,S)\langle S\rangle\in C_q(\boldsymbol{N}(\mathcal{U})),\;\; \partial\pi_q(\tau)c=0.$$

Consider the set of vertices

$$V(\tau):=\bigcup_{S\in\operatorname{supp}_q(\tau)} S$$

We deduce that the image of τ lies in all of the open sets U_t, $t\in V(\tau)$. In other words, the vertices in $V(\tau)$ span a simplex of the nerve $\boldsymbol{N}(\mathcal{U})$. The ∂-cycle $\pi_q(\tau)c$ is a cycle inside this simplex so it bounds a simplicial chain of this simplex. Hence

$$\pi_q(\tau)c=\partial\sum_{T\in\operatorname{supp}_{q+1}(\tau)} m_\tau\langle T\rangle.$$

We conclude that

$$c=\delta\Bigg(\sum_{\tau\in\mathcal{S}_{p,q+1}}\sum_{T\in\operatorname{supp}_{q+1}(\tau)} m_\tau\langle\tau,T\rangle\Bigg).\qquad\square$$

Exercise 10.3 (The generalized Mayer-Vietoris principle)**.** Suppose that we have a double complex

$$\Big(K_{\bullet,\bullet}=\bigoplus_{p,q\geq 0}K_{p,q},\;\; D_I,\;\; d_{II},\Big),$$

where

$$d_I:K_{p,q}\to K_{p-1,q},\;\; d_{II}:K_{p,q}\to K_{p,q-1},$$

satisfy the identities

$$d_I^2=d_{II}^2=d_Id_{II}+d_{II}d_I=0.$$

(see Figure 3.)

Form the *total complex*

$$(K_\bullet,D),\;\; K_m=\bigoplus_{p+q=m}K_{p,q},\;\; D=d_I+d_{II}:K_m\to K_{m-1}.$$

(a) Prove that $D^2=0$.
(b) Suppose we are given another chain complex $(C_\bullet,\partial)$, and a *surjective* morphism of chain complexes

$$\varepsilon:(K_{\bullet,0},\partial_I)\to C_\bullet,\partial),$$

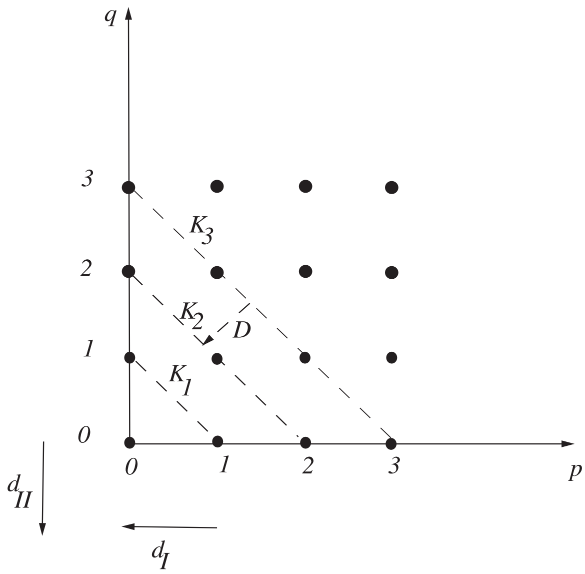


FIGURE 3. *A double chain complex*

such that

$$\varepsilon \circ d_{II} = 0.$$

Prove that ε induces a morphism of chain complexes

$$\varepsilon : (K_\bullet, D) \to (C_\bullet, \partial). \tag{10.1}$$

(c) Assume that for every $p \geq 0$, $q \geq 1$ we have

$$\operatorname{Im}\big(K_{p,q+1} \xrightarrow{d_{II}} K_{p,q}\big) = \ker\big(K_{p,q} \xrightarrow{d_{II}} K_{p,q-1}\big),$$

and

$$\operatorname{Im}\big(K_{p,1} \xrightarrow{d_{II}} K_{p,0}\big) = \ker\big(K_{p,0} \xrightarrow{\varepsilon} C_p\big).$$

Prove that the morphism (10.1) induces isomorphisms in homology.

Proof. (a) We have

$$D^2 = (d_I + d_{II})^2 = d_I^2 + d_{II}^2 + d_I d_{II} + d_{II} d_I = 0.$$

For part (b) we note that a chain $c \in K_p$ is a sum

$$c_p = \sum_{i=0}^{p} c_{i,p-i}, \;\; c_{i,p-i} \in K_{i,p-i}.$$

We define

$$\varepsilon(c_p) = \varepsilon(c_{p,0}),$$

and it is now obvious that the resulting map $\varepsilon : K_\bullet \to C_\bullet$ is a morphism of chain complexes.

To prove that ε induces an isomorphism in homology we need to prove two things.

A. For any $p \geq 0$, and any $c \in C_p$ such that $\partial c = 0$, there exists $z = \sum_{j=0}^{p} z_{j,p-j} \in K_p$ such that $Dz = 0$ and $\varepsilon(z_{p,0}) = c$. Observe that the condition $Dz = 0$ is equivalent to the collection of equalities

$$d_I z_{p-j,j} + d_{II} z_{p-j-1}, j+1 = 0, \;\; \forall j = 0, \dots p-1.$$

B. If $z \in K_p$ is a D-cycle, $Dz = 0$, and $\varepsilon(z) \in C_p$ is a ∂-boundary, i.e., exists $c \in C_{p+1}$ such that $\partial c = \varepsilon(z)$, then there exists $x \in K_{p+1}$ such that $Dx = z$.

A. We will construct by induction on $0 \leq j \leq p$ elements $z_j \in K_{p-j,j}$ such that (see Figure 4)

$$\varepsilon(z_{p,0}) = c, \ \ d_I z_{i-1} + d_{II} z_i = 0, \ \ \forall i = 1, \dots, j. \qquad (Z_j)$$

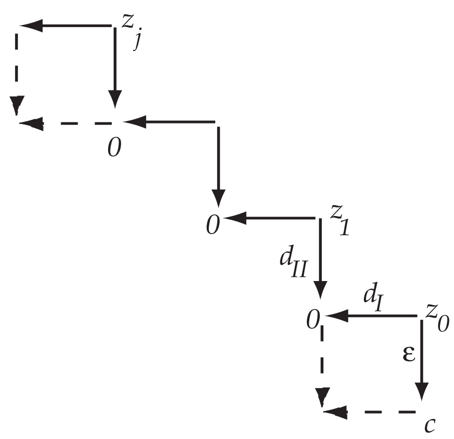


FIGURE 4. *A zig-zag*

Observe that since ε is surjective, there exists $z_0 \in K_{p,0}$ such that

$$\varepsilon(z_0) = c.$$

Since $\partial c = 0$ we deduce

$$\partial \varepsilon(z_0) = \varepsilon(d_I z_0) \Longrightarrow -d_I z_0 \in \ker \varepsilon.$$

Hence, we can find $z_1 \in K_1$ such that $d_{II} z_1 = -d_I z_0$.

Suppose that we have determined the elements $z_0, \dots, z_j$ satisfying (Z_j). We want to show that we can find $z_{j+1} \in K_{p-j-1,j+1}$ such that the extended sequence $z_0, \dots, z_{j+1}$ satisfies (Z_{j+1}).

From the equality $d_{II} z_j = -d_I z_{j-1}$ we deduce

$$d_I d_{II} z_j = -d_I^2 z_{j-1} = 0 \Longrightarrow d_{II} d_I z_j = 0.$$

Hence

$$-d_I z_j \in \ker d_{II} = \mathrm{Im}\,(d_{II}) \Longrightarrow \exists z_{j+1} \in K_{p-j-1,j+1} : \ \ d_{II} z_{j+1} = -d_I z_j.$$

This completes the proof of **A.**.

B. Suppose we have

$$z = z_{p,0} + z_{p-1,1} + \cdots + z_{0,p} \in K_p,$$

and $c \in C_{p+1}$, such that

$$\partial c = \varepsilon(Dz) = \varepsilon(z_{p,0}) \ \text{ and } \ d_I z_{p-i,i} + d_{II} z_{p-i-1,i+1} = 0, \ \ \forall i = 0, \dots, p-1.$$

For simplicity, we write $z_j = z_{p-j,j}$. Since ε is surjective we deduce that there exists $b_0 \in K_{p+1,0}$ such that $\varepsilon(b_0) = c$. We deduce

$$\varepsilon(z_0) = \partial c = \partial \varepsilon(b_0) = \varepsilon(d_I b_0)$$

Hence

$$z_0 - d_I b_0 \in \ker \varepsilon = \mathrm{Im}\,(d_{II}) \Longrightarrow \exists b_1 \in K_{p,1} : z_0 - d_I b_0 = d_{II} b_1.$$

Suppose we have determined

$$b_i \in K_{p+1-i,i}, \;\; 0 \leq i \leq j: \;\; z_i = d_{II} b_{i+1} + d_I b_i, \;\; \forall i = 0, \dots, j,$$

and we want to determine $b_{j+1} \in K_{p-j,j+1}$ such that

$$z_j = d_{II} b_{j+1} + d_I b_j.$$

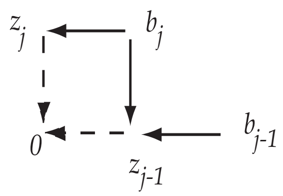


FIGURE 5. *Another zig-zag*

Observe that (see Figure 5)

$$0 = d_I z_{j-1} + d_{II} z_j \Longrightarrow d_{II} z_j = -d_I z_{j-1} = -d_I(d_{II} b_j + d_I b_{j-1}) = -d_I d_{II} b_j = d_{II} d_I b_j.$$

Hence

$$z_j - d_I b_j \in \ker d_{II} = \operatorname{Im}(d_{II})$$

so that there exists $b_{j+1} \in K_{p-j,j+1}$ such that

$$d_{II} b_{j+1} = z_j - d_I b_j.$$

This completes the proof of **B.**. □

Exercise 10.4. Obtain the usual Mayer-Vietoris theorem from the generalized Mayer-Vietoris principle.

Proof. Consider and open cover of X consisting of two open sets U_1, U_2. Denote by $K_{\bullet,\bullet}$ the double complex constructed in Exercise 10.1 determined by this cover, and by $K_\bullet$ the associated total complex constructed as in Exercise 10.3. We have the short exact sequence of complexes

$$0 \to (A_\bullet, d_I) \xrightarrow{i} (B_\bullet, D) \xrightarrow{\pi} (C_\bullet, d_I) \to 0,$$

where

$$A_m := K_{m,0}, \;\; B_n := K_n, \;\; C_p := K_{p-1,1}.$$

Observe that

$$H_m(A_\bullet) := H_m(U_1) \oplus H_m(U_2), \;\; H_m(C_* \bullet) = H_{m-1}(U_1 \cap U_2).$$

From Exercise 10.3 we deduce

$$H_m(B_\bullet) = H_m(X).$$

We get a long exact sequence

$$\cdots \to H_m(U_1) \oplus H_m(U_2) \stackrel{i_*}{\to} H_m(X) \stackrel{\pi_*}{\to} H_{m-1}(U_1 \cap U_2) \stackrel{\partial_*}{\to} H_{m-1}(U_1) \oplus H_{m-1}(U_2) \to \cdots$$

One can easily verify that π_* coincides with the connecting morphism in the Mayer-Vietoris long exact sequence. □