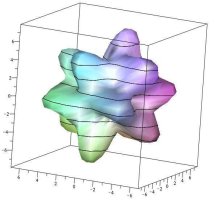

Counting Morse Functions

Liviu I. Nicolaescu

2011

Snakes

Snakes

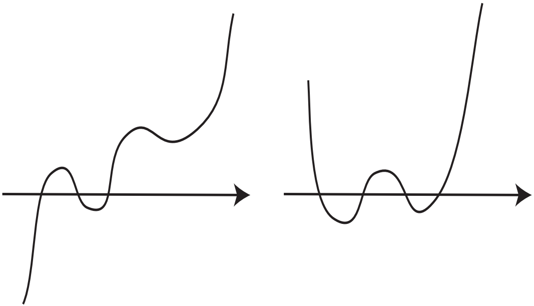

Figure: Odd/even snakes

Snakes

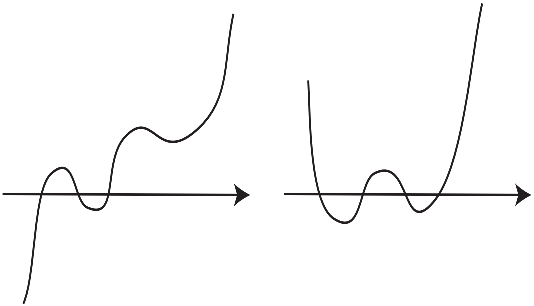

Figure: Odd/even snakes

Snake = proper Morse function $f : \mathbb{R} \to \mathbb{R}$ with a finite number of critical points, no two critical points on the same level set, $\lim_{x\to\infty} f(x) = \infty$.

Equivalence of snakes

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Definition (Arnold)

Two snakes $f, g : \mathbb{R} \to \mathbb{R}$ are called equivalent if there exist orientation preserving diffeomorphisms $L, R : \mathbb{R} \to \mathbb{R}$ such that

$$L \circ f \circ R = g.$$

We denote by ζ_n the number of snakes with n critical points.

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- The induced map $g : C_g \to D_g$ is an up/down permutation of $\boldsymbol{I}_n$, i.e.,

$$\big(\, g(i-1) - g(i)\,\big)\big(\, g(i) - g(i+1)\,\big) < 0, \;\; \forall i = 2, \dots, n-1$$

and $g(n) < g(n-1)$.
- Elementary fact. $\zeta_n =$ the number of up/down permutations of $\boldsymbol{I}_n$.

Snakes with 3 critical points

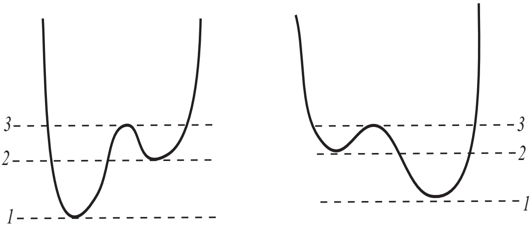

Figure: The only two snakes with 3 critical points corespond to the up/down permutations 132 and 231.

Snakes with 4 critical points

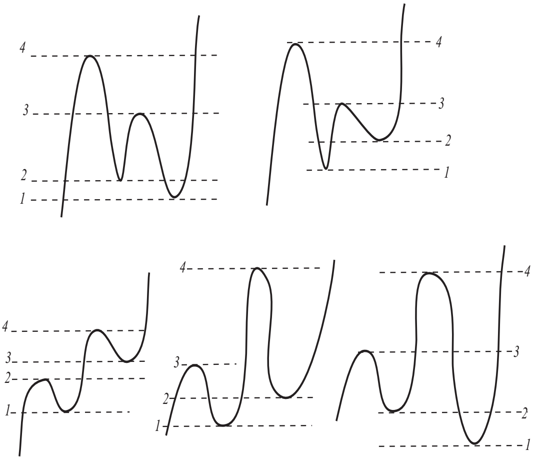

Figure: The up/down permutations corresponding to the snakes with 4 critical points are: 4231, 4132, 2143, 3142, 3241.

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- For any finite subset $S \subset \mathbb{R}$ denote by $\zeta(S)$ the number of up/down permutations of S. Note that $\zeta(S) = \zeta(S')$ if $|S| = |S'|$, $|A| :=$ the cardinality of A.

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  - $i_\varphi := \varphi^{-1}(n)$ so that $\varphi(i_\varphi) = \max_i \varphi(i)$.
  - $S_\varphi^- := \{\varphi(i);\ \ i < i_\varphi\}$.
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  - $S_\varphi^+ := \{\varphi(i);\ \ i > i_\varphi\}$.
- Observe that S_φ^+ has odd cardinality for any φ.

Computing ζ_n

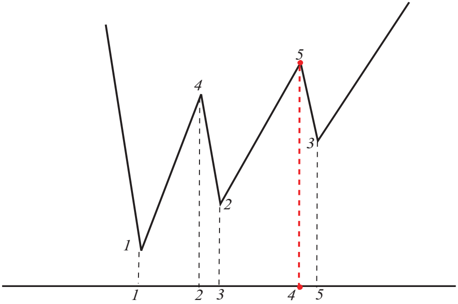

Figure: The up/down permutation $\varphi = 14253$

Computing ζ_n

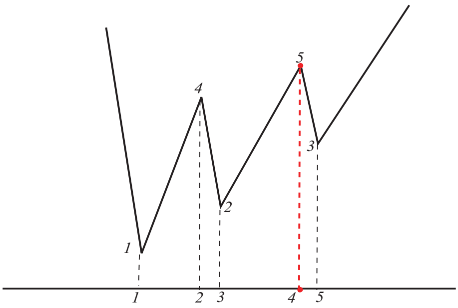

Figure: The up/down permutation $\varphi = 14253$

For the above permutation φ we have

$$i_\varphi = 4, \;\; S_\varphi^- = \{1, 4, 2\}, \;\; S_\varphi^+ = \{3\}.$$

Computing ζ_n, $n = 2k + 1$.

Set $x_k := \zeta_{2k+1}$, $k = 0, 1, \ldots$.

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- An up/down permutation φ^+ of S_φ^+.
- The sets $S_\varphi^\pm$ form a partition of $\boldsymbol{I}_{2k}$, and S_φ^+ has odd cardinality.
- The permutation φ can be reconstructed from $\varphi^\pm$ via the concatenation $\varphi = (\varphi^-, 2k + 1, \varphi^+)$.

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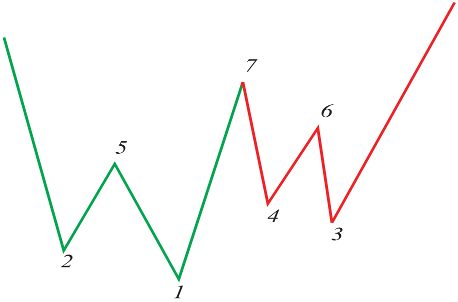

Figure: The up/down permutation $\varphi = 2517463$

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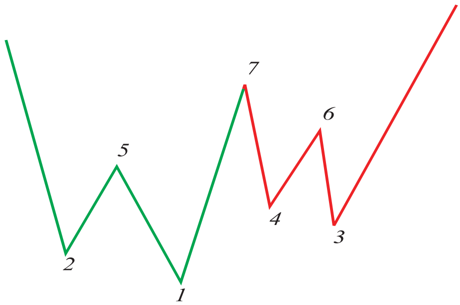

Figure: The up/down permutation $\varphi = 2517463$

In the above figure we have

$$\varphi^- = 251, \;\; \varphi^+ = 463, \;\; \varphi = \underbrace{251}_{\varphi^-} \,|\, 7 \,|\, \underbrace{463}_{\varphi^+}$$

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We deduce

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We can rewrite this equality as follows

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$$x'(t) = 1 + x(t)^2, \quad x(0) = 0.$$

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Hence $x(t)$ is the inverse of the rational integral

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Thus x_k is roughly of size $(2k)!$.

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$$\mu_1(n) = 0, \ \ \forall n \in 2\mathbb{Z} + 1.$$

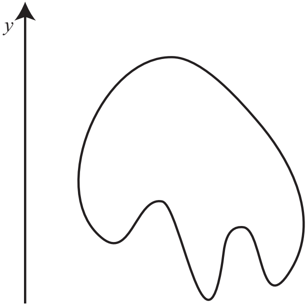

Figure: The height function is a Morse function on S^1 with 6 critical points.

Morse functions on S^1: $\mu_1(2k+2) = \zeta_{2k+1}$

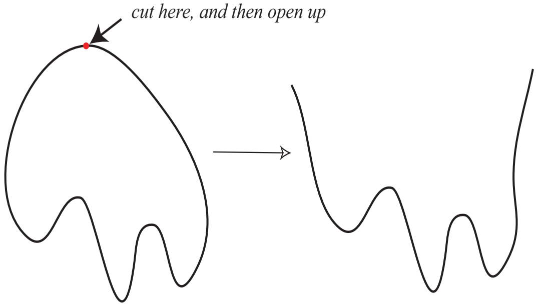

Figure: Transforming a Morse function on S^1 with 6 critical points into a snake with 5 critical points

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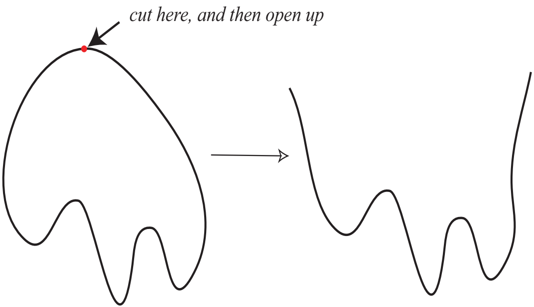

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We deduce

$$\sum_{k\geq 0} \frac{\mu_1(2k+2)}{(2k+1)!} t^{2k+1} = \tan t.$$

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- There are three types of handles: 0-, 1- and 2-handles.

Attaching 0- and 2-handles

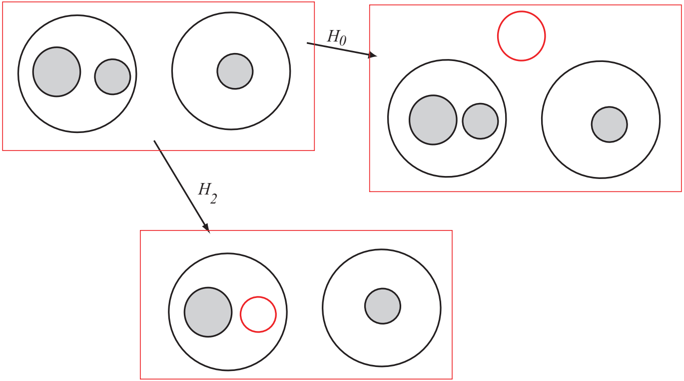

Figure: Attaching 0- and 2-handles. The shaded areas indicate holes.

Attaching 1-handles

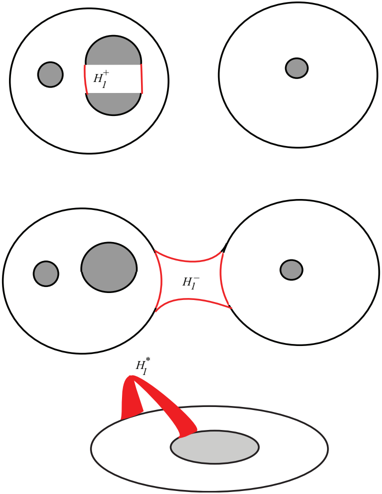

Figure: There are only three ways of attaching 1-handles while preserving orientability.

The topology types of the regular sublevel sets

- For any regular value r we denote by d_r the dimension of the cokernel of the natural morphism $H_1(\partial\Sigma_r, \mathbb{R}) \to H_1(\Sigma_r, \mathbb{R})$.

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- The operation H_1^* increases $d(r)$ by 1 so it cannot appear for a Morse function $f : S^2 \to \mathbb{R}$.
- In particular, all the regular sublevel sets Σ_r are disjoint unions of planar disks with holes in them.

Morse perestroikas

- A Morse perestroika is a succession of handle attachments of types $H_0, H_1^{\pm}, H_2$ that starts from a disk and whose end result is also a disk. (To get a sphere from the final disk, cap it off with another disk.)
- We see that a stable Morse function on S^2 leads to a Morse perestroika.

Morse perestroikas

- A Morse perestroika is a succession of handle attachments of types $H_0, H_1^{\pm}, H_2$ that starts from a disk and whose end result is also a disk. (To get a sphere from the final disk, cap it off with another disk.)
- We see that a stable Morse function on S^2 leads to a Morse perestroika.
- Conversely, any Morse perestroika can be obtained in this fashion.

Reeb graphs

Suppose that $f : S^2 \to \mathbb{R}$ is a stable Morse function.

Reeb graphs

Suppose that $f : S^2 \to \mathbb{R}$ is a stable Morse function. The Reeb graph of f is the quotient $\Gamma_f := S^2/\sim_f$, where $p \sim_f q$ iff $\exists r \in \mathbb{R}$ such that p and q belong to the same component of the level set $\{f = r\}$.

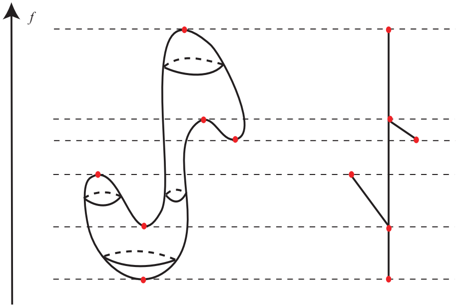

Figure: A Morse function and its Reeb graph.

Morse trees

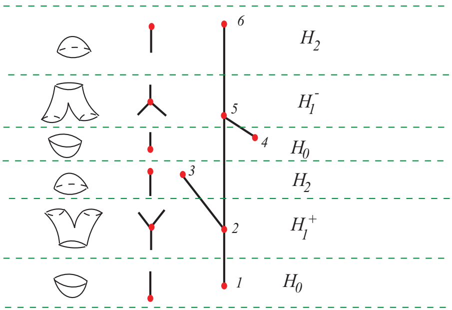

Figure: A Reeb tree.

The Reeb graph of a stable Morse function $f : S^2 \to \mathbb{R}$ is a tree.

Morse trees

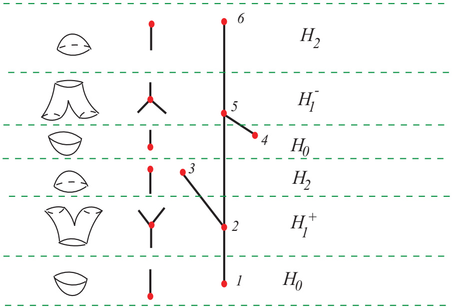

Figure: A Reeb tree.

The Reeb graph of a stable Morse function $f : S^2 \to \mathbb{R}$ is a tree.The natural map $S^2 \to \Gamma_f$ induces a bijection from the critical set of f to the set of vertices of Γ_f.

Morse trees

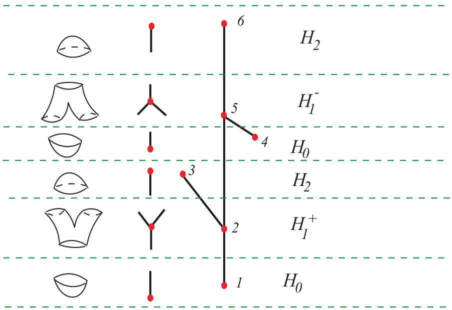

Figure: A Reeb tree.

The function f descends to a function $f : \Gamma_f \to \mathbb{R}$ whose restriction to the set of vertices is injective.

Morse trees

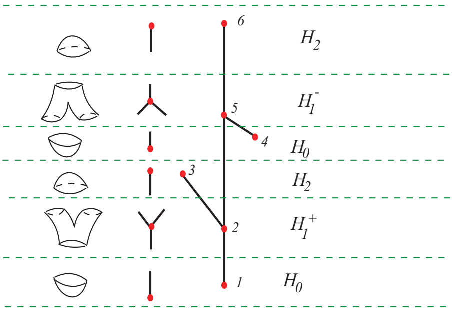

Figure: A Reeb tree.

The function f descends to a function $f : \Gamma_f \to \mathbb{R}$ whose restriction to the set of vertices is injective. The vertices are of two types: nodes (three neighbors), or leaves (a single neighbor).

Morse trees

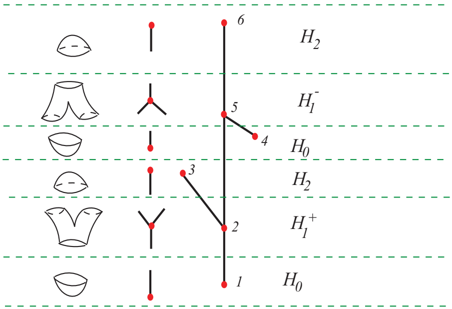

Figure: A Reeb tree.

For any node v there exist two neighbors $v^{\pm}$ such that $f(v^{+}) - f(v) > 0$ and $f(v^{-}) - f(v) < 0$.

Morse trees

Definition (Morse trees)

A Morse tree is a pair (Γ, f) with the following properties.

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Two Morse trees (Γ_0, f_0) and (Γ_1, f_1) are called equivalent if there exists a bijection $\Psi : V_{\Gamma_0} \to V_{\Gamma_1}$, $v \mapsto \bar{v}$, with the following properties.

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2. For any $u \neq v \in V_{\Gamma_0}$ we have

$$f_0(v) - f_0(u) > 0 \Longleftrightarrow f_1(\bar{v}) - f_1(\bar{u}) > 0.$$

Morse trees

- The previous discussion shows that the Reeb graph associated to a stable Morse function $f : S^2 \to \mathbb{R}$ is a Morse tree.

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- Moreover if two stable Morse functions are equivalent, then so are their associated Morse trees.

Morse trees

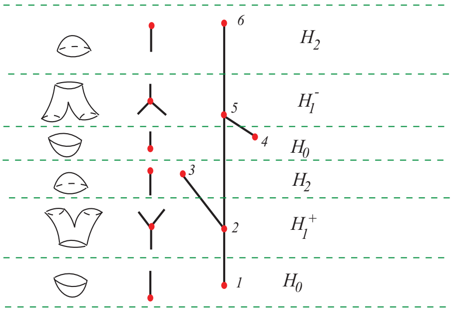

Figure: Every Morse tree is the Reeb graph of a stable Morse function $f : S^2 \to \mathbb{R}$.

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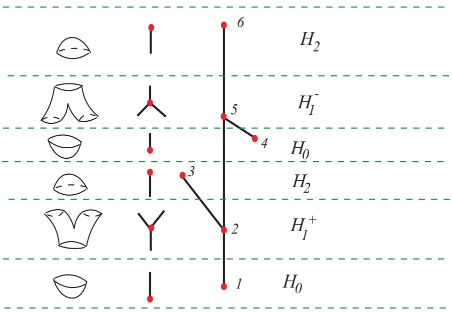

Figure: Every Morse tree is the Reeb graph of a stable Morse function $f : S^2 \to \mathbb{R}$.

Every Morse tree encodes a Morse perestroika and thus a stable Morse function on S^2. The Reeb graph of this Morse function is equivalent to the original Morse tree.

Morse trees

We have constructed a map from the space of equivalence classes of stable Morse functions with k saddle points to the space of equivalence classes of Morse trees with k-nodes.

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This map is a bijection. Thus, the Morse function count is equivalent with the Morse tree count.

Counting Morse trees

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Counting Morse trees

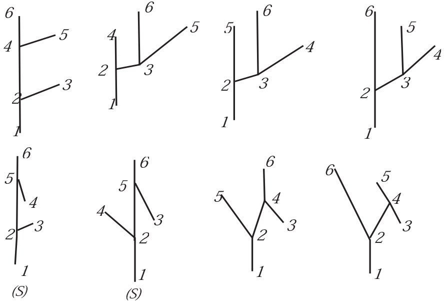

Figure: There are 8 Morse functions with 6 critical points, and the second is a saddle point.

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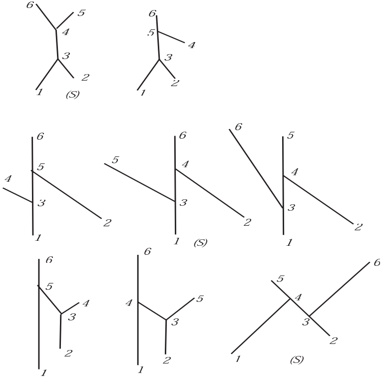

Figure: There are 8 Morse functions with 6 critical points such that the lowest two are local minima.

Counting Morse trees

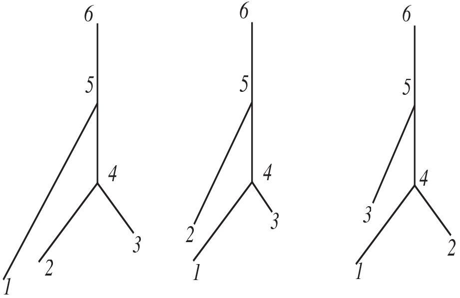

Figure: There are 3 Morse functions with 6 critical points, 3 local minima.

Counting Morse trees

Theorem (Nico., 2007)

The function $\xi(u,v)$ is the solution of the following first-order quasilinear Cauchy problem.

$$\begin{cases} -(1+u\xi+\frac{u^2}{2})\partial_u\xi+\partial_v\xi & = & (1+u\xi+\frac{1}{2}\xi^2) \\ \xi(u,0) & = & 0. \end{cases}$$

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A. $x > 0$.

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$$= \frac{x+1}{2}H(x-1,y) + \frac{x+1}{2}\sum_{(x_1,y_1)\in R_{x,y-1}} H(x_1,y_1)H(\bar{x}_1,\bar{y}_1),$$

where

$$R_{x,y-1} = \{(a,b)\in\mathbb{Z}^2;\ \ 0\leq a\leq x,\ \ 0\leq b\leq y-1\},$$

and for every $(a,b)\in R_{x,y-1}$ we denoted by $(\bar{a},\bar{b})$ the symmetric of (a,b) with respect to the center of the rectangle $R_{x,y-1}$.

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B. $x = 0$.

$$(2y+1)H(0,y) - H(1,y-1) = \frac{1}{2}\sum_{y_1=0}^{y-1} H(0,y_1)H(0,y-1-y_1).$$

Counting Morse trees

The above recurrences are easily implementable on a computer and lead to explicit computations.

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k	ξ_k
0	1
1	2
2	19
3	428
4	17,746
5	1,178,792
6	114,892,114
7	15,465,685,088
8	2,750,970,320,776
9	625,218,940,868,432

Counting Morse trees

Theorem (Nico. 2007)

The function

$$t \mapsto \xi(t) := \sum_{n\geq 0} \xi_k \frac{t^{2k+1}}{(2k+1)!}$$

is the inverse of the elliptic integral

$$\xi \mapsto t(\xi) := \int_0^{\xi} \frac{d\tau}{\sqrt{\frac{\tau^4}{4} - \frac{\tau^2}{2} + 2\xi\tau + 1}}.$$

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Remark

The Lagrange inversion formula applied to the above equality leads to another (less efficient) algorithm for computing ξ_k.

Idea of proof

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The solution of the Cauchy problem

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can be found via the method of characteristics.

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can be found via the method of characteristics.
More precisely, for every $s>0$ denote by $(u_s(t), v_s(t), \xi_s(t))$ the solution if the initial value problem

$$\begin{cases} \frac{du}{dt} &=& -(1+u\xi+\frac{u^2}{2}) \\ \frac{dv}{dt} &=& 1 \\ \frac{d\xi}{dt} &=& 1+u\xi+\frac{1}{2}\xi^2 \end{cases},$$

$$u_s(0)=s,\;\; v_s(0)=0,\;\; \xi=\xi_s)0)=0.$$

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The key observation is that the pair $(u_s(t), \xi_s(t))$ satisfies the Hamiltonian equations

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where $h(u,\xi)$ is the cubic polynomial

$$h(u,\xi) := \frac{1}{2}(u^2\xi + u\xi^2) + u + \xi = (u+\xi)\Big(\,\frac{1}{2}u\xi + 1\,\Big).$$

□

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$$P_s(u) := \frac{u^4}{4} - u^2 + 2su + 1,$$

and $\wp_s$ is the Weierstrass function that uniformizes the elliptic curve

$$E_s: \quad y^2 = 4x^3 - \frac{x}{3} + \frac{1}{27} + \frac{s^2}{16}.$$

Arnold's question

- Arnold proved that $\xi_k < k^{2k}$, so that

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$$\log \xi_k \sim 2k \log k \;\; \text{as } k \to \infty.$$

Arnold's guess was correct

Theorem (Nico., 2007)

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Proof sketch

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- For any formal power series with real coefficients

$$a(t) = \sum_{n \geq 0} a_n t^n, \quad b(t) = \sum_{n \geq 0} b_n t^n$$

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- Recall that there exists a solution $\xi(u, v)$ of the p.d.e.

$$-(1 + u\xi + \frac{u^2}{2})\partial_u \xi + \partial_v \xi = (1 + u\xi + \frac{1}{2}\xi^2),$$

such that

$$\xi(0, t) = \sum_{k\geq 0} \frac{\xi_k}{(2k+1)!} t^{2k+1}.$$

Proof sketch

- By construction, the function $\xi(u, v)$ has nonnegative Taylor coefficients at $(0, 0)$. We deduce that the function $t \mapsto \xi(0, t)$ satisfies
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- The solution of this initial value problem is $y(t) = \sqrt{2}\tan\left(\frac{t}{\sqrt{2}}\right)$.

Proof sketch

- The Taylor coefficients of $\tan x$ can be expressed explicitely in terms of Bernoulli numbers

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1}.$$

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- To conclude, use known asymptotics for the Bernoulli numbers

$$|B_{2k}| \sim \frac{2(2k)!}{(4\pi^2)^k} \text{ as } k \to \infty.$$