ON A BRUHAT-LIKE POSET

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ABSTRACT. We investigate the combinatorics and the topology of the poset of strata of a Schubert like stratification on the Grassmannian of hermitian lagrangian spaces in $\mathbb{C}^n \oplus \mathbb{C}^n$. We prove that this poset is a modular complemented lattice, we compute its Möbius function and we investigate the combinatorics and the topology of its order intervals.

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INTRODUCTION

The Grassmannian of the hermitian lagrangian subspaces of $E = \mathbb{C}^n \oplus \mathbb{C}^n$ is the space $\mathrm{Lag_h}(n)$ of complex subspaces $L \subset E$ such that

$$L^{\perp} = JL \;\text{ where }\; L = \begin{bmatrix} 0 & -\mathbb{1}_{\mathbb{C}^n} \\ \mathbb{1}_{\mathbb{C}^n} & 0 \end{bmatrix}.$$

This space is a real semialgebraic manifold which contains the space $\mathcal{S}_n$ of hermitian $n \times n$ matrices as a dense open subset. V.I. Arnold has shown that the Cayley transform

$$\mathcal{S}_n \to U(n),\quad A \mapsto (\mathbb{1} - \boldsymbol{i}A)(\mathbb{1} + \boldsymbol{i}A)^{-1}$$

extends to a diffeomorphism $\mathrm{Lag_h}(n) \to U(n)$ (see [2, 7]).

The Grassmannian $\mathrm{Lag_h}{}_h(n)$ is naturally a submanifold of the space $\mathcal{S}_E$ of selfadjoint operators $E \to E$. As such, it is equipped with a natural Riemann metric. In [7] we investigated a certain linear function $f : \mathcal{S}_E \to \mathbb{R}$ which induces a perfect Morse function on $\mathrm{Lag_h}(n)$. There exists a natural bijection between the collection $2^{[n]}$ of subsets of $[n] = \{1, \dots, n\}$ and $\mathbf{Cr}_f$, the set of critical points of f,

$$2^{[n]} \ni I \mapsto L_I \in \mathbf{Cr}_f,$$

The negative gradient flow of f satisfies the Smale transversality condition. If W_I^- denotes the unstable manifold of L_I then the collection

$$\mathcal{W}_n := \{ W_I^-;\;\; I \in 2^{[n]} \}$$

defines a Whitney regular stratification of $\mathrm{Lag_h}(n)$ which can be given a Schubert-like description in terms of incidence conditions.

The collection $\mathcal{W}_n$ of strata is equipped with a Bruhat-like partial order $\prec$ defined by

$$W_I^- \prec W_J^- \iff W_I^- \subset \text{closure}\,(W_I^-).$$

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In purely combinatorial terms we have,

$$W_I^- \prec W_J^- \iff \#(I\cap [k,n]) \leq \#(J\cap[k,n]),\;\; \forall k=1,\dots,n.$$

We obtain in this fashion a partial order $<_n$ on $2^{[n]}$ and we denote the resulting poset by $\mathcal{L}_n$. The Hasse diagrams of $\mathcal{L}_n$, $1\leq n\leq 4$ are depicted in Figure 2.

It is perhaps instructive to give another description of this partial order in terms of the game *beads-along-a-rod.*

Suppose are given a thin rod with points marked $1,\dots,n$ in increasing linear order, left-to-right along the rod. To any subset $I\subset [n]$ we associate a configuration of beads along this rod placed in the positions $i\in I$. The game consist of a succession of elementary moves of two types.

- Slide one bead to the left neighboring position, if that position is unoccupied by another bead.
- Remove the bead on the leftmost position 1, if there is such a bead.

We will refer to these moves as *elementary left slides.* If $I,J\in 2^{[n]}$ describe two configurations of rings, then $I\prec J$ if one can go from the configuration J to the configuration I by a sequence of elementary left slides.

The poset $\mathcal{L}_n$ is closely related to the Boolean poset $\mathcal{B}_n$ of subsets of $[n]$ ordered by inclusion. To describe this relationship we need to go back to Morse theory.

We denote by W_I^+ the *stable* manifold of the critical point $L_I\in \mathrm{Lag_h}(n)$. Then

$$W_J^- \prec W_I^- \iff W_J^+\cap W_I^-\neq\emptyset. \tag{G}$$

In [7] we have proved that the stable/unstable manifolds $W_I^\pm$ determine homology classes $[W_I^\pm]\in H_\bullet(\mathrm{Lag_h}(n),\mathbb{Z})$. The partial order on $\mathcal{B}_n$ can also be formulated in terms of *homological* intersection. More precisely

$$J\subset I \iff [W_J^+]\bullet [W_I^-]\neq 0, \tag{H}$$

where $\bullet$ denotes the intersection pairing in homology.

On complex Grassmannians the two statements (G) and (H) are equivalent and they define the Bruhat order on the set of Schubert varieties. On the Grassmannian $\mathrm{Lag_h}(n)$ we only have the implication (H) $\Rightarrow$ (G). In particular, the tautological map $\mathcal{B}_n\to\mathcal{L}_n$ is increasing, but its inverse is not.

The poset $\mathcal{L}_n$ shares many combinatorial features with the Bruhat posets. We show (Proposition 3.16, 3.18) that $\mathcal{L}_n$ is a *modular ortholattice lattice* with rank function $\boldsymbol{\rho}:\mathcal{L}_n\to\mathbb{Z}$ given by

$$\boldsymbol{\rho}(I)=\sum_{i\in I} i,$$

and the complement map $\boldsymbol{\sigma}_n:\mathcal{L}_n\to\mathcal{L}_n$ given by $S\mapsto \{1,\dots,n\}\setminus S$. As explained in [9] the modularity implies that the poset $\mathcal{L}_n$ is homotopy Cohen-Macaulay.

We define a pair $(I,J)\subset\mathcal{L}_n\times\mathcal{L}_n$ to be *elementary* if the sequence of integers

$$\delta_k=\#\big(J\cap[k,n]\big)-\#\big(I\cap[k,n]\big),\;\; k=1,2,\dots,n$$

consists only of 0's and 1's and there are no consecutive 1's. If $\boldsymbol{\mu}_n$ denotes the Möbius function of $\mathcal{L}_n$, then we show (Theorem 3.7) that

$$\boldsymbol{\mu}_n(S,T)=\begin{cases}(-1)^{\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)} & (I,J)\text{ is an elementary pair,}\\ 0 & \text{otherwise.}\end{cases}$$

Since a modular lattice is shellable we deduce from [5, Thm. 5.6] that if (I,J) is not an elementary pair then the open order interval $(I,J)_{\mathcal{L}_n}$ is a contractible poset. In Proposition 3.20 we prove a stronger statement, namely that in this case the nerve of the open order interval $(I,J)_{\mathcal{L}_n}$ is homeomorphic to the closed Euclidean ball of dimension $\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)-2$.

If (I,J) is an elementary pair, then the shellability of $\mathcal{L}_n$ together with the equality $\boldsymbol{\mu}_n(I,I)=(-1)^{\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)}$ imply via [5, Thm. 5.6] that the open interval $(I,J)_{\mathcal{L}_n}$ is homotopic to a sphere of dimension $\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)-2$. We were able to prove a slightly stronger result. Namely, we show that if (I,J) is an elementary pair then the order interval $[I,J]_{\mathcal{L}_n}$ is isomorphic to the boolean poset $\mathcal{B}_{\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)}$ (Theorem 3.15).

The key technical device that allowed us to reach these conclusions is a certain surgery-like operation $\#$ on pairs of posets which we introduce in Section 1. In this section we also describe explicitly the effect of this surgery on the Möbius functions (Theorem 1.3).

In Section 2 we investigate a special case of this surgery operation on a special class of posets we called *layered*. Given a layered poset P we define its *double* as the poset obtained by applying the surgery operation in Section 1 to two copies of P. In this section we analyze a few special features of this doubling operation.

In Section 3 we apply the general results to the poset $\mathcal{L}_n$ which has a natural layer structure. The key fact which allowed us to apply the general theory developed in the previous section is the "surgery formula" in Proposition 3.1 which states that $\mathcal{L}_{n+1}$ is the double of the layered poset $\mathcal{L}_n$.

1. The Möbius functions of "connected sums" of posets

We follow closely the poset terminology in [1, 8]. For simplicity we will concentrate exclusively on finite posets. If $(P,<)$ is such a poset, we denote by $\mathbb{C}^P$ the space of functions $P\to\mathbb{C}$, and we define a $\mathbb{Z}$-linear map (integration)

$$\boldsymbol{S}_P:\mathbb{C}^P\to\mathbb{C}^P,\;\;\mathbb{C}^P\ni f\mapsto \mathcal{I}_Pf\in\mathbb{C}^P,\;\;\mathcal{I}_Pf(x)=\sum_{y\geq x}f(y),\forall x\in P.$$

The vector space $\mathbb{C}^P$ has a canonical basis consisting of the Dirac functions

$$\delta_x\in\mathbb{C}^P,\;\;x\in P,\;\;\delta_x(y)=\begin{cases}1 & y=x\\ 0 & y\neq x.\end{cases}$$

With respect to this basis the above operator is described by its *incidence matrix* $\mathcal{S}_P\in\mathbb{C}^{P\times P}$ defined by

$$\mathcal{S}_P(x,y)=\begin{cases}1 & x\leq y\\ 0 & x\not\leq y.\end{cases}$$

The matrix $\mathcal{S}_P$ is "upper triangular" and it can be written as a sum $\mathcal{S}_P=\mathbb{1}+\mathcal{N}_P$, where $\mathcal{N}_P$ is "strictly upper triangular", i.e.,

$$\mathcal{N}_P(x,y)\neq 0\Longrightarrow x<y.$$

This shows that the linear operator $\boldsymbol{N}_P$ determined by $\mathcal{N}_P$ is nilpotent. Hence $\boldsymbol{S}_P$ is invertible and its inverse is given by

$$\boldsymbol{M}_P=\boldsymbol{S}_P^{-1}=\sum_{k\geq 0}(-1)^k\boldsymbol{N}_P^k.$$

The matrix describing $\boldsymbol{M}_P$ in the Dirac basis is called the *Möbius function of* P and it is denoted by $\boldsymbol{\mu}_P$. The equalities $\boldsymbol{M}_P\boldsymbol{S}_P=\mathbb{1}=\boldsymbol{S}_P^{\dagger}\boldsymbol{M}_P^{\dagger}$ ($\dagger$ = transpose) translate into the recursion

$$\sum_{x\leq z\leq y}\boldsymbol{\mu}(x,z)=\delta_x(y)=\sum_{x\leq z\leq y}\boldsymbol{\mu}(z,y).\tag{1.1}$$

The Möbius function appears in the Möbius inversion formula

$$s(x)=\sum_{y\geq x}f(y),\;\;\forall x\in P\Longleftrightarrow f(x)=\sum_{y\geq x}\boldsymbol{\mu}(x,y)s(y),\;\;\forall x\in P.\tag{1.2}$$

Suppose we are given three posets $(Q,<_Q)$, $(P_k,<_k)$, $k=0,1$, and injections $i_k:Q\to P_k$ such that

$$q<_Q q'\Longleftrightarrow i_k(q)<_k i_k(q'),\;\;\forall k=0,1.$$

In other words, Q is an induced subposet of both P_0 and P_1. To simplify the presentation we will write $x\leq_0 q$ instead of $x\leq_0 i_0(q)$, and $q\leq_1 y$ instead of $i_1(q)\leq_1 y$.

We can now define a partial order $\prec$ on the disjoint sum $P_0\sqcup P_1$ by setting $x\prec y$ if and only if

- either both x, y belong to the same set P_k and $x <_k y$,
- or $x \in P_0$, $y \in P_1$ and there exists $q \in Q$ such that $x \leq_0 q \leq_1 y$.

We denote this poset by $P_0{}_{Q_0}\#_{Q_1} P_1$, where $Q_k = i_k(Q) \subset P_k$, and we will refer to it as the *connect sum of* P_0 *and* P_1 *along* Q_0 *and* Q_1 (see Figure 1).

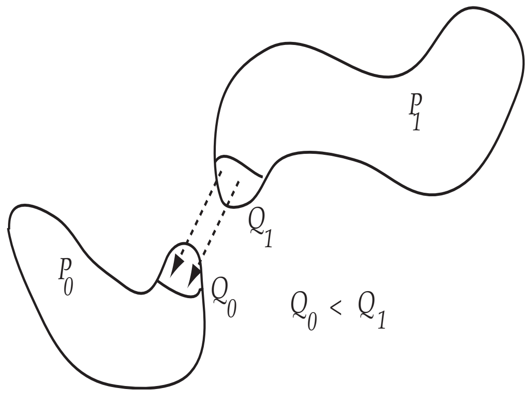

FIGURE 1. *"Gluing" P_0 to P_1 along Q_0 and Q_1.*

Example 1.1. Observe that if $P_0 = P_1 = Q$ and $i_k = \mathbb{1}_Q$ then the poset $P_0{}_{Q_0}\#_{Q_1} P_1$ is isomorphic with the poset $\{-1, 1\} \times P$ with the product order

$$(\epsilon_0, x_0) \leq (\epsilon_1, x_2) \Longleftrightarrow \epsilon_0 \leq \epsilon_1, \;\; x_0 \leq_P x_1. \qquad \square$$

Proposition 1.2. *Let $P_0 \overset{i_0}{\hookleftarrow} Q \overset{i_1}{\hookrightarrow} P_1$ be injective increasing. We set $Q_k := i_k(Q) \subset P_k$, and form the connect sum $P = P_0{}_{Q_0}\#_{Q_1} P_1$. Denote by $\mathcal{S}_k$ and respectively $\boldsymbol{\mu}_k$ the incidence matrix and respectively the Möbius function of P_k, $k = 0, 1$. If $x, y \in P_0 \sqcup P_1$ then*

$$\boldsymbol{\mu}_P(x, y) = \begin{cases} \boldsymbol{\mu}_k(x, y) & \textit{if } \; x, y \in P_k \\ -\sum_{x \leq_0 p_0 \prec p_1 \leq_1 y} \boldsymbol{\mu}_0(x, p_0)\boldsymbol{\mu}_1(p_1, y) & \textit{if } \; x \in P_0, \; y \in P_1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. With respect to the direct sum decomposition $\mathbb{C}^{P_0 \sqcup P_0} = \mathbb{C}^{P_0} \oplus \mathbb{C}^{P_1}$ the incidence matrix $\mathcal{S}_P$ has the block decomposition

$$\mathcal{S}_P = \begin{bmatrix} \mathcal{S}_0 & \mathcal{B} \\ 0 & \mathcal{S}_1 \end{bmatrix}, \;\; \mathcal{S}_k = \mathcal{S}_{P_k},$$

where $\mathcal{B} : P_0 \times P_1 \to \mathbb{Z}$ satisfies

$$\mathcal{B}(x, y) = \begin{cases} 1 & \exists q \in Q : \;\; x \leq_0 q \leq_1 y \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\boldsymbol{\mu}_P = \mathcal{S}_P^{-1} = \begin{bmatrix} \mathcal{S}_0^{-1} & -\mathcal{S}_0^{-1}\mathcal{B}\mathcal{S}_1^{-1} \\ 0 & \mathcal{S}_1^{-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_0 & -\boldsymbol{\mu}_0\mathcal{B}\boldsymbol{\mu}_1 \\ 0 & \boldsymbol{\mu}_1 \end{bmatrix}. \tag{1.3}$$

Note that if $x \in P_0$, $y \in P_1$ we then have

$$\boldsymbol{\mu}_P(x, y) = -\sum_{p_0 \in P_0, p_1 \in P_1} \boldsymbol{\mu}_0(x, p_0)\mathcal{B}(p_0, p_1)\boldsymbol{\mu}_1(p_1, y) = -\sum_{x \leq_0 p_0 \prec p_1 \leq_1 y} \boldsymbol{\mu}_0(x, p_0)\boldsymbol{\mu}_1(p_1, y). \qquad \square$$

To describe our next special case we need to introduce some notation. For any poset P and any $X \subset P$ we set

$$P^{\geq X} = \big\{ p \in P; \;\; \exists x \in X : \;\; p \geq x \big\}, \;\; P^{\leq X} = \big\{ p \in P; \;\; \exists x \in X; \; p \leq x \big\}.$$

Suppose now that the induced subposet $Q_0 \subset P_0$ satisfies the property

$$\forall x \in P_0,\ Q_0 \cap P_0^{\geq x} \text{ is either empty or contains a unique minimal element } q_x. \tag{$\boldsymbol{M}_+$}$$

and the subposet $Q_1 \subset P_1$ satisfies

$$\forall y \in P_1,\ Q_1 \cap P_1^{\leq y} \text{ is either empty or contains a unique maximal element } q^y. \tag{$\boldsymbol{M}_-$}$$

Note that $\boldsymbol{M}_+$ implies that if $x \in P_0$ and $y \in Q_0$ then

$$x \leq y \Longleftrightarrow q_x \leq y$$

while $\boldsymbol{M}_-$ implies that if $x \in Q_1$ and $y \in P_1$ then

$$x \leq y \Longleftrightarrow x \leq q^y.$$

Theorem 1.3. *Suppose $Q_0 \subset P_0$ satisfies ($\boldsymbol{M}_+$) and $Q_1 \subset P_1$ satisfies ($\boldsymbol{M}_-$), and set $P = P_0\,{}_{Q_0}\#_{Q_1} P_1$,*

$$P_0^{\leq Q_0} = \left\{ x \in P_0;\ \ P_0^{\geq x} \cap Q_0 \neq \emptyset \right\},\ \ P_1^{\geq Q_1} = \left\{ y \in P_1;\ \ P_1^{\leq x} \cap Q_1 \neq \emptyset \right\}.$$

Then

$$\boldsymbol{\mu}_P(x,y) = \begin{cases} \boldsymbol{\mu}_k(x,y) & \text{if}\, x,y \in P_k,\ k=0,1, \\ -\boldsymbol{\mu}_Q(q,q') & \text{if } x = i_0(q) \in Q_0,\ y = i_1(q') \in Q_1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Proof. The injections i_0 and i_1 define injections $j_0 = i_0 \circ i_1^{-1} : Q_1 \to P_0$ and $j_1 : Q_0 \to P_1$. Let $x \in P_0^{\leq Q_0}$, $y \in P_1^{\geq Q_1}$. Then

$$x \prec y \Longleftrightarrow j_1(q_x) \leq q^y \leq y \Longleftrightarrow x \leq q_x \leq j_0(q^y),$$

and

$$\mathcal{B}(x,y) = \mathcal{S}_1(j_1(q_x), y) = \mathcal{S}_0(x, j_0(q^y)). \tag{1.5}$$

In particular,

$$s \in P_0^{\leq Q_0},\ \ t \in P_1^{\geq Q_1},\ \ j_1(q_s) \not\leq t \Longrightarrow \mathcal{B}(s,t) = 0. \tag{1.6}$$

Using (1.3) we deduce that if $x \prec y$ we have

$$\boldsymbol{\mu}_P(x,y) = -\sum_{x \leq s \leq j_0(q^y),\ j_1(q_x) \leq t \leq y} \boldsymbol{\mu}_0(x,s)\mathcal{B}(s,t)\boldsymbol{\mu}_1(t,y)$$

$$\stackrel{(1.6)}{=} -\sum_{x \leq s \leq j_0(q^y)} \boldsymbol{\mu}_0(x,s)\left(\sum_{j_1(q_s) \leq t \leq y} \mathcal{B}(s,t)\boldsymbol{\mu}_1(t,y)\right)$$

$$\stackrel{(1.5)}{=} -\sum_{x \leq s \leq j_0(q^y)} \boldsymbol{\mu}_0(x,s)\left(\sum_{j_1(q_s) \leq t \leq y} \mathcal{S}_1(j_1(q_s),t)\boldsymbol{\mu}_1(t,y)\right) = -\sum_{x \leq s \leq j_0(q^y)} \boldsymbol{\mu}_0(x,s)\delta_y(j_1(q_s)).$$

The last sum is nonzero if and only if there exists $s \in P_0$ such that $x \leq s \leq j_0(q^y)$ and $y = j_1(q_s) \in Q_1$. This can only happen when $y \in Q_1$, so that $y = q^y$, and s is such that $j_1(q_s) = y$.

On the other hand, we have

$$\boldsymbol{\mu}_P(x,y) \stackrel{(1.6)}{=} -\sum_{j_1(q_x) \leq t \leq y}\left(\sum_{x \leq s \leq j_0(q^t)} \boldsymbol{\mu}_0(x,s)\mathcal{B}(s,t)\right)\boldsymbol{\mu}_1(t,y)$$

$$\stackrel{(1.5)}{=} -\sum_{j_1(q_x) \leq t \leq y}\left(\sum_{x \leq s \leq j_0(q^t)} \boldsymbol{\mu}_0(x,s)\mathcal{S}_0(s,j_0(q^t))\right)\boldsymbol{\mu}_1(t,y) = -\sum_{j_1(q_x) \leq t \leq y} \delta_x(j_0(q^t))\boldsymbol{\mu}_1(t,y).$$

We see that this sum can have a nontrivial term only if $x \in Q_0$ so that $x = q_x$. Thus $x = q_0 \in Q_0,\ \ y = q_1 \in Q_1$. We have

$$\boldsymbol{\mu}_P(q_0,q_1) = -\sum_{j_1(q_0) \leq t \leq q_1} \delta_{j_1(q_0)}(q^t)\boldsymbol{\mu}_1(t,q_1) = -\sum_{q_0 \leq s \leq j_0(q_1)} \boldsymbol{\mu}_0(q_0,s)\delta_{j_0(q_1)}(q_s) \tag{1.7}$$

For $q, q' \in Q$ we set

$$A(q,q') = -\boldsymbol{\mu}_P(i_0(q), i_1(q')).$$

From (1.7) we deduce

$$A(q,q) = 1, \;\; A(q,q') = \sum_{s \geq i_0(q), q_s = i_0(q')} \boldsymbol{\mu}_0(q,s).$$

Hence

$$\sum_{q \leq q'' \leq q'} A(q,q'') = \sum_{s \geq q,\; q_s \leq q''} \boldsymbol{\mu}_0(q,s) = \sum_{q \leq s \leq q'} \boldsymbol{\mu}_0(q,s) \stackrel{(1.1)}{=} \delta_q(q'').$$

We see that the function $A : Q \times Q \to \mathbb{Z}$ satisfies the recurrence (1.1) so that

$$A(q,q') = \boldsymbol{\mu}_Q(q,q').$$

This completes the proof of Theorem 1.3. □

Remark 1.4. The conditions ($\boldsymbol{M}_\pm$) are not as restrictive as they look. For example, they are automatically satisfied if the poset Q is a lattice. □

2. Layered posets and their doubles

We define a *layer* structure on a poset P to be a pair $(\boldsymbol{\epsilon}, \boldsymbol{\lambda})$ satisfying the following conditions.

- $\boldsymbol{\epsilon}$ is an increasing map $\boldsymbol{\epsilon} : P \to \{-1, 1\}$ called the *sign map* of the layer. We set $P^\pm := \boldsymbol{\epsilon}^{-1}(\pm)$. P^- is called the *lower layer* and P^+ is called the *upper layer*.
- $\boldsymbol{\lambda}$ is a poset isomorphism $\boldsymbol{\lambda} : P^- \to P^+$ such that $x < \boldsymbol{\lambda}(x)$, $\forall x \in P^-$. The map $\boldsymbol{\lambda}$ is called the *lifting* map. Its inverse $\boldsymbol{\delta} : P^+ \to P^-$ is called the *drop map*.

If P is a lattice, we say that the layer structure is *compatible with the lattice structure* if the layers are sublattices, i.e.,

$$P^\pm \wedge P^\pm \subset P^\pm, \;\; P^\pm \vee P^\pm \subset P^\pm,$$

and the lifting map is an *isomorphism* of lattices

Example 2.1. Let n be a positive integer. Then the poset $\mathcal{L}_n$ has a natural layer structure $(\boldsymbol{\epsilon}, \boldsymbol{\lambda})$ defined by

$$\boldsymbol{\epsilon}(S) = \begin{cases} 1 & n \in S \\ -1 & n \not\in S. \end{cases}, \;\; \forall S \subset \{1, \dots, n\}, \;\; \text{and} \;\; \boldsymbol{\lambda}(S) = S \cup \{n\}, \;\; \forall S \subset \{1, \dots, n-1\}.$$

In Figure 2 we depicted the layer structures in the posets $\mathcal{L}_n$, $1 \leq n \leq 4$. The layers are separated by dotted lines. □

Suppose P is a poset equipped with a layer structure $(\boldsymbol{\epsilon}, \boldsymbol{\lambda})$. We define the *double* of P to be the poset $\mathcal{D}_P$ defined as the connected sum

$$\mathcal{D}_P = P_{Q_0}\#_{Q_1} P$$

where $Q = P^+$, $i_0 : Q \to P$ is the canonical inclusion $P^+ \hookrightarrow P$, and i_1 is the drop map $\boldsymbol{\delta} : P^+ \to P^- \hookrightarrow P$. Equivalently,

$$\mathcal{D}_P = P_{\;P^+}\#_{P^-}\, P.$$

As as set we can identify $\mathcal{D}_P$ with the disjoint union $P \sqcup P = P \times \{-1, 1\}$. We the identify Q_0 with $P^+ \times \{-1\}$ and Q_1 with $P^- \times \{1\}$. The transition map $j_1 : Q_0 \to Q_1$ is given by

$$j_1(p, -1) = (\boldsymbol{\delta}(p), 1), \;\; \forall p \in P^+.$$

We see that the double $\mathcal{D}_P$ is equipped with a canonical layer structure $(\widehat{\boldsymbol{\epsilon}}, \widehat{\boldsymbol{\lambda}})$, where

$$\widehat{\boldsymbol{\epsilon}} : P \times \{-1, 1\} \to \{-1, 1\}$$

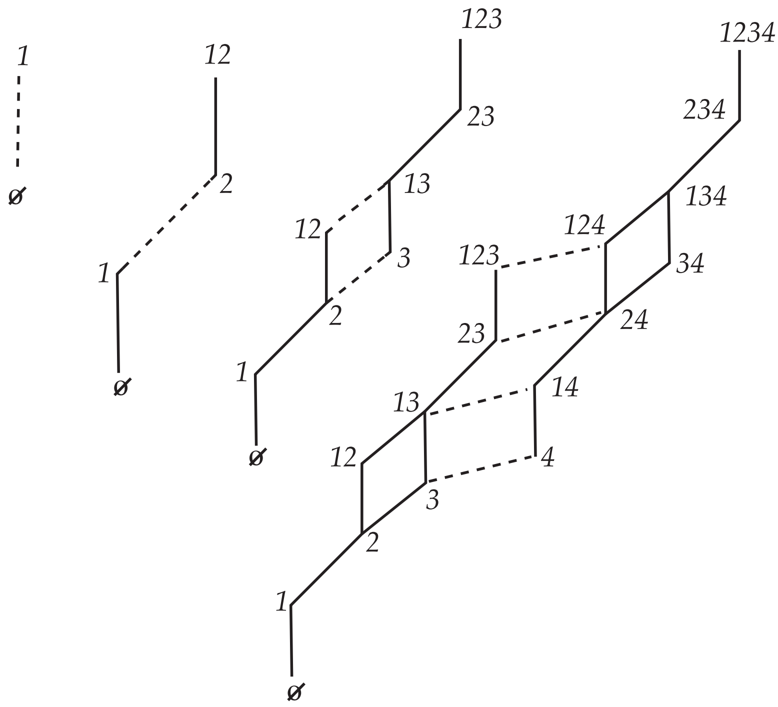

FIGURE 2. *The layered structure of $\mathcal{L}_n$.*

is the canonical projection and the lifting map $\widehat{\boldsymbol{\lambda}} : P \times \{-1\} \to P \times \{1\}$ is the tautological map,

$$\widehat{\boldsymbol{\lambda}}(x, -1) = (x, 1), \quad \forall x \in P.$$

Remark 2.2. Observe that if $(x_0, \epsilon_0), (x_1, \epsilon_1) \in \mathcal{D}_P$ $(\epsilon_i = \pm 1)$ then $(x_0, \epsilon_0) \prec_{\mathcal{D}_P} (x_1, \epsilon_1)$ if and only if

$$(\epsilon_0 = \epsilon_1 \text{ and } x_0 <_P x_1) \text{ or } (\epsilon_0 < \epsilon_1 \text{ and } \exists z \in P^+ : \; x_0 <_P z, \; \boldsymbol{\delta}(z) < x_1). \qquad \square$$

Example 2.3. Figure 2 shows that $\mathcal{L}_{n+1}$ is the double of $\mathcal{L}_n$ for $n = 1, 2, 3$. □

Definition 2.4. Suppose P is a poset equipped with the layer structure $(\boldsymbol{\epsilon}, \boldsymbol{\lambda})$. We say that the layer structure satisfies the property $(\boldsymbol{M})$ if for every $x \in P$ the sets $(P^+)^{\geq x}$ and $(P^-)^{\leq x}$ are nonempty and there exist maps

$$\boldsymbol{m}^- : P \to P^-, \quad \boldsymbol{m}^+ : P \to P^+$$

such that

$$\boldsymbol{m}^+(x) = \min(P^+)^{\geq x}, \quad \boldsymbol{m}^-(x) = \max(P^-)^{\leq x}.$$

Equivalently, this means that if $x \in P^-$ and $y \in P^+$ then $x < y$ if and only if

$$x \leq \boldsymbol{m}^-(y) \text{ and } \boldsymbol{m}^+(x) \leq y. \tag{2.1}$$

We say that $\boldsymbol{m}^{\pm}$ are the *associated maps* of the $\boldsymbol{M}$-structure. □

Lemma 2.5. *Suppose P is a poset equipped with a layer structure $(\boldsymbol{\epsilon}, \boldsymbol{\lambda})$ satisfying property $(\boldsymbol{M})$ with associated maps $\boldsymbol{m}^{\pm}$. Then the double $\mathcal{D}_P$ also satisfies the property $(\boldsymbol{M})$. Moreover, if we denote by $\widehat{\boldsymbol{m}}^{\pm} : \mathcal{D}_P \to \mathcal{D}_P^{\pm}$ the associated maps, and we identify $\mathcal{D}_P$ with $P \times \{-1, 1\}$ then*

$$\widehat{\boldsymbol{m}}^+(x, \epsilon) = \begin{cases} (x, 1) & \boldsymbol{\epsilon} = 1, \\ (\, \boldsymbol{\delta}(\, \boldsymbol{m}^+(x)\,), 1\,) & \epsilon = -1, \end{cases}, \quad \widehat{\boldsymbol{m}}^-(x, \boldsymbol{\epsilon}) = \begin{cases} (x, -1) & \epsilon = -1, \\ (\, \boldsymbol{\lambda}(\, \boldsymbol{m}^-(x)\,), -1\,) & \epsilon = 1. \end{cases}$$

Proof. Clearly $(\mathcal{D}_P^+)^{\geq(x,1)} \neq \emptyset$ and in this case $\min(\mathcal{D}_P^+)^{\geq(x,1)} = (x,1)$. Let $(x,-1) \in \mathcal{D}_P^-$. Then

$$x \leq_P \boldsymbol{m}^+(x), \ \ (\boldsymbol{m}_+(x),-1) \prec_{\mathcal{D}} (\boldsymbol{\delta}(\boldsymbol{m}^+(x)),1),$$

so that

$$(\boldsymbol{\delta}(\boldsymbol{m}^+(x)),1) \in (\mathcal{D}_P^+)^{\geq(x,-1)}.$$

Suppose $(y,1) \in \mathcal{D}_P^+$ and $(x,-1) \prec (y,1)$. Then (see Remark 2.2) there exists $z \in P^+$ and such that

$$x \leq_P z, \ \ \boldsymbol{\delta}(z) \leq_P y.$$

Hence $z \in (P^+)^{\geq x}$ so that $\boldsymbol{m}^+(x) \leq z$, and thus $\boldsymbol{\delta}(\boldsymbol{m}^+(x)) \leq \boldsymbol{\delta}(z) \leq y$. Hence

$$(\,\boldsymbol{\delta}(\boldsymbol{m}^+(x)),\,1\,) = \min(\mathcal{D}_P^+)^{\geq(x,-1)}.$$

Similarly, $(\mathcal{D}_P^-)^{\leq(x,-1)}$ is nonempty and $\max(\mathcal{D}_P^-)^{\leq(x,-1)} = (x,-1)$. Let $(x,1) \in \mathcal{D}_P^+$. Then

$$\boldsymbol{m}^-(x) <_P x, \ \ (\boldsymbol{\lambda}(\boldsymbol{m}^-(x)),-1\,) \prec_{\mathcal{D}} (x,1),$$

so that

$$(\boldsymbol{\lambda}(\boldsymbol{m}^-(x)),-1\,) \in (\mathcal{D}_P^-)^{\leq(x,1)}.$$

Suppose $(y,-1) \leq (x,1)$. Then, according to Remark 2.2 there exists $z \in P^+$ such that

$$y \leq_P z \ \text{ and } \ \boldsymbol{\delta}(z) \leq_P x.$$

Hence $\boldsymbol{\delta}(z) \in (P^-)^{\leq x}$ so that $\boldsymbol{\delta}(z) \leq \boldsymbol{m}^-(x)$. Using the monotonicity of $\boldsymbol{\lambda}$ we deduce

$$y \leq_P z \leq_P \boldsymbol{\lambda}(\boldsymbol{m}^-(x)),$$

so that

$$(\boldsymbol{\lambda}(\boldsymbol{m}^-(x)),-1) = \max(\mathcal{D}_P^-)^{\leq(x,1)}. \qquad \square$$

Using Theorem 1.3 we deduce the following consequence.

Corollary 2.6. *Suppose P is equipped with a layer structure $(\boldsymbol{\epsilon},\boldsymbol{\lambda})$ satisfying the property $\boldsymbol{M}$. Denote by $\boldsymbol{\mu}$ the Möbius function of P, by $\boldsymbol{\mu}^+$ the Möbius function of P^+ and by $\widehat{\boldsymbol{\mu}}$ the Möbius function of the double $\mathcal{D}_P$. Then, for any $x_0,x_1 \in P$, and any $\epsilon_0,\epsilon_1 = \pm 1$ we have*

$$\widehat{\boldsymbol{\mu}}\big(\,(x_0,\epsilon_0), \ \ (x_1,\epsilon_1)\,\big) = \begin{cases} \boldsymbol{\mu}(x_0,x_1) & \text{if } \ \epsilon_0 = \epsilon_1, \\ -\boldsymbol{\mu}^+(x_0,\boldsymbol{\lambda}(x_1)) & \text{if } \ \epsilon_0 < \epsilon_1, \ \ x_0 \in P^+, \ \ x_1 \in P^-, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

3. The structure of $\mathcal{L}_n$

We want to apply the abstract results proven so far to the special case of the poset $\mathcal{L}_n$. We denote by $<_n$ the partial order on $\mathcal{L}_n$, and by $\boldsymbol{\mu}_n$ the Möbius function of $\mathcal{L}_n$. The following is the key structural result.

Proposition 3.1. *For every $n \geq 1$ the map*

$$\Psi_n : \mathcal{D}_{\mathcal{L}_n} \to \mathcal{L}_{n+1}, \ \ S \mapsto \Psi_n(S,\epsilon) = \begin{cases} S & \epsilon = -1, \\ S \cup \{n+1\} & \epsilon = 1 \end{cases}$$

is an isomorphism of posets.

Proof. We denote by $\prec_n$ the partial order on $\mathcal{D}_{L_n}$. Observe first that

$$\mathcal{L}_n^+ := \{\, S \in \mathcal{L}_n;\;\; S \ni n \,\}.$$

If $\boldsymbol{\delta}_n$ denotes the drop map of $\mathcal{L}_n$ then

$$\boldsymbol{\delta}_n(S) = S \setminus \{n\},\;\; \forall S \in \mathcal{L}_n^+.$$

Clearly Ψ_n is a bijection. We first prove that it is increasing. Suppose

$$(S_0, \epsilon_0) \prec_n (S_1, \epsilon_1).$$

Using Remark 2.2 we distinguish two cases.

- $\epsilon_0 = \epsilon_1 = \epsilon$. Then $S_0 <_n S_1$ which implies immediately that $\Psi_n(S_0, \epsilon) <_{n+1} \Psi_n(S_1, \epsilon)$.
- $\epsilon_0 = -1$, $\epsilon_1 = 1$ and there exists $T \in \mathcal{L}_n^+$ such that

$$S_0 <_n T,\;\; T \setminus \{n\} <_n S_1.$$

Then

$$S_0 <_n \big(T \setminus \{n\}\big) \cup \{n+1\} <_n S_1 \cup \{n+1\} \Longrightarrow \Psi_n(S_0, -1) <_{n+1} \Psi_n(S_1, 1).$$

This proves that Ψ_n is increasing. We have to prove that the inverse map $\Phi_n = \Psi_n^{-1}$ is also increasing. For a set $S \in \mathcal{L}_{n+1}$ we define $S' \in \mathcal{L}_n$ by $S' := S \setminus \{n+1\}$. Observe that

$$\Phi_n^{-1}(S) = \begin{cases} (S', -1) & (n+1) \notin S \\ (S', 1) & (n+1) \in S. \end{cases}$$

Suppose $S <_{n+1} T$. We distinguish several cases.

- $(n+1) \in S$. Then since $S <_{n+1} T$ we must also have $(n+1) \in T$ so that $S' <_n T'$ and thus

$$(S', 1) = \Phi_n(S) \prec_n (T', 1) = \Phi_n(T).$$

- $(n+1) \notin S \cup T$. In this case again we have $S' = T'$ so that $S' <_n T'$ and thus

$$(S', -1) = \Phi_n(S) \prec_n (T', -1) = \Phi_n(T).$$

- $(n+1) \notin S$, $(n+1) \in T$. Then

$$S' = S,\;\; T' = T \setminus \{n+1\},\;\; \Phi_n(S) = (S', -1),\;\; \Phi_n(T) = (T', 1).$$

To prove that $\Phi_n(S) \prec_n \Phi_n(T)$ we need to find $U \in \mathcal{L}_n^+$ such that

$$S' \leq_n U,\;\; \boldsymbol{\delta}_n(U) = U \setminus \{n\} \leq_n T'.$$

We define

$$S'' := \begin{cases} \emptyset & \text{if } S = \emptyset \\ S \setminus \{\max S\} & \text{if } S \neq \emptyset. \end{cases}$$

Note that $S'' < T'$. Now we set $U = S'' \cup \{n\} \in \mathcal{L}_n^+$. Then

$$S' \leq_n U \;\text{ and }\; U \setminus \{n\} = S'' <_n T'. \qquad \square$$

From the above proposition we deduce inductively that we can identify $\mathcal{L}_n$ with the set of sequences $\vec{s} \in \{-1, 1\}^n$ via the map

$$\{-1, 1\}^n \ni \vec{s} \mapsto X_{\vec{s}} = \{\, i;\; s_i = 1 \,\}.$$

We then have

$$\mathcal{L}_n^{\pm} = \{\, \vec{s};\;\; s_n = \pm 1 \,\},$$

The lifting map $\boldsymbol{\lambda}_n : \mathcal{L}_n^- \to \mathcal{L}_n^+$ is given by

$$(s_1, \dots, s_{n-1}, -1) \mapsto (s_1, \dots, s_{n-1}, 1).$$

We have natural isomorphisms $\varphi_{\pm} : \mathcal{L}_{n-1} \to \mathcal{L}_n^{\pm}$ given by

$$(s_1, \dots, s_{n-1}) \mapsto (s_1, \dots, s_{n-1}, \pm 1),$$

and we also have *predecessor maps* $\boldsymbol{\pi} : \mathcal{L}_n \to \mathcal{L}_{n-1}$ given by

$$\boldsymbol{\pi}(s_1, \dots, s_{n-1}, s_n) = (s_1, \dots, s_{n-1}) \Longleftrightarrow \boldsymbol{\pi}(S) = S \setminus \{n\}.$$

Note that $\boldsymbol{\pi}\circ\varphi_\pm = \mathbb{1}$ and $\vec{s}\leq_n \vec{t}$ if and only

$$\sum_{i\geq k} s_i \leq \sum_{i\geq k} t_i,\ \ \forall k=1,\dots,n.$$

Using Corollary 2.6 and the above observations we deduce the following result.

Corollary 3.2. *For every $\vec{s},\vec{t}\in\mathfrak{L}_n$ we have*

$$\boldsymbol{\mu}_n\big(\,\vec{s},\vec{t}\,\big)=\begin{cases}\boldsymbol{\mu}_{n-1}\big(\,\boldsymbol{\pi}(\vec{s}),\boldsymbol{\pi}(\vec{t}\,)\,\big) & s_n=t_n\\ -\boldsymbol{\mu}_{n-2}\big(\,\boldsymbol{\pi}^2(\vec{s}),\boldsymbol{\pi}^2(\vec{t}\,)\,\big) & s_n<t_n,\ \ s_{n-1}=1=-t_{n-1}.\\ 0 & \textit{otherwise.}\end{cases}\qquad\square$$

We can transform the above inductive formula in a more explicit one. Given two sets $S,T\subset\{1,\dots,n\}$ we define

$$\Delta_{S,T}:\{1,\dots,n\}\to\mathbb{Z},\ \ \Delta_{S,T}(k)=\#(T\cap[k,n])-\#(S\cap[k,n]),$$

The *weight* of the pair S,T is the integer

$$w(S,T):=\sum_{k=1}^n\Delta_{S,T}(k).$$

Note that

$$S\leq_n T\iff \Delta_{S,T}(k)\geq 0,\ \ \forall k=1,\dots,n.$$

Definition 3.3. We say that a pair (S,T) of subsets of $\{1,\dots,n\}$ is *elementary* if the following two conditions hold.

- $\Delta_{S,T}(k)\in\{0,1\}$, $\forall k=1,\dots,n$.
- $\Delta_{S,T}(k)\cdot\Delta_{S,T}(k+1)=0$, $\forall k=1,\dots,n-1$.

$\square$

Thus, a pair $(S,T)\in\mathfrak{L}_n\times\mathfrak{L}_n$ is elementary if and only if in the sequence

$$\Delta_{S,T}(1),\Delta_{S,T}(2),\dots,\Delta_{S,T}(n)$$

we encounter only 0's and 1's, but we do not encounter two consecutive 1's. The weight of the elementary pair is then the number of 1's in the above sequence.

Proposition 3.4. *Let $S,T\subset\{1,\dots,n\}$. Then the following statements are equivalent.*
(a) The pair (S,T) is elementary.
(b) There exists a sequence

$$1=\nu_1<\nu_2<\cdots<\nu_k<n=\nu_{k+1},$$

such that $\nu_j-\nu_{j-1}>1$, $\forall j=2,\dots,k$, and (possibly empty) subsets

$$A_1\subset B_1\subset\{1\},\ \ C_j\subset(\nu_j+1,\nu_{j+1})\cap\mathbb{Z},\ \ j=1,\dots,k,$$

such that

$$S=A_1\cup C_1\cup\{\nu_1\}\cup C_2\cup\cdots\{\nu_k\}\cup C_k,\tag{3.1a}$$

$$T=B_1\cup C_1\cup\{\nu_2+1\}\cup C_2\cup\cdots\cup\{\nu_k+1\}\cup C_k.\tag{3.1b}$$

Remark 3.5. The technical condition (b) can be easily visualized using the beads-along-a-rod picture we described in the introduction.

We indicate a subset $T\subset\{1,\dots,n\}$ by placing beads on a rod with linearly ordered positions marked 1 through n. An element $t\in T$ corresponds to a bead located in the position t on the rod. Graphically, we depict by a "•" the positions on the rod occupied by a bead, and by a "◦" the unoccupied positions (see Figure 3).

We declare an element t in T to be mobile if either $t = 1$ or $t - 1 \notin T$. Graphically the mobile elements correspond to beads that can be moved one position to the left. (In the case of a bead located on the position 1, moving it to the left corresponds to sliding it off the rod.) In Figure 3 the mobile positions are $1, 4, 7, 10, 12$.

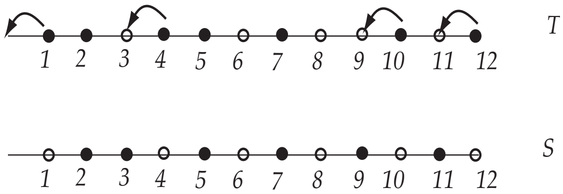

FIGURE 3. *Generating elementary pairs*

Condition (b) signifies that the distribution of beads S is obtained from the distribution T by sliding one position to the left a certain number of mobile beads of T. The number of beads that we slid to the left is precisely the weight of the pair (S, T). For the set T described in (3.1b) the mobile beads that are slid to the left are the beads located on the positions

$$(B_1 \setminus A_1) \cup \{ \nu_2 + 1, \dots, \nu_k + 1 \}.$$

The weight of the pair (S, T) is then $(k - 1) + \#(B_1 \setminus A_1)$. In Figure 3 we obtain S from T by sliding to the left the mobile T-beads located at $1, 4, 10, 12$. □

The proof of Proposition 3.4 is an elementary induction using the above remark. If we define

$$\boldsymbol{\rho} : \mathcal{L}_n \to \mathbb{Z}, \;\; \boldsymbol{\rho}(S) := \sum_{s \in S} s, \;\; \forall S \subset \{1, \dots, n\},$$

then the sliding-beads description of elementary pairs implies the following result.

Corollary 3.6. *If (S, T) is an elementary pair of subsets of $\{1, \dots, n\}$ then*

$$w(S, T) = \boldsymbol{\rho}(T) - \boldsymbol{\rho}(S).$$ □

An elementary induction based on Corollary 3.2 implies the following result.

Theorem 3.7. *Let $S, T \in \mathcal{L}_n$. Then*

$$\boldsymbol{\mu}_n(S, T) = \begin{cases} (-1)^{\boldsymbol{\rho}(T) - \boldsymbol{\rho}(S)} & (S, T) \text{ is an elementary pair,} \\ 0 & \text{otherwise.} \end{cases} \tag{$\boldsymbol{\mu}$}$$

Corollary 3.8. *Let $S, T \in \mathcal{L}_n$. If $\#T - \#S \notin \{0, 1\}$ then $\boldsymbol{\mu}_n(S, T) = 0$.*

Proof. Observe that if (S, T) is an elementary pair then $\#T - \#S \in \{0, 1\}$. □

A chain of a poset P is a linearly ordered nonempty subset $C \subset P$. The endpoints of a chain C are the elements $\min C$ and $\max C$. The length of a chain C is the integer

$$\ell(C) := \#C - 1.$$

A poset P is called *graded* if there exists an increasing function $r : P \to \mathbb{Z}$ such that for any $x \leq y$ in P, and any *maximal* chain C with endpoints x and y we have

$$\ell(C) = r(y) - r(x).$$

A function with this property is called a *rank function* for the graded poset.

Proposition 3.9. *The poset $\mathfrak{L}_n$ is graded. As rank function we can take the function $\boldsymbol{\rho}$.*

Proof. Clearly it suffices to prove that any maximal chain from $\emptyset$ to S has length $\boldsymbol{\rho}(S)$. Let $S \subset \{1, \dots, n\}$, $S \neq \emptyset$. A maximal chain from $\emptyset$ to S (of length ℓ) is a sequence

$$\emptyset \lessdot D_1 \lessdot \cdots \lessdot D_\ell = S$$

where $I \lessdot J$ means that J *covers* I, i.e., $I < J$, and there is no element $K \in \mathfrak{L}_n$ such that $I < K < J$. In terms of bead distribution the condition $I \lessdot J$ signifies that the bead distribution I is obtained from J after a single elementary left-slide. This shows that if $I \lessdot J$ then $\boldsymbol{\rho}(J) = \boldsymbol{\rho}(I) + 1$ so that every maximal chain from $\emptyset$ to S has length $\rho(S)$. □

Recall that the dual of a poset $(P, <)$ is the poset $(P^*, <^*)$ which coincides with P as a set but it is equipped with the opposite order, i.e.,

$$x <^* y \Longleftrightarrow y < x.$$

The Möbius function $\boldsymbol{\mu}^*$ of P^* is related to the Möbius function of P via the equality

$$\boldsymbol{\mu}^*(x, y) = \boldsymbol{\mu}(y, x), \;\; \forall x, y \in P.$$

A poset P is called *selfdual* if it is isomorphic to the dual poset P^*. Any poset isomorphism $P \to P^*$ is called a *self-duality* of P.

Proposition 3.10. *The map $\boldsymbol{\sigma}_n : \mathfrak{L}_n \to \mathfrak{L}_n$ given by*

$$\{-1, 1\}^n \ni \vec{s} \mapsto -\vec{s} \in \{-1, 1\}^n$$

is a selfduality of $\mathfrak{L}_n$. In particular, we deduce that

$$\boldsymbol{\mu}_n(\vec{s}, \vec{t}) = \boldsymbol{\mu}_n(-\vec{t}, -\vec{s}).$$

Proof. We have

$$\vec{s} \leq \vec{t} \Longleftrightarrow \sum_{i \geq k} s_i < \sum_{i \geq k} t_i, \;\; \forall k = 1, \dots, n$$

$$\Longleftrightarrow -\sum_{i \geq k} s_i > -\sum_{i \geq k} t_i, \;\; \forall k = 1, \dots, n \Longleftrightarrow -\vec{t} <_n -\vec{t}. \qquad \square$$

Remark 3.11. If we regard an element $S \in \mathfrak{L}_n$ as a subset of $\{1, \dots, n\}$ then $\boldsymbol{\sigma}_n(S)$ is complement of S in $\{1, \dots, n\}$. □

The selfduality $\boldsymbol{\sigma}_n$ interacts nicely with the layer structure on $\mathfrak{L}_n$. More precisely, we have

$$\boldsymbol{\sigma}_n(\mathfrak{L}_n^{\pm}) = \mathfrak{L}_n^{\mp}.$$

Lemma 2.5 and Proposition 3.1 imply inductively that the layered poset $\mathfrak{L}_n$ satisfies the property $(\boldsymbol{M})$. We denote by $\boldsymbol{m}_n^{\pm} : \mathfrak{L}_n \to \mathfrak{L}_n^{\pm}$ the associated functions. Note that

$$\boldsymbol{m}_n^{\pm} \circ \boldsymbol{\sigma}_n = \boldsymbol{\sigma}_n \circ \boldsymbol{m}_m^{\mp}.$$

Using Lemma 2.5 we deduce the following result

Lemma 3.12. *Let $\vec{s} = (s_1, \dots, s_n) \in \{-1, 1\}^n$, and set*

$$(t_1, \dots, t_{n-2}, 1) = \boldsymbol{m}_{n-1}^{+}(s_1, \dots, s_{n-1}).$$

Then

$$\boldsymbol{m}_n^{+}(\vec{s}) = \begin{cases} \vec{s} & s_n = 1 \\ \big(\boldsymbol{\delta}_{n-1} \circ \boldsymbol{m}_{n-1}^{+}(\boldsymbol{\pi}(\vec{s})\,), 1\big) & s_n = -1 \end{cases}$$

$$= \begin{cases} \vec{s} & s_n = 1 \\ (t_1, \dots, t_{n-2}, -1, 1) & s_n = -1. \end{cases} \qquad \square$$

Corollary 3.13. *Let* $S \subset \{1, \dots, n\}$ *and set*

$$S' := \begin{cases} \emptyset & \text{if } S = \emptyset \\ S \setminus \{\max S\} & \text{if } S \neq \emptyset. \end{cases}$$

Then

$$\boldsymbol{m}_n^+(S) = S' \cup \{n\}.$$

Proof. Define $M_n : 2^{[n]} \to 2^{[n]}$, $M_n(S) = S' \cup \{n\}$. It is easy to check that $\boldsymbol{m}_1^+ = M_1$ and that the maps M_n satisfy the recurrence in Lemma 3.12 so that $M_n = \boldsymbol{m}_n^+$, $\forall n \geq 1$. □

Remark 3.14. (a) The operation $S \mapsto \boldsymbol{m}_n^+(S)$ has an intuitive description. First $\boldsymbol{m}_n^+(\emptyset) = \{n\}$. If $n \in S$ then $\boldsymbol{m}_n^+(S) = S$, while if $n \notin S$, then $\boldsymbol{m}_n^+(S)$ is obtained by "trading" the greatest element of S for n, e.g.,

$$\boldsymbol{m}_{17}^+(\{2, 5, 7, 8, \mathbf{11}\}) = \{2, 5, 7, 8, \mathbf{17}\}.$$

In terms of bead distributions, the operation $S \mapsto \boldsymbol{m}_n^+(S)$ corresponds to sliding the leftmost bead of S all the way to the last position n on the rod.

(b) Using the selfduality $\boldsymbol{\sigma}_n$ we deduce

$$\boldsymbol{m}_n^- = \boldsymbol{\sigma}_n \circ \boldsymbol{m}_n^+ \circ \boldsymbol{\sigma}_n.$$

Using the description of $\boldsymbol{\sigma}_n$ in Remark 3.11 we can give very intuitive description of $\boldsymbol{m}_n^-$. More precisely, if $n \notin S$ then $\boldsymbol{m}_n^-(S) = S$. Next, if $S = \{n\}$ then $\boldsymbol{m}_n^-(S) = \emptyset$. Finally if $\{n\} \subsetneq S$ then $\boldsymbol{m}_n^-(S)$ is obtained from S by trading the element $n \in S$ with the greatest element *not in* S. E.g.,

$$\boldsymbol{m}_{17}^-(\{2, 5, 7, 8, 16, \mathbf{17}\}) = \{2, 5, 7, 8, \mathbf{15}, 16\}.$$ □

We define a *simplicial scheme with vertex set* V to be a family $\mathcal{S}$ of nonempty subsets $S \subset V$ such that

$$\emptyset \neq S \subset T, \ \ T \in \mathcal{S} \Longrightarrow S \in \mathcal{S}.$$

The sets $S \subset \mathcal{S}$ are called the *faces* of the simplicial scheme. To a simplicial scheme $\mathcal{S}$ we can associate in a canonical way a triangulated space $|\mathcal{S}|$ called the *geometric realization* of $\mathcal{S}$ (see [6, §2]). Recall (see [3]) that the *nerve* of a poset P is the simplicial scheme $\mathcal{N}_P$ with vertex set P defined by

$$S \in \mathcal{N}_P \Longleftrightarrow C \text{ is a chain in } P.$$

We denote by $|P|$ the geometric realization of $\mathcal{N}_P$. We say that a poset P is homeomorphic (homotopic) to a topological space X if its geometric realization is such. The Möbius function $\boldsymbol{\mu}_P$ of P is related to $|P|$ via the celebrated formula of P. Hall ([3, Eq. (9.14)], [8, §3.8])

$$1 + \boldsymbol{\mu}_P(x, y) = \chi\big(\, |\, (x, y)_P \,| \, \big),$$

where $(x, y)_P$ denotes the open interval

$$(x, y)_P := \big\{ z \in P; \ \ x < z < y \, \big\}$$

and χ denotes the Euler characteristic of a space, with $\chi(\emptyset) := 0$.

Theorem 3.15. *Suppose* $(I, J) \in \mathcal{L}_n \times \mathcal{L}_n$ *is an elementary pair. Then the closed interval* $[I, J]_{\mathcal{L}_n}$ *is isomorphic to the boolean poset* $\mathcal{B}_{\boldsymbol{\rho}(J) - \boldsymbol{\rho}(I)}$. *In particular*

$$\big| \, (I, J)_{\mathcal{L}_n} \, \big| \cong S^{\boldsymbol{\rho}(J) - \boldsymbol{\rho}(I) - 2},$$

where S^k *denotes the* k*-dimensional sphere if* $k \geq 0$*, while* $S^{-1} := \emptyset$*.*

Proof. We argue by induction on n. The result is clearly true for $n = 1$ so we assume it is true for any $k \leq n$ and we prove it for $n + 1$. Obviously, if $(n + 1) \notin J$ then $I, J \subset \{1, \dots, n\}$ and the claim follows by induction.

Similarly, if $(n + 1) \in I$ then $(n + 1) \in J$, the we have

$$(I, J)_{\mathcal{L}_{n+1}} \subset \mathcal{L}^+_{n+1} \cong \mathcal{L}_n,$$

and again we can conclude by induction. Thus we only need to consider the case $(n+1) \in J \setminus I$. Since the pair (I, J) is elementary we deduce from Proposition 3.4 that $n \in S \setminus T$. We define

$$\bar{I} := I \setminus \{n\}, \quad \bar{J} := J \setminus \{n + 1\}.$$

Note that $\bar{I}, \bar{J} \subset \{1, \dots, n - 1\}$. From Proposition 3.4 we deduce that $(\bar{I}, \bar{J}) \in \mathcal{L}_{n-1} \times \mathcal{L}_{n-1}$ is an elementary pair as well. Moreover

$$\boldsymbol{\rho}(J) - \boldsymbol{\rho}(I) = \boldsymbol{\rho}(\bar{J}) - \boldsymbol{\rho}(\bar{I}) + 1., \tag{3.2}$$

and

$$\boldsymbol{m}^+_{n+1}(I) = \bar{I} \cup \{n + 1\}, \;\; \boldsymbol{m}^-_{n+1}(J) = \bar{J} \cup \{n\} \tag{3.3}$$

Observe that if $S \in [I, J]_{\mathcal{L}_{n+1}}$ then S *contains exactly one of the numbers* n *or* $n + 1$. Now define a map

$$[\bar{I}, \bar{J}]_{\mathcal{L}_{n-1}} \times \mathcal{B}_1 \cong [\bar{I}, \bar{J}]_{\mathcal{L}_{n-1}} \times \{-1, 1\} \xrightarrow{\Xi} [I, J]_{\mathcal{L}_{n+1}},$$

$$[\bar{I}, \bar{J}]_{\mathcal{L}_{n-1}} \times \{-1, 1\} \ni (K, \epsilon) \mapsto \Xi(K) := \begin{cases} K \cup \{n\} & \epsilon = -1 \\ K \cup \{n + 1\} & \epsilon = 1. \end{cases}$$

We set

$$\bar{S} := S \setminus \{n, n + 1\} \subset \{1, \dots, n - 1\}.$$

We distinguish two cases.

A. $\underline{n \in S}$. Then $S = \bar{S} \cup \{n\}$. Using (2.1) and the inequality $S \leq_{n+1} J$ we deduce that

$$S \leq \boldsymbol{m}^-_{n+1}(J) = \bar{J} \cup \{n\}.$$

Hence $\bar{I} \cup \{n\} \leq_{n+1} S \leq_{n+1} \bar{J} \cup \{n\}$ so that $\bar{S} \in [\bar{I}, \bar{J}]_{\mathcal{L}_{n-1}}$.

B. $\underline{n + 1 \in S}$. Then $S = \bar{S} \cup \{n + 1\}$. From the inequality $I \leq_{n+1} S$ and (2.1) we deduce that

$$\bar{I} \cup \{n + 1\} = \boldsymbol{m}^+_{n+1}(I) \leq_{n+1} S = \bar{S} \cup \{n + 1\}$$

so that $\bar{S} \in [\bar{I}, \bar{J}]_{\mathcal{L}_{n-1}}$. This discussion shows that the map

$$\Gamma : [I, J]_{\mathcal{L}_{n+1}} \to [\bar{I}, J]_{\mathcal{L}_{n-1}}$$

given by

$$[I, J]_{\mathcal{L}_{n+1}} \ni S \mapsto \begin{cases} (\bar{S}, -1) & n \in S \\ (\bar{S}, 1) & n + 1 \in S. \end{cases}$$

is the inverse of the map Ξ so that Ξ is a bijection. Clearly Ξ is increasing. Let us prove that Γ is also increasing. Suppose

$$I \leq_{n+1} S <_{n+1} T \leq_{n+1} J.$$

If $(n + 1) \in S$ then $(n + 1) \in T$ and we have $S = \bar{S} \cup \{n + 1\} <_{n+1} T = \bar{T} = T \cup \{n + 1\}$. Hence

$$\Gamma(S) = (\bar{S}, 1) < (\bar{T}, 1) = \Gamma(T).$$

We deduce similarly that if $n \in T$ then $\Gamma(S) < \Gamma(T)$. Thus we need to discuss the case $n \in S$ and $(n + 1) \in T$. We have $S = \bar{S} \cup \{n\}$ and $T = \bar{T} \cup \{n + 1\}$. Using the inequality $S <_{n+1} T$ and (2.1) we deduce

$$\bar{S} \cup \{n\} \leq_{n+1} \boldsymbol{m}^-_{n+1}(T) = \bar{T} \cup \{n\} \Longrightarrow \Gamma(S) = (\bar{S}, -1) < (\bar{T}, 1) = \Gamma(T).$$

Hence Ξ is an isomorphism of posets. From the induction assumption we deduce that we have an isomorphism of posets

$$[I, J]_{\mathcal{L}_{n+1}} \cong \mathcal{B}_{\boldsymbol{\rho}(\bar{J}) - \boldsymbol{\rho}(\bar{I})} \times \mathcal{B}_1,$$

Using the isomorphism $\mathcal{B}_{k+1} \cong \mathcal{B}_k \times \mathcal{B}_1$ and the equality (3.2) we deduce that

$$[I,J]_{\mathcal{L}_{n+1}} \cong \mathcal{B}_{\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)}. \qquad \square$$

Proposition 3.16. *The poset $\mathcal{L}_n$ is a lattice. Moreover, for every $n>1$ and any $\vec{s},\vec{t} \in \{-1,1\}^n$ such that $s_n \le t_n$, we have*

$$\vec{s} \vee \vec{t} = \begin{cases} \big(\boldsymbol{\pi}(\vec{s}) \vee \boldsymbol{\pi}(\vec{t}), \epsilon\big) & \text{if } s_n = t_n = \epsilon \in \{\pm 1\} \\ \big(\boldsymbol{\pi}(\,\boldsymbol{m}_n^+(\vec{s})\,) \vee \boldsymbol{\pi}(\vec{t}), 1\big) & \text{if } s_n = -1,\ t_n = 1, \end{cases}$$

$$\vec{s} \wedge \vec{t} = \begin{cases} \big(\boldsymbol{\pi}(\vec{s}) \wedge \boldsymbol{\pi}(\vec{t}), \epsilon\big) & \text{if } s_n = t_n = \epsilon \in \{\pm 1\} \\ \big(\boldsymbol{\pi}(\vec{s}) \wedge \boldsymbol{\pi}(\,\boldsymbol{m}_n^-(\vec{t})\,), -1\big) & \text{if } s_n = -1,\ t_n = 1. \end{cases}$$

Proof. Using the selfduality $\boldsymbol{\sigma}_n$ it suffices to prove only to deal with the join operation. We argue by induction on n. Clearly $\mathcal{L}_1$ is a lattice. For the inductive step consider $\vec{s},\vec{t} \in \mathcal{L}_n$, $s_n \le t_n$. We distinguish two cases.

A. If $s_n = t_n = \epsilon = \pm 1$ then $\vec{s},\vec{t} \in \mathcal{L}_n^{\pm}$. Using the poset isomorphisms $\varphi_{\pm} : \mathcal{L}_{n-1} \to \mathcal{L}_n^{\pm}$ and $\mathcal{L}_n \cong \mathcal{D}_{\mathcal{L}_{n-1}}$ we deduce from the induction assumption that $\vec{s} \vee \vec{t}$ exists and satisfies

$$\vec{s} \vee \vec{t} = \big(\boldsymbol{\pi}(\vec{s}) \vee \boldsymbol{\pi}(\vec{t}), \epsilon\big).$$

B. $s_n = -1$, $t_n = 1$. Then $\vec{t} \in \mathcal{L}_n^+$. If $\vec{s},\vec{t} <_n \vec{v}$ then $\vec{v} \in \mathcal{L}_n^+$. In particular, since $\mathcal{L}_n$ satisfies property $(\boldsymbol{M})$ we deduce that

$$\vec{v}_n > \boldsymbol{m}_n^+(\vec{s}).$$

Hence $\vec{s} \vee \vec{t}$ exists if and only if $\boldsymbol{m}_n^+(\vec{s}) \vee \vec{t}$ exists and in this case we have

$$\vec{s} \vee \vec{t} = \boldsymbol{m}_n^+(\vec{s}) \vee \vec{t}.$$

We are now in the case **A** because both $\boldsymbol{m}_n^+(\vec{s})$ and $\vec{t}$ belong to $\mathcal{L}_n^+$. The equality

$$\boldsymbol{m}_n^+(\vec{s}) \vee \vec{t} = \big(\boldsymbol{\pi}(\,\boldsymbol{m}_n^+(\vec{s})\,) \vee \boldsymbol{\pi}(\vec{t}), 1\big)$$

follows as in case **A**. $\square$

Example 3.17. Suppose $n > 11$ and

$$S = \{1,4,6,7,11\} \in \mathcal{L}_n, \;\; T = \{2,5,9,10\} \in \mathcal{L}_n.$$

Then

$$\begin{aligned} S \vee T &= \{1,4,6,7,\mathbf{11}\} \vee \{2,5,9,\mathbf{11}\} = \{1,4,6,\mathbf{9},\mathbf{11}\} \vee \{2,5,\mathbf{9},\mathbf{11}\} \\ &= \{1,4,\mathbf{6},\mathbf{9},\mathbf{11}\} \vee \{2,\mathbf{6},\mathbf{9},\mathbf{11}\} = \{1,\mathbf{4},\mathbf{6},\mathbf{9},\mathbf{11}\} \vee \{\mathbf{4},\mathbf{6},\mathbf{9},\mathbf{11}\} \\ &= \{\mathbf{1},\mathbf{4},\mathbf{6},\mathbf{9},\mathbf{11}\}. \\ S \wedge T &= \{1,4,6,7,\mathbf{10}\} \wedge \{2,5,9,\mathbf{10}\} = \{1,4,6,\mathbf{7},\mathbf{10}\} \wedge \{2,5,\mathbf{7},\mathbf{10}\} \\ &= \{1,4,\mathbf{5},\mathbf{7},\mathbf{10}\} \wedge \{2,\mathbf{5},\mathbf{7},\mathbf{10}\} = \{1,\mathbf{2},\mathbf{5},\mathbf{7},\mathbf{10}\} \wedge \{\mathbf{2},\mathbf{5},\mathbf{7},\mathbf{10}\} \\ &= \{\mathbf{2},\mathbf{5},\mathbf{7},\mathbf{10}\} \end{aligned}$$

Observe that

$$\boldsymbol{\rho}(S \vee T) + \boldsymbol{\rho}(S \wedge T) = \boldsymbol{\rho}(S) + \boldsymbol{\rho}(T). \qquad \square$$

Proposition 3.18. *The lattice $\mathcal{L}_n$ is modular, i.e.,*

$$\boldsymbol{\rho}(S \vee T) + \boldsymbol{\rho}(S \wedge T) = \boldsymbol{\rho}(S) + \boldsymbol{\rho}(T), \;\; \forall S,T \in \mathcal{L}_n. \tag{3.4}$$

Moreover

$$S \vee \boldsymbol{\sigma}_n(S) = \{1,\dots,n\}, \;\; S \cap \boldsymbol{\sigma}_n(S) = \emptyset, \;\; \forall S \in \mathcal{L}_n, \tag{3.5}$$

$$S \vee T = \{1,\dots,n\}, \;\; S \wedge T = \emptyset \Longrightarrow T = \boldsymbol{\sigma}_n(S). \tag{3.6}$$

Proof. We argue by induction on n. Denote by $\vee_n$ and $\wedge_n$ the lattice operations on $\mathfrak{L}_n$.
For $n = 1$ the result is obvious. For the inductive step consider

$$S = \{s_1 < \ldots < s_k\} \in \mathfrak{L}_{n+1}, \;\; T = \{t_1 < \cdots < t_\ell\} \in \mathfrak{L}_n,$$

and define $S' = S \setminus \{s_k\}$, $T' := T \setminus \{T_\ell\}$. Then Proposition 3.16 implies

$$S \vee_{n+1} T = \big(\, S' \vee_n T' \,\big) \cup \{\max(s_k, t_\ell)\}$$

$$S \wedge_{n+1} T = \Big(\, S' \wedge_n T'\}\Big) \cup \{\min(s_k, t_\ell)\}.$$

We deduce

$$\rho(S \vee_{n+1} T) + \rho(S \wedge_{n+1} = \boldsymbol{\rho}(S' \vee_n T') + \boldsymbol{\rho}(S \wedge_n T') + \max(s_k, t_\ell) + \min(s_k, t\ell)$$

$$\text{(by induction)} = \rho(S') + \boldsymbol{\rho}(T') + s_k + t_\ell = \boldsymbol{\rho}(S) + \boldsymbol{\rho}(T).$$

This proves (3.4). The inductive argument proving (3.5) and (3.6) is very simple and we leave it to the reader. □

A modular lattice is *EL*-shellable . Using [5, Thm. 5.6] (see also [9, Sec. 3.2]) we obtain the following result.

Corollary 3.19. *If $I <_n J$ then the geometric realization of the open interval $(I, J)_{\mathfrak{L}_n}$ is contractible.*□

We can be much more precise about the topology of the order intervals in the above corollary.

Proposition 3.20. *If (I, J) is not an elementary pair of $\mathfrak{L}_n$ and $I <_n J$, then the (open) order interval is homeomorphic to $B^{\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)-2}$, where B^d denotes the d-dimensional closed Euclidean ball.*

Proof. We argue as in the proof of [4, Thm. 2.7.7]. It suffices to investigate the structure of order intervals of length 2.

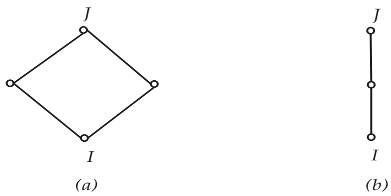

FIGURE 4. *Intervals of length* 2.

Suppose that $I <_n J$ and $\boldsymbol{\rho}(J) - \boldsymbol{\rho}(I) = 2$. In terms of distributions of beads along a rod this mens that the distribution I can be obtained from the distribution J by exactly two elementary left moves. If the two left moves involve different beads, so that the pair (I, J) is elementary, then the Hasse diagram of $[I, J]_{\mathfrak{L}_n}$ is depicted in Figure 4(a). If the two left moves involve the same bead, so that the pair (I, J) is non-elementary, then $[I, J]_{\mathfrak{L}_n}$ is a chain of length 2 (Figure 4(b)).
Using Remark 3.5 we deduce that if $I <_n J$ then the pair (I, J) is non elemntary if and only if there exists a non-elementary pair (I', J') such that $I', J' \in [I, J]_{\mathfrak{L}_n}$, $I' <_n J'$ and $\boldsymbol{\rho}(J') - \boldsymbol{\rho}(I') = 2$. This implies (see [3, Thm. 11.4] or [4, Sec. A2.4]) that the poset $(I, J)_{\mathfrak{L}_n}$ is homeomorphic to $B^{\boldsymbol{\rho}(J)-\boldsymbol{\rho}(I)-2}$. □

Recall that an element x of a lattice P is called *join reducible* if there exist $y, z < x$ such that $x = y \vee z$.

Proposition 3.21. *The set $S \subset \{1, \dots, n\}$ defines a join reducible element of the lattice $\mathfrak{L}_n$ if and only if it has a* gap, *i.e., there exists $1 < k < n$ such that $k \notin S$ and*

$$[1, k) \cap S, (k, n] \cap S \neq \emptyset.$$

Proof. Suppose S has a gap. Then S has the form

$$S = \{s_1 < \cdots < s_\ell\} \subset \{1, \dots, n\},$$

and for some $1 < j \leq \ell$ we have $s_j - s_{j-1} > 1$. We define

$$S_0 = \{s_1, \dots, s_{j-1}, s_j - 1, s_{j+1}, \dots, s_\ell\}, \;\; S_1 = \{s_j, s_j + 1 \dots, s_j + (\ell - j)\}.$$

Then

$$S_0, S_1 <_n S, \;\; S_0 \vee S_1 = S.$$

If S has no gap, so that S has the form $S = \{j, j+1, \dots, j+\ell\}$, then S covers a unique $S' \in \mathfrak{L}_n$, more precisely

$$S' = \begin{cases} \{j-1, j+1, \dots, j+\ell\} & j > 1 \\ \{j+1, \dots, j+\ell\} & j = 1. \end{cases}$$

This proves that S is join irreducible because if $S_0, S_1 <_n S$ then $S_0, S_1 \leq_n S'$ so that $S_0 \vee S_1 \leq S'$. □

Remark 3.22. We define $R, M : \mathfrak{L}_n \to \mathfrak{L}_n$ as follows. If $S = \emptyset$ we set $M(S) = R(S) = \emptyset$. If S join irreducible we set $M(S) = S$ and we define $R(S)$ to be the distribution of beads obtained from S by sliding to the left the first bead of S.

If S is join reducible, we set

$$\gamma(S) = \max\big\{1 \leq k \leq n; \;\; k \text{ is a gap of } S\,\big\}, \;\; u(S) := \#[\gamma(S), n] \cap S.$$

Note that $\gamma(S) + 1 \in S$. We define $R(S)$ to be the distribution of beads obtained from S by sliding to the left the bead on position $\gamma(S) + 1$,

$$R(S) = \big(\, S \setminus \{\, \gamma(S) + 1 \,\}\,\big) \cup \{\gamma(S)\},$$

and we set

$$M(S) := \big\{\, \gamma(S) + 1, \cdots, \gamma(S) + u(S)\,\big\}.$$

Observe that $M(S)$ is join irreducible, S covers $R(S)$ and $S = R(S) \vee M(S)$. This decomposition is minimal in the sense that if $S = A \vee B$, where B is irreducible and S covers A then $\boldsymbol{\rho}(B) \geq \boldsymbol{\rho}(\, M(S)\,)$.

We obtain in this fashion a maximal chain

$$S \gtrdot R(S) \gtrdot R^2(S) \gtrdot \cdots \gtrdot R^{\boldsymbol{\rho}(S)}(S) = \emptyset.$$

In the terminology of EL-shellings, this chain is the lexicographically minimal saturated chain from $\emptyset$ to S. □

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