PIXELATIONS OF PLANAR SEMIALGEBRAIC SETS

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ABSTRACT. We describe an algorithm that associates to each positive real number r and each finite collection C_r of planar pixels of size r a planar piecewise linear set S_r with the following additional property: if C_r is the collection of pixels of size r that touch a given compact semialgebraic set S, then the normal cycle of S_r converges to the normal cycle of S in the sense of currents. In particular, in the limit we can recover the homotopy type of S and its geometric invariants such as area, perimeter and curvature measures. At its core, this algorithm is a discretization of stratified Morse theory.

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INTRODUCTION

This paper is a natural sequel of the investigation begun by the second author in his dissertation [12]. To formulate the main problem discussed in [12] and in this paper we need to introduce a bit of terminology.

For $\varepsilon > 0$ we define an ε-*pixel* to be a square of the form

$$[(m-1)\varepsilon, m\varepsilon] \times [(n-1)\varepsilon, n\varepsilon] \subset \mathbb{R}^2,$$

where $m, n \in \mathbb{Z}$. The number ε is called the *resolution*. A *pixelation* is a union of finitely many pixels. The ε-pixelation of a set $S \subset \mathbb{R}^2$ is the union of all the pixels that touch S. We denote it by $P_\varepsilon(S)$.

The Main Problem. Produce an algorithm that associates to $P_\varepsilon(S)$ a PL-set S_ε which approximates S very well as $\varepsilon \searrow 0$. More precisely, for $\varepsilon > 0$ sufficiently small, the approximation S_ε must have the same homotopy type as S and the curvature features of S_ε must closely resemble those of S.

The pixelation $P_\varepsilon(S)$ is a PL-set. It is however very jagged and there is no hope that its curvature properties are similar to those of S. In fact there is a more insidious reason why the pixelation is a poor approximation for S.

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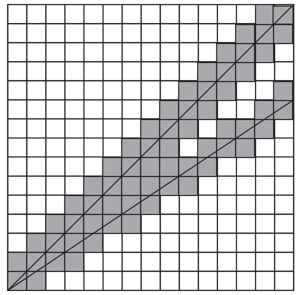

FIGURE 1. *The pixelation of the angle* $A\left(1, \frac{2}{3}\right)$ *contains two holes.*

Consider the pixelation of an angle with vertex at the origin whose edges have slopes $\frac{2}{3}$ and 1. Figure 1 shows that this pixelation is not contractible and in fact its first Betti number is 2. These two "holes" won't disappear at any resolution because all the pixelations of this angle are rescalings of each other.

In [12] the second author solved the Main Problem in the special case when S itself is a PL-set. The resulting algorithm is based on two key principles.

Principle 1. If S is the graph of a piecewise C^2-function f, then the pixelation of the graph of f has the same homotopy type as the graph, i.e., it is contractible. Moreover, every column of the pixelation is connected, where by *column* we understand the intersection of the pixelation with a vertical strip of the form $\{(m-1)\varepsilon < x < m\varepsilon\}$. We fix a function $\sigma : (0,\infty) \to \mathbb{Z}_{>0}$, called the *spread*, such that

$$\lim_{\varepsilon \searrow 0} \varepsilon\sigma(\varepsilon) = 0 \;\text{ and }\; \lim_{\varepsilon \searrow 0} \varepsilon\sigma(\varepsilon)^2 = \infty \tag{$\boldsymbol{\sigma}$}$$

For every $\varepsilon > 0$ we obtain by linear interpolation a PL function τ_ε (resp. β_ε) whose graph is produced by connecting with straight line segments the centers of the top (resp. bottom) pixels of every $\sigma(\varepsilon)$-th column of the pixelation of the graph of f.

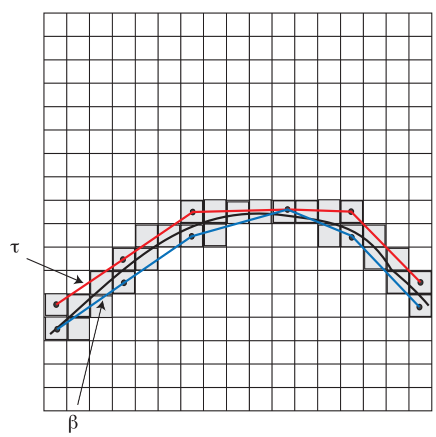

FIGURE 2. *Linear interpolations with spread* $\sigma = 3$.

The result of the algorithm is the PL-region S_ε between the graphs of β_ε and τ_ε. This is a very narrow two dimensional PL set very close to the graph of f. Moreover, the condition ($\boldsymbol{\sigma}$) gurantees that the curvature of S_ε resembles that of S.

An identical strategy works when S is a set of the form

$$S = \big\{ (x,y) \in \mathbb{R}^2;\;\; x \in [a,b],\; ;\beta(x) \le y \le \tau(x) \big\},$$

where $\beta, \tau : [a,b] \to \mathbb{R}$ are Lipschitz continuous, piecewise C^2-functions such that $\beta(x) \le \tau(x)$, $\forall x \in [a,b]$.

We will refer to these two types of sets as *elementary*. Thus, the Main Problem has a solution for elementary sets.

Principle 2. Consider the linear map $h(x,y) = x$. The Morse theoretic properties of the restriction of h to $P_\varepsilon(S)$ closely mimic the Morse theoretic properties of the restriction of h to S if ε is sufficiently small. Here are the details.

For $x_0 \in \mathbb{R}$ denote by $\boldsymbol{n}_S(x_0)$ the number of connected components of the intersection of S with the vertical line $\{x = x_0\}$ and denote by $\mathcal{J}_S$ the set of discontinuities of the function $x \mapsto \boldsymbol{n}_S(x)$. Then $\mathcal{J}_S$ is a finite subset of $\mathbb{R}$ and there exists $\gamma > 0$ such that for any $r \in (0,\gamma)$ the set S'_r obtained from S by removing the vertical strips $\{|x - j| < r\}$, $j \in \mathcal{J}_S$, is a disjoint union of elementary regions.

The set $\mathcal{J}_S$ is difficult to determine from a pixelation, but one can algorithmically produce a very small region containing it. Here is roughly the strategy.

For $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ we denote by $\boldsymbol{n}_{S,\varepsilon}(x_0)$ the number of connected components of the intersection of the vertical line $\{x = x_0\}$ with the pixelation $P_\varepsilon(S)$. We denote by $\mathcal{J}_{S,\varepsilon}$ the set of discontinuities of the function $x \mapsto \boldsymbol{n}_{S,\varepsilon}(x)$. The set $\mathcal{J}_{S,\varepsilon}$ is finite and one can prove the following fact.

(***D***) There exist $\hbar > 0$, $\nu_0 > 0$ and $\kappa_0 \in (0,1]$, depending only on S such that for $\varepsilon < \hbar$ we have

$$\operatorname{dist}(\mathcal{J}_S, \mathcal{J}_{S,\varepsilon}) < \nu_0 \varepsilon^{\kappa_0}.$$

We define the *noise region* to be the set

$$\mathcal{N}_\varepsilon := \left\{ x \in \mathbb{R};\ \operatorname{dist}(x, \mathcal{J}_{S,\varepsilon}) \le 2\nu\varepsilon^{\kappa_0} \right\}.$$

For ε sufficiently small, the noise region is a finite union of disjoint compact intervals

$$I_j(\varepsilon),\ \ j = 1, \dots, N := \#\mathcal{J}_S,$$

called *noise* intervals.

We denote by $P'_\varepsilon(S)$ the closure of the set obtained from $P_\varepsilon(S)$ by removing the vertical strips $\{x \in I_j(\varepsilon)\}$, $j = 1, \dots, N$. Each of the connected components of $P'_\varepsilon(S)$ is the pixelation of an elementary set and as such it can be PL-approximated using **Principle 1**.

The approximation of the noisy regions, i.e., the intersection of $P_\varepsilon(S)$ with the above vertical strips is rather coarse. Every component of such a region is approximated by the smallest rectangle that contains it. Here by rectangle we mean a region of the form $[a,b] \times [c,d]$, $a \le b$, $c \le d$.

It turns out that the approximation S_ε of S obtained in this fashion from $P_\varepsilon(S)$ is very good in the sense that the normal cycle of S_ε converges in the sense of currents to the normal cycle of S. For a nice introduction to the subject of normal cycles we refer to [9]. A brief description of this concept can also be found on page 20 of this paper.

The goal of this paper is to extend the above program to the more general case of compact, semi-algebraic subsets of $\mathbb{R}^2$. While **Principle 1** extends with little difficulty to the semi-algebraic case, **Principle 2** requires a more delicate analysis. This requires that S be a generic semialgebraic set in the sense that the restriction to S of the linear function $h(x,y) = x$ be a *stratified Morse function*, [4, 11]. **Principle 2** is proved using results in real algebraic geometry.

The PL approximation S_ε of S is obtained as before, using the two principles. To prove that the normal cycle of S_ε converges in the sense of currents to the normal cycle of S we rely on an approximation theorem of J. Fu, [5]. That theorem states that the convergence of the normal cycles is guaranteed once we prove two things.

- Uniform bounds for the perimeter and total curvature of S_ε.

- For almost any closed half-plane H we have

$$\lim_{\varepsilon\searrow 0}\chi\big(\, H\cap S_\varepsilon\,\big)=\chi\big(\, H\cap S\,\big),$$

where χ denotes the Euler characteristic.

Of the above two facts, the second is by far the most delicate, and its proof takes up the bulk of this paper.

Let us say a few words about the organization of the paper. In Section 1 we introduce the terminology used throughout the paper. **Principle 1** is proved in Section 2, while **Principle 2** is proved in Section 3. Section 4 is devoted to the proof of the convergence of normal cycle. The paper also contains two appendices. In Appendix A we collect a few basic facts of real algebraic geometry usd in this paper together with a few other technical results. In Appendix B we give a more formal description of the approximation algorithm in a way that makes it easily implementable on a computer. The present paper can be read independently of [12] which served both as motivation and as inspiration.

1. Basic facts

We begin by recalling some basic notions introduced in [12].

Definition 1.1. (a) Let $\varepsilon>0$. Then we define an ε-*pixel* to be the square in $\mathbb{R}^2$ of the form

$$S_{i,j}(\varepsilon)=[(i-1)\varepsilon, i\varepsilon)]\times[(j-1)\varepsilon, j\varepsilon)]\subset\mathbb{R}^2,\;\; i,j\in\mathbb{Z}.$$

(b) A union of finitely many ε-pixels is called an ε-*pixelation*. The variable ε is called the *resolution* of the pixelation.

(c) For any compact subset $S\subset\mathbb{R}^2$ we define the ε-*pixelation* of S to be the union of all the ε-pixels that intersect S. We denote the ε-pixelation of S by $P_\varepsilon(S)$. The pixelation of a function f is defined to be the pixelation of its graph $\Gamma(f)$. We will denote this pixelation by $P_\varepsilon(f)$. □

Observe that if $\|\bullet\|_\infty:\mathbb{R}^2\to[0,\infty)$ is the norm

$$\|(x,y)\|:=\max\{|x|,|y|\},$$

then the pixel $S_{i,j}(\varepsilon)$ can be identified with the $\|\bullet\|_\infty$-closed ball of radius $\varepsilon/2$ and center

$$c_{i,j}(\varepsilon):=\left(\,(i-\frac{1}{2})\varepsilon,(j-\frac{1}{2})\varepsilon,\,\right).$$

Therefore a pixelation of the compact set S can be thought of as set of points chosen near S.

Definition 1.2. Fix $\varepsilon>0$ and a compact set $S\subset\mathbb{R}^2$.

(1) A point $a\in\mathbb{R}$ will be called ε-generic if $x\in\mathbb{R}\setminus\varepsilon\mathbb{Z}$. For such a point a we denote by $I_\varepsilon(a)$ the interval of the form $(\,n\varepsilon,(n+1)\varepsilon)$, $n\in\mathbb{Z}$ that contains a.

(2) For $a<b$ we define the vertical strip

$$\mathcal{S}_{a,b}:=(a,b)\times\mathbb{R}$$

For every $k\in\mathbb{Z}$ we denote by $\mathcal{S}_{\varepsilon,k}$ the vertical strip

$$(\,k\varepsilon,(k+1)\varepsilon\,)\times\mathbb{R}=\mathcal{S}_{k\varepsilon,(k+1)\varepsilon}$$

For any ε-generic point $a\in\mathbb{R}$ we denote by $\mathcal{S}_\varepsilon(a)$ the strip

$$I_\varepsilon(a)\times\mathbb{R}=S_{\varepsilon,k},\;\; k:=\lfloor x/k\rfloor.$$

(3) A *column* of $P_\varepsilon(S)$ is the intersection of $P_\varepsilon(S)$ with a vertical strip $\mathcal{S}_{\varepsilon,k}$, $k\in\mathbb{Z}$. The connected components of a column are called *stacks*.

(4) For every ε-generic $a \in \mathbb{R}$, we define the *column* of a pixelation $P_\varepsilon(S)$ over a to be the set
$$C_\varepsilon(S,a) := \mathcal{S}_\varepsilon(a) \cap P_\varepsilon(S).$$
In other words, $C_\varepsilon(S,a)$ is the union of the pixels in $P_\varepsilon(S)$ which intersect the vertical line $\{x = a\}$. When S is the graph of a function f, we will use the notation $C_\varepsilon(f,a)$ to denote the column over a of the pixelation $P_\varepsilon(f)$.

(5) We define the *top, bottom* and respectively *height* of a column to be the quantities
$$T_\varepsilon(S,a) := \sup\{y : (a,y) \in C_\varepsilon(S,a)\},$$
$$B_\varepsilon(S,a) := \inf\{y : (a,y) \in C_\varepsilon(S,a)\},$$
and respectively
$$h_\varepsilon(S,a) := \frac{1}{\varepsilon}\big(T_\varepsilon(S,a) - B_\varepsilon(S,a) \big).$$
As in the other definitions when S is the graph of a function f we will use the notations $T_\varepsilon(f,a), B_\varepsilon(S,a)$ and $h_\varepsilon(S,a)$.

□

We have the following result,[12, Thm. 2.2]

Theorem 1.3. *If $f : [a,b] \to \mathbb{R}$ is a continuous function, then for any $\varepsilon > 0$ the columns of the ε-pixelation of the graph of f consist of single stacks.* □

In this paper we will be concerned with pixelations of generic planar semialgebraic sets, where the genericity has a very precise meaning. To describe it we need to introduce some terminology from stratified Morse theory, [4, 11].

For any subset $X \subset \mathbb{R}^2$ we denote by $\boldsymbol{cl}(X)$ its *closure* and by $\partial_{\rm top} X$ its *topological boundary*,
$$\partial_{\rm top}(X) := \boldsymbol{cl}(X) \setminus X.$$
We define a *good stratification* of a compact semialgebraic set $S \subset \mathbb{R}^2$ to be an increasing filtration
$$\mathcal{F} : \quad F^{(0)} \subset F^{(1)} \subset F^{(2)} = S$$
satisfying the following properties.

- Each of the sets $F^{(i)}$, $i = 0, 1, 2$ is closed.
- $\dim F^{(i)} \leq i$, $i = 0, 1, 2$. In particular $F^{(0)}$ is a finite collection of points called *vertices* of the good stratification.
- The connected components of $F^{(1)} \setminus F^{(0)}$ are *open real analytic arcs*, i.e., images of injective real analytic maps $(0,1) \to \mathbb{R}^2$. We will refer to these components as the *arcs* or the *edges* of the stratification.
- The connected components of $F^{(2)} \setminus F^{(1)}$ are open subsets of $\mathbb{R}^2$. They are called the *faces* of the stratification.
- $$\partial_{\rm top}\big(F^{(2)} \setminus F^{(1)} \big) \subset F^{(1)}, \;\; \partial_{\rm top}\big(F^{(1)} \setminus F^{(0)}\big) \subset F^{(0)}.$$

If v is a vertex of a good stratification of a compact semialgebraic set we define $C_\infty(v,S)$ to be the set of one dimensional subspaces of $L_\infty \subset \mathbb{R}^2$ such that there exists an arc A of the stratification with the following properties.

- $v \in \boldsymbol{cl}(A)$.
- There exists a sequence of points $v_n \in A$ such that as $n \to \infty$ we have $v_n \to v$ and the tangent spaces $T_{v_n}A$ converge to L_∞.

Suppose that $S \subset \mathbb{R}^2$ is a compact semialgebraic set equipped with a good stratification $\mathcal{F}$, and $f : \mathbb{R}^2 \to \mathbb{R}$. A point $p \in S$ is said to be a *critical point* of the restriction $f|_S$ if either p is a vertex, or p is the critical point of the restriction of f to an arc or to a face. The critical point p is said to be *nondegerate* if it satisfies one of the following conditions.

($\boldsymbol{C}_0$) The point p is a vertex and for any $L_\infty \in C_\infty(p, S)$, the differential of f at p does not vanish along L_∞.

($\boldsymbol{C}_1$) The point p belongs to an arc A of the stratification and as such it is a nondegenerate point of $f|_A$.

($\boldsymbol{C}_2$) The point p belongs to a face F of the stratification and as such it is a nondegenerate point of $f|_F$.

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to be a a *stratified Morse function* with respect to the semialgebaric set S equipped with the good stratification $\mathcal{F}$ if all its critical points are nondegenerate, and no two critical points lie on the same level set of f.

A compact semialgebraic set $S \subset \mathbb{R}^2$ is called *generic* if it admits a good stratification $\mathcal{F}$ such that projection onto the x-axis $(x, y) \mapsto x$ is a stratified Morse function with respect to $(S, \mathcal{F})$. Denote this projection by $\boldsymbol{h}$.

Observe that if $\mathcal{F}$ is a good stratification of a compact semialgebraic set $S \subset \mathbb{R}^2$, then p is a critical point of $\boldsymbol{h}$ relative to $(S, \mathcal{F})$ if either p is a vertex of the stratification, or p is a point on an arc of $\mathcal{F}$ where the the tangent space is vertical.

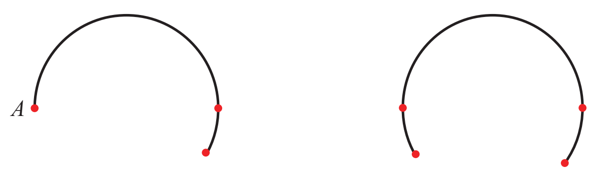

FIGURE 3. *The curve on the right is generic, while the curve on the left is not.*

In Figure 3 we have depicted two one-dimensional planar semi algebraic curves (arcs of circles). The marked points are critical points of h. The point A on the left-hand side curve is a degenerate critical point because the condition ($\boldsymbol{C}_0$) is violated: the vertical line is tangent to the curve at that point.

In the left-hand side of Figure 4 we have depicted further examples of pathologies prohibited by the genericity condition. (The pathologies involve the points with vertical tangencies.) The right-hand side depicts generic sets obtained by small perturbations from the nongeneric sets in the left-hand side.

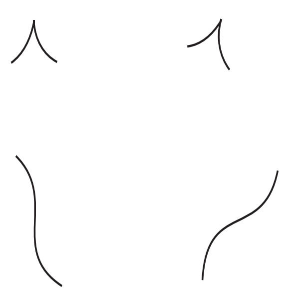

FIGURE 4. *The curves on the left-hand side are nongeneric. They become generic after a small perturbation.*

2. APPROXIMATIONS OF ELEMENTARY SETS

In this section we study the pixelations of simple two dimensional sets.

Definition 2.1. Fix $\varepsilon > 0$ and a compact set S.

(1) An *ε-profile* of S is a set Π_ε of points in the plane with the following properties.
  (a) Each point in Π_ε is the center of an ε-pixel that intersects S.
  (b) Every column of $P_\varepsilon(S)$ contains precisely one point of Π_ε.

(2) The *top/bottom ε-profile* is the profile consisting of the centers of the highest/lowest pixels in each column of $P_\varepsilon(S)$.

(3) An ε-sample of S is a subset of an ε-profile. An *upper/lower* ε-sample of S is an ε-sample of the upper/lower ε-profile of S.

□

Definition 2.2. Suppose $p_1, \dots, p_N$ is a finite sequence of points in $\mathbb{R}^2$. (The points need not be distinct). We denote by

$$\langle p_1, p_2, \dots, p_n \rangle$$

the PL curve defined as the union of the straight line segments $[p_1, p_2], \dots, [p_{n-1}, p_n]$. □

Observe that each ε-profile Π_ε of a set is equipped with a linear order $\preceq$. More precisely if p_1, p_2 are points in Π_ε, then

$$p_1 \preceq p_2 \Longleftrightarrow x(p_1) \leq x(p_2),$$

where $x : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection $(x, y) \mapsto x$. In particular, this shows that any ε-sample of S carries a natural total order.

Definition 2.3. If Ξ is an ε-sample of S, then the PL-interpolation determined by the sample Ξ is the continuous, piecewise linear function $L = L_\Xi$ obtained as follows.

- Arrange the points in Ξ in increasing order, with respect to the above total order,

$$V = \{\xi_0 \prec \xi_1 \prec \xi_2 \prec \cdots \prec \xi_n\}, \quad n+1 = |\Xi|.$$

- The graph of L_Ξ is the PL-curve $\langle \xi_0, \xi_1, \dots, \xi_n \rangle$.

□

In applications, the sample sets Ξ will be chosen to satisfy certain regularity.

Definition 2.4. (1) A *spread function* is an increasing function $(0, \infty) \ni \varepsilon \mapsto \sigma(\varepsilon) \in \mathbb{Z}_{>0}$ with the following properties.

$$\lim_{\varepsilon \searrow 0} \sigma(\varepsilon) = \infty, \tag{2.1a}$$

$$\lim_{\varepsilon \searrow 0} \varepsilon\sigma(\varepsilon) = 0. \tag{2.1b}$$

(2) If σ is a positive integer and Π_ε is an ε-profile, then an *ε-sample with spread* σ is a subset

$$\Xi = \{\xi_0 \prec \cdots \prec \xi_n\} \subset \Pi_\varepsilon(S)$$

such that the following hold.
  - The points ξ_0 and ξ_n are the left and rightmost points in the profile. (That is for each $p \in \Pi_\varepsilon$, $x(\xi_0) \leq x(p) \leq x(\xi_n)$.)
  - For any $p \in \Pi_\varepsilon$, there exists $\xi \in \Xi$ such that $|x(p) - x(\xi)| \leq \varepsilon\sigma$.

- $$\frac{1}{2}\varepsilon\sigma \leq |x(\xi_k) - x(\xi_{k-1})| \leq \varepsilon\sigma, \;\; \forall k = 1, \dots, n.$$

□

Definition 2.5. A subset $S \subset \mathbb{R}^2$ is said to be *elementary* (with respect to the x-axis) if it can be defined as

$$S = S(\beta, \tau) := \big\{ (x, y) : x \in [a, b], \beta(x) \leq x \leq \tau(x) \big\},$$

where $\beta, \tau : [a, b] \to \mathbb{R}$ are continuous semialgebraic functions such that $\beta(x) \leq \tau(x)$, $\forall x \in [a, b]$. The function β is called the *bottom* of S while τ is called the *top* of S. If

$$\beta(x) < \tau(x), \;\; \forall x \in (a, b), \tag{2.2}$$

then the elementary set is said to be *nondegenerate*. If

$$\beta(x) = \tau(x), \;\; \forall x \in [a, b], \tag{2.3}$$

then the set S is called *degenerate*. The elementary set is called *mixed* if both sets

$$\big\{ x \in (a, b); \;\; \tau(x) - \beta(x) > 0 \big\} \;\text{ and }\; \big\{ x \in (a, b); \;\; \tau(x) - \beta(x) = 0 \big\}$$

are nonempty. □

Observe that an elementary set $S(\beta, \tau)$, $\beta, \tau : [a, b] \to \mathbb{R}$ admits *good partitions*, i.e., partitions

$$a = c_0 < c_1 < c_1 \cdots < c_n = b, \;\; n \geq 2$$

such that each of the elementary sets $[c_{i-1}, c_i] \times \mathbb{R} \cap S(\beta, \tau)$ is either degenerate or nondegenerate. The good partition with minimal n is called the *minimal good partition*; see Figure 5.

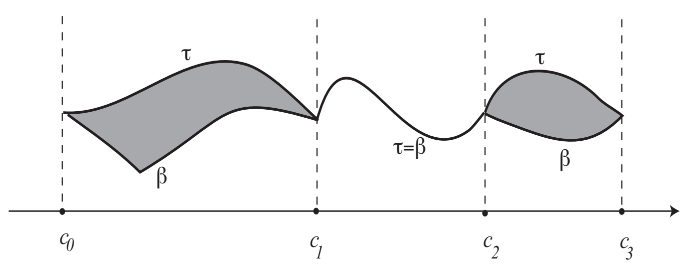

FIGURE 5. *The minimal good partition of a mixed elementary set.*

In the remainder of this section S will indicate an elementary set. We first note that like the pixelation of a function, each column of $P_\varepsilon(S)$ contains only one stack.

Proposition 2.6. *If $S = S(\beta, \tau)$ is an elementary set, then for every $x \in [a, b] \setminus \varepsilon\mathbb{Z}$, the column $C_\varepsilon(S, x)$ consists of exactly one stack.*

Proof. Fix an ε-generic $x \in [a, b]$. By Theorem 1.3 the columns $C_\varepsilon(\beta, x)$ and $C_\varepsilon(\tau, x)$ consist of single stacks. If these two columns intersect, then the conclusion is obvious. If they do not intersect, then any pixel in the strip $S_\varepsilon(x)$ situated below the stack $C_\varepsilon(\tau, x)$ and above the stack $C_\varepsilon(\beta, x)$ is a pixel of $P_\varepsilon(S)$. This proves that the column $C_\varepsilon(S, x)$ consists of a single stack. □

Definition 2.7. Fix $\varepsilon > 0$ and an elementary set $S = S(\beta, \tau)$. Suppose that $\Xi_\varepsilon^\pm$ are compatible upper/lower samples of S

$$\Xi_\varepsilon^\pm = \{\xi_0^\pm \prec \xi_1^\pm \prec \cdots \prec \xi_n^\pm\}.$$

The PL-*approximation* of S determined by these two samples is the compact PL-set bounded by the closed PL-curve

$$\langle \xi_0^-, \xi_1^-, \dots, \xi_n^-, \xi_n^+, \xi_{n-1}^+, \dots, \xi_0^+, \xi_0^- \rangle.$$

□

The total curvature of a C^2 immersion $\gamma : [a, b] \to \mathbb{R}^2$ is defined as follows. Define a C^1-function $\theta : [a, b] \to \mathbb{R}$ such that

$$\frac{1}{|\dot\gamma(t)|}\dot\gamma(t) = (\cos(\theta(t), \sin(\theta(t))),$$

where a dot $\dot{}$ denotes the t-derivative. We set

$$\kappa(t) := |\dot\theta(t)|.$$

The scalar $\kappa(t)$ is called the *curvature* of γ at the point $\gamma(t)$. We define the *total curvature* of γ to be

$$K(\gamma) := \int_a^b \kappa(t)\, dt.$$

Suppose now that $\gamma : [a, b] \to \mathbb{R}^2$ continuous and piecewise C^2-immersion, i.e., there exist a finite subset $\{t_0, \dots, t_\nu\} \subset [a, b]$, such that

$$a = t_0 < t_1 < \cdots < t_{\nu-1} < t_\nu = b,$$

and the restriction $\gamma_i := \gamma|_{[[t_{i-1}, t_i]}$ is a C^2-immersion for any $i = 1, \dots, \nu$. The curvature of γ at a jump point $\gamma(t_i)$ is the quantity

$$\kappa(t_i) = |\theta(t_i^+) - \theta(t_i^-)| := |\lim_{t \searrow t_i} \theta(t) - \lim_{t \nearrow t_i} \theta(t)|.$$

We define the total curvature of γ to be

$$K(\gamma) := \sum_{i=1}^{\nu} K(\gamma_i) + \sum_{i=1}^{\nu-1} \kappa(t_i) + \begin{cases} 0, & \gamma(b) \neq \gamma(a) \\ |\theta(t_0^+) - \theta(t_\nu^-)|, & \gamma(b) = \gamma(a). \end{cases}$$

For more details we refer to [8] and [9, §2.2].

We define a *semialgebraic arc* to be the image of a continuous, injective semialgebraic map

$$\varphi : [a, b] \to \mathbb{R}^2.$$

Suppose that $\varphi : [a, b] \to \mathbb{R}^2$ is a continuous, injective, semialgebraic map whose image is the semialgebraic arc $\boldsymbol{C}$. Set $A := \varphi(a)$ and $B := \varphi(b)$ so that $\boldsymbol{C}$ connects A to B.

An *ordered sampling* of $\boldsymbol{C}$ is an ordered collection of points

$$\mathcal{P} := \{P_1, \dots, P_n\} \subset \boldsymbol{C},$$

such that the collection

$$\varphi^{-1}(\mathcal{P}) := \{t_1 = \varphi^{-1}(P_1), \dots, t_n := \varphi^{-1}(P_n)\} \subset [a, b]$$

satisfies

$$t_1 < t_2 < \cdots < t_n.$$

The *mesh* of the ordered sampling $\mathcal{P}$ is the positive number

$$\|\mathcal{P}\| := \max\{\operatorname{dist}(A, P_1), \operatorname{dist}(P_1, P_2), \dots, \operatorname{dist}(P_{n-1}, P_n), \operatorname{dist}(P_n, B)\}.$$

We denote by $\boldsymbol{C}(\mathcal{P})$ the PL-curve $\langle P_1, \dots, P_n\rangle$. We will need the following result whose proof is delegated to Appendix A.

Proposition 2.8. *Suppose that $\boldsymbol{C}$ is a semialgebraic arc and for very $\varepsilon > 0$ we are given and ordered sampling $\mathcal{P}_\varepsilon$ of $\boldsymbol{C}$. Denote by L (resp. K) the length (resp. total curvature) of $\boldsymbol{C}$ and by L_ε (resp. K_ε) the length (resp. total curvature) of $\boldsymbol{C}(\mathcal{P}_\varepsilon)$. If*

$$\lim_{\varepsilon\searrow 0} \|\mathcal{P}_\varepsilon\| = 0,$$

then

$$\lim_{\varepsilon\searrow 0} L_\varepsilon = L \ \ \text{and} \ \ \lim_{\varepsilon\searrow 0} K_\varepsilon = K.$$

□

Theorem 2.9. *Suppose that $h : [a, b] \to \mathbb{R}$ is a continuous semialgebraic function and $\varepsilon \to \sigma(\varepsilon)$ is a spread function satisfying the additional condition*

$$\lim_{\varepsilon\searrow 0} \varepsilon\sigma(\varepsilon)^2 = \infty. \tag{2.4}$$

For every $\varepsilon > 0$ we choose an ε-sample Ξ_ε with spread $\sigma(\varepsilon)$ of the graph Γ of h. Denote by $\boldsymbol{C}_\varepsilon$ the graph of the PL-function L_{Ξ_ε} described in Definition 2.3. Then, as $\varepsilon \searrow 0$ we have

$$\operatorname{length}(\boldsymbol{C}_\varepsilon) \to \operatorname{length}(\Gamma) \ \ \text{and} \ \ K(\boldsymbol{C}_\varepsilon) \to K(\Gamma).$$

Proof. We use a simple strategy. More precisely for every $\varepsilon > 0$ we construct an ordered sampling $\mathcal{P}_\varepsilon$ of Γ such that

$$\lim_{\varepsilon\searrow 0} \|\mathcal{P}_\varepsilon\| = 0, \tag{2.5}$$

and if Γ_ε denotes the PL-curve $\Gamma(\mathcal{P}_\varepsilon)$ determined by the ordered sampling $\mathcal{P}_\varepsilon$, then as $\varepsilon \searrow 0$ we have

$$\operatorname{length}(\boldsymbol{C}_\varepsilon) = \operatorname{length}(\Gamma_\varepsilon) + O\left(\frac{1}{\sigma(\varepsilon)}\right), \tag{2.6a}$$

$$K(\boldsymbol{C}_\varepsilon) = K(\Gamma_\varepsilon) + O\left(\frac{1}{\varepsilon\sigma(\varepsilon)^2}\right). \tag{2.6b}$$

The desired conclusions will then follow from Proposition 2.8.

Suppose that Ξ_ε consists of the points $Q_0^\varepsilon, Q_1^\varepsilon, \dots, Q_{n(\varepsilon)}^\varepsilon$ arranged in the increasing order defined by their x-coordinates. Observe that since Ξ_ε has spread $\sigma(\varepsilon)$ then

$$n(\varepsilon) < \frac{2(b-a)}{\varepsilon\sigma(\varepsilon)} \tag{2.7}$$

Each of the points of Ξ_ε is the center of a pixel that touches Γ. Thus, for any $k = 0, 1, \dots, n(\varepsilon)$ there exists a point $P_k^\varepsilon \in \Gamma$ that lies in the same pixel as Q_k^ε. We obtain in this fashion an ordered sampling

$$\mathcal{P}_\varepsilon = \{P_0^\varepsilon, P_1^\varepsilon, \dots, P_{n(\varepsilon)}^\varepsilon\}$$

of Γ. The function h is continuous and semialgebraic and thus it is Hölder continuous with some Hölder exponent $\alpha \in (0, 1]$. This proves that

$$\|\mathcal{P}_\varepsilon\| = O\big((\varepsilon\sigma(\varepsilon))^\alpha\big).$$

The condition (2.5) now follows from the property (2.1b) of a spread function.

From the choice of the points P_k^ε we deduce that for any $k = 1, \dots, n(\varepsilon)$ we have

$$-\varepsilon\sqrt{2} < \operatorname{dist}(P_{k-1}^\varepsilon, P_k^\varepsilon) - \operatorname{dist}(Q_{k-1}^\varepsilon, Q_k^\varepsilon) < \varepsilon\sqrt{2}$$

so that by summing over k we deduce

$$-\frac{2\sqrt{2}(b-a)}{\sigma(\varepsilon)} \overset{(2.7)}{\leq} -n(\varepsilon)\varepsilon\sqrt{2} < \operatorname{length}(\Gamma_\varepsilon) - \operatorname{length}(\boldsymbol{C}_\varepsilon) \leq n(\varepsilon)\varepsilon\sqrt{2} \overset{(2.7)}{\leq} \frac{2\sqrt{2}(b-a)}{\sigma(\varepsilon)}.$$

The equality (2.6a) now follows from the property (2.1a) of a spread function.

Now we turn to total curvature. For $k = 1, \dots, n(\varepsilon)$ we denote by m_k^ε the slope of the segment $[P_{k-1}^\varepsilon, P_k^\varepsilon]$ and by $\bar{m}_k^\varepsilon$ the slope of the segment $[Q_{k-1}^\varepsilon, Q_k^\varepsilon]$. Note that for any $\varepsilon > 0$ and any $k = 1, \dots, n(\varepsilon)$ we have from the definition of a spread that

$$\operatorname{dist}(Q_{k-1}^\varepsilon, Q_k^\varepsilon) \leq \varepsilon\sigma(\varepsilon).$$

Furthermore we have shown that

$$\operatorname{dist}(Q_k^\varepsilon, P_k^\varepsilon) \leq \varepsilon\sqrt{2}.$$

These two inequalities imply that

$$|m_k^\varepsilon - \bar{m}_k^\varepsilon| = O\left(\frac{1}{\sigma(\varepsilon)}\right)$$

There exist $\theta_k^\varepsilon, \bar{\theta}_k^\varepsilon \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that

$$m_k^\varepsilon = \tan\theta_k^\varepsilon, \quad \bar{m}_k^\varepsilon = \tan\bar{\theta}_k^\varepsilon.$$

The difference formula of tangent implies that:

$$\theta_k^\varepsilon - \bar{\theta}_k^\varepsilon = \arctan\left(\frac{m_k^\varepsilon - \bar{m}_k^\varepsilon}{1 + m_k^\varepsilon \bar{m}_k^\varepsilon}\right)$$

Using the above equation and the fact that $|m_k^\varepsilon - \bar{m}_k^\varepsilon| = O\left(\frac{1}{\sigma(\varepsilon)}\right)$, we see that there is a constant C independent of ε such that

$$|\theta_k^\varepsilon - \bar{\theta}_k^\varepsilon| \leq \frac{C}{\sigma(\varepsilon)}$$

and therefore

$$\big|\, K(\Gamma_\varepsilon) - K(\boldsymbol{C}_\varepsilon) \,\big| \leq \frac{Cn(\varepsilon)}{\sigma(\varepsilon)} \overset{(2.7)}{\leq} \frac{C(b-a)}{\varepsilon\sigma(\varepsilon)^2}.$$

The equality (2.6b) now follows from (2.4).

□

Corollary 2.10. *Let $S(\beta, \tau)$ be an elementary set. Fix a spread $\sigma(\varepsilon)$ function satisfying the condition (2.4). For each ε we choose compatible ε-upper/lower profiles $\Xi_\varepsilon^\pm$ of S with spread $\sigma(\varepsilon)$. We denote by S_ε the PL approximation of S defined by these samples. Then*

$$\lim_{\varepsilon \searrow 0} \operatorname{length}(\partial S_\varepsilon) = \operatorname{length}(\partial S_\varepsilon) \tag{2.8a}$$

$$\lim_{\varepsilon \searrow 0} K(\partial S_\varepsilon) = K(\partial S). \tag{2.8b}$$

Proof. The semialgebraic functions β and τ are differentiable outside a finite subset of (a,b) and the limits

$$\beta'(a) := \lim_{x\searrow a}\beta'(x),\quad \tau'(a) := \lim_{x\searrow a}\tau'(x),$$
$$\beta'(b) := \lim_{x\nearrow b}\beta'(x),\quad \tau'(b) := \lim_{x\nearrow b}\tau'(x)$$

exist in $[-\infty,\infty]$. Let

$$\Xi^{\pm}_{\varepsilon} = \bigl\{\xi^{\pm}_0 \prec \xi^{\pm}_1 \prec \cdots \prec \xi^{\pm}_{n(\varepsilon)}\bigr\},\quad \xi^{\pm}_k =: (x^{\pm}_k, y^{\pm}_k).$$

The compatibility condition implies that

$$x^-_k = x^+_k =: x_k,\quad y^-_k \leq y^+_k,\quad \forall k = 0,1,\dots,n.$$

Let β_ε be the PL function whose graph is $L_{\Xi^-_\varepsilon}$ and τ_ε be the PL function whose graph is $L_{\Xi^+_\varepsilon}$. Let $m^\beta_i(\varepsilon)$ indicate the slope of the i-th line segment of the graph of β_ε and similarly let $m^\tau_i(\varepsilon)$ indicate the slope of the i-th line segment of the graph of τ_ε. We have

$$\mathrm{length}(\partial S_\varepsilon) = \mathrm{length}\,(\Gamma_{\beta_\varepsilon}) + \mathrm{length}\,(\Gamma_{\tau_\varepsilon}) + \mathrm{dist}(\xi^-_0,\xi^+_0) + \mathrm{dist}(\xi^-_{n(\varepsilon)},\xi^+_{n(\varepsilon)}\,).$$

Theorem 2.9 implies that as $\varepsilon \searrow 0$ we have

$$\mathrm{length}\,(\Gamma_{\beta_\varepsilon}) \to \mathrm{length}\,(\Gamma_\beta),\quad \mathrm{length}\,(\Gamma_{\tau_\varepsilon}) \to \mathrm{length}\,(\Gamma_\tau).$$

Moreover, as $\varepsilon \searrow 0$ we have

$$\mathrm{dist}(\xi^-_0,\xi^+_0) \to \mathrm{dist}(\beta(a),\tau(a)\,),\quad \mathrm{dist}(\xi^-_{n(\varepsilon)},\xi^+_{n(\varepsilon)}\,) \to \mathrm{dist}(\,\beta(b),\tau(b)\,).$$

This proves (2.8a).

Similarly

$$\begin{aligned} K(\partial S_\varepsilon) &= |\pi - \arctan(m^\beta_1(\varepsilon)\,)| \\ &\quad + \sum_{i=2}^n |\arctan(m^\beta_i(\varepsilon)\,) - \arctan(m^\beta_{i-1}(\varepsilon)\,)| \\ &\quad + |\arctan(m^\beta_n(\varepsilon)) - \pi| + |\pi - \arctan(m^\tau_1)| \\ &\quad + \sum_{i=2}^n |\arctan(m^\tau_i(\varepsilon)\,) - \arctan(m^\tau_{i-1}(\varepsilon)\,)| \\ &\quad + |\arctan(m^\tau_n(\varepsilon)\,) - \pi| \end{aligned}$$

which can be rewritten as

$$\begin{aligned} K(\partial S_\varepsilon) &= |\pi - \arctan(m^\beta_1(\varepsilon)\,)| + |\arctan(m^\beta_n(\varepsilon)\,) - \pi| \\ &\quad + |\pi - \arctan(m^\tau_1(\varepsilon)\,)| + |\arctan(m^\tau_n(\varepsilon)\,) - \pi| \\ &\quad + K(\beta_\varepsilon) + K(\tau_\varepsilon) \end{aligned} \tag{2.9}$$

Theorem 2.9 implies

$$\lim_{\varepsilon\searrow 0} K(\beta_\varepsilon) = K(\beta) \ \text{ and } \ \lim_{\varepsilon\searrow 0} K(\tau_\varepsilon) = K(\tau). \tag{2.10}$$

Now note that each line segment is defined by connecting two points in the pixelation of β or τ over an interval of at most $\varepsilon\sigma(\varepsilon)$. As $\varepsilon \to 0$ we have

$$\begin{aligned} &\lim_{\varepsilon\searrow 0} m^\beta_1(\varepsilon) = \beta'(a),\quad \lim_{\varepsilon\searrow 0} m^\beta_n(\varepsilon) = \beta'(b), \\ &\lim_{\varepsilon\searrow 0} m^\tau_1(\varepsilon) = \tau'(a),\quad \lim_{\varepsilon\searrow 0} m^\tau_n(\varepsilon) = \beta'(b). \end{aligned} \tag{2.11}$$

Combining (2.9), (2.10), (2.11) we find that

$$\begin{aligned}\lim_{\varepsilon\searrow 0} K(\partial S_\varepsilon) = &|\pi - \arctan(\beta'(a))| + |\arctan(\beta'(b)) - \pi| \\ &+ |\pi - \arctan(\tau'(a))| + |\arctan(\tau'(b)) - \pi| \\ &+ K(\beta) + K(\tau).\end{aligned} \tag{2.12}$$

Note that $|\pi - \arctan(\beta'(a))|$ is the value of the angle between the vertical line $x = a$ and the tangent line to the graph of β at $(a, \beta(a))$. Similarly each other difference on the right hand side of (2.12) corresponds to an angle at one of the corners of ∂S. Therefore the right hand side of the (2.12) is equal to the $K(\partial S)$, so the corollary holds. □

3. SEPARATION RESULTS

In the previous section we have dealt only with the elementary regions and we investigated mainly *geometric* properties of these regions and their pixelation. In this section we turn our attention to the relationship between the topologies of a semialgebraic set and those of its pixelations.

Surprisingly, this is a nontrivial matter. In general, the homotopy type of a planar set may be quite different from those of its pixelations. This can happen even for a simple PL case. Consider the set S composed of the rays starting from the origin and proceeding in the positive direction with slopes $\frac{1}{2}$ and $\frac{1}{7}$ (see Figure 6). A careful examination of $P_\varepsilon(S)$ reveals the existence of cycle. Worse yet, the pixelations for smaller values of ε are simply rescalings of the larger pixelations. This means that $P_\varepsilon(S)$ contains a cycle for all small ε. For a taste of how much worse this can get we refer to [12, Appendix A].

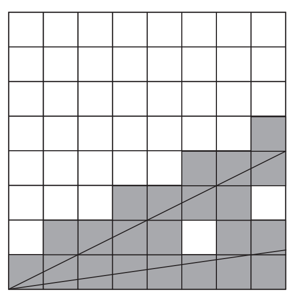

FIGURE 6. *The pixelation of the lines with slope $\frac{1}{2}$ and $\frac{1}{7}$. After this point the pixelations of the upper and lower lines diverge permanently and no more holes are formed.*

The next result provides a first ray of hope. For any compact set $X \subset \mathbb{R}^2$ we denote by $\mathcal{C}(X)$ the set of connected components of X.

Proposition 3.1. *Let $S \subset \mathbb{R}^2$ be a compact semialgebraic set. Then for sufficiently small ε, the number of connected components of $P_\varepsilon(S)$ agrees with the number of connected components of S.*

Proof. We have a natural map $\mathcal{C}(S) \to \mathcal{C}(\,P_\varepsilon(S)\,)$ that associates to each connected component C of S the unique connected component of $P_\varepsilon(S)$ containing C. For ε sufficiently small this map is injective. Since $P_\varepsilon(S)$ contains only pixels that intersect S, we deduce that each connected component of $P_\varepsilon(S)$ contains at least one connected component of S. Thus, $P_\varepsilon(S)$ has at most as many components as S. □

The above result guarantees that the zeroth Betti number of a compact semialgebraic set coincides with those of its sufficiently fine pixelations. Proposition 3.1 also suggests that, for small ε, the only way that the homotopy type of $P_\varepsilon(S)$ can disagree with that of S is if $P_\varepsilon(S)$ has *holes*, i.e., cycles in $P_\varepsilon(S)$ that are not contained in the image of the inclusion induced morphism

$$H_1(S,\mathbb{Z}) \to H_1\big(\,P_\varepsilon(S),\mathbb{Z}\,\big).$$

Thus recovery of S from $P_\varepsilon(S)$ will depend on distinguishing the cycles of $P_\varepsilon(S)$ that correspond to real cycles from S from those that are merely artifacts of the pixelation. To discard these holes, we adopt a strategy inspired from Morse theory.

Definition 3.2. Let $S \subset \mathbb{R}^2$ be a compact set and $\varepsilon > 0$

(1) For every ε-generic x we set

$$\boldsymbol{n}_\varepsilon(x) = \boldsymbol{n}_{S,\varepsilon} := \#\text{ of connected components of } C_\varepsilon(S,x).$$

(When the set S is understood from context we use the simpler notation $\boldsymbol{n}_\varepsilon$ instead of $\boldsymbol{n}_{S,\varepsilon}$.) We will refer to $\boldsymbol{n}_\varepsilon(x)$ as the *stack counter function* of S.

(2) For any $x_0 \in \mathbb{R}$ we define

$$\boldsymbol{n}(x_0) = \boldsymbol{n}_S(x) := \#\text{ of components of the intersection of } S \text{ with the vertical line } x = x_0.$$

We will refer to $\boldsymbol{n}_S$ the *component counter* of S.

(3) A *jumping point* of $\boldsymbol{n}_S$ is a real number x_0 such that

$$\boldsymbol{n}_S(x_0) \neq \boldsymbol{n}_S(x_0^-) := \lim_{x\nearrow x_0} \boldsymbol{n}_S(x) \;\text{ or }\; \boldsymbol{n}_S(x) \neq \boldsymbol{n}_S(x_0^+) := \lim_{x\searrow x_0} \boldsymbol{n}_S(x).$$

We denote by $\mathcal{J}_S$ the set of jumping points of $\boldsymbol{n}_S$. We will refer to $\mathcal{J}_S$ as the *jumping set* of S.

(4) A *jumping point* of $\boldsymbol{n}_\varepsilon$ is areal number $x_0 \in \varepsilon\mathbb{Z} + \frac{\varepsilon}{2}$ such that

$$\boldsymbol{n}_\varepsilon(x_0-\varepsilon) \neq \boldsymbol{n}(x_0).$$

We denote by $\mathcal{J}_{S,\varepsilon}$ the set of jumping points of $\boldsymbol{n}_{S,\varepsilon}$. We will refer to it as the ε*-jumping set* of S.

□

Let us point out that if S is semialgebraic, then its jumping set $\mathcal{J}_S$ is finite and it is contained in the set of critical values of the restriction to S of the function $h(x,y) = x$. The function $\boldsymbol{n}_\varepsilon$ tells us how many stacks are in a column. The jumps of $\boldsymbol{n}_\varepsilon$ are a first indicator of the presence of cycles in $P_\varepsilon(S)$. To decide whether they are holes, as opposed to cycles coming from S we will rely on the next key technical result.

Theorem 3.3 (Separation Theorem)**.** *Let $f, g : [a,b] \to \mathbb{R}$ be two semialgebraic continuous functions such that $f(x) < g(x)$, $\forall x \in [a,b]$. Denote by G the union of the graphs of f and g. Fix $L > 0$, $\alpha \in (0,1]$ and $x_0 \in [a,b]$ such that either*

$$|g(x) - g(y)| \leq L|x-y|^\alpha, \quad \textit{or } |f(x) - f(y) \leq L|x-y|^\alpha \forall x, y \in [a,b] \tag{3.1}$$

and

$$g(x_0) - f(x_0) \leq g(x) - f(x), \;\; \forall x \in [a,b], \tag{3.2}$$

Then for any $\varepsilon > 0$ such that

$$3\varepsilon + L\varepsilon^\alpha < g(x_0) - f(x_0) \tag{3.3}$$

and any ε-generic $x \in [a,b] \setminus \varepsilon\mathbb{Z}$ the column $C_\varepsilon(G, x_0)$ has two components. In other words, if

$$\min_{x\in[a,b]}\big(\,g(x) - f(x)\,\big) \geq 3\varepsilon + L\varepsilon^\alpha,$$

then for any ε-generic $x \in [a,b]$ we have

$$\boldsymbol{n}_{G,\varepsilon}(x) = \boldsymbol{n}_G(x).$$

Proof. We deal with the case that $|g(x) - g(y)| \leq L|x-y|^\alpha$. Note that

$$C_\varepsilon(G,x) = C_\varepsilon(f,x) \cup C_\varepsilon(g,x),$$

and furthermore, Theorem 1.3 implies that each of these columns is connected. Therefore $C_\varepsilon(G,x)$ will have two components exactly when $C_\varepsilon(f,x)$ and $C_\varepsilon(g,x)$ do not intersect. Since $f \leq g$ and x is ε-generic, this will occur when

$$T_\varepsilon(f,x) < B_\varepsilon(g,x)$$

or equivalently,

$$B_\varepsilon(g,x) - T_\varepsilon(f,x) > \varepsilon.$$

Now fix $x \in [a,b]$ and let $i \in \mathbb{Z}$ such that $i\varepsilon < x < (i+1)\varepsilon$. Choose $x_f, x_g \in [i\varepsilon, (i+1)\varepsilon)]$ such that

$$f(x_f) = \max_{x \in [i\varepsilon,(i+1)\varepsilon]} f(x), \quad g(x_g) = \min_{x \in [i\varepsilon,(i+1)\varepsilon]} g(x).$$

Therefore we have

$$B_\varepsilon(g,x) \geq g(x_g) - \varepsilon, \quad T_\varepsilon(f,x) \leq f(\,x_f\,) + \varepsilon,$$

so that

$$B_\varepsilon(g,x) - T_\varepsilon(f,x) \geq g(x_g) - f(x_f) - 2\varepsilon.$$

Thus

$$\begin{aligned} B_\varepsilon(g,x) - T_\varepsilon(f,x) &= g(x_g) - g(x_f) + g(\,x_f\,) - f(\,x_f\,) - 2\varepsilon \\ &\overset{(3.2)}{\geq} g(x_g) - g(x_f) + g(x_0) - f(x_0) - 2\varepsilon \\ &\overset{(3.1)}{\geq} g(x_0) - f(x_0) - L\varepsilon^\alpha - 2\varepsilon \overset{(3.3)}{>} \varepsilon. \end{aligned}$$

which completes the proof for the case that $|g(x) - g(y)| \leq L|x-y|^\alpha$. If instead we are given that $|f(x) - f(y)| \leq L|x-y|^\alpha$ the proof is similar, except that the roles of f and g exchanged. □

This important Separation Theorem can be used to prove the following two results, which will tell us exactly when $\boldsymbol{n}$ and $\boldsymbol{n}_\varepsilon$ correspond, using only information from the pixelation. With these theorems we will be able to distinguish real cycles from the original set from fake cycles created by the pixelation.

Theorem 3.4. *Let $S \subset \mathbb{R}^2$ be a compact semialgebraic set with jumping set $\mathfrak{J}_S$. Then there exist $\kappa_0 = \kappa_0(S) \in (0,1]$, $\nu_0 = \nu_0(S) > 0$ and $\varepsilon_0 = \varepsilon_0(S) > 0$,* depending only on S, *such that, if $0 < \varepsilon < \varepsilon_0$ and x is an ε-generic value such that*

$$\operatorname{dist}(x, \mathfrak{J}_S) \geq \nu_0 \varepsilon^{\kappa_0},$$

then $\boldsymbol{n}_{S,\varepsilon}(x) = \boldsymbol{n}_S(x)$.

Proof. Let $x_0 < x_1 < \cdots < x_\ell$ be the jumping points of $\boldsymbol{n} = \boldsymbol{n}_S$. We set

$$\Delta x_i := x_i - x_{i-1}, \quad \forall i = 1, \ldots, \ell, \quad \Delta := \min_{1 \leq i \leq \ell} \Delta x_i.$$

Note that $\boldsymbol{n}(x)$ is constant on each of the intervals (x_{i-1}, x_i). For $i = 1, \ldots, \ell$ we set

$$S_i := \big\{ (x,y) \in S; \;\; x \in [x_{i-1}, x_i] \,\big\}.$$

The set S_i is a disjoint union of elementary sets (see Definition 2.5 for notations)

$$S(\beta_{i,j}, \tau_{i,j}), \quad j = 0, \ldots, p_i,$$

"stacked one above the other", i.e.,

$$\beta_{i,0}(x) \leq \tau_{i,0}(x) < \beta_{i,1}(x) \leq \tau_{i,1}(x) < \cdots < \beta_{i,p_i}(x) \leq \tau_{i,p_i}(x), \quad \forall x \in (x_{i-1}, x_i). \tag{3.4}$$

From Proposition 2.6 we deduce that for any ε-generic $x \in (x_{i-1}, x_i)$ we have $\boldsymbol{n}(x) = p_i$.

Both of the functions $\beta_{i,j}$ and $\tau_{i,j}$ are continuous and semialgebraic. For any $i = 1, \dots, \ell$, any $j = 1, \dots p_i$ and any $\hbar \in (0, \frac{\Delta}{4})$ we denote by $\gamma_{i,j}(\hbar)$ the minimum of $\beta_{i,j} - \tau_{i,j-1}$ on the interval $[x_{i-1} + \hbar, x_i - \hbar]$. From Łojasewicz's inequality we deduce that there exists $C = C(S) > 0$ and $r = r(S) \in \mathbb{Z}_{>0}$ such that for any $i = 1, \dots, \ell$, any $j = 1, \dots, p_i$ and any $\hbar \in (0, \frac{\Delta}{4})$ we have

$$\gamma_{i,j}(\hbar) > C\hbar^r. \tag{3.5}$$

Fix $L > 0$ and $\alpha > 0$ such that for any $i = 1, \dots, \ell$, $j = 1, \dots, p_i$ and any $x, y \in [x_{i-1}, x_i]$ we have

$$|\beta_{i,j}(x) - \beta_{i,j}(y)| + |\tau_{i,j}(x) - \tau_{i,j}(y)| \leq L|x - y|^\alpha.$$

Now choose $\varepsilon_0 > 0$, $\nu_0 > 0$ and $\kappa_0 \in (0, 1]$ such that

$$C\big(\nu_0 \varepsilon^{\kappa_0}\big)^r > 3\varepsilon + L\varepsilon^\alpha, \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.6}$$

The conclusion now follows from (3.5), (3.6) and Theorem 3.3. □

Definition 3.5. The constant $\kappa_0(S)$ guaranteed by Theorem 3.4 is called the *separation constant* of the set S.

This theorem tells us that the jumps of $\boldsymbol{n}_\varepsilon$ occur within $\nu_0\varepsilon^{\kappa_0}$ pixels from the jumps in $\boldsymbol{n}$. A priori, it could be possible that, given a jumping point x_0 of $\boldsymbol{n}$, there is no jump in $\boldsymbol{n}_\varepsilon$ within $\nu_0\varepsilon^{\kappa_0}$ pixels of x_0. This next theorem shows that in fact this cannot happen.

Theorem 3.6. *Let S be a generic compact semialgebraic set, and $\varepsilon_0 = \varepsilon_0(S)$, $\nu_0 = \nu_0(S)$ as in Theorem 3.4. Let $\kappa_0 = \kappa_0(S)$ be the separation constant of S. Then, there exist $\varepsilon_1 = \varepsilon_1(S) > 0$ such that if $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$ and x_0 is a jumping point of $\boldsymbol{n} = \boldsymbol{n}_S$, then $\boldsymbol{n}_\varepsilon = \boldsymbol{n}_{S,\varepsilon}$ has at least one jumping point in the interval $[x_0 - \nu_0\varepsilon^{\kappa_0}, x_0 + \nu_0\varepsilon^{\kappa_0}]$.*

Proof. Fix a good stratification $\mathcal{F}$ of S such that the function $\boldsymbol{h}(x, y) = x$ is a stratified Morse function with respect to $(S, \mathcal{F})$. Then there exists exactly one critical point $p_0 \in S$ of $\boldsymbol{h}$ such that $\boldsymbol{h}(p_0) = x_0$. Let $p_0 = (x_0, y_0)$.

Since x_0 is a jumping point of $\boldsymbol{n}$ we have

$$\boldsymbol{n}(x_0^+) \neq \boldsymbol{n}(x_0) \ \text{ or } \ \boldsymbol{n}(x_0) \neq \boldsymbol{n}(x_0^-).$$

We discuss only the case $\boldsymbol{n}(x_0^-) \neq \boldsymbol{n}(x_0)$ because the other case reduces to this case applied to the region obtained from S via a reflection in the y-axis. For every $\varepsilon > 0$ fix an ε-generic point $x_0'(\varepsilon)$ such that

$$x_0'(\varepsilon) = \begin{cases} x_0, & \text{if } x_0 \in \mathbb{R} \setminus \varepsilon\mathbb{Z} \\ x_0 - \frac{\varepsilon}{2} & \text{if } x_0 \in \varepsilon\mathbb{Z}. \end{cases}$$

We distinguish several cases cases.

Case 1. $\boldsymbol{n}(x_0^-) > \boldsymbol{n}(x_0)$. We can find $\delta > 0$ sufficiently small such that the interval $[x_0 - \delta, x_0]$ will contain no new jumping points of S. The set

$$S_{[x_0 - \delta, x_0]} := \big\{ (x, y) \in S; \ \ x \in [x_0 - \delta, x_0] \,\big\}$$

is disjoint union of elementary regions

$$S(\beta_j, \tau_j), \quad j = 0, \dots, m = \boldsymbol{n}(x_0^-) - 1,$$

"stacked one above the other", i.e.,

$$\beta_0(x) \leq \tau_0(x) < \beta_1(x) \leq \tau_1(x) < \cdots < \beta_m(x) \leq \tau_m(x), \quad \forall x \in (x_0 - \delta, x_0),$$

where β_j, τ_j are are continuous semialgebraic functions. Since $\boldsymbol{n}(x_0) < \boldsymbol{n}(x_0^-)$ we deduce that there exist $j_0, j_1 = 1, \dots, m$ such that $j_0 \leq j_1$ and

$$\beta_{j_1}(x_0) = \tau_{j_0-1}(x_0) \;\text{ and }\; \gamma_j := \beta_j(x_0) - \tau_{j-1}(x_0) > 0, \;\; \forall j \notin [j_0, j_1].$$

In particular, for any $\varepsilon > 0$, the ε-stacks over $x_0'(\varepsilon)$ of

$$S(\beta_{j_1}, \tau_{j_1}), \dots, S(\beta_{j_0}, \tau_{j_0}), S(\beta_{j_0-1}, \tau_{j_0-1})$$

have a point in common.

Now choose ε_1 sufficiently small so that for $j \notin [j_0, j_1]$ and $\varepsilon < \varepsilon_1$, the ε-stack of $S(\beta_j, \tau_j)$ over $x_0'(\varepsilon)$ is disjoint form the ε-stack of $S(\beta_{j-1}, \tau_{j-1})$ over $x_0'(\varepsilon)$. Fix $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$. The above discussion shows that

$$\boldsymbol{n}(x_0) = \boldsymbol{n}_\varepsilon(x_0'(\varepsilon)).$$

Theorem 3.4 now implies that

$$\boldsymbol{n}_\varepsilon(x_0 - \nu_0\varepsilon^{\kappa_0}) = \boldsymbol{n}(x_0 - \nu_0\varepsilon^{\kappa_0}) > \boldsymbol{n}(x_0).$$

This proves that the interval $[x_0 - \nu_0\varepsilon^{\kappa_0}, x_0]$ contains a jumping point of $\boldsymbol{n}_\varepsilon$.

Case 2. $\boldsymbol{n}(x_0^-) < \boldsymbol{n}(x_0)$. The critical point p_0 has the property that there exists a tiny disk D centered at p_0 such that the intersection of D with the open half-plane $\{x < x_0\} \subset \mathbb{R}^2$ is empty. In particular, this shows that p_0 is an isolated point of the set

$$S_{x \leq x_0} = \big\{ (x, y) \in S; \;\; x \leq x_0 \,\big\}.$$

If $\boldsymbol{n}(x_0^-) = 0$, the conclusion is obvious. We assume that $\boldsymbol{n}(x_0^-) > 0$. Choose $\delta > 0$ such that the interval $[x_0 - \delta, x_0)$ contains no jumping point of S. Set

$$R := \boldsymbol{cl}\big(\, S_{[x_0-\delta, x_0]} \setminus \{p_0\}\,\big).$$

Then R is a union of simple regions

$$S(\beta_j, \tau_j), \;\; j = 0, 1 \dots, m = \boldsymbol{n}(x_0^-) - 1,$$

where β_j and τ_j are continuous semialgebraic functions functions such that

$$\beta_0(x) \leq \tau_0(x) < \beta_1(x) \leq \tau_1(x) < \cdots < \beta_m(x) \leq \tau_m(x), \;\; \forall x \in [x_0 - \delta, x_0].$$

We can find $\varepsilon_1 = \varepsilon_1(S)$ such that for any $\varepsilon < \varepsilon_1$ and any ε-generic $x \in [x_0 - \delta, x_0]$ we have:

- $\boldsymbol{n}_{R,\varepsilon}(x) = \boldsymbol{n}_S(x) = m + 1 = \boldsymbol{n}_S(x_0^-)$, and
- the ε-column of $S_{[x_0-\delta, x_0]}$ over x_0 consists of $\boldsymbol{n}(x_0) = m + 2$ stacks.

Theorem 3.4 implies that

$$\boldsymbol{n}_{S,\varepsilon}(x) = \boldsymbol{n}_S(x) = \boldsymbol{n}_S(x_0^-) = m + 1 \;\; \forall x \in [x_0 - \delta, x_0 - \nu_0\varepsilon^{\kappa_0}] \setminus \mathbb{Z}\varepsilon.$$

On the other hand, $\boldsymbol{n}_{S,\varepsilon}(x_0^-) = m + 2$. Thus the interval $[x_0 - \nu_0\varepsilon^{\kappa_0}, x_0]$ must contain a jumping point of $\boldsymbol{n}_{S,\varepsilon}$.

□

Remark 3.7. Theorem 3.4 states that the two functions $\boldsymbol{n}$ and $\boldsymbol{n}_\varepsilon$ coincide at points situated at a distance at least $\nu_0(S)\varepsilon^{\kappa_0 - 1}$ pixels away from the jumping points of $\boldsymbol{n}$. On the other hand, Theorem 3.6 shows that, for a generic semialgebraic set, then within $\nu_0(S)\varepsilon^{\kappa_0 - 1}$ pixels from a jumping point of $\boldsymbol{n}$ there must be jumping points of $\boldsymbol{n}_\varepsilon$. □

Definition 3.8. Let S be a generic semialgebraic set in $\mathbb{R}^2$ and the constants $\varepsilon_0(S)$ and $\varepsilon_1(S)$ as defined in Theorems 3.4 and 3.6. We set

$$\hbar(S) := \min\big(\, \varepsilon_0(S),\ \varepsilon_1(S)\,\big),$$

and we will refer to it as the *critical resolution* of S.

□

4. Approximation of generic semi-algebraic sets

This section is the heart of the paper. Here we will create an algorithm which will approximate a generic semi-algebraic set using only its pixelations, and then prove a very strong convergence result for this approximation. This algorithm is based on the central algorithm of [12], updated to handle the additional complexities of semi-algebraic sets.

We first observe that when narrow vertical strips around the jumping set $\boldsymbol{J}_S$ are removed from a semi-algebraic set S, the remainder is a disjoint union of elementary sets. Corollary 2.10 indicates a good way to approximate continuous semi-algebraic functions, and Theorems 3.4 and 3.6 indicate that for small ε, the jumping points of S become close to the jumping points of $P_\varepsilon(S)$. Therefore a viable approximation technique is to treat parts of $P_\varepsilon(S)$ which occur near jumping points as noise (to be approximated crudely) and to approximate outside of this noise by means of Corollary 2.10.

There are two quantities which must be used in this approximation. The first is the previously mentioned spread function σ which determines the width of line segments to be used in approximating outside of noise. From Corollary 2.10 we know that this spread function should satisfy the following limits:

$$\lim_{\varepsilon\searrow 0} \varepsilon\sigma(\varepsilon) = 0,\ \ \lim_{\varepsilon\searrow 0} \varepsilon\big(\sigma(\varepsilon)\big)^2 = \infty.$$

The second quantity determines the width of the noise, measured in pixels, about jumping points. We will call this quantity ν and refer to it as the *noise width*. It is defined as follows:

Definition 4.1. Let S be a semi-algebraic set and κ_0 be its separation constant. Then a *noise width* ν is a function $\nu : \mathbb{R}^+ \to \mathbb{Z}^+$ which satisfies the following equations:

$$\lim_{\varepsilon\searrow 0} \varepsilon\nu(\varepsilon) = 0, \tag{4.1a}$$

$$\lim_{\varepsilon\searrow 0} \frac{\varepsilon\nu(\varepsilon)}{\varepsilon^{\kappa_0}} = \infty, \tag{4.1b}$$

The first property in this definition ensures that noise is a small phenomenon. The second property implies that $\varepsilon\nu(\varepsilon)$ increases faster than ε^{κ_0} so that the noise will eventually contain all fake cycles (consult Remark 3.7).

For a reasonable approximation we must have an estimate of κ_0. For example if $\kappa_0 = \frac{1}{2}$ we could set $\nu(\varepsilon) = \lceil \varepsilon^{\frac{-2}{3}} \rceil$.

Using these constants together with the techniques developed in the first three sections we create an algorithm to approximate a semi-algebraic set from its ε-pixelations. For ease of discussion we will use the following language in the algorithm:

Definition 4.2. Let S be a semi-algebraic set and $P_\varepsilon(S)$ its ε-pixelation. If $A \subset \mathbb{R}$ then the *part of* P_ε *over* A is the set

$$P_\varepsilon(S) \cap (A \times \mathbb{R})$$

Algorithm 4.3. (1) Choose a spread σ such that $\varepsilon\sigma(\varepsilon)^2 \to \infty$ and $\varepsilon\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$.

(2) Choose a noise width ν such that $\frac{\varepsilon\nu(\varepsilon)}{\varepsilon^{\kappa_0}} \to \infty$ and $\varepsilon\nu(\varepsilon) \to 0$ as $\varepsilon \to 0$.
(3) (a) For each point $p \in \boldsymbol{J}_\varepsilon$ let $\zeta_\varepsilon^-(p)$ be the largest odd multiple of $\frac{\varepsilon}{2}$ such that $p - \zeta_\varepsilon^-(p) > \varepsilon\nu(\varepsilon)$. Similarly let $\zeta_\varepsilon^+(p)$ be the smallest odd multiple of $\frac{\varepsilon}{2}$ such that $p + \zeta_\varepsilon^+(p) < \varepsilon\nu(\varepsilon)$.
  (b) For each $p \in \boldsymbol{J}_\varepsilon$ let $\Delta_\varepsilon(p)$ be the interval $[\zeta_\varepsilon^-(p), \zeta_\varepsilon^+(p)]$.
  (c) Let
$$\Delta_\varepsilon = \bigcup_{p \in \boldsymbol{J}_\varepsilon} \Delta_\varepsilon(p).$$
  The set Δ_ε is called the *noise* set of $P_\varepsilon(S)$ and its connected components are called the *noise* intervals of $P_\varepsilon(S)$.
(4) Define $\mathcal{R}_\varepsilon$ to be the closure of $\mathbb{R} \setminus \Delta_\varepsilon$. We call $\mathcal{R}_\varepsilon$ the *regular* set of $P_\varepsilon(S)$, and its connected components are called *regular* intervals of $P_\varepsilon(S)$.
(5) For each bounded regular interval $I \subset \mathcal{R}_\varepsilon$ and each connected component $\mathcal{C}$ of $P_\varepsilon(S) \cap (I \times \mathbb{R})$, the part of $P_\varepsilon(S)$ over the regular regular intervals I do the following:
  (a) Choose compatible upper and lower profiles Π_ε^+ and Π_ε^- on the component $\mathcal{C}$.
  (b) Choose compatible upper and lower samples Ξ_ε^+ and Ξ_ε^- with spread σ.
  (c) Generate the PL-approximation determined by the above upper and lower samples.
(6) The union of all PL-approximations found in the above step is called the *regular approximation* which we denote by S_ε^{reg}.
(7) For each noise interval $I \subset \Delta_\varepsilon$ denote by $\mathcal{C}_\varepsilon(I)$ the set of connected components of $P_\varepsilon(S)$ over I
  (a) Let $I = [a, b] \subset \Delta_\varepsilon$ be a noise interval of $P_\varepsilon(S)$.
  (b) For every $C \in C_\varepsilon(I)$ we denote by U_C (resp. L_C) the highest (resp. lowest) y-coordinate of the center of a pixel in C.
  (c) Denote by $\mathcal{P}_\varepsilon(C)$ the rectangle determined by the inequalities
$$x \geq a, x \leq b, y \geq L_C, y \leq U_C$$
  (see Figure 7).

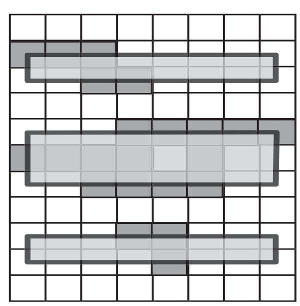

FIGURE 7. *Covering up the noise.*

  (d) The *noise approximation over I*, which we denote by $P_\varepsilon(I)$, is the union
$$P_\varepsilon(I) := \bigcup_{C \in C_\varepsilon(I)} \mathcal{P}_\varepsilon(C)$$
(8) The union of all $N_\varepsilon(I)$ where I are the noise intervals in Δ_ε is called the *noise approximation* which we denote by S_ε^{noise}.
(9) The final approximation S_ε is simply the union of the noise and regular approximations, i.e.
$$S_\varepsilon = S_\varepsilon^{noise} \cup S_\varepsilon^{reg}$$

This final set S_ε will be piecewise linear by construction, and will be a good approximation of the set. □

Here "good approximation" means that it captures both topological and geometric information such as area, perimeter, and curvature measures. The precise notion of "good approximation" relies on the concept of *normal cycle*.

The normal cycle is a correspondence that associates to each compact planar semialgebraic set X a 1-dimensional current $\boldsymbol{N}^X$ on the unit sphere tangent bundle of $\mathbb{R}^2$

$$\boldsymbol{S}(T\mathbb{R}^2) = \big\{ (\boldsymbol{v}, \boldsymbol{p}) \in \mathbb{R}^2;\ \ |\boldsymbol{v}| = 1 \,\big\}.$$

For a precise definition of this object we refer to [1, 6, 9, 10]. Here we will content ourselves with a brief description of its construction.

For a semialgebraic domain $D \subset \mathbb{R}^2$ with C^2-boundary the normal cycle $\boldsymbol{N}^D$ has a simple description. It is the current of integration given by the closed curve $\mathcal{G}_D \subset \boldsymbol{S}(TR^2)$

$$\mathcal{G}_D = \big\{ (\boldsymbol{n}(\boldsymbol{p}), \boldsymbol{p}) \in \boldsymbol{S}(T\mathbb{R}^2);\ \ \boldsymbol{p} \in \partial D \,\big\}$$

where $\boldsymbol{n}(\boldsymbol{p})$ denotes the unit outer normal to ∂D at $\boldsymbol{p} \in \partial D$. Equivalently, $\mathcal{G}_D$ is the graph the the Gauss map

$$\partial D \ni \boldsymbol{p} \mapsto \boldsymbol{n}(\boldsymbol{p}) \in S^1.$$

Clearly, in this case, the normal cycle contains all the curvature information concerning the boundary of D.

More generally, if S is a compact semialgebraic set, the we can find a C^3 semialgebraic function $f : \mathbb{R}^2 \to [0, \infty)$ such that $S = f^{-1}(0)$. For all $\varepsilon > 0$ sufficiently small the region $S_\varepsilon := \{ f \le \varepsilon \}$ is a semialgebraic domain with C^2-boundary so we can define the normal cycle $\boldsymbol{N}^{S_\varepsilon}$ as above. One can show that as $\varepsilon \to 0$ the currents $\boldsymbol{N}^{S_\varepsilon}$ converge weakly to a current which by definition is the normal cycle of S. The hard part is to prove that this current is independent of the choice of defining function f. This current is a current of integration along a finite number of oriented semialgebraic arcs in $\boldsymbol{S}(T\mathbb{R}^2)$. We refer to [9] to a more in depth description of the normal cycle of planar semialgebraic sets. In particular, in [9] one can see how this current captures the various curvature properties of S.

Theorem 4.4. *Let S be a generic compact semi-algebraic subset of the plane. For each ε, let S_ε be the PL set constructed through the above approximation algorithm. Then N^S converges to $\boldsymbol{N}^{S_\varepsilon}$ weakly, where $\boldsymbol{N}^S$ and $\boldsymbol{N}^{S_\varepsilon}$ indicate the normal cycles of S and S_ε respectively.*

Proof. First we note that the approximation converges in the Hausdorff metric to the original set. This is because each vertex of a line segment is taken from a pixel which contains a piece of the boundary of the original set. Since every pixel of the pixelation can be at most $\varepsilon\sqrt{2}$ far from the original set, this forces the approximation into a tube around the original set which becomes arbitrarily small as ε goes to 0.

From here the strategy of the proof will make heavy use of the inclusion-exclusion principle satisfied by the normal cycle correspondence $X \mapsto N^X$. More explicitly, this means that for any compact semialgebraic sets X and Y, we have

$$\boldsymbol{N}^{X \cup Y} = \boldsymbol{N}^X + \boldsymbol{N}^Y - \boldsymbol{N}^{X \cap Y}.$$

We will use this principle to reduce the calculation of the normal cycle of S_ε into calculations on simpler subsets of S.

First, we need to introduce some more notation. We set $S_0 = S$. For each $\varepsilon > 0$ and each $c \in \boldsymbol{J}_{S,\varepsilon}$ we indicate by $\mathcal{I}_\varepsilon(c)$ the ε-noise interval containing c. For each $c \in \boldsymbol{J}_S$ we set $\mathcal{I}_0(c) := \{c\}$. We then define, for each $\varepsilon \geq 0$, the noise strip $\mathcal{N}_\varepsilon(c)$ as

$$\mathcal{N}_\varepsilon(c) := \{(x,y); x \in \mathcal{I}_\varepsilon(c)\}$$

finally we set

$$\mathcal{N}_\varepsilon = \bigcup_{c \in \boldsymbol{J}_\varepsilon} \mathcal{N}_\varepsilon(c)$$

and $\mathcal{R}_\varepsilon$ as $\mathbb{R}^2 \setminus \mathcal{N}_\varepsilon$.

For each $\varepsilon \geq 0$ we construct a graph Γ_ε as follows. The vertex set is the set of connected components of $\mathcal{N}_\varepsilon \cap S_\varepsilon$. The edge set is the set of connected components of $\mathcal{R}_\varepsilon \cap S_\varepsilon$, so that two vertices are connected if and only if there is a component of $\mathcal{R}_\varepsilon \cap S_\varepsilon$ which connects the two appropriate vertices of $\mathcal{N}_\varepsilon \cap S_\varepsilon$. This graph is the Reeb graph of the projection of S_ε onto the x-axis. Furthermore there is a $\delta > 0$ independent of S such that for all $\varepsilon \in (0, \delta]$ the graph Γ_ε is isomorphic to the graph Γ_0. Let

$$\varepsilon_2 := \min\{\hbar(S), \delta\}$$

where $\hbar(S)$ is the critical resolution defined in Definition 3.8. For the remainder of the proof we will deal only with $\varepsilon \in (0, \varepsilon_2]$.

Let $\mathcal{V}_\varepsilon$ be the set of vertices of Γ_ε and $\mathcal{E}_\varepsilon$ be the set of edges of Γ_ε. Note that, by the above equivalence of Reeb graphs, for $\varepsilon \in [0, \varepsilon_2]$, there is a natural bijection between the vertices of Γ_ε with those of Γ_0 and similarly for the edges. For any vertex $\boldsymbol{v}$ of $\mathcal{V}_0$ and $\varepsilon \in [0, \varepsilon_2]$ we indicate by $C_{\boldsymbol{v},\varepsilon}$ the connected component of $\mathcal{N}_\varepsilon \cap S_\varepsilon$ corresponding to the vertex. Similarly, for any edge $\boldsymbol{e}$ of Γ_0 we indicate by $C_{\boldsymbol{e},\varepsilon}$ the *closure* of the connected component of $\mathcal{R}_\varepsilon \cap S_\varepsilon$ corresponding to $\boldsymbol{e}$. We have the following result, (compare [12, Lemma 5.3]).

Lemma 4.5. *For any $\varepsilon \in [0, \varepsilon_2]$ we have*

$$\boldsymbol{N}^{S_{\boldsymbol{e}}} = \sum_{\boldsymbol{v} \in \mathcal{V}_0} \boldsymbol{N}^{C_{\boldsymbol{v},\varepsilon}} + \sum_{\boldsymbol{e} \in \mathcal{E}_0} \boldsymbol{N}^{C_{\boldsymbol{e},\varepsilon}} - \sum_{\boldsymbol{v} \in \mathcal{V}_0} \sum_{\boldsymbol{e} \in \mathcal{E}_0} \boldsymbol{N}^{C_{\boldsymbol{v},\varepsilon} \cap C_{\boldsymbol{e},\varepsilon}}. \tag{4.2}$$

Proof. Note that we have a decomposition

$$S_\varepsilon = \left(\bigcup_{\boldsymbol{v} \in \mathcal{V}_0} C_{\boldsymbol{v},\varepsilon} \right) \cup \left(\bigcup_{\boldsymbol{e} \in \mathcal{E}_0} C_{\boldsymbol{e},\varepsilon} \right). \tag{4.3}$$

We need to discuss separately the cases $\varepsilon > 0$ and $\varepsilon = 0$.

1. Assume that $\varepsilon \in (0, \varepsilon_2]$. In this case we have

$$C_{\boldsymbol{v},\varepsilon} \cap C_{\boldsymbol{v}',\varepsilon} = \emptyset = C_{\boldsymbol{e},\varepsilon} \cap C_{\boldsymbol{e}',\varepsilon}, \ \ \forall \boldsymbol{v} \neq \boldsymbol{v}', \ \ \boldsymbol{e} \neq \boldsymbol{e}'. \tag{4.4}$$

The equality (4.2) now follows from inclusion-exclusion principle applied to the decomposition (4.3) satisfying the overlap conditions (4.4).

2. Assume that $\varepsilon = 0$. In this case the overlap conditions are more complicated. We have

$$C_{\boldsymbol{v},0} \cap C_{\boldsymbol{v}',0} = \emptyset, \ \ \forall \boldsymbol{v} \neq \boldsymbol{v}', \tag{4.5a}$$

$$C_{\boldsymbol{e},0} \cap C_{\boldsymbol{e}',0} = \emptyset \iff \boldsymbol{e} \cap \boldsymbol{e}' = \emptyset, \tag{4.5b}$$

where the condition $\boldsymbol{e} \cap \boldsymbol{e}' = \emptyset$ signifies that the edges $\boldsymbol{e}$ and $\boldsymbol{e}'$ have no vertex in common. Moreover,

$$\bigcap_{\boldsymbol{e} \in A} C_{\boldsymbol{e},0} = C_{\boldsymbol{v},0}, \ \ \forall \boldsymbol{v} \in \mathcal{V}_0, \ \ A \subset E_{\boldsymbol{v}}. \tag{4.6}$$

Using (4.3), (4.5a), (4.5b), (4.6) and the inclusion-exclusion principle we deduce

$$\begin{aligned}\boldsymbol{N}^S &= \sum_{\boldsymbol{v}\in\mathcal{V}_0}\boldsymbol{N}^{C_{\boldsymbol{v},0}} + \sum_{\boldsymbol{e}\in\mathcal{E}_0}\boldsymbol{N}^{C_{\boldsymbol{e},0}} - \sum_{\boldsymbol{v}\in\mathcal{V}_0}\sum_{\boldsymbol{e}\in E_{\boldsymbol{v}}}\boldsymbol{N}^{C_{\boldsymbol{v},0}\cap C_{\boldsymbol{e},0}} \\ &\quad + \sum_{\boldsymbol{v}\in\mathcal{V}_0}\sum_{\emptyset\neq A\subset E_{\boldsymbol{v}}}(-1)^{|A|+1}\boldsymbol{N}^{C_{\boldsymbol{v}}\cap(\bigcap_{\boldsymbol{e}\in A})C_{\boldsymbol{e},0}} + \sum_{\boldsymbol{v}\in\mathcal{V}_0}\sum_{\emptyset\neq A\subset E_{\boldsymbol{v}}}(-1)^{|A|}\boldsymbol{N}^{\bigcap_{\boldsymbol{e}\in A}C_{\boldsymbol{e},0}} \\ &= \sum_{\boldsymbol{v}\in\mathcal{V}_0}\boldsymbol{N}^{C_{\boldsymbol{v},0}} + \sum_{\boldsymbol{e}\in\mathcal{E}_0}\boldsymbol{N}^{C_{\boldsymbol{e},0}} - \sum_{\boldsymbol{v}\in\mathcal{V}_0}\sum_{\boldsymbol{e}\in E_{\boldsymbol{v}}}\boldsymbol{N}^{C_{\boldsymbol{v},0}\cap C_{\boldsymbol{e},0}} \\ &\quad + \sum_{\boldsymbol{v}\in\mathcal{V}_0}\underbrace{\left(\sum_{\emptyset\neq A\subset E_{\boldsymbol{v}}}\big((-1)^{|A|+1}+(-1)^{|A|}\big)\right)}_{=0}\boldsymbol{N}^{C_{\boldsymbol{v},0}}\end{aligned}$$

□

The above lemma means that the proof will follow if the following three equations are satisfied for every edge $\boldsymbol{e}$ and vertex ε in Γ_ε:

$$\lim_{\varepsilon\searrow 0}\boldsymbol{N}^{C_{\boldsymbol{v},\varepsilon}} = N^{C_{\boldsymbol{v},0}} \tag{4.7a}$$

$$\lim_{\varepsilon\searrow 0}\boldsymbol{N}^{C_{\boldsymbol{v},\varepsilon}\cap C_{\boldsymbol{e},\varepsilon}} = \boldsymbol{N}^{C_{\boldsymbol{v},0}\cap C_{\boldsymbol{e},0}} \tag{4.7b}$$

$$\lim_{\varepsilon\searrow 0}\boldsymbol{N}^{C_{\boldsymbol{e},\varepsilon}} = \boldsymbol{N}^{C_{\boldsymbol{e},0}}, \tag{4.7c}$$

where convergence of each limit is meant in the weak sense of currents.

Each of these equations will rely on an approximation result of normal cycles proved by Joseph Fu in [5]. A restricted version of this theorem, which shall suffice for the purposes of this paper, is stated below:

Theorem 4.6 (Approximation Theorem)**.** *Suppose S is a compact semialgebraic subset of the plane and for each $\varepsilon > 0$ we are given a compact semialgebraic subset S_ε of the plane with the following properties.*

(1) *There is a compact set $K \subset \mathbb{R}^2$ which contains each S_ε.*
(2) *There is a $M \in \mathbb{R}$ such that*

$$\operatorname{mass}\big(\boldsymbol{N}^{S_\varepsilon}\big) \leq M, \quad \forall\varepsilon.$$

(3) *For almost every $\xi \in \operatorname{Hom}(\mathbb{R}^2,\mathbb{R})$ and almost every $c \in \mathbb{R}$ we have*

$$\lim_{\varepsilon\searrow 0}\chi(S_\varepsilon\cap\{\xi\geq c\}) = \chi(S\cap\{\xi\geq c\})$$

Then $\boldsymbol{N}^{S_\varepsilon}$ converges to $\boldsymbol{N}^S$ as $\varepsilon\to 0$ weakly and in the flat metric. □

Equations (4.7a) and (4.7b) will both follow from applying this approximation theorem to the case of rectangles. Therefore, the following lemma will be useful:

Lemma 4.7. *Suppose $(S_\varepsilon)_{\varepsilon>0}$ is a family of convex polygons in the plane that converge in the Hausdorff metric to a convex polygon S. Then $\boldsymbol{N}^{S_\varepsilon}$ converges weakly to $\boldsymbol{N}^S$ as $\varepsilon\to 0$.*

Proof. We argue by proving the conditions of Fu's Theorem. Observe first that there exists $R > 0$ such that

$$\operatorname{dist}(S_\varepsilon, S) < R, \;\; \forall \varepsilon$$

and thus the condition (1) of the Approximation Theorem. The computations of [9, Chap. 23] show that mass of the normal cycle of a convex polygon P is equal to $2\pi + L(P)$. From Hadwiger's characterization theorem [7, Thm. 9.1.1] we deduce that

$$\lim_{\varepsilon \to 0} L(S_\varepsilon) = L(S)$$

and thus condition (2) is also satisfied.

Therefore we must show that for almost every $\xi \in \operatorname{Hom}(\mathbb{R}^2, \mathbb{R})$ and almost every $c \in \mathbb{R}$ we have

$$\lim_{\varepsilon \searrow 0} \chi(S_\varepsilon \cap \{\xi \le c\}) = \chi(S \cap \{\xi \le c\})$$

Note that S and each S_ε are all convex subsets of the plane. Therefore any intersection with a half-plane will either be empty or be a contractible set. Therefore to prove the convergence of Euler characteristic on half-planes we need only prove that a half plane H will only intersect S_ε for small ε if and only if it intersects S. This is true since $H \cap S_\varepsilon$ converges in the Hausdorff metric to $H \cap S$. □

Proof of (4.7a) Fix a vertex $\boldsymbol{v} \in \mathcal{V}_0$. The set $C_{\boldsymbol{v},0}$ is a subset of a vertical line over a jumping point, and is so either a point or a line segment. For every $\varepsilon \in [0, \varepsilon_2]$, the set $C_{\boldsymbol{v},\varepsilon}$ is a rectangle which spans a noise interval I_ε and contains $C_{\boldsymbol{v},0}$. The width of I_ε (and so the width of $C_{\boldsymbol{v},\varepsilon}$) is proportional to $\varepsilon\nu(\varepsilon)$, and so vanishes as $\varepsilon \to 0$ (by choice of ν).

The rectangle $C_{\boldsymbol{v},\varepsilon}$ is constructed by choosing the highest and lowest pixels from the component of $I_\varepsilon \cap P_\varepsilon(S)$ containing $C_{\boldsymbol{v},0}$. For sufficiently small ε, the noise interval I_ε will be thin enough so that $C_{\boldsymbol{v},\varepsilon} \cap S$ can be described as a number of regions lying between the graphs of functions which are C^2 everywhere except possibly at the jumping point. This implies that for small ε, the height of $C_{\boldsymbol{v},\varepsilon}$ differs from the height of $C_{\varepsilon,0}$ be an arbitrarily small amount.

Since the height and width of $C_{\boldsymbol{v},\varepsilon}$ converge to the height and width of $C_{\boldsymbol{v},0}$ and since each $C_{\boldsymbol{v},\varepsilon}$ contains $C_{\boldsymbol{v},0}$, the rectangles $C_{bv,\varepsilon}$ converge in the Hausdorff metric to $C_{\boldsymbol{v},0}$. Therefore by Lemma 4.7 $\lim_{\varepsilon \searrow 0} \boldsymbol{N}^{C_{\boldsymbol{v},\varepsilon}} = \boldsymbol{N}^{C_{\boldsymbol{v},0}}$.

Proof of (4.7b) Fix a vertex $\boldsymbol{e} \in \mathcal{E}_0$ and a vertex $\boldsymbol{v} \in \mathcal{V}_0$. If $C_{\boldsymbol{e},0} \cap C_{\boldsymbol{v},0} = \emptyset$, then for sufficiently small ε the component $C_{\boldsymbol{e},\varepsilon}$ will also not intersect $C_{\boldsymbol{v},\varepsilon}$ and so the convergence in normal cycle follows.

If the $C_{\boldsymbol{e},0}$ and $C_{\boldsymbol{v},0}$ do in fact intersect, then note that $C_{\boldsymbol{e},0} \cap C_{\boldsymbol{v},0} = C_{\boldsymbol{v},0}$ (since the vertex is a connected component over a point, and the edges is a connected component over an interval which overlaps that point).

The intersection $C_{\boldsymbol{e},\varepsilon} \cap C_{\boldsymbol{v},\varepsilon}$ is a vertical line segment. In fact it is either the right or left edge of $C_{\boldsymbol{v},\varepsilon}$. However, since $C_{\boldsymbol{v},\varepsilon}$ converges to a vertical line segment $C_{\boldsymbol{e},0}$, it follows that its left right and right edges converge to the same line segment. Therefore (4.7b) follows from (4.7a).

Proof of (4.7c) We again plan to use the Approximation Theorem. Condition (1) of the theorem is plainly satisfied while condition (2) follows from Corollary 2.10 and the explicit description of the mass of the normal cycle of a planar set given in [9, Chap. 23]. All that is left to do is to verify condition (3) of the Approximation Theorem.

The component $C_{\boldsymbol{e},0}$ is an elementary region defined by continuous semialgebraic functions

$$\beta_{\boldsymbol{e}}, \tau_{\boldsymbol{e}} : [a, b] \to \mathbb{R}.$$

More precisely, this means that

$$\beta_{\boldsymbol{e}}(x) \le \tau_{\boldsymbol{e}}(x), \;\; \forall x \in [a, b],$$

and

$$C_{\boldsymbol{e},0} = \big\{ (x, y) \in \mathbb{R}^2; \;\; x \in [a, b], \;\; \beta_{\boldsymbol{e}}(x) \le y \le \tau_{\boldsymbol{e}}(x) \big\}.$$

There exists an integer $n > 0$ and points

$$a = c_0 < c_1 < \cdots < c_n = b$$

such that for any $i = 1, \dots, n$ the restrictions of $\beta_{\boldsymbol{e}}$ and $\tau_{\boldsymbol{e}}$ to (c_{i-1}, c_i) are real analytic. Moreover, since the set S is generic, the derivatives $\beta_{\boldsymbol{e}}'$ and $\tau_{\boldsymbol{e}}'$ are bounded near $c_1, \dots, c_{n-1}$. In particular, the functions $\beta_{\boldsymbol{e}}$ and $\tau_{\boldsymbol{e}}$ are locally Lipschitz on the open interval (a, b). We will refer to the points

$$\big(c_j, \beta_{\boldsymbol{e}}(c_j)\big), \ \ \big(c_j, \tau_{\boldsymbol{e}}(c_j)\big), \ \ j = 0, 1, \dots, n,$$

as the *vertices* of $C_{\boldsymbol{e},0}$. The other points on these graphs are called *regular*. Now fix a constant $c \in \mathbb{R}$ and a linear map $\xi : \mathbb{R}^2 \to \mathbb{R}$, $\xi(x, y) = ux + vy$, with the following generic properties:

$\mathbf{G_1}$. The restriction of ξ to $C_{\boldsymbol{e},0}$ is a stratified Morse function, and $v \neq 0$, i.e., the level sets of ξ are not vertical lines.

$\mathbf{G_2}$. The constant c is not a critical value of $\xi|_{C_{\boldsymbol{e},0}}$.

$\mathbf{G_3}$. The line $L_{\xi,c} := \{\xi = c\}$ does not contain any of the vertices of $C_{\boldsymbol{e},0}$.

We will show that

$$\lim_{\varepsilon \searrow 0} \chi\Big(C_{\boldsymbol{e},\varepsilon} \cap \{\xi \geq c\}\Big) = \chi\Big(C_{\boldsymbol{e},0} \cap \{\xi \geq c\}\Big). \tag{4.8}$$

We denote by $C_{\boldsymbol{e},0}^{\varepsilon}$ the intersection of $C_{\boldsymbol{e},0}$ with the vertical strip

$$a_\varepsilon := a + \nu(\varepsilon)\varepsilon \leq x \leq b - \nu(\varepsilon)\varepsilon =: b_\varepsilon.$$

for all $\varepsilon > 0$ small enough so that the above inequalities make sense. Let us observe that the conditions $\mathbf{G_1}$,$\mathbf{G_2}$ and $\mathbf{G_3}$ imply that for ε sufficiently small we have

$$\chi\Big(C_{\boldsymbol{e},0} \cap \{\xi \geq c\}\Big) = \chi\Big(C_{\boldsymbol{e},0}^{\varepsilon} \cap \{\xi \geq c\}\Big).$$

Thus, to prove (4.8) it suffices to show that

$$\chi\Big(C_{\boldsymbol{e},\varepsilon} \cap \{\xi \geq c\}\Big) = \chi\Big(C_{\boldsymbol{e},0}^{\varepsilon} \cap \{\xi \geq c\}\Big), \ \ \forall \varepsilon \ll 1. \tag{4.9}$$

The region $C_{\boldsymbol{e},\varepsilon}$ is an elementary region defined by the PL functions

$$\beta_{\boldsymbol{e},\varepsilon}, \tau_{\boldsymbol{e},\varepsilon} : [a_\varepsilon, b_\varepsilon] \to \mathbb{R}, \ \ \beta_{\boldsymbol{e},\varepsilon}(x) \leq \tau_{\boldsymbol{e},\varepsilon}(x), \ \ \forall x \in [a_\varepsilon, b_\varepsilon].$$

Here we develop some terminology to handle the intersections of these PL boundary functions. If $f : [s, t] \to \mathbb{R}$ is a piecewise C^2 function, then we say that the line $L_{\xi,c}$ intersects the graph of f transversally at a point $\boldsymbol{p}_0 = (x_0, f(x_0))$ if there exists a $\delta > 0$ such that the function

$$x \mapsto \xi(x, f(x))$$

is differentiable on the set $0 < |x - x_0| \leq \delta$ and it has constant sign on this set. We will denote by $\mathsf{sign}(\boldsymbol{p}_0, f) \in \{\pm 1\}$ this sign. Thus, if $\mathsf{sign}(\boldsymbol{p}_0, f) = 1$, then the curve

$$x \mapsto (x, f(x)), \ \ |x - x_0| \leq \delta,$$

intersects the line $L_{\xi,c}$ at $\boldsymbol{p}_0$ coming from the half-plane $\{\xi < c\}$ and entering the half-plane $\{\xi > c\}$.

For any point $\boldsymbol{p}_0 = (x_0, y_0)$ and any $r > 0$ we denote by $\Sigma_r(\boldsymbol{p})$ the closed square

$$\Sigma_r(\boldsymbol{p}) := \Big\{ (x, y) \in \mathbb{R}^2; \ \ |x - x_0|, |y - y_0| \leq r \Big\}.$$

We need to discuss separately three cases.

Case 1. *The elementary set $C_{\boldsymbol{e},0}$ is nondegenerate*, i.e., $\beta_{\boldsymbol{e}}(x) < \tau_{\boldsymbol{e}}(x)$, $\forall x \in (a, b)$. The conditions $\mathbf{G_1}, \mathbf{G_2}, \mathbf{G_3}$ imply that there exists a compact subinterval $I = [a_*, b_*] \subset (a, b)$ such that the the

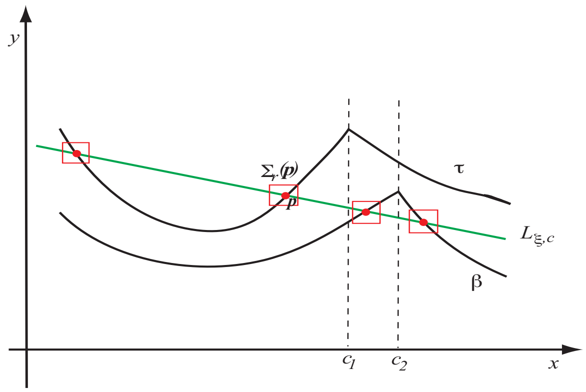

FIGURE 8. *Isolating the intersection points of $L_{\xi,c}$ with the graphs of β and τ so that the squares $\Sigma_r(\boldsymbol{p})$ do not touch any of the vertical lines containing the singular points of these graphs.*

line $L_{\xi,c}$ intersects the graphs of $\beta_{\boldsymbol{e}}$ and $\tau_{\boldsymbol{e}}$ transversally in *regular* points on these graphs whose x-coordinates are contained in the interval I. Denote by $\boldsymbol{I}^0_\beta$ (resp. $\mathcal{I}^0_\tau$) the intersection of $L_{\xi,c}$ with the graph of $\beta_{\boldsymbol{e}}$ (resp. $\tau_{\boldsymbol{e}}$.) The superscript of 0 in $\boldsymbol{I}^0_\beta$ comes our convention that $S = S_0$. Finally we set

$$\boldsymbol{I}^0 := \boldsymbol{I}^0_\beta \cup \boldsymbol{I}^0_\tau$$

Fix a small positive real number r with the following properties (see Figure 8).

- The closed squares $\Sigma_r(\boldsymbol{p})$, $\boldsymbol{p} \in \boldsymbol{I}^0$ are pairwise disjoint.
- For each point $\boldsymbol{p} \in \boldsymbol{I}^0$, there exists $i = 0, 1, \dots, n$ such that the projection of $\Sigma_r(\boldsymbol{p})$ onto the x-axis is contained in a compact sub-interval $J_{\boldsymbol{p}} \subset (c_{i-1}, c_i)$.

We set

$$\Sigma_r(\boldsymbol{I}^0_\beta) := \bigcup_{\boldsymbol{p}\in \boldsymbol{I}_\beta} \Sigma_r(\boldsymbol{p}),\;\; \Sigma_r(\boldsymbol{I}^0_\tau) := \bigcup_{\boldsymbol{p}\in \boldsymbol{I}_\tau} \Sigma_r(\boldsymbol{p}).$$

Lemma 4.8. *(a) Denote by $\boldsymbol{I}^\varepsilon_\beta$ the intersection of $L_{\xi,c}$ with the graph of $\beta_{\boldsymbol{e},\varepsilon}$. There exists $\varepsilon_\beta > 0$ with the following properties.*

(a1) *For any $\varepsilon \leq \varepsilon_\beta$ we have*

$$\boldsymbol{I}^\varepsilon_\beta \subset \Sigma_r(\boldsymbol{I}_\beta),$$

(a2) *For any $\boldsymbol{p} \in \boldsymbol{I}^0_\beta$ and any $\varepsilon \leq \varepsilon_\beta$ the line $L_{\xi,c}$ intersects the portion of the graph of $\beta_{\boldsymbol{e},\varepsilon}$ inside $D_r(\boldsymbol{p})$ in a unique point $\boldsymbol{p}(\varepsilon)$. This intersection is transversal and*

$$\mathsf{sign}(\boldsymbol{p}, \beta_{\boldsymbol{e},0}) = \mathsf{sign}(\boldsymbol{p}(\varepsilon), \beta_{\boldsymbol{e},\varepsilon}). \tag{4.10}$$

(b) Similar statements are true with the bottom functions $\beta_{\boldsymbol{e},\varepsilon}$ replaced with the top functions $\tau_{\boldsymbol{e},\varepsilon}$. □

We defer the proof of this result to the end of this section.

Set $\boldsymbol{I}^\varepsilon = \boldsymbol{I}^\varepsilon_\beta \cup \boldsymbol{I}^\varepsilon_\tau$. We will refer to the intersection of $L_{\xi,c}$ with $\partial C^\varepsilon_{\boldsymbol{e},0}$ as the 0-crossing set and, for $\varepsilon > 0$, we we will refer to the intersection of $L_{\xi,c}$ with $\partial C_{\boldsymbol{e},\varepsilon}$ as the ε-crossing set. For $\varepsilon \geq 0$ we denote by $\dagger_\varepsilon$ the ε-crossing set.

Observe that the sets $\boldsymbol{I}^\varepsilon$ is contained in $\dagger_\varepsilon$ but the ε-crossing set may contain additional points, namely, the intersection of $L_{\xi,c}$ with the vertical lines $x = a_\varepsilon, b_\varepsilon$. The Hausdorff distance between the

$\dagger_\varepsilon$ and $\dagger_0$ goes to zero as $\varepsilon \searrow 0$. Moreover, Lemma 4.8 implies that for any ε sufficiently small there exists a bijection

$$\Psi_\varepsilon : \dagger_0 \to \dagger_\varepsilon$$

defined by

$$\Psi_\varepsilon(\boldsymbol{p}) = \dagger_\varepsilon \cap \Sigma_r(\boldsymbol{p}).$$

For $\varepsilon > 0$ we denote by $C^+_{\boldsymbol{e},\varepsilon}$ the intersection of $C_{\boldsymbol{e},\varepsilon}$ with the half-plane $\{\xi \geq c\}$. Similarly, we define C^+_0 to be the intersection of $C^\varepsilon_{\boldsymbol{e},0}$ with the same half-plane. We have to prove that

$$\chi(C^+_\varepsilon) = \chi(C^+_0), \;\; \forall \varepsilon \ll 1. \tag{4.11}$$

For $\varepsilon \geq 0$ the connected components of $C^+_{\boldsymbol{e},\varepsilon}$ are all homeomorphic to closed 2-dimensional disks so that the Euler characteristic of $C^+_{\boldsymbol{e},\varepsilon}$ is equal to the number of boundary components of ∂C^+_ε.

Observe that if $\dagger_0 = \emptyset$, then $\dagger_\varepsilon = \emptyset$ for all ε sufficiently small. In this case $C^+_{\boldsymbol{e},\varepsilon}$ is homeomorphic to a closed disk for all ε sufficiently small and (4.11) is obviously true. We need to investigate the case $\dagger_0 \neq \emptyset$.

For $\varepsilon > 0$ we define an equivalence relation $\sim_\varepsilon$ on $\dagger_\varepsilon$ by declaring $\boldsymbol{p} \sim_\varepsilon \boldsymbol{q}$ if and only if $\boldsymbol{p}$ and $\boldsymbol{q}$ belong to the same component of ∂C^+_ε. Similarly, we define an equivalence relation $\sim_0$ on $\dagger_0$ by declaring $\boldsymbol{p} \sim_0 \boldsymbol{q}$ if and only $\boldsymbol{p}$ and $\boldsymbol{q}$ belong to the same connected component of ∂C^+_0. Thus, for $\varepsilon \geq 0$ the number of connected components of ∂C^+_ε is equal to the number of equivalence classes of $\sim_\varepsilon$. To prove the equality (4.11) it suffices to show that

$$\boldsymbol{p} \sim_0 \boldsymbol{q} \Rightarrow \Psi_\varepsilon(\boldsymbol{p}) \sim_\varepsilon \Psi_\varepsilon(\boldsymbol{q}). \tag{4.12}$$

Indeed, if (4.12) holds, then we deduce that the number of equivalence classes of $\sim_\varepsilon$ is not larger than the number of equivalence classes of $\sim_0$. Since the number of connected components of C^+_0 is independent of ε if ε is small and

$$\operatorname{dist}(C^+_\varepsilon, C^+_0) \to 0 \;\text{ as }\; \varepsilon \searrow 0$$

we deduce that $C^+_{\boldsymbol{e},\varepsilon}$ has at least as many components as C^+_0.

Fix a component R of C^+_0 and $\boldsymbol{p}, \boldsymbol{q} \in \partial R$. We denote by $[\boldsymbol{p}, \boldsymbol{q}]_R$ the arc of ∂R obtained by traveling counterclockwise from $\boldsymbol{p}$ to $\boldsymbol{q}$. Along this arc there could be another crossing points $\boldsymbol{p}_0 = \boldsymbol{p}, \dots, \boldsymbol{p}_k = \boldsymbol{q} \in \dagger_0$, arrange in counterclockwise order. We set

$$\boldsymbol{p}^\varepsilon_j := \Psi_\varepsilon(\boldsymbol{p}_j).$$

Each of the arcs $[\boldsymbol{p}_{j-1}, \boldsymbol{p}_j]_R$ can is of one of the following two types

I. A line segment contained in $L_{\xi,c}$.
II. A sub-arc of $\partial C^\varepsilon_{\boldsymbol{e},0}$ that intersects $L_{\xi,c}$ only at its endpoints.

If $[\boldsymbol{p}_{j-1}, \boldsymbol{p}_j]_R$ is of type I so that it is contained in $L_{\xi,c}$, then the points $\boldsymbol{p}^\varepsilon_{j-1}$ and $\boldsymbol{p}^\varepsilon_j$ are also contained in $L_{\xi,c}$ and we denote by $[\boldsymbol{p}^\varepsilon_{j-1}, \boldsymbol{p}^\varepsilon_j]_R$ the oriented line segment going from $\boldsymbol{p}^\varepsilon_{j-1}$ to $\boldsymbol{p}^\varepsilon_j$. Clearly $p^\varepsilon_{ji1} \sim \boldsymbol{p}^\varepsilon_j$.

Suppose now that $[\boldsymbol{p}_{j-1}, \boldsymbol{p}_j]_R$ is of type II. The points $\boldsymbol{p}^\varepsilon_{j-1}$ and $\boldsymbol{p}^\varepsilon_j$ divide the boundary $\partial C_{\boldsymbol{e},\varepsilon}$ into two arcs, one of which approaches $[\boldsymbol{p}_{j-1}, \boldsymbol{p}_j]_R$ in the Hausdorff distance as $\varepsilon \to 0$. We denote this arc by $[\boldsymbol{p}^\varepsilon_{j-1}, \boldsymbol{p}^\varepsilon_j]_R$. Lemma 4.8 implies that the arc $[\boldsymbol{p}^\varepsilon_{j-1}, \boldsymbol{p}^\varepsilon_j]_R$ intersects $L_{\xi,c}$ only at its end points if ε is sufficiently small. For such ε's the arc $[\boldsymbol{p}^\varepsilon_{j-1}, \boldsymbol{p}^\varepsilon_j]_R$ lies on the same side of $L_{\xi,c}$ as $[\boldsymbol{p}_{j-1}, \boldsymbol{p}_j]_R$ so that $p^\varepsilon_{ji1} \sim \boldsymbol{p}^\varepsilon_j$. By transitivity we now deduce that

$$\Psi_\varepsilon(\boldsymbol{p}) = \boldsymbol{p}^\varepsilon_0 \sim_\varepsilon \boldsymbol{p}^\varepsilon_k = \Psi_\varepsilon(\boldsymbol{q}).$$

This proves (4.12) and thus proves (4.9) in the case when the elementary set $C_{\boldsymbol{e},0}$ is nondegenerate.

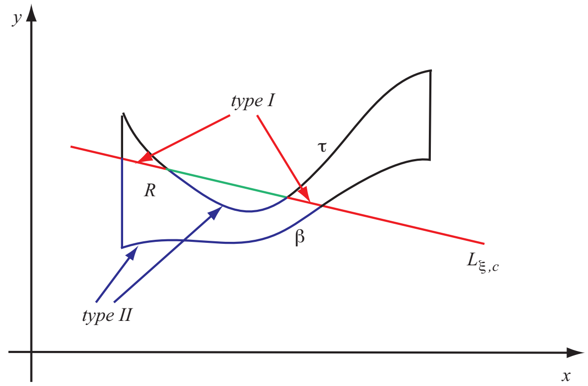

FIGURE 9. *An elementary region $S(\beta,\tau)$ cut by a line $L_{\xi,c}$. The intersection of this region with the lower half-plane determined by $L_{\xi,c}$ has one component R whose boundary is decomposed in arcs of two types.*

Case 2. *The elementary set $C_{\boldsymbol{e},0}$ is degenerate*, i.e., $\beta_{\boldsymbol{e},0} = \tau_{\boldsymbol{e},0}$. We denote by $\boldsymbol{J}^0$ the set consisting of the endpoints of the graph of $\beta_{\boldsymbol{e},0}$ and the intersection of this graph with $L_{\xi,c}$. Similarly, denote by $\boldsymbol{J}^\varepsilon_\beta$ (resp. $\boldsymbol{J}^\varepsilon_\tau$) the set consisting of the endpoints of the graph of $\beta_{\boldsymbol{e},\varepsilon}$ (resp. $\tau_{\boldsymbol{e},\varepsilon}$) and the intersection of this graph with the line $L_{\xi,c}$. As in **Case 1** we can invoke Lemma 4.8 to obtain bijections

$$\Psi^\beta_\varepsilon : \boldsymbol{J}^0 \to \boldsymbol{J}^\varepsilon_\beta, \quad \Psi^\tau_\varepsilon : \boldsymbol{J}^0_+ \to \boldsymbol{J}^\varepsilon_\tau.$$

We continue to use the notations C_0^+ and $C^+_{\boldsymbol{e},\varepsilon}$ introduced in the proof of **Case 1**. In this case C_0^+ is a finite union of subarcs of the graph of $\beta_{\boldsymbol{e},0}$. Let these arcs be $A_1, \dots, A_k$. Each of these arcs carry a natural orientation. Denote by $\boldsymbol{p}_j$ the initial point of A_j and by $\boldsymbol{q}_j$ the final point of A_j.

We set

$$\boldsymbol{p}^\beta_j(\varepsilon) := \Psi^\beta_\varepsilon(\boldsymbol{p}_j), \quad \boldsymbol{p}^\tau_j(\varepsilon) := \Psi^\tau_\varepsilon(\boldsymbol{p}_j).$$

We define $\boldsymbol{q}^\beta_j(\varepsilon)$ and $\boldsymbol{q}^\tau_j(\varepsilon)$ in a similar fashion. Consider the simple closed curve A^ε_j which is the union of the following four arcs; see Figure 10

- The line segment from $\boldsymbol{p}^\tau_j(\varepsilon)$ to $\boldsymbol{p}^\beta_j(\varepsilon)$.
- The arc of $\beta_{\boldsymbol{e},\varepsilon}$ from $\boldsymbol{p}^\beta_j(\varepsilon)$ to $\boldsymbol{q}^\beta_j(\varepsilon)$.
- the line segment from $\boldsymbol{q}^\beta_j(\varepsilon)$ to $\boldsymbol{q}^\tau_j(\varepsilon)$.
- The arc of $\tau_{\boldsymbol{e},\varepsilon}$ from $\boldsymbol{q}^\tau_j(\varepsilon)$ to $\boldsymbol{p}^\tau_j(\varepsilon)$.

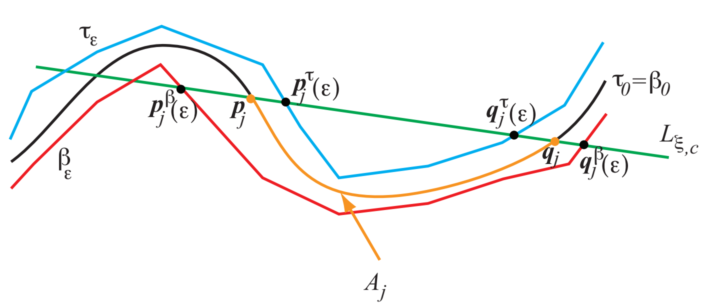

FIGURE 10. *The arc A_j and the simple closed curve A^ε_j, $\boldsymbol{p}^\tau_j(\varepsilon) \to \boldsymbol{p}^\beta_j(\varepsilon) \to \boldsymbol{q}^\beta_j(\varepsilon) \to \boldsymbol{q}^\tau_j(\varepsilon) \to \boldsymbol{p}^\tau_j(\varepsilon)$.*

Lemma 4.8 implies that for ε sufficiently small the closed curve A_j^ε is contained entirely in the half-plane $\{\xi \geq c\}$ so the bounded region it surrounds is contained in this half-plane as well. The region $C^+_{\boldsymbol{e},\varepsilon}$ consists precisely of the regions surrounded by the closed curves A_j^ε, $j = 1, \dots, k$ so that $\chi(C_0^+) = \chi(C_\varepsilon^+) = k$ for all ε sufficiently small. This proves (4.9) in **Case 2**.

Case 3. *$C_{\boldsymbol{e},0}$ is a mixed elementary set.* It has a *minimal* good partition

$$a = t_0 < t_1 < \cdots < t_n = b, \;\; n \geq 2,$$

where for each $j = 1, \dots, n$ the intersection of $C_{\boldsymbol{e},0}$ with the strip $[t_{j-1}, t_j] \times \mathbb{R}$ is either degenerate or nondegenerate. The intersection of the graphs of β and τ with each of the vertical lines $x = t_j$, $j = 0, \dots, n$, is a singular point of $C_{\boldsymbol{e},0}$; see Figure 11. Since the c is not a critical value of the restriction of ξ to $C_{\boldsymbol{e},0}$, we denote that the line $L_{\xi,c}$ does not contain any of these singular points.

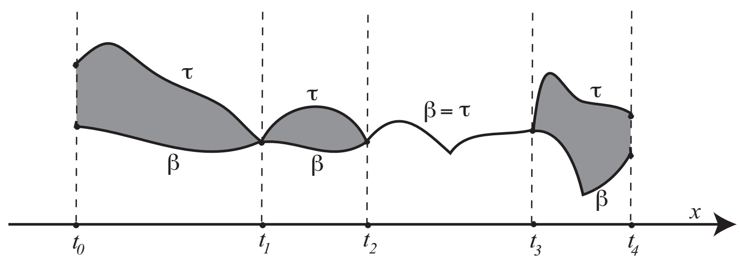

FIGURE 11. *The minimal good partition of a mixed elementary sets.*

Set $H^+_{\xi,c} := \{\xi \geq c\}$. For $j = 1, \dots, n$ and $\varepsilon > 0$ we set

$$R_i := \big([t_{j-1}, t_j] \times \mathbb{R}\big) \cap C^\varepsilon_{\boldsymbol{e},0}, \;\; R_{i,\varepsilon} := \big([t_{j-1}, t_j] \times \mathbb{R}\big) \cap C_{\boldsymbol{e},\varepsilon}.$$

For $k = 1, \dots, n-1$ and $\varepsilon > 0$ we set

$$V_k := \{x = t_k\} \cap C^\varepsilon_{\boldsymbol{e},0}, \;\; V_{k,\varepsilon} := \{x = t_k\} \cap C_{\boldsymbol{e},\varepsilon}.$$

To prove (4.9) it suffices to show that

$$\chi(R_j \cap H^+_{\xi,c}) = \chi(R_{j,\varepsilon} \cap H^+_{\xi,c}), \;\; \forall \varepsilon \ll 1, \;\; j = 1, \dots, n, \tag{4.13a}$$

$$\chi(V_k \cap H^+_{\xi,c}) = \chi(V_{k,\varepsilon} \cap H^+_{\xi,c}), \;\; \forall \varepsilon \ll 1, \;\; k = 1, \dots, n-1, \tag{4.13b}$$

The equalities (4.13a) follow from the Cases 1 and 2 investigated above. The equalities (4.13b) are consequences of the following simple facts.

- For any $k = 1, \dots, n-1$, the set V_k consists of a single point that does not line on the line $L_{\xi,c}$.
- For any $k = 1, \dots, n-1$, and $\varepsilon > 0$ the set $V_{k,\varepsilon}$ consists of a single vertical line segment.
- For any $k = 1, \dots, n-1$, the set $V_{k,\varepsilon}$ converges in the Hausdorff metric to the set V_k. In particular, for $\varepsilon \ll 1$ we have

$$V_k \subset H^+_{\xi,c} \Longleftrightarrow V_{k,\varepsilon} \subset H^+_{\sigma,c}.$$

This completes the proof of Theorem 4.4.

□

Proof of Lemma 4.8. The inclusion (a1) follows from the fact that the distance between the graph of $\beta_{\boldsymbol{e},\varepsilon}$ and the graph of $\beta_{\boldsymbol{e},0}$ approaches zero as $\varepsilon \to 0$. To prove (a2) let us denote by (x_0, y_0) the coordinates of $\boldsymbol{p}$. From the choice of r we deduce that for ε sufficiently small the interval

$$J_\varepsilon := [x_0 - r - \sigma(\varepsilon)\varepsilon, x_0 + r + \sigma(\varepsilon)\varepsilon]$$

is contained entirely in an interval of the form (c_{j-1}, c_j) for some $j = 1, \dots, n$ (where c_j were defined much earlier in the proof to be the x-coordinates such that either β_ε or τ_ε fail to be real analytic) so that $\beta_{\boldsymbol{e},0}$ is C^2 on J_ε. We set

$$K_1 = \sup_{x\in J_\varepsilon} |\beta'_{\boldsymbol{e},0}(x)|, \quad K_2 = \sup_{x\in J_\varepsilon} |\beta''_{\boldsymbol{e},0}(x)|.$$

We denote by m_ξ the slope of $L_{\xi,c}$ and by m_0 the slope of the tangent to the graph of $\beta_{\boldsymbol{e},0}$ at $\boldsymbol{p}$,

$$m_0 = \beta'_{\boldsymbol{e},0}(x_0).$$

Because $L_{\xi,c}$ intersects the graph of $\beta_{\boldsymbol{e},0}$ transversally at $\boldsymbol{p}$ we deduce $m_0 \neq m_\xi$. We deduce that for every $x \in J_\varepsilon$ we have

$$|\beta'_{\boldsymbol{e},0}(x) - m_0| \leq K_2|x - x_0|. \tag{4.14}$$

The function $\beta_{\boldsymbol{e},\varepsilon}$ is piecewise linear. Consider the portion of this graph

$$\langle \boldsymbol{p}_0^\varepsilon, \dots, \boldsymbol{p}_{\ell(\varepsilon)}^\varepsilon \rangle$$

with the property that $\boldsymbol{p}_0^\varepsilon, \dots, \boldsymbol{p}_{\ell(\varepsilon)-1}^\varepsilon$ are successive vertices on the graph of $\beta_{\boldsymbol{e},\varepsilon}$ such that

$$\langle \boldsymbol{p}_1^\varepsilon, \dots, \boldsymbol{p}_{\ell(\varepsilon)}^\varepsilon \rangle \subset \Sigma_r(\boldsymbol{p}), \quad \boldsymbol{p}_0^\varepsilon, \boldsymbol{p}_{\ell(\varepsilon)} \notin \Sigma_r(\boldsymbol{p}).$$

Denote by $(x_j^\varepsilon, y_j^\varepsilon)$ the coordinates of $\boldsymbol{p}_j^\varepsilon$, $j = 0, \dots, \ell(\varepsilon)$, and set $z_j^\varepsilon = \beta_{\boldsymbol{e},0}(x_j^\varepsilon)$. Observe that

$$|y_j^\varepsilon - z_j^\varepsilon| \leq (K_1 + 4)\varepsilon, \quad \forall j = 0, \dots, \ell(\varepsilon).$$

We deduce that

$$m_j^\varepsilon := \frac{y_j^\varepsilon - y_{j-1}^\varepsilon}{x_j^\varepsilon - x_{j-1}^\varepsilon} = \frac{z_j^\varepsilon - z_{j-1}^\varepsilon}{x_j^\varepsilon - x_{j-1}^\varepsilon} + O\left(\frac{1}{\sigma(\varepsilon)}\right),$$

where the constant implied by the O-symbol is independent of ε. The mean value theorem implies that the difference quotient in the right-hand side of the above equality is equal to the derivative of $\beta_{\boldsymbol{e},0}$ ate a point $\eta_j^\varepsilon \in (x_{j-1}^\varepsilon, x_j^\varepsilon)$. Using (4.14) we deduce that

$$\left|m_j^\varepsilon - m_0\right| = O\left(\frac{1}{\sigma(\varepsilon)} + |x_{j-1} - x_0| + |x_j^\varepsilon - x_0|\right). \tag{4.15}$$

Since $\sigma(\varepsilon) \to \infty$ we deduce that given

$$\gamma < \min\left\{r, \frac{1}{4}|m_\xi - m_0|\right\}$$

there exist constants $\delta = \delta(\gamma) > 0$ and $\varepsilon(\gamma) > 0$ such that, for any $\varepsilon < \varepsilon(\gamma)$ the segments of the graph of $\beta_{\boldsymbol{e},\varepsilon}$ situated in the strip $|x - x_0| \leq \gamma$ have slopes m_j^ε located in the range $(m_0 - \gamma, m_0 + \gamma)$. In particular, none of these slopes can be equal to m_ξ, and they are all situated on the same side of m_ξ as m_0.

If we fix γ as above we can find $\varepsilon_1(\gamma) > 0$ such that, for $\varepsilon < \varepsilon_1(\gamma)$ all the intersection points of $L_{\xi,c}$ with the graph of $\beta_{\boldsymbol{e},\varepsilon}$ located in $\Sigma_r(\boldsymbol{p})$ are in fact located in the narrow strip $|x - x_0| < \gamma$. The above discussion then shows that all these intersections must be transversal and they all have the same sign, $\mathsf{sign}(\boldsymbol{p}, \beta_{\boldsymbol{e},0})$. Denote by $N_\varepsilon(\gamma)$ the number of such intersections. Set

$$P_0^\pm = \big(x_0 \pm \gamma, \beta_{\boldsymbol{e},0}(x_0 \pm \gamma)\big), \quad P_\varepsilon^\pm = \big(x_0 \pm \gamma, \beta_{\boldsymbol{e},\varepsilon}(x_0 \pm \gamma)\big).$$

Consider now the closed curve C_γ^ε obtained as follows.

- Travel from the point P_ε^- to P_ε^+ along the graph of $\beta_{\boldsymbol{e},\varepsilon}$.

- Next, travel on the vertical segment connecting P_ε^+ to P_0^+.
- Travel along the graph of $\beta_{\boldsymbol{e},0}$ from P_0^+ to P_0^-.
- Finally, travel along the vertical segment connecting P_0^- to P_ε^-.

The above discussion shows that the intersection number between the line $L_{\xi,c}$ and the curve C_γ^ε is $\pm\big(N_\varepsilon(\gamma)-1\big)$. On the other hand, since this curve is homologous to a circle, we have that the intersection number $L_{\xi,c}$ and C_γ^ε is 0. Therefore we conclude that $N_\varepsilon(\gamma)=1$, which completes the lemma in the case of the function $\beta_{\boldsymbol{e},\varepsilon}$.

The above proof can be repeteated replacing $\beta_{\boldsymbol{e},\varepsilon}$ with $\tau_{\boldsymbol{e},\varepsilon}$ for the upper boundary case. □

APPENDIX A. SEMIALGEBRAIC GEOMETRY

A set $X\subset\mathbb{R}^n$ is called *semialgebraic* if it can be written as a finite union

$$X=X_1\cup\cdots\cup X_N,$$

where each of the sets X_i is described by a finite system of polynomial inequalities.

A map $F:X_0\to X_1$ between two semialgebraic sets $X_i\in\mathbb{R}^{n_i}$, $i=0,1$, is called *semialgebraic* if its graph Γ_F is a semialgebraic subset of $\mathbb{R}^{n_0+n_1}$.

Here is a list of basic properties of semialgebraic sets and functions. For proofs and more details we refer to [2, 3, 13].

- The union, the intersection and the Cartesian product of two semialgebraic sets are semialgebraic.
- If X,Y are semialgebraic subsets of $\mathbb{R}^n$ then so is their difference.
- A subset of $\mathbb{R}$ is semialgebraic if and only if it is a finite union of open interval and points.
- (*Tarski-Seidenberg*) The image and preimage of a semialgebraic set via a semialgebraic map are semialgebraic sets.
- If I is an interval of the real axis and $f:I\to\mathbb{R}$ is semialgebraic, then there exists a finite subset $F\subset I$ such that the restriction of F to any component of $I\setminus F$ is monotone and real analytic.
- (*Curve selection*) If X is a semialgebraic subset of $\mathbb{R}^n$ and $x_0\in\boldsymbol{cl}(X)\setminus X$, then there exists a continuous semialgebraic map $\gamma:(0,1)\to X$ such that
$$\lim_{t\searrow 0}\gamma(t)=x_0.$$
- (*Łojasewicz' inequality*) Suppose that X is a compact semialgebraic set and $f,g:X\to\mathbb{R}$ are continuous semialgebraic functions such that
$$\{f=0\}\subset\{g=0\}.$$
Then there exists a positive integer N and a positive real number C such that
$$|g(x)|^N\leq C|f(x)|,\ \ \forall x\in X.$$
- Suppose that X is a compact semialgebraic set and $f:X\to\mathbb{R}$ is a continuous semialgebraic function. Then the function $\mathbb{R}\to\mathbb{Z}$ that associates to each $t\in\mathbb{R}$ the Euler characteristic of the level set $\{f=t\}$ is a semialgebraic function.
- A semialgebraic set is connected if and only if it is path connected.
- A semialgebraic set has finitely many connected components and each of them is also a semialgebraic set.

Proof of Proposition 2.8 We prove only the statement about the total curvature. The statement about the perimeter follows the same pattern and has fewer complications. First some terminology.

A continuous function $f : [a, b] \to \mathbb{R}$ is said to be piecewise C^2 if there exists a finite set

$$S = \{ a = s_0 < s_1 < \cdots < s_\ell = b \},$$

such that for any $j = 1, \dots, \ell$, and any $k = 1, 2$ the restriction of f to the open interval (s_{j-1}, s_j) is a C^2 function and the limits

$$\lim_{x \searrow s_{j-1}} f^{(k)}(x), \quad \lim_{x \nearrow s_j} f^{(k)}(x)$$

exist and are finite.

We say that the arc $\boldsymbol{C}$ is *convenient* if there exists a piecewise C^2-function $f : [a, b] \to \mathbb{R}$ such that either $\boldsymbol{C}$ is a graph of the function

$$y = f(x)$$

or $\boldsymbol{C}$ is the graph of the function $x = f(y)$. When $\boldsymbol{C}$ is convenient, Proposition 2.8 is a special case of [12, Prop. 3.6].

To deal with the general case let us observe that since $\boldsymbol{C}$ is semialgebraic there exists an ordered sampling of $\boldsymbol{C}$

$$\mathcal{Q} = \{ Q_0, \dots, Q_N \}$$

with the following properties.

(a) The arc $\boldsymbol{C}$ starts at Q_0 and ends at Q_N.
(b) The arc $\boldsymbol{C}$ is smooth at each of the points $Q_1, \dots, Q_{N-1}$.
(c) For any $j = 1, \dots, N$, the portion of $\boldsymbol{C}$ between Q_{j-1} and Q_j is convenient. We denote by $\boldsymbol{C}_j$ this portion.

Denote by $\mathcal{P}^j_\varepsilon$ the ordered sampling of $\boldsymbol{C}_j$ determined by the points in $\mathcal{P}_\varepsilon$ contained in $\boldsymbol{C}_j$. We denote by K^j_ε the total curvature of the PL-curve $\boldsymbol{C}(\mathcal{P}^j)$. Since each of the curves $\boldsymbol{C}_j$ is convenient we have

$$\lim_{\varepsilon \searrow 0} K^j_\varepsilon = K(\boldsymbol{C}_j), \quad \forall j = 1, \dots, N,$$

so that

$$\lim_{\varepsilon \searrow 0} \sum_{j=1}^N K^j_\varepsilon = K(\boldsymbol{C}).$$

On the other hand, since $\boldsymbol{C}$ is smooth at the points $Q_1, \dots, Q_{N-1}$ we deduce that

$$\lim_{\varepsilon \searrow 0} \left(K_\varepsilon - \sum_{j=1}^N K^j_\varepsilon \right) = 0.$$

□

Appendix B. The approximation algorithm

In this section we give a more formal description of the approximation algorithm. We convert $P_\varepsilon(S)$ into an $m \times m$ matrix A (where m depends on ε) of 1's and 0's, where $A[i, j] = 1$ if and only if the pixel of center $c_{i,j}(\varepsilon)$ touches S.

Given this matrix A we will generate a PL set S_ε which approximates the original set S. We will assume that ε is fixed throughout the description of the algorithm.

The algorithm depends on two parameters, both positive integers: the spread σ and the noise width ν. These should be chosen using functions $\sigma(\varepsilon)$ and $\nu(\varepsilon)$ which have the properties:

$$\lim_{\varepsilon \searrow 0} \varepsilon \sigma(\varepsilon) = 0, \lim_{\varepsilon \searrow 0} \varepsilon (\sigma(\varepsilon))^2 = \infty$$

$$\lim_{\varepsilon \searrow 0} \varepsilon \nu(\varepsilon) = 0, \lim_{\varepsilon \searrow 0} \nu(\varepsilon) \varepsilon^{1-\kappa_0} = \infty$$

where κ_0 is a constant dependent on S introduced in Theorem 3.4. However, for purpose of performing the algorithm it is only relevant that we have chosen these two integers σ and ν.

The algorithm uses several smaller subroutines. The first subroutine stack obtains information about the various columns of A which will be used to determine both the noise intervals as well as to select the vertices of S_ε. The input of stack is a list

$$C = c_1, \dots c_m, \;\; c_i = 0, 1,$$

where C is one of the columns of A. The output of stack is a list of nonnegative integers

$$\boldsymbol{n}(C); \;\; b_1 \leq t_1 < b_2 \leq t_2 < \cdots < b_{\boldsymbol{n}(C)} \leq t_{\boldsymbol{n}(C)},$$

where $\boldsymbol{n}(C)$ is the number of stacks in the column encoded by C, and the location of the bottom and top pixel in the j-th stack is determined by the integers b_j, t_j. More formally

$$c_k = 1 \Longleftrightarrow \exists 1 \leq j \leq \boldsymbol{n}(C): \;\; b_j \leq k \leq t_j.$$

If $C = C_i$, the i-th column of A, i.e.,

$$C_i = a_{i,1}, \dots, a_{i,m},$$

then we will denote the output $\mathsf{stack}(C_i)$ by

$$\boldsymbol{n}_i, \;\; b_{i,1} \leq t_{i,1} < \cdots < b_{i,\boldsymbol{n}_i} \leq t_{i,\boldsymbol{n}_i}.$$

A number $1 \leq i \leq m-1$ is called a *jump point* if

$$\boldsymbol{n}_i \neq \boldsymbol{n}_{i+1}.$$

The next subroutine is called jump. Its input is an integer $k \in [1, m)$ and the output is an integer $j_k = \mathsf{jump}(k)$ where j_k is the next jump point, i.e.,

$$\big\{ i \in [k, m) \cap \mathbb{Z}; \;\; i \text{ is a jump point} \big\} = \emptyset,$$

then we set

$$\mathsf{jump}(k) := m + 1.$$

Otherwise

$$\mathsf{jump}(k) = \min \big\{ i \in [k, m) \cap \mathbb{Z}; \;\; i \text{ is a jump point} \big\}.$$

Using these subroutines we can construct the *noise regions* of the approximation. These are simply the columns which are within 2ν columns of a jump point. Specifically we create a certain number of intervals:

$$[\ell_1, r_1], \dots, [\ell_\alpha, r_\alpha] \subset [1, m]$$

where the integers ℓ_k, r_k are determined inductively as follows.

$$\ell_1 = \max\big(\mathsf{jump}(1) - 2\nu, 1 \big), \;] \; r_1 = \min\big(m, \mathsf{jump}(1) + 2\nu \big).$$

Suppose that $\ell_1, r_1, \dots, \ell_j, r_j$ are determined. If $\mathsf{jump}(r_j) > m$ we stop. Otherwise we set

$$\ell_{j+1} = \max\big(\mathsf{jump}(r_j) - 2\nu, 1 \big), \;\; r_{j+1} = \min\big(m, \mathsf{jump}(r_j) + 2\nu \big).$$

The intervals $[\ell_1, r_1], \dots, [\ell_\alpha, r_\alpha]$ may not be disjoint, but their union is a *disjoint* union of intervals

$$[a_1, b_1], \dots, [a_J, b_J], \;\; b_i < a_{i+1}.$$

The intervals $[a_j, b_j], 1 \leq j \leq J$ are the *noise intervals*. The intervals

$$[1, a_1], [b_1, a_2], \dots, [b_{J-1}, a_J], [b_J, m]$$

are the *regular intervals*.

Now that we have determined the noise and regular intervals, we can create the approximation S_ε. We do this with separate procedures on the noise or regular intervals. In either case the approximation will be formed by (possibly degenerate) trapezoids whose bases are vertical.We call any set which is a union of finitely many such trapezoids a *polytrapezoid*. The approximations on the regular and noise intervals will both be polytrapezoids, and S_ε itself will also be a polytrapezoid.

First some notation. Given a collection of points

$$B_0, T_0, \dots, B_N, T_N \in \mathbb{R}^2$$

such that

$$x(B_i) = x(T_i),\ \ y(B_i) \le y(T_i),\ \ \forall i = 0, \dots, N,$$
$$x(B_{j-1}) < x(B_j),\ \ \forall 1 \le j \le N,$$

we denote by $\mathsf{polygon}(B_0, T_0, \dots, B_N, T_N)$ the region surrounded by the simple closed PL-curve obtained as the union of line segments

$$[B_0, B_1], \dots, [B_{N-1}, B_N],$$
$$[B_N, T_N], \dots, [T_1, T_0], [T_0, B_0].$$

Note that each of the quadrilaterals $B_{i-1}B_iT_iT_{i-1}$ is a (possibly degenerate) trapezoid with vertical bases.

Consider first the regular intervals. Given a regular interval $I := [p, q]$ we observe that the number of stacks $\boldsymbol{n}_i$ is independent of $i \in [p, q]$. We denote this shared number by $\boldsymbol{n} = \boldsymbol{n}(I)$.

We construct inductively a sequence of numbers $i_0 < \cdots < i_N$ as follows:

- We set $i_0 = p$.
- If $q - p < 2\sigma$ we set $N = 1$ and $i_1 = q$.
- If $i_0, \dots, i_k$ are already constructed, then, if $q - i_k < 2\sigma$ we set $N = k + 1$ and $i_{k+1} = q$, else $i_{k+1} = i_k + \sigma$.

Note that if $q - p > \sigma$, then $N \ge 1$, $i_0 = p$, $i_N = q$ and

$$N = 1 \ \text{ if } \ q - p < \sigma.$$

We have

$$\mathsf{stack}(C_{i_k}) = \boldsymbol{n},\ \ b_{i_k,1}, t_{i_k,1}, \dots, b_{i_k,\boldsymbol{n}}, t_{i_k,\boldsymbol{n}}.$$

For $j = 1, \dots, \boldsymbol{n}$, and $k = 0, \dots, N$ we denote by $B_{k,j}$ the center of the ε-pixel corresponding to the element entry $b_{i_k,j}$ in the column C_{i_k}. Similarly we denote by $T_{k,j}$ the center of the pixel corresponding to the entry $t_{i_k,j}$ of the column C_{i_k}. For $1 \le j \le \boldsymbol{n}(I)$, we set

$$\mathcal{P}_j(I) := \mathsf{polygon}(B_{0,j}, T_{0,j}, \dots, B_{N,j}, T_{N,j}).$$

Define

$$\mathcal{P}(I) = \bigcup_{j=1}^{\boldsymbol{n}(I)} P_j(I),\ \ \mathcal{P}_{\text{reg}} := \bigcup_{I \text{ regular interval}} P(I).$$

Suppose now that $I = [p, q]$ is a noise interval. We modify the column

$$C_p = a_{p,1}, \dots, a_{p,m}$$

to a column

$$C'_p = a'_{p,1}, \dots, a'_{p,m},$$

by setting

$$a'_{p,k} := \begin{cases} 1, & \text{if } \sum_{i=p}^{q} a_{i,k} > 0 \\ \\ 0, & \text{if } \sum_{i=p}^{q} a_{i,k} > 0. \end{cases}$$

We apply the subroutine stack to the new column $C_p'$and the output is

$$\mathsf{stack}(C_p') = \boldsymbol{n}(I),\ \ b_1 \leq t_1 < \cdots < b_n \leq t_n.$$

For $j = 1, \dots, \boldsymbol{n}(I)$ we set

$$B_{0,j} := A[p, b_j],\ \ T_{0,j} := A[p, t_j],$$
$$B_{1,j} := A[q, b_j],\ \ T_{0,j} := A[q, t_j],$$

(recall that $A[i, j]$ is defined as the center of the pixel associated to $(a_{i,j})$. Next, for $j = 1, \dots, \boldsymbol{n}(I)$ we define the rectangle

$$\mathcal{R}_j(I) := \mathsf{polygon}(B_{0,j}, T_{0,j}, B_{1,j}, T_{1,j}),$$

and we set

$$\mathcal{R}(I) = \bigcup_{j=1}^{\boldsymbol{n}(I)} \mathcal{R}_j(I),\ \ \mathcal{P}_{\text{noise}} := \bigcup_{I \text{ noise interval}} R(I).$$

The output of the algorithm is the polytrapezoid

$$\mathcal{P}_\varepsilon(A) := \mathcal{P}_{\text{regular}} \cup \mathcal{P}_{\text{noise}}.$$

References

[1] A. Bernig: *The normal cycle of compact definable sets*, Israel J. Math., **159**(2007), 373-411.

[2] J. Bochnak, M. Coste, M.-F.. Roy: *Real Algebraic Geometry*, Translated from the 1987 French original. Revised by the authors. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36, Springer-Verlag, Berlin, 1998.

[3] M. Coste: *An Introduction to Semialgebraic Geometry*, Dip. Mat. Univ. Pisa, Dottorato di Ricerca in Matematica, Istituti Editoriali e Poligrafici Internazionali, Pisa (2000). http://perso.univ-rennes1.fr/michel.coste/polyens/SAG.pdf

[4] M. Gorseky, R. MacPherson: *Stratified Morse Theory*, Springer Verlag, 1988.

[5] J. Fu: *Convergence of curvatures in secant approximations*, J. Diff. Geom. **37**(1993), 177-190.

[6] J. Fu: *Curvature measures of subanalytic sets*, Am. J. Math. **116**(1994), 819-890.

[7] D.A. Klain, G.-C. Rota: *Introduction to Geometric Probability*, Cambridge University Press, 1997.

[8] J.W. Milnor: *On the total curvature of knots*, Ann. Math., **52**(1950), 248-257.

[9] J.M. Morvan: *Generalized Curvatures*, Springer Verlag, 2008.

[10] L. Nicolaescu *On the Normal Cycles of Subanalytic Sets*, Ann. Glob. Anal. Geom., **39**(2011), 427-454.

[11] R. Pignoni: *Density and stability of Morse functions on a stratified space*, Ann. Scuola Norm. Sup. Pisa, series IV, **6**(1979), 593-608.

[12] B. Rowekamp: *Planar pixelations and image reconstruction*, arXiv: 1105.2831

[13] L. van den Dries: *Tame Topology and o-minimal Structures*, London Math. Soc. Lectures Notes Series, vol. 248, Cambridge University Press, 1998.

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