Microlocal Investigations of Shape

Notes for "Topics in topology" class, Fall 2006.

Last updated: December 6, 2006

Liviu I. Nicolaescu

Contents

Chapter 1

Grassmannians

1.1. Linear Grassmannians

Suppose V is a real vector space of dimension n. For every $0 \leq k \leq n$ we denote by $\mathbf{Gr}_k(V)$ the set of k-dimensional vector subspaces of V. We will say that $\mathbf{Gr}_k(V)$ is the *linear Grassmannian of k-planes in E*. When $V = \mathbb{R}^n$ we will write $\mathbf{Gr}(n, k)$ instead of $\mathbf{Gr}_k(\mathbb{R}^n)$.

We would like to give several equivalent descriptions of the natural structure of smooth manifold on $\mathbf{Gr}_k(V)$. To do this it is very convenient to fix an Euclidean metric on V. We will denote the corresponding inner product by $\bullet$.

Any k-dimensional subspace $L \subset V$ is uniquely determined by the orthogonal projection onto L which we will denote by P_L. Thus we can identify $\mathbf{Gr}_k(V)$ with the set of rank k projectors

$$\mathbf{Proj}_k(V) := \{ P : V \to V;\ \ P^* = P = P^2,\ \ \operatorname{rank} P = k \}.$$

We have a natural map

$$P : \mathbf{Gr}_k(V) \to \mathbf{Proj}_k(V),\ \ L \mapsto P_L$$

with inverse

$$P \mapsto \operatorname{Range}(P).$$

$\mathbf{Proj}_k(V)$ is a subset of the vector space of symmetric endomorphisms

$$\operatorname{End}^+(V) := \{ A \in \operatorname{End}(V),\ \ A^* = A \}.$$

$\operatorname{End}^+(V)$ is equipped with a natural inner product

$$(A, B) := \frac{1}{2} \operatorname{tr}(AB),\ \ \forall A, B \in \operatorname{End}^+(V). \tag{1.1}$$

The norm on $\operatorname{End}^+(V)$ induced by this inner product is 1/2 the norm of a symmetric operator viewed as a bounded operator between Hilbert spaces.

$\mathbf{Proj}_k(V)$ is a closed and bounded subset of $\operatorname{End}^+(V)$. The bijection $P : \mathbf{Gr}_k(V) \to \mathbf{Proj}_k(V)$ induces a topology on $\mathbf{Gr}_k(V)$. We want to show that $\mathbf{Gr}_k(V)$ has a natural structure of smooth manifold compatible with this topology. To see this we define for every $L \subset \mathbf{Gr}_k(V)$ the set

$$\mathbf{Gr}_k(V, L) := \{ U \in \mathbf{Gr}_k(V); \; U \cap L^\perp = 0 \}.$$

Lemma 1.1.1. *(a) Let $L \in \mathbf{Gr}_k(V)$. Then*

$$U \cap L^\perp = 0 \Longleftrightarrow \mathbb{1} - P_L + P_U : V \to V \text{ is an isomorphism.} \tag{1.2}$$

(b) $\mathbf{Gr}_k(V, L)$ is an open subset of $\mathbf{Gr}_k(V)$.

Proof. (a) Note first that a dimension count implies that

$$U \cap L^\perp = 0 \Longleftrightarrow U + L^\perp = V \Longleftrightarrow U^\perp \cap L = 0.$$

Let us show that $U \cap L^\perp = 0$ implies that $\mathbb{1} - P_L + P_L$ is an isomorphism. It suffices to show that

$$\ker(\mathbb{1} - P_L + P_U) = 0.$$

Suppose $v \in \ker(\mathbb{1} - P_L + P_U)$. Then

$$0 = P_L(\mathbb{1} - P_L + P_U)v = P_L P_U v = 0 \Longrightarrow P_U v \in U \cap \ker P_L = U \cap L^\perp = 0.$$

Hence $P_U v = 0$ so that $v \in U^\perp$. From the equality $(\mathbb{1} - P_L - P_U)v = 0$ we also deduce $(\mathbb{1} - P_L)v = 0$ so that $v \in L$. Hence

$$v \in U^\perp \cap L = 0.$$

Conversely, we will show that if $\mathbb{1} - P_L + P_U = P_{L^\perp} + P_U$ onto then $U + L^\perp = V$. Indeed let $v \in V$. Then there exists $x \in V$ such that

$$v = P_{L^\perp} x + P_U x \in L^\perp + U.$$

(b) We have to show that for every $K \in \mathbf{Gr}_k(V, L)$ there exists $\varepsilon > 0$ such that any U satisfying

$$\|P_U - P_K\| < \varepsilon$$

intersects $L^\perp$ trivially. Since $K \in \mathbf{Gr}_k(V, L)$ we deduce from (a) that

$$\mathbb{1} - P_L - P_K : V \to V$$

is an isomorphism. Note that

$$\|(\mathbb{1} - P_L - P_K) - (\mathbb{1} - P_L - P_U)\| = \|P_K - P_U\|.$$

The space of isomorphisms of V is an open subset of $\mathrm{End}(V)$. Hence there exists $\varepsilon > 0$ such that for any U satisfying

$$\|P_U - P_K\| < \varepsilon$$

the endomorphism $(\mathbb{1} - P_L - P_U)$ is an isomorphism. We now conclude using part (a). $\square$

Since $L \in \mathbf{Gr}_k(V, L)$, $\forall L \in \mathbf{Gr}_k(V)$ we have an open cover of $\mathbf{Gr}_k(V)$

$$\mathbf{Gr}_k(V) = \bigcup_{L \in \mathbf{Gr}_k(V)} \mathbf{Gr}_k(V, L).$$

Note that for every $L \in \mathbf{Gr}_k(V)$ we have a natural map

$$\Gamma : \mathrm{Hom}(L, L^\perp) \to \mathbf{Gr}_k(V, L),$$

which associates to each linear map $S : L \to L^\perp$ its graph (see Figure 1.1)

$$\Gamma_S = \{x + Sx \in L + L^\perp = V;\ \ x \in L\}.$$

We will show that this is a homeomorphism by providing an explicit description of the orthogonal projection P_{Γ_S}

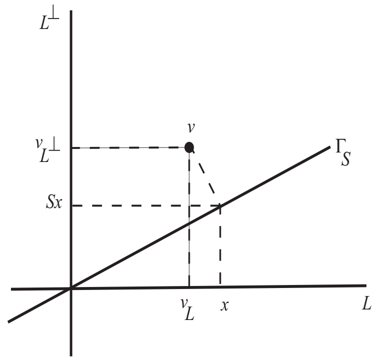

Figure 1.1. *Subspaces as graphs of linear operators.*

Observe first that the orthogonal complement of Γ_S is the graph of $-S^* : L^\perp \to L$. More precisely

$$\Gamma_S^\perp = \Gamma_{-S^*} = \big\{ y - S^* y \in L^\perp + L = V;\ \ y \in L^\perp \big\}.$$

Let $v = P_L v + P_{L^\perp} v = v_L + v_{L^+} \in V$ (see Figure 1.1). Then

$$P_{\Gamma_S} v = x + Sx,\ \ x \in L \Longleftrightarrow v - (x + Sx) \in \Gamma_S^\perp$$

$$\Longleftrightarrow \exists x \in L,\ \ y \in L^\perp \ \text{ such that } \left\{ \begin{array}{rcl} x + S^* y & = & v_L \\ Sx - y & = & v_{L^\perp} \end{array} \right. .$$

Consider the operator $\mathcal{S} : L \oplus L^\perp \to L \oplus L^\perp$ which has the block decomposition

$$\mathcal{S} = \left[\begin{array}{cc} \mathbb{1}_L & S^* \\ S & -\mathbb{1}_L^\perp \end{array} \right].$$

Then the above linear system can be rewritten as

$$\mathcal{S} \cdot \left[\begin{array}{c} x \\ y \end{array} \right] = \left[\begin{array}{c} v_L \\ v_{L^\perp} \end{array} \right].$$

Now observe that

$$\mathcal{S}^2 = \left[\begin{array}{cc} \mathbb{1}_L + S^* S & 0 \\ 0 & \mathbb{1}_{L^\perp} + SS^* \end{array} \right].$$

Hence $\mathcal{S}$ is invertible and

$$\mathcal{S}^{-1} = \left[\begin{array}{cc} (\mathbb{1}_L + S^* S)^{-1} & 0 \\ 0 & (\mathbb{1}_{L^\perp} + SS^*)^{-1} \end{array} \right] \cdot \mathcal{S}$$

$$= \left[\begin{array}{cc} (\mathbb{1}_L + S^* S)^{-1} & (\mathbb{1}_L + S^* S)^{-1} S^* \\ (\mathbb{1}_{L^\perp} + SS^*)^{-1} S & -(\mathbb{1}_{L^\perp} + SS^*)^{-1} \end{array} \right].$$

We deduce

$$x = (\mathbb{1}_L + S^* S)^{-1} v_L + (\mathbb{1}_L + S^* S)^{-1} S^* v_{L^\perp}$$

and

$$P_{\Gamma_S} v = \begin{bmatrix} x \\ Sx \end{bmatrix}.$$

Hence P_{Γ_S} has the block decomposition

$$P_{\Gamma_S} = \begin{bmatrix} \mathbb{1}_L \\ S \end{bmatrix} \cdot [(\mathbb{1}_L + S^*S)^{-1} \quad (\mathbb{1}_L + S^*S)^{-1}S^*]$$
$$= \begin{bmatrix} (\mathbb{1}_L + S^*S)^{-1} & (\mathbb{1}_L + S^*S)^{-1}S^* \\ S(\mathbb{1}_L + S^*S)^{-1} & S(\mathbb{1}_L + S^*S)^{-1}S^* \end{bmatrix}.$$

Note that if $U \in \mathbf{Gr}_k(V, L)$ and with respect to the decomposition $V = L + L^\perp$ the projector P_U has the block form

$$P_U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} P_L P_U I_L & P_L P_U I_{L^\perp} \\ P_{L^\perp} P_U I_L & P_L L^\perp P_U I_{L^\perp} \end{bmatrix}$$

where for every subspace $K \hookrightarrow V$ we denoted by $I_K : K \to V$ the canonical inclusion, then $U = \Gamma_S$, where $S = CA^{-1}$. This shows that the graph map

$$\mathrm{Hom}(L, L^\perp) \ni S \mapsto \Gamma_S \in \mathbf{Gr}_k(V)$$

is a homeomorphism. Moreover, the above formulæ show that if $U \in \mathbf{Gr}_k(V, L_0) \cap \mathbf{Gr}_k(V, L_1)$ then we can represent U in two was,

$$U = \Gamma_{S_0} = \Gamma_{S_1}, \quad S_i \in \mathrm{Hom}(L_i, L_i^\perp), \quad i = 0, 1$$

and the map

$$S_0 \to S_1$$

is smooth. This shows that $\mathbf{Gr}_k(V)$ has a natural structure of smooth manifold of dimension

$$\dim \mathbf{Gr}_k(V) = \dim \mathrm{Hom}(L, L^\perp) = k(n-k).$$

The above considerations shows that via the projection map $U \mapsto P_U$ we can regard $\mathbf{Gr}_k(V)$ as a submanifold of $\mathrm{End}^+(V)$. The Euclidean metric (1.1) on $\mathrm{End}(V)$ induces a metric $h = h_{n,k}$ on $\mathbf{Gr}_k(V)$.

Denote by $O(V)$ the group of orthogonal transformations of V. The group $O(V)$ acts smoothly and transitively on $\mathbf{Gr}_k(V)$

$$O(V) \times \mathbf{Gr}_k(V) \ni (g, L) \mapsto g(L) \in \mathbf{Gr}_k(V).$$

Note that

$$P_{gL} = g P_L g^{-1}.$$

The action of $O(V)$ on $\mathrm{End}^+(V)$ by conjugation preserves the inner product on $\mathrm{End}^-(V)$ and thus we deduce the action of $O(V)$ on $\mathbf{Gr}_k(V)$ preserves the metric $\hat{h}$.

We would like to express this metric in the graph coordinates. Consider $L \in \mathbf{Gr}_k(V)$ and $S \in \mathrm{Hom}(L, L^\perp)$. Then, for every $t \in \mathbb{R}$, we have

$$U_t := \Gamma_{tS} \in \mathbf{Gr}_k(V, L).$$

If $\dot{U}_0$ denotes the tangent to the path $t \mapsto U_t$ at $t = 0$, then

$$\hat{h}(\dot{U}_0, \dot{U}_0) = \frac{1}{2} \mathrm{tr}(\dot{P}_0^2), \quad \dot{P}_0 := \frac{d}{dt}|_{t=0} P_{\Gamma_{tS}},$$

If we write $P_t := P_{\Gamma_{tS}}$ we deduce

$$P_t = \left[\begin{array}{cc} (\mathbb{1}_L + t^2 S^*S)^{-1} & t(\mathbb{1}_L + t^2S^*S)^{-1}S^* \\ tS(\mathbb{1}_L + t^2S^*S)^{-1} & t^2S(\mathbb{1}_L + t^2S^*S)^{-1}S^* \end{array}\right].$$

Hence

$$\dot{P}_{t=0} = \left[\begin{array}{cc} 0 & S^* \\ S & 0 \end{array}\right] = S^*P_{L^\perp} + SP_L. \tag{1.3}$$

so that

$$\hat{h}(\dot{U}_0, \dot{U}_0) = \frac{1}{2}\big(\operatorname{tr}(SS^*) + \operatorname{tr}(S^*S)\big) = \operatorname{tr}(SS^*).$$

We can be even more concrete by choosing an orthonormal basis $(\vec{e}_i)_{1\leq i\leq k}$ of L and an orthonormal basis $(\boldsymbol{e}_\alpha)_{k<\alpha\leq n}$ of $L^\perp$.

With respect to these bases the map $S : L \to L^\perp$ is described by a matrix $(s_{\alpha i})_{1\leq i\leq k<\alpha\leq n}$ and then

$$\operatorname{tr}(SS^*) = \operatorname{tr}(S^*S) = \sum_{i,\alpha} |s_{\alpha i}|^2.$$

We can think of the collection $(s_{\alpha i})$ as defining local coordinates on $\mathbf{Gr}_k(V, L)$. Hence

$$\hat{h}(\dot{U}_0, \dot{U}_0) = \sum_{i,\alpha} |s_{\alpha i}|^2. \tag{1.4}$$

In integral geometric computations we will find convenient to relate the above coordinates to the classical language of moving frames. In the sequel we make the following notational conventions.

- We will use small Latin letters $i, j, k, ...$ to denote indices in the range $\{1, ..., k\}$.
- We will use the Greek letters $\alpha, \beta, \gamma, ...$ do denote indices in the range $\{k + 1, \cdots, n\}$.
- We will use Latin letters $A, B, C, ...$ to denote indices in the range $\{1, \cdots, n\}$.

Suppose we have a smooth 1-parameter family of orthonormal frames $(\boldsymbol{e}_A) = (\boldsymbol{e}_A(t))$, $|t| \ll 1$. This defines a smooth path

$$t \mapsto L_t = \operatorname{span}(\boldsymbol{e}_i(t)) \in \mathbf{Gr}_k(V).$$

We would like to compute

$$\hat{h}(\dot{L}_0, \dot{L}_0).$$

Observe that we have a smooth path

$$t \mapsto g_t \in O(V),$$

defined by

$$g_t\boldsymbol{e}_A(0) = \boldsymbol{e}_A(t).$$

With respect to the fixed frame $(\boldsymbol{e}_A(0))$ the orthogonal transformation g_t is given my a matrix $(s_{AB}(t))$, where

$$s_{AB} = \boldsymbol{e}_A(0) \bullet \big(g_t\boldsymbol{e}_B(0)\big).$$

Observe that $g_0 = \mathbb{1}_V$. Let P_t denote the projection onto L_t. Then

$$P_t = g_tP_0g_t^{-1}$$

so that if we set $X = \frac{d}{dt}|_{t=0} g_t$ we have

$$\dot{P}_0 = [X, P_0].$$

With respect to the decomposition $V = L_0 + L_0^\perp$ the projector P_0 has the block decomposition

$$P_0 = \begin{bmatrix} \mathbb{1}_{L_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

X is represented by a skew-symmetric matrix with entries

$$x_{AB} = \dot{s}_{AB} = \boldsymbol{e}_A \bullet \dot{\boldsymbol{e}}_B$$

which has the block form

$$X = \begin{bmatrix} X_{L_0, L_0} & -X^*_{L_0^\perp, L_0} \\ X_{L_0^\perp, L_0} & X_{L_0^\perp, L_0^\perp} \end{bmatrix},$$

where $X_{L_0^\perp, L_0}$ denotes a map $L_0 \to L_0^\perp$ etc. We deduce

$$[X, P_0] = \begin{bmatrix} 0 & X^*_{L_0^\perp, L_0} \\ X_{L_0^\perp, L_0} & 0 \end{bmatrix}.$$

We deduce

$$\hat{h}(\dot{L}_0, \dot{L}_0) = \frac{1}{2}\operatorname{tr}(\dot{P}_0, \dot{P}_0) = \operatorname{tr}(X_{L_0^\perp, L_0} X^*_{L_0^\perp, L_0}) = \sum_{\alpha, i} |\dot{s}_{\alpha i}|^2. \tag{1.5}$$

We will find it convenient later on to interpret the above computations in the language of moving frames.

Suppose M is a smooth m-dimensional manifold and $L : M \to \mathbf{Gr}_k(V)$ is a smooth map. Fix a point $p_0 \in M$ and local coordinates $(u^i)_{1 \le i \le m}$ near p_0 such that $u^i(p_0) = 0$.

The map L_t can be described near p_0 via a moving frame, i.e. an orthonormal frame $(\boldsymbol{e}_A)$ depending smoothly on (u^i) such that

$$L(u) = \operatorname{span}(\, \boldsymbol{e}_i(u)\,).$$

The above computations show that the differential of L at p_0 is described by the $(n-k) \times k$ matrix of 1-forms on M

$$D_{p_0} L = (\theta_{\alpha i}), \quad \theta_{\alpha i} = \boldsymbol{e}_\alpha \bullet d\boldsymbol{e}_i.$$

This means that if $X = (\dot{u}^a) \in T_{p_0} M$ then

$$D_{p_0} L(X) = (x_{\alpha i}) \in T_{L(0)}\, \mathbf{Gr}_k(V), \;\; x_{\alpha i} = \sum_a \boldsymbol{e}_\alpha \bullet \frac{\partial \boldsymbol{e}_i}{\partial u^a} \dot{u}_a. \tag{1.6}$$

If M happens to be an open subset of $\mathbf{Gr}_k(V)$ then we can use the forms $\theta_{\alpha i}$ to describe the metric h. More precisely, the equalities (1.4) and (1.5) show that

$$h = \sum_{\alpha, i} \theta_{\alpha i} \otimes \theta_{\alpha i}. \tag{1.7}$$

1.2. Densities and integration

On an orientable manifold M we can obtain Borel measures in a simple fashion. We fix an orientation on M. Then, any nowhere vanishing, top dimensional differential form $\boldsymbol{\omega}$ on M compatible with the orientation (which maps positively oriented frames to positive numbers) defines a positive Borel measure μ_ω on M via the equality

$$\mu_\omega(U) := \int_U \omega.$$

Unfortunately, this trick does not work on Grassmannians since many of them are not orientable. To produce Borel measures we must abandon working with differential forms and instead work with densities.

If V is a finite dimensional real vector space we denote by $\det V$ the top exterior power of V, i.e. the one dimensional space $\Lambda^{\dim V} V$. Given a real number s we define an *s-density on V* to be a map

$$\lambda : \det V \to \mathbb{R}, \;\; \lambda(t\Omega) = |t|^s \lambda(\Omega), \;\; \forall t \in \mathbb{R}^*, \;\; \Omega \in \det V.$$

We denote by $|\Lambda|_s(V)$ the one dimensional space of s-densities. Note that we have a canonical identification $|\Lambda|_0(E) = \mathbb{R}$. We will refer to 1-densities simply as *densities* and denote the corresponding space by $|\Lambda|(V)$. We say that an s-density $\lambda : \det V \to \mathbb{R}$ is *positive* if

$$\lambda(\det V \setminus 0) \subset (0, \infty).$$

We denote by $|\Lambda|_s^+(V)$ the cone of positive densities.

Note that any basis $(v_1, \cdots, v_n)$ of V defines linear isomorphisms

$$|\Lambda|_s V \to \mathbb{R}, \;\; \lambda \mapsto \lambda(v_1 \wedge \cdots \wedge v_n).$$

In particular, we have a canonical identification

$$|\Lambda|_s(\mathbb{R}^n) \cong \mathbb{R}, \;\; \lambda \mapsto \lambda(e_1 \wedge \cdots \wedge e_n)$$

where $(e_1, \cdots, e_n)$ is the canonical basis of $\mathbb{R}^n$.

If V_0 and V_1 are vector spaces of the same dimension n and $g : V_0 \to V_1$ is a linear isomorphism then we get a linear map

$$g^* : |\Lambda|_s(V_1) \to |\Lambda|_s(V_0), \;\; |\Lambda|_s(V_1) \ni \lambda \mapsto g^*\lambda,$$

where

$$(g^*\lambda)\,(\wedge_i v_i) = \lambda\big(\wedge_i (g v_i)\big), \;\; \forall v_1, \cdots, v_n \in V_0.$$

If $V_0 = V_1 = V$ so that $g \in \mathrm{Aut}(EV)$ then

$$g^*\lambda = |\det g|^s \lambda.$$

For every $g, h \in \mathrm{Aut}(V)$ we gave $(gh)^* = h^* g^*$ and thus we have a *left* action of $\mathrm{Aut}(V)$ on $|\Lambda|_s(V)$

$$\mathrm{Aut}(V) \times |\Lambda|_s(V) \to |\Lambda|_s(V),$$
$$\mathrm{Aut}(V) \times |\Lambda|_s(V) \ni (g, \lambda) \mapsto g_*\lambda = (g^{-1})^*\lambda = |\det g|^{-s}\lambda.$$

Observe that we have bilinear maps

$$|\Lambda|_s(V) \otimes |\Lambda|_t(V) \to |\Lambda|_{s+t}(V), \;\; (\lambda, \mu) \mapsto \lambda \cdot \mu.$$

Note that to any short exact sequence of vector spaces

$$0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0, \;\; \dim U = m, \;\; \dim V = m, \;\; \dim W = p$$

we can associate maps

$$\backslash : |\Lambda|_s^+(U) \otimes |\Lambda|_s(V) \to |\Lambda|_s(W),$$
$$/ : |\Lambda|_s(V) \otimes |\Lambda|_s^+(W) \to |\Lambda|_s(U),$$

and

$$\times : |\Lambda|_s(U) \otimes |\Lambda|_s(W) \to |\Lambda|_s(V)$$

as follows.

- Let $\mu \in |\Lambda|_s^+(U)$ and $\lambda \in |\Lambda|_s(V)$ and suppose $(w_j)_{1\leq j\leq p}$ is a basis of W. Now choose lifts $v_j \in V$ of w_j such that $\beta(v_j) = w_j$ and a basis $(u_i)_{1\leq i\leq m}$ of U such that
$$\{u_1, \dots, u_m, \tilde{w}_1, \dots, \tilde{w}_p, \}$$
is a basis of V and we set
$$(\mu\backslash\lambda)\big(\wedge_j w_j\big) := \frac{\lambda\big((\wedge_i u_i) \wedge (\wedge_j v_j)\big)}{\mu\big(\wedge_i u_i\big)}.$$
It is easily seen that the above definition is independent of the choices of v's and u's.
- Let $\lambda \in |\Lambda|_s(V)$ and $\nu \in |\Lambda|_s^+(W)$. Given a basis $(u_i)_{1\leq i\leq m}$ of U, extend the linearly independent set $\big(\alpha(u_i)\big) \subset V$ to a basis
$$\{\,\alpha(u_1), \dots, \alpha(u_m), v_1, \cdots, v_p, \}$$
of V and now define
$$(\lambda/\nu)\big(\wedge_i u_i\big) := \frac{\lambda\big((\wedge_i \alpha(u_i)) \wedge (\wedge_j v_j)\big)}{\nu\big(\wedge_j \beta(v_j)\big)}.$$
Again it is easily verified that the above definition is independent of the various choices.
- Let $\mu \in |\Lambda|_s(U)$ and $\nu \in |\Lambda|_s(W)$. To define $\mu\times\nu : \det V \to \mathbb{R}$ it suffices to indicate its value on a single nonzero vector of the line $\det V$. Fix a basis $(u_i)_{1\leq i\leq m}$ of U and a basis $(w_j)_{1\leq j\leq p}$ of W. Choose lifts (v_j) of w_j to V. Then we set
$$(\mu \times \nu)\big((\wedge_i u_i) \wedge (\wedge_j w_j)\big) = \mu(\wedge_i u_i)\nu(\wedge_j v_j).$$
Note that for a different choice of lifts v_j' of w_j we have
$$(\wedge_i u_i) \wedge (\wedge_j v_j) = (\wedge_i u_i) \wedge (\wedge_j v_j').$$
Again one can check that this is independent of the various bases (u_i) and (w_j).

Example 1.2.1. Consider the short exact sequence

$$0 \to U = \mathbb{R} \xrightarrow{\alpha} V = \mathbb{R}^2 \xrightarrow{\beta} W = \mathbb{R} \to 0$$

given by

$$\alpha(s) = (4s, 10s), \;\; \beta(x, y) = 5x - 2y.$$

Denote by $\boldsymbol{e}$ the canonical basis of U, by $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ the canonical basis of V and by $\boldsymbol{f}$ the canonical basis of W. We obtain canonical densities λ_U on U, λ_V on V and λ_W on W given by

$$\lambda_U(\boldsymbol{e}) = \lambda_V(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) = \lambda_W(\boldsymbol{f}) = 1.$$

We would like to describe the density $\lambda_V/\beta^*\lambda_W$ on V. Set

$$\boldsymbol{f}_1 = \alpha(\boldsymbol{e}) = (2,3).$$

We choose $\boldsymbol{f}_2 \in V$ such that $\beta(\boldsymbol{f}_2) = \boldsymbol{f}$, for example, $\boldsymbol{f}_2 = (1,2)$. Then

$$\lambda_V/\beta^*\lambda_W(\boldsymbol{e}) = \lambda_V(\boldsymbol{f}_1 \wedge \boldsymbol{f}_2)/\lambda_W(\boldsymbol{f}) = \left|\det \begin{bmatrix} 4 & 10 \\ 1 & 2 \end{bmatrix}\right| = 2.$$

Hence $\lambda_V/\beta^*\lambda_W = 2\lambda_U$. □

Suppose now that $E \to M$ is a real vector bundle of rank n over the smooth manifold M. Assume it is given by the open cover (U_α) and gluing cocycle

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{Aut}(V),$$

where V is a *fixed* real vector space of dimension n. Then the bundle of s-densities associated to E is the real line bundle $|\Lambda|_s E$ given by the open cover (U_α) and gluing cocycle

$$|\det g_{\beta\alpha}|^{-s} : U_{\alpha\beta} \to \mathrm{Aut}(\,|\Lambda|_s(V)\,).$$

We denote by $C^\infty(|\Lambda|_s E\,)$ the space of smooth sections of $|\Lambda|_s E$. Such a section is given by a collection of smooth maps

$$\lambda_\alpha : U_\alpha \to |\Lambda|_s(V)$$

satisfying the gluing conditions

$$\lambda_\beta(x) = |\det g_{\beta\alpha}|^{-s}\lambda_\beta(x), \ \ \forall \alpha, \beta, \ \ x \in U_{\alpha\beta}.$$

Let us point out that if $V = \mathbb{R}^n$ then we have a canonical identification $|\Lambda|_s(\mathbb{R}^n) \to \mathbb{R}$ and in this case a density can be regarded as a collection of smooth functions $\lambda_\alpha : U_\alpha \to \mathbb{R}$ satisfying the above gluing conditions.

An s-density $\lambda \in C^\infty(|\Lambda|_s E)$ is called *positive* if for every $x \in M$ we have $\lambda(x) \in |\Lambda|_s^+(E_x)$.

If $\phi : N \to M$ is a smooth map and $E \to M$ is a smooth real vector bundle, we obtain the pullback bundle $\pi^*E \to N$. We have canonical isomorphisms

$$|\Lambda|_s \pi^* E \cong \pi^* |\Lambda|_s E$$

and a natural pullback map

$$\phi^* : C^\infty(|\Lambda|_s E) \to C^\infty(\pi^* |\Lambda|_s E) \cong C^\infty(|\Lambda|_s \pi^* E).$$

Given a short exact sequence of vector bundles

$$0 \to E_0 \to E_1 \to E_2 \to 0$$

over E we obtain maps

$$\backslash : C^\infty(|\Lambda|_s^+ E_0) \times C^\infty(|\Lambda|_s E_1) \to C^\infty(|\Lambda|_s E_2)$$
$$/ : C^\infty(|\Lambda|_s E_1) \times C^\infty(|\Lambda|_s E_2) \to C^\infty(|\Lambda|_s E_0)$$
$$\times : C^\infty(|\Lambda|_s E_0) \times C^\infty(|\Lambda|_s E_2) \to C^\infty(|\Lambda|_s E_1).$$

Observe that for every positive smooth function $f : M \to (0,\infty)$ we have

$$(f\mu)\backslash\lambda = (f^{-1}(\mu\backslash\lambda),\ \ \lambda/(f\nu) = (f^{-1})(\mu/\lambda).$$

Moreover, for $\mu \in C^\infty(|\Lambda|_s^+ E_0)$, $\nu \in C^\infty(|\Lambda|_s^+ E_2)$ we have

$$\mu\backslash(\mu\times\nu) = \nu,\ \ (\mu\times\nu)/\nu = \mu.$$

In the sequel we will almost exclusively need a special case of the above construction, when E is the tangent bundle of the smooth manifold M. We will denote by $|\Lambda|_s(M)$ the line bundle $|\Lambda|_s(TM)$ and we will refer to its sections as (smooth) s-densities on M. When $s = 1$ we will use the simpler notation $|\Lambda|_M$ to denote $|\Lambda|_1(M)$.

To give a local description of s-densities we first fix a coordinate atlas $(U_\alpha,\ \ (x^i_\alpha)\,)$ where

$$x^i_\alpha : U_\alpha \to \mathbb{R},\ \ i = 1,\cdots,n = \dim M$$

are local coordinates on U_α. Suppose $p \in U_{\alpha\beta}$. A tangent vector $v \in T_pM$ has coordinate decompositions

$$\sum_i X^i_\alpha \partial_{x^i_\alpha} = v = \sum_j X^j_\beta \partial_{x^j_\beta}.$$

Using the identity

$$\partial_{x^i_\alpha} = \sum_j \frac{\partial x^j_\beta}{\partial x^i_\alpha}\partial_{x^j_\beta}$$

we deduce

$$\sum_j X^j_\beta \partial_{x^j_\beta} = \sum_j \Big(\sum_i X^i_\alpha \frac{\partial x^j_\beta}{\partial x^i_\alpha}\Big)\partial_{x^j_\beta} \Longrightarrow X^j_\beta = \sum_i \frac{\partial x^j_\beta}{\partial x^i_\alpha} X^i_\alpha.$$

This proves that the tangent bundle TM is given by the open cover (U_α) and gluing maps

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(n,\mathbb{R}),\ \ g_{\beta\alpha} = \Big(\frac{\partial x^j_\beta}{\partial x^i_\alpha}\Big)_{1\le i,j\le n} \in \mathrm{GL}(n,\mathbb{R})$$

We deduce that an s-density on M is described by a coordinate atlas

$$\big(U_\alpha,\ \ (x^i_\alpha)\big),$$

and smooth functions

$$\lambda_\alpha : U_\alpha \to \mathbb{R}$$

satisfying the conditions

$$\lambda_\beta = |d_{\beta\alpha}|^{-s}\lambda_\alpha,\ \ \text{where}\ \ d_{\beta\alpha} = \det\left(\frac{\partial x^j_\beta}{\partial x^i_\alpha}\right)_{1\le i,j\le n}. \tag{1.8}$$

We deduce that the smooth 0-densities on M are precisely the smooth functions.

Example 1.2.2. (a) Suppose $\omega \in \Omega^n(M)$ is a top degree differential form on M. Then in a coordinate atlas $(U_\alpha,\ \ (x^i_\alpha)\,)$ this form is described by a collection of forms

$$\omega_\alpha = \lambda_\alpha dx^1_\alpha \wedge \cdots \wedge sx^n_\alpha.$$

The functions λ_α satisfy the gluing conditions

$$\lambda_\beta = d^{-1}_{\beta\alpha}\lambda_\alpha$$

and thus we deduce that the collection of functions $|\lambda_\alpha|^s$ defines an s-density on M which we will denote by $|\omega|^s$. Because of this fact the densities are traditionally described as collections

$$\lambda_\alpha |dx_\alpha|^s, \;\; dx_\alpha := dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n.$$

(b) Suppose M is an orientable manifold. By fixing an orientation we choose an atlas $(U_\alpha, \;\; (x_\alpha^i)\,)$ so that all the determinants $d_{\beta\alpha}$ are positive. If ω is a top dimensional form on M described locally by forms

$$\omega_\alpha = \lambda_\alpha dx_\alpha$$

then the collection of functions λ_α defines a density on M. Thus a choice of orientation produces an linear map

$$\Omega^n(M) \to C^\infty(|\Lambda|(M)\,).$$

As explained in [**N**, §3.4.2] this map is a bijection.

(c) Any Riemann metric g on M defines a canonical density on M denoted by $|dV_g|$ and called the volume density. It is locally described by

$$\sqrt{|g_\alpha|}|dx_\alpha|,$$

where g_α denotes the symmetric matrix representing the metric g in the coordinates (x_α^i). □

Observe that we cannot speak of its value of a given density at a point $p \in M$. However, as any section of a vector bundle, a density has a well defined zero set. The support of density is by definition the closure of the complement of its zero set. We then denote by $C_0(|\Lambda|(M)\,)$ the space of continuous densities with compact support.

The densities on a manifold serve a major purpose: they can be integrated. More precisely there is a natural linear map

$$\int_M : C_0(|\Lambda|(M)\,) \to \mathbb{R}, \;\; |d\mu| \mapsto \int |d\mu|$$

defined as follows. Represent $|d\mu|$ as a collection

$$\mu_\alpha |dx_\alpha|$$

associated to a coordinate atlas $(U_\alpha, \;\; (x_\alpha^i)\,)$. Next, choose a partition of unity subordinated to the cover (U_α), i.e. a collection of compactly supported smooth functions

$$\eta_k : C_0^\infty(M) \to [0,1],$$

such that for every k there exists $\alpha = \alpha(k)$ so that $\operatorname{supp} \eta_k \subset U_{\alpha(ik)}$ and

$$\sum_k \eta_k = 1.$$

The density $\eta_k |d\mu|$ is supported in $U_\alpha = U_{\alpha(k)}$ where it is described by

$$\eta_k \mu_\alpha |dx_\alpha|.$$

Now regard U_α as an open subset of the Euclidean space $\mathbb{R}^n$ with Euclidean coordinates (x_α^i). Then interpret $|dx_\alpha|$ as the Lebesgue measure on $\mathbb{R}^n$ and then

$$\int \eta_k \mu_\alpha |dx_\alpha|$$

as the Lebesgue integral of the function

$$\eta_k\mu_\alpha : U_\alpha \subset \mathbb{R}^n \to \mathbb{R}.$$

Define

$$\int_M |d\mu| := \sum_k \int \eta_k \mu_{\alpha(k)} |dx_{\alpha(k)}|.$$

We refer to [**N**, §3.4.1] For a proof that the above definition is independent of the various choices.

Note that if f is a continuous, compactly supported function on M and $|d\mu|$ is a density then $f|d\mu|$ is a continuous compactly supported density and thus there is a well defined integral

$$\int_M f|d\mu|.$$

Thus there is a natural pairing

$$C_0(M) \times C(|\Lambda|_M), \ \ (f, |d\mu|) \mapsto \int_M f|d\mu|.$$

Let us observe that if $|d\rho|$ and $|d\tau|$ are two positive densities then there exists a positive function f such that

$$|d\rho| = f|d\tau|.$$

The existence of this function follows from the Radon-Nicodym theorem and for every $x \in M$ we have

$$f(x) = \lim_{U \to \{x\}} \frac{\int_U |d\rho}{\int_U |d\tau|},$$

where the above limit is taken the open sets shrinking to x. We will use the notation

$$f = \frac{|d\rho|}{|d\tau|}.$$

If $\phi : M \to N$ is a diffeomorphism and $|d\rho| = (U_\alpha, \rho_\alpha |dy_\alpha|)$ is a density on N, then we define the *pullback* of $|d\rho|$ by ϕ to be the density $\phi^*|d\rho|$ on M defined by

$$\phi^*|d\rho| = \big(\phi^{-1}(U_\alpha), \rho_\alpha |dy_\alpha|\big), \ \ y_\alpha^i = x_\alpha^i \circ \phi.$$

The classical change in variables formula now takes the form

$$\int_N |d\rho| = \int_M \phi^*|d\rho|.$$

Example 1.2.3. Suppose $\phi : M \to N$ is a diffeomorphism between two smooth m-dimensional manifolds, $\omega \in \Omega^m(M)$ and $|\omega|$ is the associated density. Then

$$\phi^*|\omega| = |\phi^*\omega|.$$

□

Suppose a Lie group acts smoothly on M. Then for every $g \in G$ and any density $|d\rho|$ we get a new density $g^*|d\rho|$. The density $|d\rho|$ is called G-invariant if

$$g^*|d\rho| = |d\rho, \ \ \forall g \in G.$$

Note that a density is invariant if and only if the associated Borel measure is g-invariant. A positive density is invariant if the jacobian $\frac{|d\rho|}{|d\rho|_g}$ is identically equal to 1.

Proposition 1.2.4. *Suppose $|d\rho|$ and $|d\tau|$ are two G-invariant positive densities. Then the jacobian $\frac{|d\rho|}{|d\tau|}$ is a G-invariant smooth, positive function on G.*

Proof. Let $x \in M$ and $g \in G$. Then for every open neighborhood U of x we have

$$\int_U |d\rho| = \int_{g(U)} |d\rho|, \ \int_U |d\tau| = \int_{g(U)} |d\tau| \Longrightarrow \frac{\int_U |d\tau|}{\int_U |d\rho|} = \frac{\int_{g(U)} |d\tau|}{\int_{g(U)} |d\rho|}$$

and then letting $U \to \{x\}$ we deduce

$$J(x) = J(gx), \ \ \forall x \in M, g \in G.$$

□

Corollary 1.2.5. *If G acts smoothly transitively on the smooth manifold M then, up to a positive multiplicative constant there exists at most one invariant positive density.* □

Suppose $\Phi : M \to B$ is a submersion. The kernels of the differentials of Φ form a vector subbundle $T^V M \hookrightarrow TM$ consisting of the planes tangent to the fibers of Φ. We will refer to it as the *vertical bundle*. Since Φ is a submersion we have a short exact sequence of bundles over M.

$$0 \to T^V M \xrightarrow{D\Phi} \Phi^* TB \to 0.$$

Observe that any (positive) density $|d\nu|$ on B defines by pullback a (positive) density $\Phi^*|d\nu|$ on the bundle $\Phi^* TB \to M$. If λ is a density on $T^V M$ then we obtain a density $\lambda \times \Phi^*|d\nu|$ on M.

Suppose $|d\mu|$ is a density on M such that Φ is proper on the support of $|d\mu|$. Set $k = \dim B$, $r = \dim M - \dim B$. We would like to describe a density $\Phi_*|d\mu|$ on B called the *pushforward* of $|d\mu|$ by Φ. Intuitively, $\Phi_*|d\mu|$ is the unique density on B such that for any open subset $U \subset B$ we have

$$\int_U \Phi_*|d\mu| = \int_{\Phi^{-1}(U)} |d\mu|.$$

Proposition 1.2.6. *There exists a smooth density $\Phi_*|d\mu|$ on B uniquely characterized by the following condition. For every density $|d\nu|$ on B we have*

$$\Phi_*|d\mu| = V_\nu |d\nu|$$

where $V_\nu \in C^\infty(B)$ is given by

$$V_\nu(b) = \int_{\Phi^{-1}(b)} |d\mu| / \Phi^*|d\nu|.$$

Proof. Fix a positive density $|d\nu|$ on B. Along every fiber M_b we have a density $|d\mu|_b / \Phi^*|d\nu|$. To understand this density fix $x \in M_b$. Then we can find local coordinates $(y_j)_{1 \le j \le k}$ near $b \in B$ and smooth functions $(x^i)_{1 \le i \le r}$ defined in a neighborhood V of x in M such that the collection of functions (x^i, y^j) defines local coordinates near x on M and in these coordinates the map Φ is given by the projection $(x, y) \mapsto y$.

In the coordinates y on B we can write

$$|d\nu| = \rho_B(y)|dy| \text{ and } |d\mu| = \rho_M(x,y)|dx \wedge dy|.$$

Then along the fibers $y = const.$ we have

$$|d\mu|_b/\Phi^*|d\nu| = \frac{\rho_M(x,y)}{\rho_B(y)}|dx|.$$

We set

$$V_\nu(b) := \int_{M_b} |d\mu|_b/\Phi^*|d\nu|$$

V_ν is a smooth function on B. We can form the density $V_\nu|d\nu|$ which a priori depends on ν.

Observe that if $|d\hat{\nu}|$ is another density, then there exists a positive smooth function $w : B \to \mathbb{R}$ such that

$$|d\hat{\nu}| = w|d\nu|.$$

Then

$$|d\mu|_b/\Phi^*|d\hat{\nu}| = w^{-1}|d\mu|_b/\Phi^*|d\nu|, \quad V_{\hat{\nu}} = w^{-1}V_\nu$$

so that

$$V_{\hat{\nu}}|d\hat{\nu}| = V_\nu|d\nu|.$$

In other words the density $V_\nu|d\nu|$ on B is independent on ν. It depends only on $|d\mu|$. □

Using partitions of unity and the classical Fubini theorem we obtain the identity

$$\int_{\Phi^{-1}(U)} |d\mu| = \int_U \Phi_*|d\mu|, \text{ for any open subset } U \subset B. \tag{1.9}$$

Remark 1.2.7. Very often the submersion $\Phi : M \to B$ satisfies the following condition.

For every point on the base $b \in B$ there exist an open neighborhood U of b in B, a nowhere vanishing form $\omega \in \Omega^k(U)$, a nowhere vanishing form $\Omega \in \Omega^{k+r}(M_U)$, ($M_U := \Phi^{-1}(U)$), and a form $\eta \in \Omega^r(U)$ such that

$$\Omega = \eta \wedge \pi^*\omega.$$

Then we can write

$$|d\mu| = \rho|\Omega|$$

for some $\rho \in C^\infty(M_U)$

$$\Phi_*|d\mu| = f|\omega|$$

and then for every $u \in U \subset B$ we have

$$f(u) = \int_{M_u} \rho|\eta|.$$

In particular

$$\int_{M_U} |d\mu| = \int_U f(u)|\omega| = \int_U \Big(\int_{M_u} \rho|\eta|\Big)|\omega|.$$

□

Example 1.2.8 (Co-area formula)**.** Suppose (M, g) is a Riemann manifold of dimension $m+1$, and $f : M \to \mathbb{R}$ is a smooth function without critical points. On M we have a volume density $|dV_g|$. We would like to compute the pushforward density $f_*|dV_g|$ on $\mathbb{R}$. We seek $f_*|dV_g|$ in the form

$$f_*|dV_g| = \rho(t)|dt|$$

where $|dt|$ is the Euclidean volume density on $\mathbb{R}$, and ρ is a smooth function.

For $t \in \mathbb{R}$ we set $M_t := f^{-1}(t)$. M_t is a codimension 1 submanifold of M. We denote by $|dV_t|$ the volume density on M_t defined by the induced metric $g_t := g|_{M_t}$. We denote by ∇f the g-gradient of f, and we set $\boldsymbol{n} := \frac{1}{|\nabla f|}\nabla f$.

Fix $t_0 \in \mathbb{R}$. For every point $p \in M_{t_0}$ we have $df(p) \neq 0$, and from the implicit function theorem we deduce that we can find an open neighborhood U, and smooth function $x^1, \dots, x^m$ such that $(f, x^1, \dots, x^m)$ are local coordinates on U. Then along U we can write

$$|dV_g| = \omega|df \wedge dx^1 \wedge \cdots \wedge dx^m|.$$

is a unit normal vector field along $M_{t_0} \cap U$, and we have

$$|dV_{t_0}| \mid_{U\cap M_{t_0}} = \omega|\boldsymbol{n} \lrcorner (df \wedge dx^1 \wedge \cdots \wedge dx^m) \mid_{U\cap M_{t_0}} | = \omega|\nabla f||(dx^1 \wedge \cdots \wedge dx^m) \mid_{U\cap M_{t_0}} |$$

Now observe that along U we have

$$|dV_g|/f^*|dt| = \omega|(dx^1 \wedge \cdots \wedge dx^m)|$$

so that

$$|dV_{t_0}| \mid_{U\cap M_{t_0}} = |\nabla f||dV_g|/f^*|dt|$$

so that

$$|dV_g|/f^*|dt| = \frac{1}{|\nabla f|}|dV_{t_0}| \mid_{U\cap M_{t_0}} .$$

Hence

$$f_*|dV_g| = \rho(t)|dt|, \;\; \rho(t) = \int_{M_t} \frac{1}{|\nabla f|}|dV_t|, \tag{1.10}$$

and obtain in this fashion the *co-area formula*

$$\int_M |dV_g| = \int_{\mathbb{R}} \Big(\int_{M_t} \frac{1}{|\nabla f|}|dV_t| \Big)|dt| \tag{1.11}$$

To see how this works in practice consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$. We denote the coordinates in $\mathbb{R}^{n+1}$ by $(t, x^1, \dots, x^n)$. We let $P_\pm \in S^n$ denote the poles given $t = \pm 1$.

We denote by $|dV_n|$ the volume density on S^n and by $\pi : S^n \to \mathbb{R}$ the natural projection given by

$$(t, x^1, \dots, x^n) \mapsto t.$$

π is a submersion on $M = S^n \setminus \{P_\pm\}$ and $\pi(M) = (-1, 1)$. We want to compute $\pi_*|dV_n|$. Observe that $\pi^{-1}(t)$ is the $(n-1)$-dimensional sphere of radius $(1-t^2)^{1/2}$. To find the gradient $\nabla\pi$ observe that for every $p \in S^n$ the tangent vector $\nabla\pi(p)$ is the projection of the vector ∂_t on the tangent space T_pS^n, because ∂_t is the gradient of the function linear function $\pi : \mathbb{R}^{n+1} \to \mathbb{R}$, $\pi(t, x^i) = t$ with respect to the Euclidean metric on $\mathbb{R}^{n+1}$. Denote by θ the angle between ∂_t and T_pS^n, set $p' = \pi(p)$ and denote by t the coordinate of p' (see

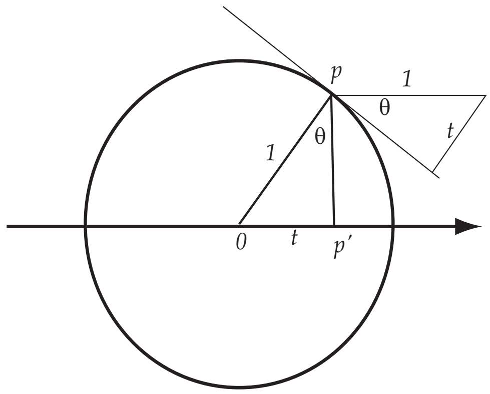

Figure 1.2. *Slicing a sphere by hyperplanes*

Figure 3.1). Then θ is equal to the angle at p between the radius $[0,p]$ and the segment $[p,p']$. We deduce

$$\cos\theta = \text{length}\,[p,p'] = (1-t^2)^{1/2}.$$

Hence

$$|\nabla\pi(p)| = (1-t^2)^{1/2}.$$

Hence

$$\int_{\pi^{-1}(t)} \frac{1}{|\nabla\pi|}|dV_t| = (1-t^2)^{-1/2}\int_{\pi^{-1}(t)}|dV_t| = \sigma_{n-1}(1-t^2)^{\frac{n-2}{2}},$$

where σ_m denotes the m-dimensional area of the unit m-dimensional. The last formula implies

$$\sigma_n = 2\sigma_n\int_0^1(1-t^2)^{\frac{n-2}{2}}|dt| = \sigma_{n-1}\int_0^1(1-s)^{\frac{n-2}{2}}s^{-1/2}|ds| = \sigma_{n-1}B\left(\frac{1}{2},\frac{n}{2}\right),$$

where B denotes the *Beta function*

$$B(p,q) = \int_0^1 s^{p-1}(1-s)^{q-1}ds,\ \ p,q>0. \tag{1.12}$$

It is known (see) that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{1.13}$$

where $\Gamma(x)$ denotes Euler's gamma function

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt.$$

We deduce

$$\boldsymbol{\sigma}_n = \frac{2\Gamma(\frac{1}{2})^{n+1}}{\Gamma(\frac{n+1}{2})}. \tag{1.14}$$

□

1.3. Invariant measures on linear Grassmannians

On the Grassmannian $\mathbf{Gr}_k(V)$ we have an $O(V)$-invariant metric $h = h_{n,k}$. The associated Riemannian volume defines an invariant density. We will denote by $|d\gamma_{n,k}|$, where $n = \dim V$. It is called the *kinematic density on* $\mathbf{Gr}_k(V)$. Since the action of the group $O(V)$ is transitive, we deduce that any other invariant density is equivalent to a constant multiple of this metric density. We would like to give a local description of $|d\gamma_{n,k}|$.

Fix consider a small open set $\mathcal{O} \subset \mathbf{Gr}_k(V)$. Set $n := \dim V$. If $\mathcal{O}$ is sufficiently small we can find smooth maps

$$\boldsymbol{e}_A : \mathcal{O} \to V, \quad A = 1, \cdots, n$$

with the following properties.

- For every $L \in \mathcal{O}$ the collection $(\boldsymbol{e}_A(L))_{1 \leq A \leq n}$ is an orthonormal frame of V.
- For every $L \in \mathcal{O}$ the collection $(\boldsymbol{e}_i(L))_{1 \leq i \leq k}$ is an orthonormal frame of L.

For every $1 \leq i \leq k$ and every $k+1 \leq \alpha \leq 1$ we have a 1-form

$$\theta_{\alpha i} \in \Omega^1(\mathcal{O}), \quad \theta_{\alpha i} = d\boldsymbol{e}_\alpha \bullet \boldsymbol{e}_i.$$

As explained in the previous subsection, the metric h is described along $\mathcal{O}$ by

$$h = \sum_{\alpha,i} \theta_{\alpha i} \otimes \theta_{\alpha i}$$

and the associated volume density is described by

$$|d\gamma_{n,k}| = \Big|\prod_{\alpha,i} \theta_{\alpha i}\Big| := \Big|\bigwedge_{\alpha,i} \theta_{\alpha i}\Big|.$$

Example 1.3.1. To understand the above construction it is helpful to consider a special case, $\mathbf{Gr}_1(\mathbb{R}^2)$, the Grassmannian of lines through the origin in $\mathbb{R}^2$. This space is also known as the *real projective line* and as such it is also denoted by $\mathbb{RP}^1$.

A line L in $\mathbb{R}^2$ is uniquely determined by the angle $\theta \in [0, \pi]$ it forms with the x axis. For such an angle θ we denote by L_θ the corresponding line. L_θ is also represented by the the orthonormal frame

$$\boldsymbol{e}_1(\theta) = (\cos\theta, \sin\theta), \quad \boldsymbol{e}_2(\theta) = (-\sin\theta, \cos\theta), \quad L_\theta = \operatorname{span}(\,\boldsymbol{e}_1(\theta)\,).$$

Then

$$\theta_{21} = \boldsymbol{e}_2 \bullet d\boldsymbol{e}_1 = d\theta$$

and

$$|d\gamma_{2,1}| = |d\theta|. \qquad \square$$

1.4. The volumes of the linear Grassmannians

We would like to compute the volumes of the Grassmannian $\mathbf{Gr}_k(V)$, $\dim V = n$ with respect to the density $d\nu_{n,k}$, i.e. we would like to compute

$$C_{n,k} := \int_{\mathbf{Gr}_k(V)} |d\gamma_{n,k}|.$$

Denote by $\boldsymbol{\omega}_n$ the volume of the unit ball $\boldsymbol{B}^n \subset \mathbb{R}^n$ and by $\boldsymbol{\sigma}_{n-1}$ the $(n-1)$-dimensional "surface area" of the unit sphere $\boldsymbol{S}^{n-1}$. Then

$$\boldsymbol{\sigma}_{n-1} = n\boldsymbol{\omega}_n$$

and

$$\boldsymbol{\omega}_n = \frac{\Gamma(1/2)^n}{\Gamma(1+n/2)} = \begin{cases} \dfrac{\pi^k}{k!} & n = 2k \\ \\ \dfrac{2^{2k+1}\pi^k k!}{(2k+1)!} & n = 2k+1 \end{cases},$$

where $\Gamma(x)$ is the gamma function. We list below the values of $\boldsymbol{\omega}_n$ for small n.

n	0	1	2	3	4
$\boldsymbol{\omega}_n$	1	2	π	$\frac{4\pi}{3}$	$\frac{\pi^2}{2}$

To compute the volume of the Grassmannians we need to give yet another description for the Grassmannians, as a homogeneous space.

Fix $L_0 \in \mathbf{Gr}_k(V)$. Then the stabilizer of L_0 with respect to the action of $O(V)$ on $\mathbf{Gr}_k(V)$ is the subgroup $O(L_0) \times O(L_0^\perp)$ and thus we can identify $\mathbf{Gr}_k(V)$ with the homogeneous space $O(V)/O(L_0) \times O(L_0^\perp)$.

The computation of $C_{n,k}$ is carried out in three steps.

Step 1. We equip the orthogonal groups $O(\mathbb{R}^n)$ with a canonical invariant density $|d\gamma_n|$ called the *kinematic density on* $O(n)$. Set

$$C_n := \int_{O(\mathbb{R}^n)} |d\gamma_n|.$$

Step 2. We show that

$$C_{n,k} = \frac{C_n}{C_k C_{n-k}}.$$

Step 3. We show that

$$C_{n,1} = \frac{1}{2}\boldsymbol{\sigma}_{n-1} = \frac{n\boldsymbol{\omega}_n}{2}.$$

and then compute C_n inductively using the recurrence relation from Step 2

$$C_{n+1} = (C_1 C_{n,1})C_n,$$

and the initial condition

$$C_2 = \mathrm{vol}\,(\,O(2)\,) = 2\boldsymbol{\sigma}_1.$$

Step 1. The group $O(V)$ is a submanifold of $\mathrm{End}(V)$ consisting of endomorphisms S satisfying $SS^* = S^*S = \mathbb{1}_V$. We equip $\mathrm{End}(V)$ with the inner product

$$\langle A, B\rangle = \frac{1}{2}\,\mathrm{tr}(AB^*).$$

This metric induces an invariant metric h on $O(V)$. We would like to give a more concrete description of this metric.

Denote by $\operatorname{End}^-(V)$ the subspace of $\operatorname{End}(V)$ consisting of skew-symmetric operators. If For $S_0 \in O(V)$ we have a map

$$\exp_{S_0} : \operatorname{End}^-(V) \to O(V), \;\; \operatorname{End}^-(V) \longmapsto S_0 \cdot \exp(X).$$

This defines a diffeomorphism from a neighborhood of 0 in $\operatorname{End}^-(V)$ to a neighborhood of S_0 in $O(V)$. Two skewsymmetric endomorphisms $X, Y \in \operatorname{End}^-(V)$ define paths

$$\gamma_X, \gamma_Y : \mathbb{R} \to O(V), \;\; \gamma_X(t) = S_0 \exp(tX), \;\; \gamma_Y(t) = S_0 \exp(tY)$$

originating at S_0. We set

$$\dot{X} = \dot{\gamma}_X(0) \in T_{S_0} O(V) \subset \operatorname{End}(V), \;\; \dot{Y} = \dot{\gamma}_Y(0) \in T_{S_0} O(V) \subset \operatorname{End}(V).$$

Then

$$\dot{X} = S_0 X, \;\; \dot{Y} = S_0 Y,$$

$$h(\dot{X}, \dot{Y}) = \frac{1}{2} \operatorname{tr}\big((S_0 X)(S_0 Y)^* \big) = \frac{1}{2} \operatorname{tr}\big(S_0 X Y^* S_0^* \big) = \frac{1}{2} \operatorname{tr}\big(S_0^* S_0 X Y^* \big)$$

$$= \frac{1}{2} \operatorname{tr}\big(XY^* \big)$$

If we choose an orthonormal basis $(\boldsymbol{e}_A)$ of V so that X and Y are given by the skew symmetric matrices (x_{AB}), (y_{AB}) then we deduce

$$h(\dot{X}, \dot{Y}) = \sum_{A>B} x_{AB} y_{AB}.$$

If we set

$$\boldsymbol{f}_A(t) := \exp(tX) \boldsymbol{e}_A$$

then we deduce

$$x_{AB} = \boldsymbol{e}_A \bullet \dot{\boldsymbol{f}}_B(0) = \boldsymbol{f}_A(0) \bullet \dot{\boldsymbol{f}}_B(0).$$

More generally, we define

$$\boldsymbol{f}_A : O(V) \to V, \;\; \boldsymbol{f}_A(S) = S \boldsymbol{e}_A$$

we obtain the angular forms

$$\theta_{AB} = \boldsymbol{f}_A \bullet d\boldsymbol{f}_B.$$

Then the above metric has the description

$$h = \sum_{A>B} \theta_{AB} \otimes \theta_{AB}.$$

The associated volume density is

$$|d\gamma_n| = \Big| \bigwedge_{A>B} \theta_{AB} \Big|.$$

Step 2. *Fix an orthonormal frame* $(\boldsymbol{e}_A)$ of V such that $L_0 = \operatorname{span}(\boldsymbol{e}_i;\;\; 1 \le i \le k)$. We can identify V with $\mathbb{R}^n$, $O(V)$ with $O(n)$ and L_0 with the subspace $\mathbb{R}^n \oplus 0_{n-k} \subset \mathbb{R}^n$. An orthogonal $n \times n$ matrix T is uniquely determined by the orthonormal frame $(T\boldsymbol{e}_A)$ via the equalities

$$T_{AB} = \boldsymbol{e}_A \bullet T\boldsymbol{e}_B.$$

Define

$$p : O(n) \to \mathbf{Gr}_k(V), \;\; p(T) = T(L_0).$$

More explicitly we have

$$p(T) = \operatorname{span}\,(T\boldsymbol{e}_i)_{1\leq i\leq k}.$$

We will prove that we have a principal fibration

$$\begin{array}{ccc} O(k)\times O(n-k) & \hookrightarrow & O(n) \\ & & \downarrow p \\ & & \mathbf{Gr}_k(\mathbb{R}^n) \end{array},$$

and that

$$p_*|d\gamma_n| = C_k C_{n-k}|d\gamma_{n,k}|.$$

Once we have this we deduce from the Fubini theorem that

$$C_n = C_k C_{n-k} C_{n,k}.$$

Let us prove the above facts.

For every sufficiently small open subset $U \subset \mathbf{Gr}_k(V)$ we can find a smooth section

$$\phi : U \to O(n)$$

of $p : O(n) \to \mathbf{Gr}_k(V)$. The section can be identified with a smooth family of frames $(\boldsymbol{\phi}_A(L),\ \ L\in U)_{1\leq A\leq n}$ such that

$$L = \operatorname{span}\,(\boldsymbol{\phi}_i(L);\ \ 1\leq i\leq k).$$

To such a frame we associate the orthogonal matrix $\boldsymbol{\phi}(L) \in O(n)$ which maps the fixed frame $\big(\boldsymbol{e}_A\big)$ to the frame $\big(\boldsymbol{\phi}_B\big)$. It is a given by a matrix with entries

$$\boldsymbol{\phi}(L)_{AB} = \boldsymbol{e}_A \bullet \boldsymbol{\phi}_B.$$

Then we have a diffeomorphism

$$\Psi : O(k)\times O(n-k)\times U \to O(n)$$

defined as follows.

- Given $(s,t,L)\in O(k)\times O(n-k)\times U$ express s as a $k\times k$ matrix $s = (s^i_j)$ and t as a $(n-k)\times(n-k)$ matrix (t^α_β).
- Define the frame of V.

  $$(\,\boldsymbol{f}_A\,) = (\boldsymbol{\phi}_B) * (s,t),$$

  via the equalities

  $$\boldsymbol{f}_i = \boldsymbol{f}_i(s,L) = \sum_j s^j_i \boldsymbol{\phi}_j(L) \in L,\ \ 1\leq i\leq k \tag{1.15}$$

  $$\boldsymbol{f}_\alpha = \boldsymbol{f}_\alpha(t,L) = \sum_\beta t^\beta_\alpha \boldsymbol{\phi}_\beta(L) \in L^\perp,\ \ k+1\leq \alpha\leq n. \tag{1.16}$$

- Now define $\Psi = \Psi(s,t,L)$ to be the orthogonal transformation of V which maps the frame $(\boldsymbol{e}_A)$ to the frame $(\boldsymbol{f}_B)$, i.e.

  $$\boldsymbol{f}_A = \Psi\boldsymbol{e}_A,\ \ \forall A.$$

The map Ψ is a homeomorphism with inverse

$$O(n) \ni T \mapsto \Psi^{-1}(T) = (s,t;L) \in O(k) \times O(n-k) \times L$$

defined as follows. We set $\boldsymbol{f}_A = \boldsymbol{f}_A(T) = T\boldsymbol{e}_A$, $1 \leq A \leq n$. Then

$$L = L_T = \operatorname{span}(\boldsymbol{f}_i)_{1\leq i\leq k}$$

while the matrices (s^i_j) and (t^α_β) are obtained via (1.15) and (1.16). More precisely, we have

$$s^i_j = \boldsymbol{\phi}_i(L_T) \bullet \boldsymbol{f}_j, \;\; s^\alpha_\beta = \boldsymbol{\phi}_\alpha(L_T) \bullet \boldsymbol{f}_\beta.$$

Observe that, $\forall s_0, s_1 \in O(k), \;\; t_0, t_1 \in O(n-k)$, we have

$$\big((\boldsymbol{\phi}_B) * (s_0, t_0) \big) * (s_1, t_1) = (\boldsymbol{\phi}_B) * (s_0 s_1, t_0 t_1).$$

This means that Ψ is equivariant with respect to the right actions of $O(k) \times O(n-k)$ on $O(k) \times O(n-k) \times U$ and $O(n)$. We have a commutative diagram

$$\begin{tikzcd} O(k) \times O(n-k) \times U \arrow[rr, "\Psi"] \arrow[dr, "\pi"'] & & p^{-1}(U) \arrow[dl, "p"] \\ & U \subset \mathbf{Gr}_k(V) & \end{tikzcd}$$

In particular, this shows that p defines a principal $O(k) \times O(n-k)$-bundle.

Observe now that $p_*|d\gamma_n|$ is an invariant density on $\mathbf{Gr}_k(\mathbb{R}^n)$ and thus there exists a constant c such that

$$p_*|d\gamma_n| = c|d\gamma_{n,k}|.$$

This constant is given by the integral of the density $|d\gamma_n|/p^*|d\gamma_{n,k}|$ along the fiber $p^{-1}(L_0)$.

Recall the if we define $\boldsymbol{f}_A : O(n) \to \mathbb{R}^n$ by

$$\boldsymbol{f}_A(T) := T\boldsymbol{e}_A,$$

and

$$\theta_{AB} := \boldsymbol{f}_A \bullet d\boldsymbol{f}_B,$$

then

$$|d\gamma_n| = |\bigwedge_{A>B} \theta_{AB}|.$$

We write this as

$$\Big| \Big(\bigwedge_{i>j} \theta_{ij} \Big) \wedge \Big(\bigwedge_{\alpha>\beta} \theta_{\alpha\beta} \Big) \wedge \Big(\bigwedge_{\alpha,i} \theta_{\alpha,i} \Big) \Big|.$$

The form $\Big(\bigwedge_{\alpha,i} \theta_{\alpha,i} \Big)$ is the pullback of a nowhere vanishing form defined in a neighborhood of L_0 in $\mathbb{R}^k$ whose associated density id $|d\gamma_{n,k}|$. We now find ourselves in the situation described in Remark 1.2.7. We deduce

$$\begin{aligned} c &= \int_{p^{-1}(L_0)} \Big| \Big(\bigwedge_{i>j} \theta_{ij} \Big) \wedge \Big(\bigwedge_{\alpha>\beta} \theta_{\alpha\beta} \Big) \Big| \\ &= \Big(\int_{O(k)} |d\gamma_k| \Big) \Big(\int_{O(n-k)} |d\gamma_{n-k}| = C_k C_{n-k}. \end{aligned}$$

Step 3. Fix an orthonormal basis $\{\boldsymbol{e}_A\}$ of V and denote by $\boldsymbol{S}_+^{n-1}$ the open hemisphere

$$\boldsymbol{S}_+^{n-1} = \big\{ \vec{v} \in V;\ \ |\vec{v}| = 1,\ \ \vec{v} \bullet \vec{1} > 0 \big\}.$$

Note that $\mathbf{Gr}_1(V) \cong \mathbb{RP}^{n-1}$ is the Grassmannian of lines in V. The set of lines which do not intersect $\boldsymbol{S}_+^{n-1}$ is a smooth hypersurface of $\mathbf{Gr}_1(V)$ diffeomorphic to $\mathbb{RP}^{n-2}$ and thus has kinematic measure zero. We denote by $\mathbf{Gr}_1(V)^*$ the open subset consisting of lines intersecting $\boldsymbol{S}_+^{n-1}$. We thus have a map

$$\psi : \mathbf{Gr}_1^*(V) \to \boldsymbol{S}_+^{n-1},\ \ \ell \mapsto \ell \cap \boldsymbol{S}_+^{n-1}.$$

This map is a diffeomorphism and we have

$$C_{n,1} = \int_{\mathbf{Gr}_1(V)} |d\gamma_{n,1}| = \int_{\mathbf{Gr}_1^*(V)} |d\gamma_{n,1}| = \int_{\boldsymbol{S}_+^{n-1}} |d\gamma_{n,1}|_\psi.$$

Now observe that ψ is in fact an *isometry* and thus we deduce

$$C_{n,1} = \frac{1}{2}\text{area}\,(\boldsymbol{S}^{n-1}) = \frac{\boldsymbol{\sigma}_{n-1}}{2}\frac{n\boldsymbol{\omega}_n}{2}.$$

Hence

$$C_{n+1} = C_n C_1 C_{n+1,1} = \boldsymbol{\sigma}_n C_n.$$

And we deduce

$$C_n = \boldsymbol{\sigma}_{n-1} \cdots \boldsymbol{\sigma}_2 C_2 = 2\prod_{k=1}^{n-1} \boldsymbol{\sigma}_k = \prod_{j=0}^{n-1} \boldsymbol{\sigma}_j.$$

In particular, we deduce the following result.

Proposition 1.4.1. *For every $1 \leq k < n$ we have*

$$C_{n,k} = \int_{\mathbf{Gr}_k(\mathbb{R}^n)} |d\gamma_{n,k}| = \frac{\prod_{j=0}^{n-1}\boldsymbol{\sigma}_j}{\big(\prod_{i=0}^{k-1}\boldsymbol{\sigma}_i\big)\cdot\big(\prod_{j=0}^{n-k-1}\boldsymbol{\sigma}_j\big)} = \binom{n}{k}\frac{\prod_{j=1}^{n}\boldsymbol{\omega}_j}{\big(\prod_{i=1}^{k}\boldsymbol{\omega}_i\big)\cdot\big(\prod_{j=1}^{n-k}\boldsymbol{\omega}_j\big)}.$$

Following [**KR**] we set

$$[n] := \frac{1}{2}\frac{\boldsymbol{\sigma}_{n-1}}{\boldsymbol{\omega}_{n-1}} = \frac{n\boldsymbol{\omega}_n}{2\boldsymbol{\omega}_{n-1}},\ \ [n]! := \prod_{k=1}^{n}[k] = \frac{\boldsymbol{\omega}_n n!}{2^n},$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{([k]!)([n-k]!)} = \binom{n}{k}\frac{\boldsymbol{\omega}_n}{\boldsymbol{\omega}_k\boldsymbol{\omega}_{n-k}}. \tag{1.17}$$

Denote by $|d\nu_{n,k}|$ the unique invariant density on $\mathbf{Gr}_k(V)$, $\dim V = n$ such that

$$\int_{\mathbf{Gr}_k(V)} |d\nu_{n,k}| = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{1.18}$$

We have

$$|d\nu_{n,k}| = \frac{\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]}{C_{n,k}}|d\gamma_{n_k}|.$$

Example 1.4.2. Using the computation in Example 1.3.1 we deduce

$$|d\gamma_{2,1}| = |d\theta|, \;\; 0 \leq \theta < \pi.$$

and we deduce

$$C_{2,1} = \int_0^\pi |d\theta| = 2\frac{\omega_2}{\omega_1^2},$$

as predicted by Proposition 1.4.1. We have

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\frac{\omega_2}{\omega_1^2} = C_{2,1},$$

so that $|d\nu_{2,1}| = |d\gamma_{2,1}|$. □

1.5. Affine Grassmannians

We denote by $\mathbf{Graff}_k(V)$ the set of k-dimensional affine subspaces of V. We would like to describe a natural structure of smooth manifold on $\mathbf{Graff}_k(V)$.

Note that we have tautological vector bundle $\mathcal{U} = \mathcal{U}_{n,k} \to \mathbf{Gr}_k(V)$. It is naturally a subbundle of the trivial vector bundle $\underline{V} = V \times \mathbf{Gr}_k(V) \to \mathbf{Gr}_k(V)$ whose fiber over $\in \mathbf{Gr}_k(V)$ is the vector subspace L. The trivial vector bundle $\underline{V}$ is equipped with a natural metric and we denote by $\mathcal{U}^\perp \to \mathbf{Gr}_k(V)$ the orthogonal complement of $\mathcal{U}$ in $\underline{V}$.

The fiber of $\mathcal{U}^\perp$ over $L \in \mathbf{Gr}_k(V)$ is canonically identified with the orthogonal complement $L^\perp$ of L in V. The points of $\mathcal{U}^\perp$ are pairs $(\vec{c}, L)$, where $L \in \mathbf{Gr}_k(V)$, and $\vec{c}$ is a vector in $L^\perp$.

Observe that we have a natural map $\mathcal{A} : \mathcal{U}^\perp \to \mathbf{Graff}_k(V)$ given by

$$(\vec{c}, L) \mapsto \vec{r} + L.$$

This map is a bijection with inverse

$$\mathbf{Graff}_k(V) \ni S \mapsto (S \cap [S]^\perp, [S]),$$

where $[S] \in \mathbf{Gr}_k(V)$ denotes the vector subspace $S - S$ parallel to S. We set

$$\vec{c}(S) := S \cap [S]^\perp$$

and we say that $\vec{c}(S)$ is the *center* of the affine plane S.

We equip $\mathbf{Graff}_k(V)$ with the structure of smooth manifold which makes $\mathcal{A}$ a diffeomorphism. Thus, we identify $\mathbf{Graff}_k(V)$ with a vector subbundle of the *trivial bundle* $V \times \mathbf{Gr}_k(V)$ described by

$$\mathcal{U}^\perp = \big\{ (\vec{c}, L) \in V \times \mathbf{Gr}_k(V); \;\; P_L\vec{c} = 0 \big\},$$

where P_L denotes the orthogonal projection onto L.

The projection $\pi : \mathcal{U}^\perp \to \mathbf{Gr}_k(V)$ is a submersion. The fiber of this submersion over $L \in \mathbf{Gr}_k(V)$ is canonically identified with the vector subspace $L^\perp \subset V$. As such is equipped with a volume density $dV_{L^\perp}$. We obtain in this fashion a density $dV_{L^\perp}$ on the horizontal subbundle $\ker D\pi \subset T\mathcal{U}^\perp$.

The base $\mathbf{Gr}_k(V)$ of the submersion π is equipped with a density $|d\gamma_{n,k}|$ and thus we obtain a density

$$|d\tilde{\gamma}_{n,k}| = |dv_{L^\perp}| \times \pi^*|d\gamma_{n,k}|.$$

Let us provide a local description for this density. Fix a small open subset $\mathcal{O} \subset \mathbf{Gr}_k(V)$ and denote by $\tilde{\mathcal{O}}$ its preimage in $\mathbf{Graff}_k(V)$ via the projection π. Then we can find smooth maps

$$\boldsymbol{e}_A : \mathcal{O} \to V, \ \ \vec{r} : \tilde{\mathcal{O}} \to V$$

with the following properties

- For every $S \in \tilde{O}$, $(\boldsymbol{e}_A(S)\,)$ is an orthonormal frame of V and
$$[L] = \operatorname{span}\big(\,\boldsymbol{e}_i([L])\,\big).$$
- For every $S \in \tilde{O}$ we have
$$S = \vec{r}(S) + [S].$$

We rewrite the last equality as

$$S = S(\vec{r},\, \boldsymbol{e}_i).$$

Observe that the center of this affine plane is the projection of $\vec{r}$ onto $[S]^{\perp}$

$$\vec{c}(S) = \sum_{\alpha} (\,\boldsymbol{e}_\alpha \bullet \vec{r}\,)\boldsymbol{e}_\alpha.$$

Following the tradition we introduce the (locally defined) 1-forms

$$\theta_\alpha := \boldsymbol{e}_\alpha \bullet d\vec{r}, \ \ \theta_{\alpha i} := \boldsymbol{e}_\alpha \bullet d\boldsymbol{e}_i.$$

For fixed $L \in \mathbf{Gr}_k(V)$ the density on the fiber $\mathcal{U}_L^{\perp} = L^{\perp}$ is given by

$$dV_{L^\perp} = \Big|\bigwedge \theta_\alpha\Big|.$$

The volume density on $\mathbf{Graff}_k(V)$ is described along $\tilde{\mathcal{O}}$ by

$$|d\tilde{\gamma}_{n,k}| = \Big|\,\Big(\bigwedge_{\alpha} \theta_\alpha\Big) \wedge \Big(\bigwedge_{\alpha,i} \theta_{\alpha i}\Big)\,\Big|.$$

Theorem 1.5.1. *Suppose $f : \mathbf{Graff}_k(V) \to \mathbb{R}$ is a compactly supported $|d\hat{\gamma}_{n,k}|$-integrable function. Then*

$$\int_{\mathbf{Graff}_k(V)} f(S)|d\tilde{\gamma}_{n,k}(S)| = \int_{\mathbf{Gr}_k(V)} \left(\int_{L^\perp} f(p+L)dV_{L^\perp}(p) \right)|d\gamma_{n,k}(L)|,$$

where $dV_{L^\perp}$ denotes the Euclidean volume density on $L^\perp$.

Denote by $\operatorname{Iso}(V)$ the group of affine isometries of V, i.e. the subgroup of the group of affine transformations generated by translations and rotations about a fixed point. Any affine isometry $T : V \to V$ is described by a unique pair $(t, S) \in V \times V$ so that

$$T(v) = Sv + t, \ \ \forall v \in V.$$

The group $\operatorname{Iso}(V)$ acts in an obvious fashion on $\mathbf{Graff}_k(V)$ and a simple computation shows that the associated volume density $|d\tilde{\gamma}_{n,k}|$ is $\operatorname{Iso}(V)$ invariant.

If instead of the density $|d\gamma_{n,k}|$ on $\mathbf{Gr}_k(V)$ we use the density $|d\nu_{n,k}|$, we obtain a density $|d\tilde{\nu}_{n,k}|$ on $\mathbf{Graff}_k(V)$ which is a constant multiple of $|d\tilde{\gamma}_{n,k}|$.

$$|d\tilde{\nu}_{n,k}| = \frac{\left[{n \atop k}\right]}{C_{n,k}}|d\tilde{\gamma}_{n_k}|. \tag{1.19}$$

Example 1.5.2. Let us unravel the above definition in the special case $\mathbf{Graff}_1(\mathbb{R}^2)$, the Grassmannians of affine lines in $\mathbb{R}^2$. Such a line L is determined by two quantities: the angle $\theta \in [0, \pi)$ is makes with the x-axis, and the *signed* distance $\rho \in (-\infty, \infty)$ from the origin. More precisely, for every $\rho \in \mathbb{R}$ and $\theta \in [0, \pi)$ we denote by $L_{\theta,\rho}$ the line is given in Euclidean coordinates by the equation

$$x \sin \theta - y \cos \theta = \rho$$

As a manifold, the Grassmannian $\mathbf{Graff}_1(\mathbb{R}^2)$ is diffeomorphic to the interior of the Möbius band. The Fubini formula in Theorem 1.5.1 can now be rewritten

$$\int_{\mathbf{Graff}_1(\mathbb{R}^2)} f(L)\, |d\tilde{\gamma}_{2,1}|(L) = \int_{-\infty}^{\infty} \left(\int_0^{\pi} f(L_{\theta,\rho}) |d\theta| \right) |d\rho|,$$

$\forall f \in C^{\infty}_{cpt}(\mathbf{Graff}_1(\mathbb{R}^2)\,)$. □

Chapter 2

A brief survey of Riemannian geometry

2.1. The Levi-Civita connection and its curvature

Let M be a smooth, connected manifold. We denote by $\mathrm{Vect}(M)$ the space of smooth vector fields on M. For any smooth vector bundle $E \to M$ we denote and by $\operatorname{End} E$ the vector bundle whose fiber over $x \in M$ is $\operatorname{End}(E_x)$, by $C^\infty(E)$ the space of smooth sections of E and by $\Omega^p(E)$ the space of smooth differential forms of degree p on M with coefficients in E i.e., the space of smooth sections of $\Lambda^p T^*M \otimes E$.

Definition 2.1.1. (a) A *connection* on the smooth vector bundle $E \to M$ is an $\mathbb{R}$-bilinear map

$$\nabla : \mathrm{Vect}(M) \times C^\infty(E) \to C^\infty(E), \;\; \mathrm{Vect}(M) \times C^\infty(E) \ni (X, u) \mapsto \nabla_X u$$

satisfying

$$\nabla_{fX} u = f(\nabla_X u), \;\; \nabla_X(fu) = (Xf)Yu + f\nabla_X u,$$

$\forall f \in C^\infty(M), \;\; X \in \mathrm{Vect}(M), \;\; u \in C^\infty(E)$. $\nabla_X u$ is called the *covariant derivative* of the section u in the direction of X.

(b) The *torsion* of a connection ∇ on TM is the $\mathbb{R}$-bilinear map

$$T = T_\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M),$$

$$(X, Y) \mapsto T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y], \;\; \forall X, Y \in \mathrm{Vect}(M).$$

The connection is called *symmetric* if its torsion is zero.

(c) The *curvature* of a connection ∇ on the vectur bundle $E \to M$ is the $\mathbb{R}$-bilinear map

$$R = R_\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to C^\infty(\operatorname{End} E),$$

$$(X, Y) \mapsto R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \;\; \forall X, Y \in \mathrm{Vect}(M).$$

More precisely

$$R(X, Y)u := \nabla_X \nabla_Y u - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} u, \;\; u \in C^\infty(E).$$

(d) A connection ∇ on TM is called *compatible with a Riemann metric* g on M if

$$X \cdot g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),\ \ \forall X,Y,Z \in \mathrm{Vect}(M).$$

When the Riemann metric is clear from the context, we will simply say that ∇ is a *metric connection*.

We denote by $\mathcal{C}(M)$ the space of connections on TM and by $\mathcal{C}(M,g) \subset \mathcal{C}(M,g)$ the space of connections compatible with the Riemann metric g. The following facts are immediate consequences of the above definitions.

Proposition 2.1.2. *Suppose $\nabla \in \mathcal{C}(M)$. Then the following hold.*

(a) The torsion $T = T_\nabla$ is a tensor $T \in \Omega^2(TM)$ *i.e.,*

$$T(X,Y) = -T(Y,X),\ \ T(fX,Y) = T(X,fY) = fT(X,Y),$$

$\forall X,Y \in \mathrm{Vect}(M)$, $f \in C^\infty(M)$.

(b) The curvature $R = R_\nabla$ is a tensor $R \in \Omega^2(\mathrm{End}\,TM)$ *i.e.,*

$$R(X,Y) = -R(Y,X),\ \ R(fX,Y) = R(X,fY) = fR(X,Y),$$

$\forall X,Y \in \mathrm{Vect}(M)$, $f \in C^\infty(M)$. □

Example 2.1.3. The Euclidean space is equipped with a natural connection $\boldsymbol{D} = \boldsymbol{D}^{\mathbb{R}^n}$. If $(x^1,\dots,x^n)$ are the natural coordinates on $\mathbb{R}^n$, and we set $\partial_i := \partial_{x^i}$ then for any vector fields

$$X = \sum_i X^i \partial_i,\ \ Y = \sum_j Y^j \partial_j$$

we have

$$\boldsymbol{D}_X Y = \sum_i X^i \boldsymbol{D}_{\partial_i} Y = \sum_j \Big(\sum_i X^i \partial_i Y^j \Big) \partial_j.$$

Both the torsion, and the curvature of $\boldsymbol{D}$ are equal to zero. We will refer to $\boldsymbol{D}$ as the *trivial connection* on $\mathbb{R}^n$. □

For any Riemann metric g on M we denote by $\mathrm{End}_g^-\,TM$ the bundle of skew-symmetric endomorphisms of TM. The fiber of $\mathrm{End}_g^-\,TM$ over $x \in M$ consists of endomorphisms of T_xM which are skew-symmetric with respect to the inner product g_x on T_xM.

Proposition 2.1.4. *(a) If nonempty, $\mathcal{C}(M)$ is affine space modelled by* $\Omega^1(\mathrm{End}\,TM)$.

(b) If nonempty, $\mathcal{C}(M,g)$ is an affine space modelled by $\Omega^1(\mathrm{End}_g^-\,TM)$.

Proof. Let $\nabla^0, \nabla^1 \in \mathcal{C}(M)$ define

$$A = \nabla^1 - \nabla^0 : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$$

by

$$\mathrm{Vect}(M) \times \mathrm{Vect}(M) \ni (X,Y) \mapsto A_X Y := \nabla^1_X Y - \nabla^0_X Y.$$

Observe that

$$A_X(fY) = fA_XY,\ \ \forall f \in C^\infty(M) \Longrightarrow A_X \in C^\infty(\mathrm{End}\,TM).$$

Clearly $A_{fX} = fA_X$, $\forall f \in C^\infty(M)$ which shows that $A \in \Omega^1(\operatorname{End} TM)$. Conversely let $\nabla \in \mathcal{C}(M)$ and $A \in \Omega^1(\operatorname{End} TM)$. For $X, Y \in \operatorname{Vect}(M)$ define

$$\nabla'_X Y := \nabla_X^Y + A_X Y.$$

Then ∇' is a connection on TM.

(b) Let $\nabla^0, \nabla^1 \in \mathcal{C}(M, g)$ and set $A = \nabla^1 - \nabla^0$. We want to show that for every $X \in \operatorname{Vect}(M)$ the endomorphism $A_X \in C^\infty(\operatorname{End} TM)$ is skew-symmetric i.e.,

$$g(A_X Y, Z) + g(Y A_X Z) = 0, \ \ \forall Y, Z \in \operatorname{Vect}(M).$$

To see this note that

$$0 = Xg(Y,Z) - Xg(Y,Z) = g(\nabla^1_X Y, Z) + g(Y \nabla^1_X Z) - g(\nabla^0_X Y, Z) - g(Y, \nabla^0_X Z)$$
$$= g(A_X Y, Z) + g(Y A_X Z).$$

Conversely, if $\nabla \in \mathcal{C}(M, g)$ and $A \in \Omega^1(\operatorname{End}^-_g TM)$ the if we define

$$\nabla'_X Y := \nabla_X Y + A_X Y,$$

and one can check easily that $\nabla' \in \mathcal{C}(M, g)$. □

Example 2.1.5. Suppose $\nabla \in \mathcal{C}(M)$. Fix local coordinates $(x^1, \dots, x^m)$, $m = \dim M$ defined on an open subset $U \subset M$. We can then regard U as an subset of $\mathbb{R}^m$. We write $\partial_i := \partial_{x^i}$ and we observe that the collection $(\partial_i)_{1 \le i \le m}$ trivializes $TU = TM|_U$.

The trivial connection on $T\mathbb{R}^m$ defines a connection D on TU and we can write

$$\nabla = D + \Gamma, \ \ \Gamma \in \Omega^1(\operatorname{End} TU) \cong \Omega^1(U) \otimes \operatorname{End}(\mathbb{R}^m).$$

More precisely, if we set

$$\nabla_i := \nabla_{\partial_i}, \ \ D_i := D_{\partial_i}, \ \ \Gamma_i := \Gamma_{\partial_i}$$

then Γ_i can be identified with an $m \times m$ matrix $\Gamma_i = (\Gamma^j_{ik})_{1 \le j,k \le m}$ and we have

$$\nabla_i \partial_k = D_i \partial_k + \sum_j \Gamma^j_{ik} \partial_j = \sum_j \Gamma^j_{ik} \partial_j.$$

The quantities Γ^j_{ik} are known classically as the *Christoffel symbols* and uniquely determine the action of the connection ∇ over U.

If T denotes the torsion of ∇ then

$$T(\partial_i, \partial_j) = \nabla_i \partial_j - \nabla_j \partial_i - [\partial_i, \partial_j] = \nabla_i \partial_j - \nabla_j \partial_i = \sum_k (\Gamma^k_{ij} - \Gamma^k_{ji}) \partial_k.$$

The connection is symmetric (i.e. $T = 0$) if and only if $\Gamma^k_{ij} = \Gamma^k_{ji}$, $\forall k$. If R denotes the curvature of ∇ then using the equality $[\partial_i, \partial_j] = 0$ we deduce

$$R(\partial_i, \partial_j)\partial_k = [\nabla_i, \nabla_j]\partial_k = \nabla_i \Big(\sum_s \Gamma^s_{jk} \partial_s \Big) - \nabla_j \Big(\sum_s \Gamma^s_{ik} \partial_s \Big)$$
$$= \sum_s \Big((\partial_i \Gamma^s_{jk}) \partial_s + \Gamma^s_{jk} (\nabla_i \partial_s) \Big) - \sum_s \Big((\partial_j \Gamma^s_{ik}) \partial_s + \Gamma^s_{ij} (\nabla_j \partial_s) \Big)$$
$$= \sum_s (\partial_i \Gamma^s_{jk} - \partial_j \Gamma^s_{ik}) \partial_s + \sum_{s,\ell} \Big(\Gamma^s_{jk} \Gamma^\ell_{is} - \Gamma^s_{ij} \Gamma^\ell_{js} \Big) \partial_\ell$$

$$= \sum_\ell \underbrace{\left((\partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik}) + \sum_s \left(\Gamma^s_{jk}\Gamma^\ell_{is} - \Gamma^s_{ij}\Gamma^\ell_{js} \right) \right)}_{=:R^\ell_{kij}} \partial_\ell = \sum_\ell R^\ell_{kij}\partial_\ell.$$

□

Einstein Convention To reduce the notational overload when operating with tensors we will use *Einstein's convention.* Thus, when summing over a parameter which appears twice, as a subscript and as a superscript we will omit the summation symbol. For example, with this convention the expression

$$\sum_\ell (\partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik})\partial_\ell + \sum_{s,\ell} \left(\Gamma^s_{jk}\Gamma^\ell_{is} - \Gamma^s_{ij}\Gamma^\ell_{js} \right)\partial_\ell$$

can be rewritten as

$$(\partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik})\partial_\ell + \left(\Gamma^s_{jk}\Gamma^\ell_{is} - \Gamma^s_{ij}\Gamma^\ell_{js} \right)\partial_\ell.$$ □

A connection $\nabla \in \mathcal{C}(M)$ can be used to derivate vector fields along a smooth path in M. Suppose $\gamma : (a, b) \to M$ is a smooth path. Then a vector field along γ is a section of the pullback bundle $\gamma^*TM \to (a, b)$. For example, the velocity $\dot\gamma$ is a vector field along γ. For any vector field X along γ, its derivative along γ is another vector field along γ denoted by $\nabla_{\dot\gamma} X$. Its value at $t_0 \in (a, b)$ is a vector $\nabla_{\dot\gamma} X|_{t=t_0} \in T_{\gamma(t_0)}M$ determined as follows.

• Choose local coordinates (x^i) on M near $\gamma(t_0)$ such that $x^i(\gamma(t_0)) = 0$, $\forall i$.

• In these coordinates, γ is described by a collection of smooth functions $x^i(t)$ defined in an open neighborhood of t_0. The vector field X has the local description

$$X = \sum_i X^i \partial_i,$$

where X^i are smooth function of t, while the velocity $\dot\gamma$ has the local description

$$\dot\gamma = \sum_i \dot x^i \partial_i.$$

• If Γ^i_{jk} denote the Christoffel symbols of ∇ in the coordinates (x^i) then

$$\nabla_{\dot\gamma} X = \sum_k \left(\dot X^k + \sum_{i,j} \Gamma^k_{ij} \dot x^i X^j \right) \partial_k.$$

One can check that $\nabla_{\dot\gamma} X$ defined as above is independent of coordinates. Note alos that the above system of ODE's is *linear* in the unknown X^k.

Definition 2.1.6. Let $\nabla \in \mathcal{C}(M)$. A smooth path γ in M is called *autoparallel* with respect to ∇ if $\nabla_{\dot\gamma}\dot\gamma = 0$. □

Example 2.1.7. Suppose ∇ is a connection on M, described in local coordinates (x^i) by the Christoffel symbols (Γ^i_{jk}). Then a smooth path *gamma* : $[0, 1] \to M$ described in local coordinates by smooth functions $(x^i(t))$ is autoparallel if and only if the functions $X^i(t)$ satisfy the *nonlinear* system of second order ODE's

$$\ddot{x}^i + \sum_{j,k} \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \ \ \forall i. \qquad \square$$

Remark 2.1.8. A connection $\nabla \in \mathcal{C}(M)$ defines connections on all the tensor bundles of M by requiring that the product rule be satisfied for any natural product between tensors. For simplicity will all be denoted by ∇. For example, the covariant derivative of a 1-form $\alpha \in \Omega^1(M)$ along the vector field X is defined so that the product rule is satisfied

$$X \cdot \big(\alpha(Y) \big) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y)$$

so that

$$(\nabla_X \alpha)(Y) = X \cdot \big(\alpha(Y) \big) - \alpha(\nabla_X Y), \ \ X, Y \in \mathrm{Vect}(M). \tag{2.1}$$

If $S \in C^\infty(\mathrm{End}\, TM)$ is an endomorphism of TM, and $X \in \mathrm{Vect}(M)$ then the covaraint derivative of S along S is the endomorphism $\nabla_X S$ defined by product rule requirement

$$\nabla_X(SY) = (\nabla_X S)Y + S(\nabla_X Y, \ \ \forall Y \in \mathrm{Vect}(M)$$

so that

$$(\nabla_X S)Y = \nabla_X(SY) - S(\nabla_X Y) \ \ \forall X, Y \in \mathrm{Vect}(M). \tag{2.2}$$

This last equality is often written in the commutator form

$$\nabla_X S = [\nabla_X, S]. \qquad \square$$

Proposition 2.1.9. *Let* $\nabla^0, \nabla^1 \in \mathcal{C}(M, g)$*. Then*

$$\nabla^0 = \nabla^1 \Longleftrightarrow T_{\nabla^0} = T_{\nabla^0}.$$

Proof. Let

$$T := T_{\nabla^1} - T_{\nabla^0}, \ \ A = \nabla^1 - \nabla^0.$$

Then

$$T(X,Y) = \nabla^1_X Y - \nabla^0_X Y - \big(\nabla^1_Y X - \nabla^0_Y X \big) = A_X Y - A_Y X.$$

On the other hand for every $X, Y, Z \in \mathrm{Vect}(M)$ we have

$$\begin{aligned} 0 &= g(A_Z X, Y) + g(X, A_Z Y) - \Big(g(A_Y X, Z) + g(X, A_Y Z) \Big) + g(A_X, Y) + g(Y, A_X Z) \\ &= g(X, A_Z Y - A_Y Z) + g(A_X Y - A_Y X, Z) + g(Y, A_Z X + A_X Z) \\ &= g(X, T(Z,Y)) + g(T(X,Y), Z) + g(Y, A_Z X + A_X Z). \\ &= g(Y, A_Z X + A_X Z) + g(T(X,Y), Z) - g(X, T(Y,Z)). \end{aligned}$$

On the other hand

$$A_X Z - A_Z X = T(X,Z) \Longrightarrow g(Y, T(X,Z)) = g(Y, A_X Z - A_Z Y)$$

so that

$$\begin{aligned} g(Y, T(X,Z)) &= g(Y, A_X Z - A_Z X) + g(Y, A_Z X + A_X Z) + g(T(X,Y), Z) - g(X, T(Y,Z)\,) \\ &= 2g(Y, A_X Z) + g(T(X,Y), Z) - g(X, T(Y,Z)\,). \end{aligned}$$

We deduce

$$2g(Y, A_X Z) = g(Y, T(X,Z)) + g(X, T(Y,Z)) - g(Z, T(X,Y)).$$

This proves that $A = 0 \Longleftrightarrow T = 0$. $\square$

The above result implies that that there exists at most one symmetric, metric connection with trivial torsion.

Proposition 2.1.10. *Suppose g is a Riemann metric on the smooth manifold. Then there exists precisely one symmetric metric connection ∇ on M. This is known as the* Levi-Civita *connection associated to the metric g.*

Proof. Suppose ∇ is a symmetric metric connection on TM. For $X, Y, Z \in \mathrm{Vect}(M)$ we have

$$
\begin{aligned}
& Xg(Y,Z) - Zg(X,Y) + Yg(Z,X) \\
&= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\
&= g(\nabla_X Y + \nabla_Y X, Z) + g(\nabla_X Z - \nabla_Z X, Y) + g(\nabla_Y Z - \nabla_Z Y, X) \\
&= g(2\nabla_X Y - [X,Y], Z) + g([X,Z], Y) + g([Y,Z], X),
\end{aligned}
$$

so that

$$
\begin{aligned}
g(\nabla_X Y, Z) &:= \frac{1}{2}\Big\{ Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \Big\} \\
&- \frac{1}{2}\Big\{ g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) \Big\}, \ \ \forall Z \in \mathrm{Vect}(M)
\end{aligned}
$$

A simple computation shows that this defines a symmetric metric connection. □

Remark 2.1.11. We can use the above identity to produce local descriptions of the Levi-Civita connection. If (x^i) are local coordinates on M such that in these coordinate the metric g has the form

$$g = g_{ij} dx^i dx^j$$

then the Christoffel symbols of ∇ in these coordinates are determined as follows. We have (using Einstein's convention)

$$\nabla_i \partial_j = \Gamma_{ij}^k \partial_k,$$

so that

$$g_{k\ell}\Gamma_{ij}^k = g(\nabla_i \partial_j, \partial_\ell) = \frac{1}{2}\big(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \big).$$

We conclude

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}\big(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij} \big).$$

From a computational standpoint this formula is too complex and we will rarely use it. □

Definition 2.1.12. (a) A *geodesic* on a Riemann manifold is a path autoparalel with respect to the Levi-Civita connection.

(b) The *Riemann curvature* of a Riemann metric is the curvature of the associated Levi-Civita connection. □

We see that the Riemann curvature is locally described by the metric tensor g_{ij} and its partial derivatives up to order two.

In the sequel, unless otherwise indicated, we will work only with the Levi-Civita connection of a metric.

Associated to the Riemann curvature R of a Riemann metric g on M is the *Riemann tensor*. This is a section $R^\dagger$ of the tensor bundle $T^*M^{\otimes 4}$ defined by

$$R^\dagger(U,V;X,Y) := g(U, R(X,Y)V),\ \ \forall U,V,X,Y \in \mathrm{Vect}(M).$$

In local coordinates X^i) on M, if $g = (g_{ij})$, then

$$R(\partial_k, \partial_\ell)\partial_j = R^i_{jk\ell}\partial_i$$

and $R^\dagger = (R_{ijk\ell})$, where

$$R_{ijk\ell} = g_{is}R^s_{jk\ell} = g(\partial_i, R(\partial_k, \partial_\ell)\partial_j)$$

The Riemann curvature and tensor enjoy have many symmetries. The next result list the fundamental symmetries relations of these tensors. For a proof we refer to [**N**, §4.2.1].

Proposition 2.1.13. *The Riemann curvature R of a Riemann metric g on M satisfy the following identities, for any $U,V,X,Y,Z \in \mathrm{Vect}(M)$.*

(a)

$$R^\dagger(U,V;X,Y) = R^\dagger(U,V;Y,X) = -R^\dagger(V,U;X,Y).$$

(b) (The first Bianchi identity)

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$$

(c)

$$R^\dagger(U,V;X,Y) = R^\dagger(X,Y;U,V).$$

(d) (The second Bianchi identity)

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0. \qquad \square$$

In local coordinates the above identities take the form

$$R_{k\ell ij} = R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k},$$

$$R^i_{jk\ell} + R^i_{k\ell j} + R^i_{\ell jk} = 0.$$

Note that the Riemann curvature defines for every $X, Y \in \mathrm{Vect}(M)$ and endomorphism of TM described by

$$\mathrm{Vect}(M) \longmapsto R(U,X)Y.$$

The trace of this endomorphism is a section Ricci $= \mathrm{Ricci}_g$ of $T^*M^{\otimes 2}$ called the *Ricci curvature* of g. The symmetry of the Riemann curvature implies that Ricci is a symmetric tensor, i.e.

$$\mathrm{Ricci}(X,Y) = \mathrm{Ricci}(Y,X),\ \ \forall X,Y \in \mathrm{Vect}(M).$$

In local coordinates the Ricci curvature is described by the quantities (R_{ij}), where

$$R_{ij} := \sum_\ell R^\ell_{j\ell i} = \sum_\ell R^\ell_{i\ell j} = \sum_\ell R_{\ell i\ell j}.$$

Using the metric we obtain isomorphisms

$$T^*M \cong TM,\ \ T^*M^{\otimes 2} \cong T^*M \otimes TM \cong \mathrm{End}\, TM$$

and thus we can regard Ricci as an endomorphism of TM. As such, its trace is called the *scalar curvature* and it is denoted by $s = s_g$. In local coordinates s is described by

$$s = \sum_{i,j} g^{ij} R_{ij} = \sum_{i,j} R_{ijij}, \tag{2.3}$$

where (g^{ij}) denotes the inverse of the matrix (g_{ij}).

Fix a point p_0 on the Riemann manifold M, $\gamma : [0,1] \to M$ a smooth path staring at p_0, and $X_0 \in T_{p_0}M$. A *parallel transport* of X_0 along γ is a parallel vector field $X = X(t)$ along γ such that $X(0) = X_0$.

The existence and uniqueness results for initial value problems of linear ODE's with smooth coefficients implies that there exists a unique parallel transport of $X_0 \in T_{p_0}M$ along γ.

A connected Riemann manifold (M, g) is naturally a metric space, where the distance between two points $p, q \in M$ is defined by

$$d_g(p,q) := \inf_{\gamma \in \mathcal{P}_{p,q}} L_g(\gamma),$$

where $\mathcal{P}_{p,q}$ is the set of continuous, piecewise smooth paths $\gamma : [0,1] \to M$ connecting p to q, and for $\gamma \in \mathcal{P}_{p,q}$ we denoted by $L_g(\gamma)$ its length

$$L_g(\gamma) := \int_0^1 |\dot\gamma(t)|_g dt.$$

The topology induced by this metric coincides with the natural topology on M. For every $x \in M$ and $r > 0$ we set

$$B_M(x,r) := \{\, y \in M;\ \ d_g(x,y) < r \,\}.$$

Suppose $\gamma : [a,b] \to \mathbb{R}$ is a geodesic on M then

$$\frac{d}{dt} g(\,\dot\gamma, \dot\gamma\,) = 2g(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) = 0$$

so that the speed $|\dot\gamma(t)|$ is a constant, $\sigma = \sigma(\gamma)$. Hence the length of γ is

$$L_g(\gamma) = \sigma(\gamma)(b-a).$$

In particular

$$d_g\big(\,\gamma(b), \gamma(a)\,\big) \le \sigma(\gamma)(b-a). \tag{2.4}$$

Observe that if $\gamma : [a,b] \to M$ is a geodesic with speed σ then the rescaled path

$$\tilde\gamma(s) := \gamma(s/\sigma),\ \ \sigma a \le s \le \sigma b$$

is also geodesic, but with unit speed. We will refer to the geodesics with unit speed as *geodesics parameterized by arclength.*

The geodesics are locally defined by second order ode's. The displacement inequality (2.4) and the standard existence and uniqueness results for such equations imply the following result.

Proposition 2.1.14. *Let (M, g) be a connected Riemann manifold and $p_0 \in M$. Then there exists $r = r(p_0) > 0$ such that for every tangent vector $X \in T_{p_0}M$ of length $|X|_g \leq r$, there exists a unique geodesic*

$$\gamma = \gamma_{p_0,X} : [0, 1] \to M$$

satisfying the initial conditions

$$\gamma(0) = p_0, \;\; \dot{\gamma}(0) = X. \qquad \square$$

The endpoint of the geodesic $\gamma_{p_0,X}$ postulated by the above result is denoted by $\exp_{p_0}(X)$. The smooth dependence on initial conditions of solutions of ode's implies that this map is actually smooth shows that we have a smooth map

$$\exp TM \to M, \;\; \exp(X) = \exp_p(X_p), \;\; \forall p \in M, \;\; X \in \mathrm{Vect}(M).$$

We denote by $\mathbb{D}_M(p_0, r)$ the open disk in $T_{p_0}M$ of radius r centered at the origin. If r is sufficiently small then the displacement inequality (2.4) implies

$$\exp_{p_0}\big(\,\mathbb{D}_M(p_0, r)\,\big) \subset B_M(p_0, r).$$

The differential of $\exp_{p_0} : \mathbb{D}(p_0, r) \to M$ at $0 \in T_{p_0}M$ is a linear map

$$D \exp_{p_0} : T_0(T_{p_0}M) \to T_{p_0}M.$$

A simple computation shows that via the tautological identification $T_0(T_{p_0}M) \cong T_{p_0}M$ we can identify this linear map with the identity $\mathbb{1}_{T_{p_0}M}$. The implicit function theorem then implies the following result.

Proposition 2.1.15. *There exists $\rho = \rho(p_0) > 0$ such that the exponential map*

$$\exp_{p_0} : \mathbb{D}_M(p_0, \rho) \to B_M p_0, \rho)$$

is a diffeomorphism onto an open neighborhood of $p_0 \in M$. $\square$

A much more refined result is true.

Proposition 2.1.16. *If ρ is as in Proposition 2.1.15 then* $\exp_{p_0}$ *defines a diffeomorphism*

$$\exp_{p_0}\big(\,\mathbb{D}_M(p_0, \rho)\,\big) = B_M(p_0, \rho).$$

Note that any choice of Euclidean coordinates (x^i) in the tangent space $T_{p_0}M$ produces local coordinates in some neighborhood of p_0 which we continue to denote by (x^i). The local coordinates obtained in this fashion are called *normal coordinates* at p_0. In these coordinates we have

$$x^i(p_0) = 0, \;\; \forall i.$$

$$\Gamma^i_{jk}(p_0) = 0, \;\; g_{ij}(x) = \delta_{ij} + O(2), \;\; \forall i, j, k$$

where δ_{ij} denotes the Kronecker symbol, and for positive real number ν we denote by $O(\nu)$ denotes a quantity bounded from above by *const.* $\sum_i |x^i|^\nu$ near p_0.

Example 2.1.17. Consider the vector space $\mathbb{R}^n$ equipped with the Euclidean metric. Then the exponential map

$$\exp : T\mathbb{R}^n \to \mathbb{R}^n$$

has the simple form

$$\exp(v, x) = x + v, \ \ \forall x \in \mathbb{R}^n, \ \ v \in T_x\mathbb{R}^n = \mathbb{R}^n.$$

In the sequel we will denote this Euclidean exponential map by $\mathbb{E}$. □

2.2. Cartan's moving frames method

We want to describe a very useful method for computing the Riemann curvature in concrete situations.

Suppose (M, g) is a Riemann manifold of dimension m. A *local* (or moving) *frame* on M is a collection $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ of smooth vector fields defined on an open subset $U \subset M$ such that for every $u \in U$ the collection of vectors $\{\boldsymbol{e}_1(u), \dots, \boldsymbol{e}_m(u)\} \subset T_uM$ is an *orthonormal* basis of T_uM.

To any local frame $(\boldsymbol{e}_i)$ defined on an open set $U \subset M$ we can associate the *dual coframe* which is the collection of 1-forms $\boldsymbol{\theta}^i \in \Omega^1(U)$, $1 \leq i \leq m = \dim M$ uniquely determined by the requirements

$$\boldsymbol{\theta}^i(e_j) = \delta^i_j \ \text{ on } \ U, \ \ \forall i, j.$$

Following E. Cartan we want to explain how to extract information about the Riemann curvature from the knowledge of the exterior derivatives $d\boldsymbol{\theta}^i$. In the sequel we will use Einstein's convention.

Note the Levi-Civita connection determines 1-forms $\omega^i_j \in \Omega^1(M)$ uniquely defined by the equalities

$$\nabla_k e_j = (\boldsymbol{e}_k \lrcorner \, \omega^i_j) e_i = \omega^i_j(e_k) e_i, \ \ \forall j, k$$

where $\lrcorner$ denotes the contraction of a differential form with a vector field. We can rewrite the above equalities in the more compact form

$$\nabla \boldsymbol{e}_j = \omega^i_j \boldsymbol{e}_i, \ \ \forall j \tag{2.5}$$

We denote by ω the $m \times m$ matrix with entries (ω^i_j). We will refer to these forms as *the* 1-*forms associated to* ∇ *by the frame* $(\boldsymbol{e}_i)$. Observe that because the Levi-Civita connection is compatible with the metric the matrix ω is skew-symmetric,

$$\omega^i_j = -\omega^j_i.$$

We set

$$\omega^i_{kj} := \boldsymbol{e}_k \lrcorner \, \omega^i_j = \omega^i_j(e_k).$$

If we now denote by $\vec{\boldsymbol{e}}$ the matrix $[\boldsymbol{e}_1, \dots, \boldsymbol{e}_m]$, then the equality (2.5) simplifies some more to the equality

$$\nabla \vec{\boldsymbol{e}} = \vec{\boldsymbol{e}} \cdot \omega.$$

The Levi-Civita connection enters into the differentials $d\boldsymbol{\theta}^i$ through the back door. More precisely we have the identity.

$$d\boldsymbol{\theta}^i(\boldsymbol{e}_j, \boldsymbol{e}_k) = \boldsymbol{e}_j \boldsymbol{\theta}^i(\boldsymbol{e}_k) - \boldsymbol{e}_k \boldsymbol{\theta}^i(\boldsymbol{e}_j) - \boldsymbol{\theta}^i([\boldsymbol{e}_j, \boldsymbol{e}_k]) = -\boldsymbol{\theta}^i([\boldsymbol{e}_j, \boldsymbol{e}_k]). \tag{2.6}$$

On the other hand, since the torsion of the Levi-Civita connection is zero we deduce

$$[\boldsymbol{e}_j, \boldsymbol{e}_k] = \nabla_{\boldsymbol{e}_j} \boldsymbol{e}_k - \nabla_{\boldsymbol{e}_k} \boldsymbol{e}_j = \omega^s_{jk} \boldsymbol{e}_s - \omega^s_{kj} \boldsymbol{e}_s$$

Hence

$$d\boldsymbol{\theta}^i(\boldsymbol{e}_j, \boldsymbol{e}_k) = -\delta^i_s \omega^s_{jk} + \delta^i_s \omega^s_{kj} = -\omega^i_{jk} + \omega^i_{kj}$$

If we consider the 2-form

$$\eta^i = \boldsymbol{\theta}^s \wedge \omega^i_s$$

then we observe that

$$\eta^i(\boldsymbol{e}_j, \boldsymbol{e}_k) = \boldsymbol{\theta}^s(\boldsymbol{e}_j)\omega^i_s(\boldsymbol{e}_k) - \boldsymbol{\theta}^s(\boldsymbol{e}_k)\omega^i_s(\boldsymbol{e}_j) = \delta^s_j \omega^i_s(\boldsymbol{e}_k) - \delta^s_k \omega^i_s(\boldsymbol{e}_j)$$
$$= \omega^i_{kj} - \omega^i_{jk} = d\boldsymbol{\theta}^i(\boldsymbol{e}_j, \boldsymbol{e}_k).$$

Hence

$$d\boldsymbol{\theta}^i = \boldsymbol{\theta}^s \wedge \omega^i_s = -\omega^i_s \wedge \boldsymbol{\theta}^s.$$

If we introduce the column vector

$$\vec{\boldsymbol{\theta}} = \begin{bmatrix} \boldsymbol{\theta}^1 \\ \vdots \\ \boldsymbol{\theta}^m \end{bmatrix}$$

then we can rewrite the above equality as

$$d\vec{\boldsymbol{\theta}} = -\omega \wedge \vec{\boldsymbol{\theta}}. \tag{2.7}$$

The last equality uniquely determines ω. More precisely, we have the following result.

Theorem 2.2.1 (Cartan). *There exists a unique, skew-symmetric matrix ω with entries 1-forms on U such that*

$$d\vec{\boldsymbol{\theta}} = -\omega \wedge \vec{\boldsymbol{\theta}}. \tag{2.8}$$

Moreover, the curvature 2-form $R \in \Omega^2(\mathrm{End}^-_g TM)$ is given by the equality

$$R = d\omega + \omega \wedge \omega, \tag{2.9}$$

where

$$(\omega \wedge \omega)^i_j = \sum_k \omega^i_k \wedge \omega^k_j.$$

The equalities (2.8) and (2.9) are known as *Cartan's structural equations.*

Proof. We have already established the existence of such a matrix. Let us establish its uniqueness. Observe first that there exist smooth functions g^i_{jk} on U uniquely determined by the equalities

$$d\boldsymbol{\theta}^i = \frac{1}{2} g^i_{jk} \boldsymbol{\theta}^j \wedge \boldsymbol{\theta}^k, \;\; g^i_{jk} = -g^i_{kj}.$$

Suppose $\omega = (\omega^i_j)$ is a skew-symmetric matrix of 1-forms on U satisfying (2.7). Then we can write each entry ω^i_j as a linear combination of $\boldsymbol{\theta}$'s,

$$\omega^i_j = f^i_{jk} \boldsymbol{\theta}^k, \;\; \forall i, j.$$

Since ω satisfies (2.7) we deduce

$$\boldsymbol{\theta}^j \wedge f^i_{jk} = \boldsymbol{\theta}^j \wedge \omega^i_j = d\boldsymbol{\theta}^i = g^i_{jk} \boldsymbol{\theta}^j \wedge \boldsymbol{\theta}^k$$

If we write

$$\bar{f}^i_{jk} = (f^i_{jk} - f^i_{kj})$$

then we deduce

$$\bar{f}^i_{jk} = g^i_{jk}.$$

Since ω is skew symmetric we deduce

$$f^i_{jk} = -f^j_{ik}.$$

We have

$$g^i_{jk} + g^j_{ki} - g^k_{ij} = f^i_{jk} - f^i_{kj} + f^j_{ki} - f^j_{ik} - f^k_{ij} + f^k_{ji} = 2f^i_{jk}.$$

Hence, the coefficients f^i_{jk} are uniquely determined by the $g's$ via

$$f^i_{jk} = \frac{1}{2}\big(g^i_{jk} + g^j_{ki} - g^k_{ij} \big).$$

To prove (2.9) we note that

$$R(\boldsymbol{e}_\ell, \boldsymbol{e}_m)\boldsymbol{e}_j = [\nabla_{\boldsymbol{e}_\ell}, \nabla_{\boldsymbol{e}_m}]\boldsymbol{e}_j - \nabla_{[\boldsymbol{e}_\ell,\boldsymbol{e}_m]}e_j$$

Using the identity (2.6) we deduce that

$$[\boldsymbol{e}_\ell, \boldsymbol{e}_m] = \sum_s \boldsymbol{\theta}^s([\boldsymbol{e}_\ell, \boldsymbol{e}_m])\boldsymbol{e}_s = -\sum_s d\boldsymbol{\theta}^s([\boldsymbol{e}_\ell, \boldsymbol{e}_m])\boldsymbol{e}_s$$

$$\stackrel{(2.8)}{=} \sum_s \sum_k \omega^s_k \wedge \boldsymbol{\theta}^k([\boldsymbol{e}_\ell, \boldsymbol{e}_m])\boldsymbol{e}_s = \sum_s \sum_k \big(\omega^s_{\ell k}\delta^k_m - \omega^s_{mk}\delta^k_\ell\big)\boldsymbol{e}_s = \sum_s \big(\omega^s_{\ell m} - \omega^s_{m\ell}\big)\boldsymbol{e}_s.$$

Hence

$$\nabla_{[\boldsymbol{e}_\ell,\boldsymbol{e}_m]}\boldsymbol{e}_j = \sum_s \big(\omega^s_{\ell m} - \omega^s_{m\ell}\big)\nabla_{\boldsymbol{e}_s}\boldsymbol{e}_j = \sum_i \sum_s \big(\omega^s_{\ell m} - \omega^s_{m\ell}\big)\omega^i_{sj}\boldsymbol{e}_i$$

Next, if we denote by $L_{\boldsymbol{e}_i}$ the Lie derivative along $\boldsymbol{e}_i$ we deduce

$$[\nabla_{\boldsymbol{e}_\ell}, \nabla_{\boldsymbol{e}_m}]\boldsymbol{e}_j = \nabla_{\boldsymbol{e}_\ell}(\nabla_{\boldsymbol{e}_m}\boldsymbol{e}_j) - \nabla_{\boldsymbol{e}_m}(\nabla_{\boldsymbol{e}_\ell}\boldsymbol{e}_j) = \nabla_{\boldsymbol{e}_\ell}\sum_s \omega^s_{mj}\boldsymbol{e}_s - \nabla_{\boldsymbol{e}_m}\sum_s \omega^s_{\ell j}\boldsymbol{e}_s$$

$$= \sum_s \big(L_{\boldsymbol{e}_\ell}\omega^s_{mj} - L_{\boldsymbol{e}_m}\omega^s_{m\ell}\big)\boldsymbol{e}_s + \sum_s \sum_i \Big(\omega^s_{mj}\omega^i_{\ell s} - \omega^s_{\ell j}\omega^i_{ms}\Big)\boldsymbol{e}_i$$

$$= \sum_i \big(L_{\boldsymbol{e}_\ell}\omega^i_{mj} - L_{\boldsymbol{e}_m}\omega^i_{m\ell}\big)\boldsymbol{e}_i + \sum_i \sum_s \omega^i_s \wedge \omega^j_j(\boldsymbol{e}_\ell, \boldsymbol{e}_m)\boldsymbol{e}_j$$

On the other hand, using the formula

$$d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta([X,Y]),\ \ \forall \eta \in \Omega^1(M),\ \ \forall X, Y \in \mathrm{Vect}(M)$$

we deduce

$$d\omega^i_j(\boldsymbol{e}_\ell, \boldsymbol{e}_m) = L_{\boldsymbol{e}_\ell}\omega^i_{mj} - L_{\boldsymbol{e}_m}\omega^i_{m\ell} - \omega^i_j([\boldsymbol{e}_\ell, \boldsymbol{e}_m])$$

From the equality

$$[\boldsymbol{e}_\ell, \boldsymbol{e}_m] = \sum_s \big(\omega^s_{\ell m} - \omega^s_{m\ell}\big)\boldsymbol{e}_s$$

we get

$$\omega^i_j([\boldsymbol{e}_\ell, \boldsymbol{e}_m]) = \sum_s \omega^i_{sj}\big(\omega^s_{\ell m} - \omega^s_{m\ell}\big).$$

Putting together all the above equalities we obtain Cartan's structural equation (2.9). □

Example 2.2.2. Suppose U is an open set in the plane $\mathbb{R}^2$ and $w : U \to (0, \infty)$ is a smooth function. Consider the Riemann metric g on U defined by

$$g = w^2(dx^2 + dy^2).$$

Then $\boldsymbol{e}_1 = \frac{1}{w}\partial_x$, $\boldsymbol{e}_2 = \frac{1}{w}\partial_y$ is an orthonormal fram with dual coframe

$$\boldsymbol{\theta}^1 = wdx, \quad \boldsymbol{\theta}^2 = wdy.$$

Then

$$\vec{\boldsymbol{\theta}} = \begin{bmatrix} wdx \\ wdy \end{bmatrix}, \quad d\vec{\boldsymbol{\theta}} = \begin{bmatrix} -w_y' dx \wedge dy \\ w_x' dx \wedge dy \end{bmatrix} = \begin{bmatrix} 0 & -\frac{w_y'}{w}dx \\ \frac{w_x'}{w}dy & 0 \end{bmatrix} \wedge \begin{bmatrix} wdx \\ wdy \end{bmatrix}$$

$$= \frac{1}{w}\begin{bmatrix} 0 & -w_y'dx + w_x'dy \\ w_y'dx - w_x'dy \end{bmatrix} \wedge \begin{bmatrix} wdx \\ wdy \end{bmatrix}$$

We deduce that

$$\omega = \frac{1}{w}\begin{bmatrix} 0 & w_y'dx - w_x'dy \\ -w_y'dx + w_x'dy \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \partial_y(\log w)dx - \partial_x(\log w)dy \\ -\partial_y(\log w)dx + \partial_x(\log w)dy \end{bmatrix}$$

The curvature 2-form is

$$R = d\omega + \omega \wedge \omega = d\omega = \begin{bmatrix} 0 & -(\Delta \log w)dx \wedge dy \\ (\Delta \log w)dx \wedge dy & 0 \end{bmatrix}, \quad \Delta := \partial_x^2 + \partial_y^2.$$

Then the sectional curvature is

$$R_{1212} = g(\boldsymbol{e}_1, R(\boldsymbol{e}_1, \boldsymbol{e}_2)\boldsymbol{e}_2) = -(\Delta \log w)dx \wedge dy(\boldsymbol{e}_1, \boldsymbol{e}_2) = -\frac{\Delta \log w}{w^2}.$$

For example, if U is the half-plane $y > 0$ and $w(x, y) = y^{-1}$ then the corresponding metric has constant sectional curvature $= -1$. This half-plane equipped with the metric $h = \frac{1}{y^2}(dx^2 + y^2)$ is known as the *hyperbolic plane*. □

2.3. The shape operator and the second fundamental form of a submanifold in $\mathbb{R}^n$

Suppose M is an m-dimensional smooth submanifold of $\mathbb{R}^n$. The Euclidean metric on $\mathbb{R}^n$ induces a metric g on M. We would like to determine the Levi-Civita connection ∇^M and the curvature tensor of g. In the sequel, we will use without mentioning the Einstein summation convention. We will use the following *indexing conventions*.

- We will use small Latin letters $i, j, \dots$ to denote indices in the range

$$1 \leq i, j \leq m = \dim M.$$

- We will use small Greek letters $\alpha, \beta, \dots$ to denote indices in the range

$$m < \alpha, \beta, \dots \leq n.$$

- We will use the capital Latin letters A, B, C to denote indices in the range

$$1 \leq A, B, C \leq n.$$

To compute the Levi-Civita connection we will use Cartan's moving frame method. Denote by $\boldsymbol{D}$ the Levi-Civita connection of the Euclidean metric on $\mathbb{R}^n$. Let us recall its definition since it will come in handy a bit later.

Every vector field on $\mathbb{R}^n$ can be regarded as an n-uple of functions

$$X = \begin{bmatrix} X^1 \\ \vdots \\ X^n \end{bmatrix}.$$

Then

$$\boldsymbol{D}\, X = \begin{bmatrix} dX^1 \\ \vdots \\ dX^n \end{bmatrix} = dX.$$

The restriction to M of the tangent bundle $T\mathbb{R}^n$ admits an orthogonal decomposition

$$(T\mathbb{R}^n)|_M = TM \oplus (TM)^{\perp}.$$

Correspondingly, a section X of $(T\mathbb{R}^n)|_M$ decomposes into a tangential part X^τ and a normal part X^ν. Fix a a point $p_0 \in M$, an open neighborhood U of p_0 in $\mathbb{R}^n$, and a local orhonormal frame $(\vec{e}_A)$ of $T\mathbb{R}^n$ along U. We denote by $(\boldsymbol{\theta}_A)$ the dual coframe of $(\boldsymbol{e}_A)$, i.e.

$$\boldsymbol{\theta}_A(\boldsymbol{e}_B) = \boldsymbol{e}_A \bullet \boldsymbol{e}_B = \delta_{AB}.$$

If X is a section of $T\mathbb{R}^n\big|_M$ then

$$X^\tau = \boldsymbol{\theta}^i(X)\boldsymbol{e}_i, \;\; X^\nu = \boldsymbol{\theta}^\alpha(X)\boldsymbol{e}_\alpha.$$

We denote by Θ^A_B the 1-forms associated to $\boldsymbol{D}$ by the frame $(\boldsymbol{e}_A)$. They satisfy Cartan's equations

$$d\boldsymbol{\theta}^A = -\Theta^A_B \wedge \boldsymbol{\theta}^B, \;\; D\boldsymbol{e}_B = \Theta^A_B\boldsymbol{e}_A, \;\; \Theta^A_B = -\Theta^B_A.$$

Now observe that

$$\boldsymbol{\theta}^\alpha|_M = 0$$

from which we conclude that

$$(d\boldsymbol{\theta}^i)|_M = -(\Theta^i_j \wedge \boldsymbol{\theta}^j)|_M.$$

If we write

$$\phi^A := \boldsymbol{\theta}^A|_M, \;\; \Phi^A_B := \Theta^A_B|_M$$

we deduce from the equalities

$$d\phi^i = -\Phi^i_j \wedge \phi^j, \;\; \Phi^i_j = -\Phi^j_i,$$

and Cartan's theorem that (Φ^i_j) are the 1-forms associated to the Levi-Civita connection ∇^M by the local orthonormal frame $(\boldsymbol{e}_i|_M)$. This implies that

$$\nabla^M \boldsymbol{e}_j = \Phi^i_j \boldsymbol{e}_i = \text{the tangential component of } \; \Phi^A_j e_A = \boldsymbol{D}\, e_j.$$

We have thus obtained the following result

Proposition 2.3.1. *For every $X, Y \in \mathrm{Vect}(M)$ we have the equality*

$$\nabla^M_X Y = (\boldsymbol{D}_X\, Y)^\tau. \qquad \square$$

Consider the *Gauss map*

$$\mathcal{G} = \mathcal{G}_M : M \to \mathbf{Gr}_m(\mathbb{R}^n), \quad x \mapsto T_x M.$$

The *shape operator* of the submanifold $M \hookrightarrow \mathbb{R}^n$ is, by definition, the differential of the Gauss map. We denote it by S^M and we would like to relate it to the structural coefficients Φ^A_B.

As explained in 1.1, in the neighborhood U of p_0, the "moving plane" $x \mapsto T_x M$ can be represented by the orthonormal frame $(\boldsymbol{e}_A)$ which has the property that the first m vectors $\boldsymbol{e}_1(x), \dots, \boldsymbol{e}_m(x)$ span $T_x M$. The differential of the Gauss map at $x \in U \cap M$ is a linear map

$$D\mathcal{G} : T_x M \to T_{\mathcal{G}(x)}\, \mathbf{Gr}_m(\mathbb{R}^m) = \mathrm{Hom}(T_x M, (T_x M)^{\perp}).$$

As explained in (1.6), this differential described by the $(n-m) \times m$ matrix of 1-forms

$$\big(\, \boldsymbol{e}_\alpha \bullet \boldsymbol{D}\, \boldsymbol{e}_j, \big)_{\alpha,i}$$

On the other hand

$$\boldsymbol{D}\, \boldsymbol{e}_j = \Phi^A_j \boldsymbol{e}_A$$

so that

$$\boldsymbol{e}_\alpha \bullet \boldsymbol{D}\, \boldsymbol{e}_j = \Phi^\alpha_j.$$

Define

$$\Phi^A_{ij} := \boldsymbol{e}_i \lrcorner\, \Phi^A_j \in \Omega^0(M \cap U),$$

so that

$$\Phi^A_j = \Phi^A_{ij} \wedge \phi^i.$$

We have thus obtained the following result.

Proposition 2.3.2. *The shape operator of M, that is the differential of the Gauss map, is locally described by the matrix of 1-forms $(\Phi^\alpha_i)_{1 \le i \le m < \alpha < n}$. More precisely the operator*

$$S^M(\boldsymbol{e}_i) \in \mathrm{Hom}(T_x M, (T_x M)^{\perp})$$

is given by

$$S^M(\boldsymbol{e}_i)\boldsymbol{e}_j = (\boldsymbol{D}_{\boldsymbol{e}_i}\, \boldsymbol{e}_j)^{\nu} = \Phi^\alpha_{ij} \boldsymbol{e}_\alpha, \quad \forall i, j. \qquad \square$$

The torsion of $\boldsymbol{D}$ is trivial so that,

$$\boldsymbol{D}_{\boldsymbol{e}_i}\, \boldsymbol{e}_j - \boldsymbol{D}_{\boldsymbol{e}_j}\, \boldsymbol{e}_i = [\boldsymbol{e}_i, \boldsymbol{e}_j].$$

Since the vector fields $\boldsymbol{e}_i, \boldsymbol{e}_j$ are tangent to M, so is their bracket $[\boldsymbol{e}_i, \boldsymbol{e}_j]$ so that

$$[\boldsymbol{e}_i, \boldsymbol{e}_j] \bullet \boldsymbol{e}_\alpha = 0, \quad \forall 1 \le i, j \le m < \alpha < n.$$

We deduce

$$(\boldsymbol{D}_{\boldsymbol{e}_i}\, \boldsymbol{e}_j) \bullet \boldsymbol{e}_\alpha = (\boldsymbol{D}_{\boldsymbol{e}_j}\, \boldsymbol{e}_i) \bullet \boldsymbol{e}_\alpha, \quad \forall i, j, \alpha$$

The last equality can be rewritten as

$$(\boldsymbol{D}_{\boldsymbol{e}_i}\, \boldsymbol{e}_j)^{\nu} = (\boldsymbol{D}_{\boldsymbol{e}_j}\, \boldsymbol{e}_i)^{\nu}.$$

If we observe that

$$(\boldsymbol{D}_{f\boldsymbol{e}_i}\, \boldsymbol{e}_j)^{\nu} = \big(\, \boldsymbol{D}_{\boldsymbol{e}_i}(f\boldsymbol{e}_j)\,\big)^{\nu} = f(\boldsymbol{D}_{\boldsymbol{e}_i}\, \boldsymbol{e}_j)^{\nu}, \quad \forall i, j, \quad f \in C^\infty(U \cap M)$$

we deduce that the map

$$\mathrm{Vect}(M)\times\mathrm{Vect}(M)\to C^\infty\big((TM)^\perp\big),\ \ \mathrm{Vect}(M)\times\mathrm{Vect}(M)\ni(X,Y)\mapsto(\boldsymbol{D}_X\,Y)^\nu$$

is $C^\infty(M)$ bilinear and symmetric, i.e., $\forall X,Y\in\mathrm{Vect}(M),\ \ f\in C^\infty(M)$ we have

$$(\boldsymbol{D}_{fX}\,Y)^\nu=(\boldsymbol{D}_X(fY))^\nu=f(\boldsymbol{D}_X\,Y)^\nu=f(\boldsymbol{D}_Y\,X)^\nu.$$

This symmetric bilinear form is called the *second fundamental form* of the submanifold $M\hookrightarrow\mathbb{R}^n$, and we will denote it by S_M. Note that

$$S_M(\boldsymbol{e}_i,\boldsymbol{e}_j)=\phi^\alpha(\boldsymbol{D}_{\boldsymbol{e}_i}\,\boldsymbol{e}_j)\boldsymbol{e}_\alpha=\Phi^\alpha_{ij}\boldsymbol{e}_\alpha=S^M(\boldsymbol{e}_i)\boldsymbol{e}_j. \tag{2.10}$$

From the equalities

$$0=\boldsymbol{e}_i(\boldsymbol{e}_j\bullet\boldsymbol{e}_\alpha)=(\boldsymbol{D}_{\boldsymbol{e}_i}\,\boldsymbol{e}_j)\bullet\boldsymbol{e}_\alpha+\boldsymbol{e}_j\bullet(\boldsymbol{D}_{\boldsymbol{e}_i}\,\boldsymbol{e}_\alpha),$$

we deduce

$$\boldsymbol{e}_j\bullet(\boldsymbol{D}_{\boldsymbol{e}_i}\,\boldsymbol{e}_\alpha)=-(\boldsymbol{D}_{\boldsymbol{e}_i}\,\boldsymbol{e}_j)\bullet\boldsymbol{e}_\alpha=-\boldsymbol{e}_\alpha\bullet S_M(\boldsymbol{e}_i,\boldsymbol{e}_j),\ \ \forall i,j,\alpha. \tag{2.11}$$

From Cartan's structural equations (2.9) we deduce that the Riemann tensor satisfies

$$R=d\Phi+\Phi\wedge\Phi\iff R^i_j=d\Phi^i_j+\Phi^i_k\wedge\Phi^k_j,\ \ \forall i,j.$$

Since the curvature of the Euclidean metric on $\mathbb{R}^n$ is trivial, we deduce from Cartan's structural equations that

$$d\Theta^A_B+\Theta^A_C\wedge\Theta^C_B=0,\ \ \forall A,B.$$

Restricting this equality to M we deduce

$$d\Phi^i_j=-\Phi^j_C\wedge\Phi^C_j=-\Phi^i_k\wedge\Phi^k_j-\Phi^i_\alpha\wedge\Phi^\alpha_j$$

so that

$$R^i_j=d\Phi^i_j+\Phi^i_k\wedge\Phi^k_j=-\Phi^i_\alpha\wedge\Phi^\alpha_j.$$

In particular, we deduce

$$g\big(\,\boldsymbol{e}_i,R(\boldsymbol{e}_k,\boldsymbol{e}_\ell)\boldsymbol{e}_j\,\big)=R^i_j(\boldsymbol{e}_k,\boldsymbol{e}_\ell)=-\sum_\alpha\left|\begin{array}{cc}\Phi^i_{k\alpha}&\Phi^i_{\ell\alpha}\\ \Phi^\alpha_{kj}&\Phi^\alpha_{\ell j}\end{array}\right|$$

$$=\sum_\alpha\left|\begin{array}{cc}\Phi^\alpha_{ki}&\Phi^\alpha_{\ell i}\\ \Phi^\alpha_{kj}&\Phi^\alpha_{\ell j}\end{array}\right|=S_M(\boldsymbol{e}_k,\boldsymbol{e}_i)\bullet S_M(\boldsymbol{e}_\ell,\boldsymbol{e}_j)-S_M(\boldsymbol{e}_k,\boldsymbol{e}_j)\bullet S_M(\boldsymbol{e}_\ell,\boldsymbol{e}_i).$$

This implies the following result.

Theorem 2.3.3 (Gauss' Golden Theorem)**.** *Suppose M is a submanifold of $\mathbb{R}^n$. We denote by g the induced metric on M and by S_M the second fundamental form of the embedding $M\hookrightarrow\mathbb{R}^n$. Denote by R the Riemann curvature of M with the induced metric. Then for any $X_1,\dots,X_4\in\mathrm{Vect}(M)$ we have*

$$g(X_1,R(X_3,X_4)X_2)=S_M(X_3,X_1)\bullet S_M(X_4,X_2)-S_M(X_3,X_2)\bullet S_M(X_4,X_1),$$

where $\bullet$ denotes the inner product in $\mathbb{R}^n$. □

2.4. The Gauss-Bonnet theorem for hypersurfaces of an Euclidean space.

The results in the previous subsection have very surprising consequences.

Suppose M is a compact, *orientable* hypersurface of $\mathbb{R}^{m+1}$. If fix on orientation on M then we obtain a normal vector field

$$\boldsymbol{n} : M \to \mathbb{R}^{m+1}, \;\; \boldsymbol{n}(x) \perp T_xM, \;\; |\boldsymbol{n}(x)| = 1, \;\; \forall x \in M.$$

If we choose a local *oriented* orthonormal frame $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ of TM then $\boldsymbol{n}(x), \boldsymbol{e}_1(x), \dots, \boldsymbol{e}_m$ is an oriented orthonormal frame of $\mathbb{R}^{m+1}$. In this case we can identify the second fundamental form with a genuine symmetric bilinear form

$$S_M \in C^\infty(T^*M^{\otimes 2}, \;\; S_M(X, Y) = \boldsymbol{n} \bullet (\boldsymbol{D}_X Y).$$

The Gauss map of $M \hookrightarrow \mathbb{R}^{m+1}$ can be given the description

$$\mathcal{G}_M : M \to \mathbf{Gr}_m(\mathbb{R}^{m+1}), \;\; M \ni x \mapsto \langle \boldsymbol{n}(x) \rangle^\perp := \text{the vector subspace orthogonal to } \boldsymbol{n}(x).$$

On the other hand, we have an *oriented* Gauss map

$$\vec{\mathcal{G}}_M : M \to S^m, \;\; x \to \boldsymbol{n}(x),$$

and a double cover

$$\pi : S^m \to \mathbf{Gr}_m(\mathbb{R}^{m+1}), \;\; S^m \ni \vec{u} \mapsto \langle \vec{u} \rangle^\perp,$$

so that the diagram below is commutative

$$\begin{tikzcd}
M \arrow[r, "\vec{\mathcal{G}}"] \arrow[dr, "\mathcal{G}"'] & S^m \arrow[d, "\pi"] \\
 & \mathbf{Gr}_m(\mathbb{R}^{m+1})
\end{tikzcd}$$

We fix an oriented orthonormal frame $(\vec{f}_0, \vec{f}_1, \dots, \vec{f}_m)$ of $\mathbb{R}^{m+1}$, and we orient the unit sphere $S^m \subset \mathbb{R}^{m+1}$ so that the orientation of $T_{\vec{f}_0} S^m$ is given by the ordered frame

$$(\vec{f}_1, \dots, \vec{f}_m).$$

Theorem 2.4.1 (Poincaré-Hopf-Morse)**.** *Suppose $m = \dim M$ is even. Then*

$$\deg \vec{\mathcal{G}}_M = \frac{1}{2}\chi(M) = \textit{Euler charactersitic of } M.$$

Proof. Pick a regular value $h_0 \in \mathbf{Gr}_m(\mathbb{R}^{m+1})$ of $\mathcal{G}_M$. Then $\pi^{-1}(h_0)$ consists of two unit vectors $\pm u_0 \in S^m$ which are both regular values of $\vec{\mathcal{G}}_M$. For every regular point $x \in M$ of $\vec{\mathcal{G}}_M$ we set

$$\epsilon_x : \begin{cases} 1 & D_x\vec{\mathcal{G}}_M : T_xM \to T_{\boldsymbol{n}(x)}S^m \text{ preserves orientations,} \\ -1 & D_x\vec{\mathcal{G}}_M : T_xM \to T_{\boldsymbol{n}(x)}S^m \text{ reverses orientations.} \end{cases}$$

Then

$$\deg \vec{\mathcal{G}}_M = \sum_{\boldsymbol{n}(x) = u_0} \epsilon_x = \sum_{\boldsymbol{n}(x') = -u_0} \epsilon_{x'}.$$

Consider now the function

$$\hat{\ell} : \mathbb{R}^{m+1} \to \mathbb{R}, \ \ \ell(x) = u_0 \bullet x.$$

and denote by ℓ its restriction to M.

Lemma 2.4.2. *(a) A point $x \in M$ is a critical point of ℓ if and only if $\boldsymbol{n}(x) = \pm u_0$.*
(b) The function ℓ to M is a Morse function. Moreover, if x is a critical point of ℓ and $\mu(x)$ is its Morse index then

$$\epsilon_x = (-1)^{\mu(x)}.$$

Proof. (a) Observe that u_0 is the gradient of $\hat{\ell}$. We deduce that $x \in M$ is a critical point of $L|_M$ if and only if $u_0 \perp T_x M$, i.e., $\boldsymbol{n}(x) = \pm u_0$.
(b) Suppose $x_0 \in M$ is a critical point of ℓ, that is, $\boldsymbol{n}(x_0) = \pm u_0$. Choose a local, oriented orthonormal moving frame $x \mapsto \vec{\boldsymbol{e}}(x) := (\boldsymbol{e}_1(x), \dots, \boldsymbol{e}_m(x))$ of TM defined in a neighborhood of x_0. Then

$$\vec{\boldsymbol{e}}(x_0) := (\, \boldsymbol{e}_1(x_0), \dots, \boldsymbol{e}_m(x_0)\,)$$

is a *positively* oriented orthonormal frame of $T_{\boldsymbol{n}(x_0)} S^m$.

The Hessian of ℓ at x_0 is the symmetric bilinear form $H_{x_0} : T_{x_0} M \times T_{x_0} M \to \mathbb{R}$ defined by

$$H_{x_0}(\, \boldsymbol{e}_i, \boldsymbol{e}_j\,) = (L_{\boldsymbol{e}_i} L_{\boldsymbol{e}_j} \ell)(x_0),$$

where $L_{\boldsymbol{e}_i}$ denotes the Lie derivative along the vector field $\boldsymbol{e}_i$. We have

$$L_{\boldsymbol{e}_j} \ell = L_{\boldsymbol{e}_j}(u_0 \bullet x) = u_0 \bullet \boldsymbol{e}_j,$$

$$L_{\boldsymbol{e}_i} L_{\boldsymbol{e}_j} \ell = L_{\boldsymbol{e}_i}(u_0 \bullet \boldsymbol{e}_j) = u_0 \bullet (\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_j) = \begin{cases} S_M(\boldsymbol{e}_i, \boldsymbol{e}_j) & \text{if } \ \boldsymbol{n}(x_0) = u_0 \\ -S_M(\boldsymbol{e}_i, \boldsymbol{e}_j) & \text{if } \ \boldsymbol{n}(x_0) = -u_0. \end{cases}$$

On the other hand, the differential at x_0 of the oriented Gauss map is the linear map

$$D_{x_0} \vec{\mathcal{G}}_M : T_{x_0} M \to T_{\boldsymbol{n}(x_0)} S^m$$

which associates to $\boldsymbol{e}_i(x_0) \in T_{x_0} M$ the orthogonal projection of the vector $(\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{n})_{x_0}$ onto $T_{\boldsymbol{n}(x_0)} S^m = \operatorname{span} \vec{\boldsymbol{e}}(x_0)$. In other words,

$$D_{x_0} \vec{\mathcal{G}}_M \boldsymbol{e}_i = \sum_j (\boldsymbol{e}_j \bullet \boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{n}) \boldsymbol{e}_i \overset{(2.11)}{=} -\sum_j S_M(\boldsymbol{e}_i, \boldsymbol{e}_j) \boldsymbol{e}_j. \tag{2.12}$$

Because $\boldsymbol{n}(x_0)$ is a regular value of $\vec{\mathcal{G}}_M$ we deduce that the matrix $\big(\, S_M(\boldsymbol{e}_i, \boldsymbol{e}_j)\, \big)_{1 \leq i,j \leq m}$ is nonsingular. This implies that the Hessian of ℓ at x_0 is also nonsingular and

$$(-1)^{\mu(x_0)} = \det H_{x_0} = (\pm 1)^m \det\big(\, S_M(\boldsymbol{e}_i, \boldsymbol{e}_j)\, \big) = \det\big(\, -S_M(\boldsymbol{e}_i, \boldsymbol{e}_j)\, \big) = \epsilon_{x_0}.$$

□

From the above lemma we deduce

$$2 \deg \vec{\mathcal{G}}_M = \sum_{\boldsymbol{n}(x) = \pm x_0} \epsilon_x = \sum_{d\ell(x) = 0} (-1)^{\mu(x)} = \chi(M),$$

where at the last step we used the Morse (in)equalities for the Morse function ℓ. This concludes the proof of Theorem 2.4.1. $\square$

Remark 2.4.3. The above result has one interesting consequence, namely that the compact, orientable hypersurfaces of an odd dimensional vector space have *even* Euler characteristic. This shows for example that the complex projective plane $\mathbb{CP}^2$ cannot be embedded smoothly in $\mathbb{R}^5$ because $\chi(\mathbb{CP}^2) = 3$. $\square$

In the remainder of this subsection we will assume that m is even, $m = 2h$. Denote by dA_m the "area" form on the unit m-dimensional sphere S^m. Recall that σ_m denotes the "area" of S^m. Hence

$$\int_{S^m} \frac{1}{\sigma_m} dA_m = 1$$

so that

$$\frac{1}{\sigma_m} \int_M \vec{\mathcal{G}}_M^* dA_m = \deg \mathcal{G}_M = \frac{1}{2}\chi(M).$$

We recall that

$$\sigma_{2h} = (2h+1)\omega_{2r+1} \Longrightarrow \frac{\sigma_{2h}}{2} = \frac{2^{2h}\pi^h h!}{(2h)!}. \tag{2.13}$$

Denote by g the induced metric on M and by R the curvature of g. We would like to prove that the integrand $\vec{\mathcal{G}}_M^* dA_m$ has the form

$$\vec{\mathcal{G}}_M^* dA_m = P(R_M) dV_M,$$

where dV_M denotes the metric volume form on M and $P(R_M)$ is a universal polynomial of degree $\frac{m}{2}$ in the curvature R of M.

Fix a positively oriented orthonormal frame $\vec{\boldsymbol{e}} = (\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ of TM defined on some open set $U \subset M$ and denote by $\vec{\boldsymbol{\theta}} = (\boldsymbol{\theta}^1, \dots, \boldsymbol{\theta}^m)$ the dual coframe. Observe that

$$dV_M = \boldsymbol{\theta}^1 \wedge \cdots \wedge \boldsymbol{\theta}^m.$$

We set

$$S_{ij} := S_M(\boldsymbol{e}_i, \boldsymbol{e}_j) \bullet \boldsymbol{n}, \;\; R_{ijk\ell} := g\big(\boldsymbol{e}_i, R(\boldsymbol{e}_k, \boldsymbol{e}_\ell)\boldsymbol{e}_j\big).$$

Theorem 2.3.3 implies that

$$R_{ijk\ell} = S_{ik}S_{j\ell} - S_{i\ell}S_{jk} = \begin{vmatrix} S_{ik} & S_{i\ell} \\ S_{jk} & S_{j\ell} \end{vmatrix}. \tag{2.14}$$

Observe that $R_{ijk\ell} \neq 0 \Longrightarrow i \neq j, \;\; k \neq \ell$, and in this case the matrix

$$\begin{bmatrix} S_{ik} & S_{i\ell} \\ S_{jk} & S_{j\ell} \end{bmatrix}$$

is the 2×2 submatrix of $S = (S_{ij})_{1 \leq i,j \leq m}$ obtained by intersection the rows i, j with the columns k, ℓ. We can rephrase the equality (2.14) in a more convenient form.

First, we regard the curvature $R_{ijk\ell}$ at a point $x \in M$ as a linear map

$$\Lambda^2 T_x M \to \Lambda^2 T_x^* M, \;\; R(\boldsymbol{e}_k \wedge \boldsymbol{e}_\ell) = \sum_{i<j} R_{ijk\ell} \boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^j.$$

Next, regard S as a linear map

$$S : T_xM \to T^*M, \;\; S\boldsymbol{e}_j = S_{ij}\boldsymbol{\theta}^i.$$

Then S induces linear maps

$$\Lambda^p S : \Lambda^k T_xM \to \Lambda^p T^*M,$$

defined by

$$S(\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_p}) = (S\boldsymbol{e}_{i_1}) \wedge \cdots \wedge (S\boldsymbol{e}_{i_p}), \;\; \forall 1 \leq i_1 < \cdots < i_p \leq m.$$

The equality (2.14) can now be rephrased as

$$R = \Lambda^2 S. \tag{2.15}$$

Along U we have the equality

$$(\vec{\mathcal{G}}_M^* dA_M) \mid_U = \Lambda^m S(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_m) = (\det S)\boldsymbol{\theta}^1 \wedge \cdots \wedge \boldsymbol{\theta}^m = (\det S) dV_M.$$

We want to prove that $\det S$ can be described in terms of $\Lambda^2 S$. To see this observe that

$$(\Lambda^{2h}S)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \cdots \boldsymbol{e}_{2h-1} \wedge \boldsymbol{e}_{2h}) = (\Lambda^2 S)(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2) \wedge \cdots \wedge (\Lambda^2 S)(\boldsymbol{e}_{2h-1} \wedge \boldsymbol{e}_{2h})$$

$$= \bigwedge_{s=1}^{h} \Big(\sum_{i<j} R_{ij,2s-1,2s} \boldsymbol{\theta}^i \wedge \boldsymbol{\theta}^j \Big) = \sum_{\varphi \in \mathcal{S}_m'} \epsilon(\varphi) \Big(\prod_{s=1}^{h} R_{\varphi(2s-1)\varphi(2s),2s-1,2s} \Big) dV_M,$$

where $\mathcal{S}_m'$ denotes the set of permutations φ of $\{1, 2, \dots, m = 2h\}$ such that

$$\varphi(1) < \varphi(2), \dots, \varphi(2h-1) < \varphi(2h),$$

and $\epsilon(\varphi) = \pm 1$ denotes the signature of a permutation. Observe that

$$\#\mathcal{S}_m' = \binom{2h}{2} \cdot \binom{2h-2}{2} \cdots \binom{2}{2} = \frac{(2h)!}{2^h}. \tag{2.16}$$

We would like to give an alternate description of $\det S$ using the concept of *pfaffian*.

First of all, define

$$\Theta_{ij} = \sum_{k<\ell} R_{ijk\ell} \boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^\ell = \frac{1}{2} \sum_{k,\ell} R_{ijk\ell} \boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^\ell = \Lambda^2 S(\boldsymbol{e}_i \wedge \boldsymbol{e}_j) \in \Omega^2(U).$$

We obtain in this fashion a $m \times m$ skew-symmetric matrix

$$\Theta = \Theta_g := \big(\Theta_{ij} \big)_{1 \leq i,j \leq m}$$

whose entries are 2-forms on U. Note that we can also think of Θ as a 2-form whose coefficients are skew-symmetric matrices. With the latter interpretation Θ is the curvature 2-form associated to the Levi-Civita connection

$$\Theta_g = R_{\nabla^g} \in \Omega^2(\operatorname{End}_g^- TM).$$

Define the *pfaffian* of Θ by the equality

$$\boldsymbol{Pf}(\Theta) := \frac{(-1)^h}{2^h h!} \sum_{\varphi \in \mathcal{S}_m} \epsilon(\varphi) \Theta_{\varphi(1)\varphi(2)} \wedge \cdots \wedge \Theta_{\varphi(2h-1)\varphi(2h)} \in \Omega^{2h}(U),$$

where $\mathcal{S}_m$ denotes the group of permutations of $\{1, 2, \dots, m\}$. Observe that

$$\boldsymbol{Pf}(\Theta) = \frac{(-1)^h}{h!} \sum_{\varphi \in \mathcal{S}_m'} \epsilon(\varphi) \Theta_{\varphi(1)\varphi(2)} \wedge \cdots \wedge \Theta_{\varphi(2h-1)\varphi(2h)},$$

We can simplify this some more if we introduce the set $\mathcal{S}''_m$ consisting of permutations $\varphi \in \mathcal{S}'_m$ such that

$$\varphi(1) < \varphi(3) < \cdots < \varphi(2h-1).$$

Observe that

$$\#\mathcal{S}'_m = (\#\mathcal{S}''_m)h! \Longrightarrow \#\mathcal{S}''_m = \frac{\mathcal{S}'_M}{h!} = \frac{(2h)!}{2^h h!} = 1\cdot 3\cdots(2h-1) =: \boldsymbol{\gamma}(2h).$$

Then

$$\boldsymbol{Pf}(\Theta) = (-1)^h \sum_{\varphi\in\mathcal{S}''_m} \epsilon(\varphi)\Theta_{\varphi(1)\varphi(2)} \wedge \cdots \wedge \Theta_{\varphi(2h-1)\varphi(2h)}.$$

We have

$$\boldsymbol{Pf}(-\Theta) = \frac{1}{h!} \sum_{(\sigma,\varphi)\in\mathcal{S}'_m\times\mathcal{S}'_m} \epsilon(\sigma\varphi)\Big(\prod_{j=1}^{h} R_{\varphi(2j-1)\varphi(2j)\sigma(2j-1)\sigma(2j)}\Big)dV_M.$$

On the other hand

$$\begin{aligned}
&(\Lambda^{2h}S)(\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\cdots\wedge\boldsymbol{e}_{2h-1}\wedge\boldsymbol{e}_{2h})\\
&= \frac{1}{\#\mathcal{S}'_m}\sum_{\varphi\in\mathcal{S}'_m} \epsilon(\varphi)\Lambda^m S(\boldsymbol{e}_{\varphi(1)}\wedge\boldsymbol{e}_{\varphi(2)}\wedge\cdots\wedge\boldsymbol{e}_{\varphi(2h-1)}\wedge\boldsymbol{e}_{\varphi(2h)})\\
&= \frac{1}{\#\mathcal{S}'_m}\sum_{\varphi\in\mathcal{S}'_m} \epsilon(\varphi)\Theta_{\varphi(1)\varphi(2)}\wedge\cdots\wedge\Theta_{\varphi(2h-1)\varphi(2h)}\\
&= \frac{1}{(\#\mathcal{S}'_m)} \sum_{(\sigma,\varphi)\in\mathcal{S}'_m\times\mathcal{S}'_m} \epsilon(\sigma\varphi)\Big(\prod_{j=1}^{h} R_{\varphi(2j-1)\varphi(2j)\sigma(2j-1)\sigma(2j)}\Big)dV_M = \frac{h!}{\#\mathcal{S}'_m}\boldsymbol{Pf}(-\Theta).
\end{aligned}$$

Hence

$$\vec{\mathcal{G}}^*_M dA_M = (\Lambda^{2h}S)(\boldsymbol{e}_1\wedge\boldsymbol{e}_2\wedge\cdots\wedge\boldsymbol{e}_{2h-1}\wedge\boldsymbol{e}_{2h}) = \frac{h!}{\#\mathcal{S}'_m}\boldsymbol{Pf}(-\Theta),$$

so that

$$\vec{\mathcal{G}}^*_M\big(\frac{2}{\sigma_{2h}}dA_M\big) = \frac{2}{\sigma_{2h}}\frac{h!}{\#\mathcal{S}'_m}\boldsymbol{Pf}(-\Theta) \stackrel{(2.13)}{=} \frac{h!}{\#\mathcal{S}'_m}\frac{(2h)!}{2^{2h}\pi^h h!}\boldsymbol{Pf}(-\Theta) \stackrel{(2.16)}{=} \frac{1}{(2\pi)^h}\boldsymbol{Pf}(-\Theta)$$

We have thus obtained the following result.

Theorem 2.4.4 (Gauss-Bonnet)**.** *If $M^{2h}\subset\mathbb{R}^{2h+1}$ is a compact, oriented hypersurface, and g denotes the induced metric, then*

$$\chi(M) = 2\deg\vec{\mathcal{G}}_M = \frac{1}{(2\pi)^h}\int_M \boldsymbol{Pf}(-\Theta_g).$$

More explicitly, if $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_{2h})$ is a local, positively oriented orthonormal frame of TM, then

$$\boldsymbol{Pf}(-\Theta_g) = \frac{1}{h!} \sum_{(\sigma,\varphi)\in\mathcal{S}'_{2h}\times\mathcal{S}'_m} \epsilon(\sigma\varphi)\Big(\prod_{j=1}^{h} R_{\varphi(2j-1)\varphi(2j)\sigma(2j-1)\sigma(2j)}\Big)dV_M \tag{2.17a}$$

$$= \sum_{\varphi\in\mathcal{S}''_{2h}} \epsilon(\varphi)\Theta_{\varphi(1)\varphi(2)}\wedge\cdots\wedge\Theta_{\varphi(2h-1)\varphi(2h)}, \tag{2.17b}$$

where $\mathcal{S}'_{2h}$ denotes the set of permutations φ of $\{1,\dots,2h\}$ such that

$$\varphi(2j-1) < \varphi(2j),\ \ \forall 1\le j\le h,$$

$\mathcal{S}''_m$ denotes the set of permutations $\varphi \in \mathcal{S}'_m$ such that

$$\varphi(1) < \varphi(3) < \cdots < \varphi(2h-1),$$

and

$$\Theta_{ij} = \sum_{k<\ell} R_{ijk\ell}\boldsymbol{\theta}^k \wedge \boldsymbol{\theta}^\ell. \qquad \square$$

Example 2.4.5. (a) If $\dim M = 2$ then

$$\boldsymbol{Pf}(-\Theta_g) = R_{1212}dV_M = (\text{the Gaussian curvature of } M) \times dV_M$$

(b) If $\dim M = 4$ then $\mathcal{S}''_4$ consists of 3 permutations

$$\begin{aligned}
&1,2,3,4,\ \ \epsilon = 1 \longrightarrow R_{1212}R_{3434},\\
&1,3,2,4,\ \ \epsilon = -1 \longrightarrow -R_{1312}R_{2434}\\
&1,4,2,3,\ \ \epsilon = 1 \longrightarrow R_{1412}R_{2334}
\end{aligned}$$

We deduce

$$\boldsymbol{Pf}(-\Theta_g) = \Theta_{12} \wedge \Theta_{34} - \Theta_{13} \wedge \Theta_{24} + \Theta_{14} \wedge \Theta_{23}$$

(c) We can choose the *positively oriented* local orthonormal frame $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ so that it diagonalizes S_M at a given point $x \in M$. Then the eigenvalues of S_M at x are called the *principal curvatures* at x and are denoted by $\kappa_1(x), \dots, \kappa_m(x)$. Then

$$\boldsymbol{Pf}(-\Theta) = \rho dV_M, \ \ \rho \in C^\infty(M),$$

where

$$\rho(x) = (2h-1)!! \prod_{k=1}^{m} \kappa_i(x), \ \ \forall x \in M.$$

Definition 2.4.6. If (M, g) is an oriented, even dimensional, Riemann manifold, then the top dimensional form

$$\frac{1}{(2\pi)^h}\boldsymbol{Pf}(-\Theta_g) \in \Omega^{2h}(M), \ \ h = \frac{1}{2}\dim M,$$

is called the *Euler form* associated with the metric and the orientation. We will denote it by $\boldsymbol{e}(M, g)$. $\square$

Remark 2.4.7. Although $\boldsymbol{Pf}(-\Theta_g)$ was described in terms of a positively oriented *local* orthonormal frame, one can very that the definition is independent of the choice of the frame. $\square$

2.5. Gauss-Bonnet theorem for domains in an Euclidean space

Suppose D is a relatively compact open subset of an Euclidean space $\mathbb{R}^{m+1}$ with smooth boundary ∂D. We denote by $\boldsymbol{n}$ the outer normal vector field along the boundary. It defines an *oriented Gauss map*

$$\vec{\mathcal{G}}_D : \partial D \to S^m.$$

We denote by dA_m the area form on the unit sphere S^m so that

$$\deg \vec{\mathfrak{G}}_D = \frac{1}{\sigma_m}\int_{\partial D} \vec{\mathfrak{G}}_D^* dA_m.$$

If m is even then the Gauss-Bonnet theorem for the hypersurface ∂D implies

$$\frac{1}{\sigma_m}\int_{\partial D} \vec{\mathfrak{G}}_D^* dA_m = \frac{1}{2}\chi(\partial D).$$

Using the Poincaré dulaity for the oriented manifold with boundary D we deduce $\chi(\partial D) = 2\chi(D)$, so that

$$\frac{1}{\sigma_m}\int_{\partial D} \vec{\mathfrak{G}}_D^* dA_m = \chi(D), \;\; m \in 2\mathbb{Z}.$$

We want to prove that the above equality holds even when m is odd. Therefore in the remainder of this section we assume *m is odd*.

Let us first describe the integrand $\vec{\mathfrak{G}}_D^* dA_m$. Let S_D denote the second fundamental form of the hypersurface

$$S_D(X, Y) = \boldsymbol{n} \bullet (\boldsymbol{D}_X Y) = -X \bullet (\boldsymbol{D}_Y \boldsymbol{n}), \;\; \forall X, Y \in \operatorname{Vect}(\partial D).$$

We deduce

$$\frac{1}{\sigma_m}\vec{\mathfrak{G}}_D^* dA_m = \frac{1}{\sigma_m}\det(-S_D) dV_{\partial D},$$

where $dV_{\partial D}$ denotes the volume form on ∂D.

A smooth vector field on $\bar{D}$,

$$X : \bar{D} \to \mathbb{R}^{m+1}$$

is called *admissible* if along the boundary points towards the exterior of D,

$$X \bullet \boldsymbol{n} > 0, \;\; \text{on } \;\partial D.$$

For an admissible vector field X define

$$\bar{X} : \partial D \to S^m, \;\; \bar{X}(p) = \frac{1}{|X(p)|}X(p), \;\; \forall p \in \partial D.$$

Let us observe that the map $\bar{X}$ is homotopic to the map $\vec{\mathfrak{G}}_D$. Indeed, for $t \in [0, 1]$ define

$$Y_t : \partial D \to S^m, \;\; Y_t(p) = \frac{1}{|(1-t)\boldsymbol{n} + t\bar{X}|}((1-t)\boldsymbol{n} + t\bar{X}$$

Observe that this map is well defined since

$$|(1-t)\boldsymbol{n} + t\bar{X}|^2 = t^2 + (1-t)^2 + 2t(1-t)(\boldsymbol{n} \bullet \bar{X}) > 0.$$

Hence

$$\deg \vec{\mathfrak{G}}_D = \deg \bar{X}$$

for any admissible vector field X.

Suppose X is a *nondegenerate* admissible vector field which means that X has a finite numbemer of stationary points

$$\mathcal{Z}_X = \{p_1, \dots, p_\nu\}, \;\; X(p_i) = 0,$$

and all of them are nondegenerate, i.e., for any $p \in \mathcal{Z}_X$ the linear map

$$AX, p : T_p\mathbb{R}^{m+1} \to T_p\mathbb{R}^{m+1}, \;\; T_p\mathbb{R}^{m+1} \ni v \mapsto (\boldsymbol{D}_v X)(p)$$

is invertible. Define

$$\epsilon_X : \mathcal{Z}_X \to \{\pm 1\},\ \ \epsilon(p) = \operatorname{sign}\det A_{X,p}.$$

For any $\varepsilon > 0$ sufficiently small the closed balls of radius ε centered at the points in $\mathcal{Z}_X$ are disjoint. Set

$$D_\varepsilon = D \setminus \bigcup_{p \in \mathcal{Z}_X} B_\varepsilon(p).$$

X does not vanish on D_ε and we obtain a map

$$\bar{X} : \bar{D}_\varepsilon \to S^{m-1},\ \ \bar{X} = \frac{1}{|X|}X.$$

Set

$$\Omega := \frac{1}{\sigma_m}\bar{X}^* dA_m.$$

Observe that

$$d\Omega = \frac{1}{\sigma_m}\bar{X}^* d(dA_m) = 0 \ \text{ on } \ D_\varepsilon.$$

Stokes theorem then implies that

$$\int_{\partial D_\varepsilon} \Omega = \int_{D_\varepsilon} d\Omega = 0 \Longrightarrow \deg \vec{\mathcal{G}}_D = \int_{\partial D} \Omega = \sum_{p \in \mathcal{Z}_X} \int_{\partial B_\varepsilon(p)} \Omega,$$

where the spheres $\partial B_\varepsilon(p)$ are oriented as boundaries of the balls $B_\varepsilon(p)$. If we let $\varepsilon \to 0$ we deduce

$$\deg \vec{\mathcal{G}}_D = \sum_{p \in \mathcal{Z}_X} \epsilon_X(p), \tag{2.18}$$

for any nondegenerate admissible vector field X.

To give an interpretation of the right-hand side of the above equality consider the *double of D*. This is the smooth manifold $\hat{D}$ obtained by gluing D along ∂D to a copy of itself equipped with the opposite orientation,

$$\hat{D} = D \cup_{\partial D} (-D).$$

$\hat{D}$ is equipped with an orientation reversing involution $\varphi : \hat{D} \to \hat{D}$ whose fixed point set is ∂D. In particular, along $\partial D \subset \hat{D}$ we have a φ-invariant decomposition

$$T\hat{D}|_{\partial D} = T\partial D \oplus L,$$

where L is a real line bundle along which the differential of φ acts as $-\mathbb{1}_L$. The normal vector field $\boldsymbol{n}$ defines a basis of L. If X is a vector field on D which is equal to $\boldsymbol{n}$ along ∂D, then we obtain a vector field $\hat{X}$ on $\hat{D}$ by setting

$$\hat{X} := \begin{cases} X & \text{on } D \\ -\varphi_*(X) & \text{on } -D. \end{cases}$$

If X is nondegenerate, then so is $\hat{X}$, where the nondegeneracy of $\hat{X}$ is defined in terms of an arbitrary connection on $T\hat{D}$. More precisely, if ∇ is a connection on $T\hat{D}$ and $q \in \mathcal{Z}_{\hat{X}}$, then q is nondegenerate if the map

$$A_{\hat{X},q} : T_q\hat{D} \to T_q\hat{D},\ \ v \mapsto \nabla_v \hat{X}$$

is an isomorphism. This map is independent of the connection ∇, and we denote by $\epsilon_{\hat{X}}(q)$ the sign of its determinant. Moreover

$$\mathcal{Z}_{\hat{X}} = \mathcal{Z}_X \cup \varphi(\mathcal{Z}_X),$$

and, *because m is odd*, the map

$$\epsilon_{\hat{X}} : \mathcal{Z}_{\hat{X}} \to \{\pm 1\}$$

satisfies

$$\epsilon_X(p) = \epsilon_{\hat{X}}(\varphi(p)).$$

Hence

$$\sum_{q \in \mathcal{Z}_{\hat{X}}} \epsilon_{\hat{X}}(q) = 2 \sum_{p \in \mathcal{Z}_X} \epsilon_X(q) \stackrel{(2.18)}{=} 2 \deg \vec{\mathcal{G}}_D.$$

On the other hand, the general Poincaré-Hopf theorem implies that

$$\sum_{q \in \mathcal{Z}_{\hat{X}}} \epsilon_{\hat{X}}(q) = \chi(\hat{D}).$$

Using the Mayer-Vietoris theorem we deduce

$$\chi(\hat{D}) = 2\chi(D) - \chi(\partial D).$$

Since ∂D is *odd dimensional and oriented* we deduce that $\chi(\partial D) = 0$, and therefore

$$2\chi(D) = \chi(\hat{D}) = 2 \sum_{p \in \mathcal{Z}_X} \epsilon_X(q) = 2 \deg \vec{\mathcal{G}}_D.$$

We have thus proved the following result.

Theorem 2.5.1 (Gauss-Bonnet for domains)**.** *Suppose D is a relatively compact open subset of $\mathbb{R}^{m+1}$ with smooth boundary ∂D. We denote by $\vec{\mathcal{G}}_D$ the oriented Gauss map*

$$\vec{\mathcal{G}}_D : \partial D \to S^m, \;\; \partial D \ni p \mapsto \boldsymbol{n}(p) = \textit{unit outer normal},$$

and by S_D the second fundamental form of ∂D,

$$S_D(X, Y) = \boldsymbol{n} \bullet (\boldsymbol{D}_X Y), \;\; \forall X, Y \in \mathrm{Vect}(\partial D).$$

Then

$$\frac{1}{\sigma_m} \int_{\partial D} \det(-S_D) dV_{\partial D} = \deg \vec{\mathcal{G}}_D = \chi(D).$$

Chapter 3

Curvature measures

We can now formulate and prove the key result of these notes, the tube formula, which will produce some interesting metric invariants of a Riemann manifold. We will then describe their *reproducing properties*, also known as *Crofton fomulæ*.

3.1. Invariants of the orthogonal group

In the proof of the tube formula we will need to use H.Weyl's characterization of polynomials invariant under the orthogonal group.

Suppose V is a finite dimensional Euclidean space with metric $(-,-)$. We denote by $\langle -,-\rangle$ the canonical pairing

$$\langle -,-\rangle : V^* \times V \to \mathbb{R},\ \ \langle \boldsymbol{\lambda}, v\rangle = \boldsymbol{\lambda}(v),\ \ \forall v \in V,\ \ \lambda \in V^* = \mathrm{Hom}(V,\mathbb{R}).$$

We denote by $O(V)$ the group of orthogonal transformations of the Euclidean space V. By definition, an $O(V)$-module is a pair (E,ρ), where E is a finite dimensional real vector space, while ρ is a group morphism

$$\rho : O(V) \to \mathrm{Aut}(E),\ \ g \mapsto \rho(g).$$

A morphism of $O(V)$-modules (E_i,ρ_i), $i = 0,1$, is a linear map $A : E_0 \to E_1$ such that for every $g \in O(V)$ the diagram below is commutative

$$\begin{tikzcd}
E_0 \arrow[r, "A"] \arrow[d, "\rho_0(g)"'] & E_1 \arrow[d, "\rho_1(g)"] \\
E_0 \arrow[r, "A"'] & E_1
\end{tikzcd}$$

We will denote by $\mathrm{Hom}_{O(V)}(E_0,E_1)$ the spaces of morphisms of $O(V)$-modules.

The vector space V has a tautological structure of $O(V)$-module given by

$$\tau : O(V) \to \mathrm{Aut}(V),\ \ \tau(g)\boldsymbol{v} = gv,\ \ \forall g \in O(V),\ \ v \in V.$$

It induces a structure of $O(V)$-module on $V^* = \mathrm{Hom}(V, \mathbb{R})$ given by

$$\rho^\dagger : O(V) \to \mathrm{Aut}(V^*), \;\; g \mapsto \rho^\dagger(g),$$

where

$$\langle \rho^\dagger(g)\boldsymbol{\lambda}, \boldsymbol{v}\rangle = \langle \boldsymbol{\lambda}, g^{-1}\boldsymbol{v}\rangle, \forall \boldsymbol{\lambda} \in V^*, \boldsymbol{v} \in V.$$

In particular, we obtain an action on $(V^*)^{\otimes n}$,

$$(\rho^\dagger)^{\otimes n} : O(V) \to \mathrm{Aut}\big((V^*)^{\otimes n} \big), \;\; g \mapsto \rho^\dagger(g)^{\otimes n}.$$

We denote by $(V^*)^{\otimes n}_{O(V)}$ the subspace consisting of invariant tensors,

$$\omega \in (V^*)^{\otimes n}_{O(V)} \Longleftrightarrow (\rho^\dagger_g)^{\otimes n}\omega = \omega, \;\; \forall g \in O(V).$$

Observe that $(V^*)^{\otimes n}$ can be identified with the vector space of multi-linear maps

$$\omega : V^n = \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

so that $(V^*)^{\otimes n}_{O(V)}$ can be identified with the subspace of $O(V)$-invariant multilinear maps $V^n \to \mathbb{R}$.

Hermann Weyl has produced in his classic monograph [**W2**] an explicit description of $(V^*)^{\otimes n}_{O(V)}$. We would like to present here, without proof, this beautiful result of Weyl since it will play an important role in the future. We follow the elegant and more modern presentation in [**ABP**, Appendix I] to which we refer for proofs.

Observe first that the metric duality defines a natural isomorphism of vector spaces

$$D : V \to V^*, \;\; \boldsymbol{v} \mapsto \boldsymbol{v}^\dagger$$

defined by

$$\langle \boldsymbol{v}^\dagger, \boldsymbol{u}\rangle = (\boldsymbol{v}, \boldsymbol{u}), \;\; \forall \boldsymbol{u}, \boldsymbol{v} \in V.$$

This isomorphism induces an isomorphism of $O(V)$-modules

$$D : (V, \rho) \to (V^*, \rho^\dagger).$$

We conclude that for ever nonnegative integers r, s we have isomorphisms of G-modules

$$(V^*)^{\otimes(r+s)} \cong (V^* \otimes r) \otimes V^{\otimes s} \cong \mathrm{Hom}(V^{\otimes r}, V^{\otimes s})$$

In particular,

$$\big((V^*)^{\otimes(r+s)} \big)_{O(V)} \cong \big(\mathrm{Hom}(V^{\otimes r}, V^{\otimes s}) \big)_{O(V)} = \mathrm{Hom}_{O(V)}(V^{\otimes r}, V^{\otimes s}).$$

Let us observe that if we denote by $\mathcal{S}_r$ the group of permutations of $\{1, \dots, r\}$, then for every $\varphi \in \mathcal{S}_r$ we obtain a morphism of $O(V)$-modules

$$T_\phi \in \mathrm{Hom}_{O(V)}(V^{\otimes r}, V^{\otimes r}), \;\; T_\varphi(v_1 \otimes \cdots \otimes v_r) = v_{\varphi(1)} \otimes \cdots \otimes v_{\varphi(r)}.$$

Weyl's *First Main Theorem of Invariant Theory* states that

$$\mathrm{Hom}_{O(V)}(V^{\otimes r}, V^{\otimes s}) \neq 0 \Longleftrightarrow r = s,$$

and that

$$\mathrm{Hom}_{O(V)}(V^{\otimes r}, V^{\otimes r}) = \mathbb{R}[\mathcal{S}_r] := \Big\{ \sum_{\varphi \in \mathcal{S}_r} c_\varphi T_\varphi; \;\; c_\varphi \varphi \in \mathbb{R}, \Big\}.$$

We can translate this result in terms of invariant multi-linear forms. Thus

$$(V^*)^{\otimes n}_{O(V)} \neq 0 \Longleftrightarrow n = 2r, \quad r \in \mathbb{Z}_{\geq 0},$$

and $(V^*)^{\otimes 2r}_{O(V)}$ is spanned by the multilinear forms

$$P_\varphi : V^{2r} \to \mathbb{R}, \quad (\varphi \in \mathcal{S}_r),$$

defined by

$$P_\varphi(\boldsymbol{u}_1, \dots, \boldsymbol{u}_r, \boldsymbol{v}_1, \dots, \boldsymbol{v}_r) = \big(\, \boldsymbol{u}_1, \boldsymbol{v}_{\varphi(1)} \,\big) \cdots \big(\, \boldsymbol{u}_r, \boldsymbol{v}_{\varphi(r)} \,\big).$$

The above has an immediate consequence. Suppose we have a map

$$f : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}, \quad (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) \mapsto f(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n),$$

which is a homogeneous polynomial of degree d_i in the variable $\boldsymbol{v}_i$, $\forall i = 1, \dots, n$. This form determines a multilinear form

$$\mathrm{Pol}_f : V^{d_1} \times \cdots \times V^{d_n} \to \mathbb{R}$$

obtained by polarization in each variables separately,

$$\mathrm{Pol}_f(\boldsymbol{u}_1^1, \dots, \boldsymbol{u}_1^{d_1}; \dots ; \boldsymbol{v}_n^1, \dots, \boldsymbol{v}_n^{d_n})$$

$$= \text{the coefficient of the monomial } t_{11}t_{12}\cdots t_{1d_1}\cdots t_{n1}\cdots t_{nd_n} \text{ in the polynomial}$$

$$P_f(t_{11}, t_{12}, \dots, t_{1d_1}, \dots, t_{n1}, \dots, t_{nd_n}) = f\Big(\sum_{j=1}^{d_1} t_{ij}\boldsymbol{u}_1^j, \dots, \sum_{j=1}^{d_n} t_{nj}u_n^j\Big).$$

Observe that

$$f(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) = \mathrm{Pol}_f(\underbrace{\boldsymbol{v}_1, \dots, \boldsymbol{v}_1}_{d_1}, \dots, \underbrace{\boldsymbol{v}_n, \dots, \boldsymbol{v}_n}_{d_n}),$$

and f is $O(V)$-invariant if and only if Pol_f is $O(V)$ invariant.

Note that every function

$$f : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

which is polynomial in each of the variables is a linear combination of functions which is polynomial and homogeneous in each of the variables. For every $1 \leq i \leq j \leq n$ we define

$$q_{ij} : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}, \quad q_{ij}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) := (\boldsymbol{v}_i, \boldsymbol{v}_j).$$

Theorem 3.1.1 (Weyl)**.** *If $f : V \times \cdots \times V \to \mathbb{R}$ is a polynomial map then f is $O(V)$-invariant if and only if there exists a polynomial P in the $\binom{n+1}{2}$ variables q_{ij} such that*

$$f(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) = P(\, q_{ij}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)_{1 \leq i \leq j \leq n}\,). \qquad \square$$

Example 3.1.2. (a) Consider the space $E = V^{\otimes k}$. Observe that a degree n homogeneous polynomial P on E can by identified with an element in the symmetric tensor product

$$\mathrm{Sym}_d(E^*) \subset (V^*)^{\otimes 2kn}.$$

P is called a *degree d orthogonal invariant* of tensors $T \in V^{\otimes k}$ if it is invariant as an element of $(V^*)^{\otimes kn}$. For example, Weyl's theorem implies that the only degree 1 invariant of a tensor

$$T = \sum_{i,j} T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \in V^{\otimes 2}$$

is the *trace*

$$\operatorname{tr}(T) = \sum_{i,j} T_{i,j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sum_i T_{ii}.$$

The space of degree 2 invariants is spanned by the polynomials

$$(\operatorname{tr}(T))^2, \;\; Q(T) = \sum_{ij} T_{ij}^2, \;\; \tilde{Q}(T) = \sum_{i,j} T_{ij} T_{ji}. \qquad \square$$

3.2. The tube formula and the curvature measures of closed submanifolds of an Euclidean space

Suppose M is an m-dimensional submanifold of $\mathbb{R}^n$. We set

$$c := \operatorname{codim} M = n - m.$$

In this section we will assume that M is *compact and without boundary* but *we will not assume that it is orientable*. For $r > 0$ we define the tube of radius r around M to be the closed set

$$\mathbb{T}_r(M) := \bigl\{ x \in M; \;\; \operatorname{dist}(x, M) \leq r \bigr\},$$

and we denote by $V(M, r)$ its volume.

Let $\mathcal{N}(M)$ denote the orthogonal complement of TM in $(T\mathbb{R}^n)|_M$, and we will call it the *normal bundle* of $M \hookrightarrow \mathbb{R}^n$. We define

$$\mathbb{D}_r(\mathbb{R}^n) := \bigl\{ (v, p); \;\; p \in \mathbb{R}^n, \;\; v \in T_p\mathbb{R}^n, \;\; |v| \leq r \bigr\} \subset T\mathbb{R}^n$$

and we set

$$\mathcal{N}_r(M) := \mathcal{N}(M) \cap \mathbb{D}_r(\mathbb{R}^n).$$

$\mathbb{D}_r(\mathbb{R}^n)$ is a bundle of n-dimensional disks over $\mathbb{R}^n$, while $\mathcal{N}_r(M)$ is a bundle of c-dimensional disks over M.

The exponential map $\mathbb{E} : T\mathbb{R}^n \to \mathbb{R}^n$ restricts to an exponential map

$$\mathbb{E}_M : \mathcal{N}(M) \to \mathbb{R}^n.$$

Observe that because M is compact there exists $r_0 = r_0(M) > 0$ such that for every $r \in (0, r_0)$ the exponential map $\mathbb{E}_M$ induces a diffeomorphism

$$\mathbb{E}_M : \mathcal{N}_r(M) \to \mathbb{T}_r(M).$$

If we denote by $|dV_n|$ the volume density on $\mathbb{R}^n$ we deduce

$$V(M, r) = \operatorname{vol}\bigl(\mathbb{T}_r(M)\bigr) = \int_{\mathbb{T}_r(M)} |dV_n| = \int_{\mathcal{N}_r(M)} \mathbb{E}_M^* |dV_n|.$$

If $\pi : \mathcal{N}_r(M) \to M$ denotes the canonical projection, then we deduce from Fubini's theorem that

$$V(M, r) = \int_M \pi_* \mathbb{E}_M^* |dV_n|. \tag{3.1}$$

We want to give a more explicit description of the density $\pi_*\mathbb{E}_M^*|dV_n|$.

Fix a local orthonormal frame $(\boldsymbol{e}_A)$ of $(T\mathbb{R}^n)|_M$ defined in a neighborhood $U \subset M$ of a point $p_0 \in M$ such that for all $1 \leq i \leq m$ vector field $\boldsymbol{e}_i$ is tangent to U. We assume that the orientation of $\mathbb{R}^n$ is given by the ordered frame

$$(\boldsymbol{e}_{m+1}, \dots, \boldsymbol{e}_n; \boldsymbol{e}_1, \dots, \boldsymbol{e}_m).$$

$$D_r^c := \bigl\{ \vec{t} = (t^\alpha) = (t^{m+1}, \dots, t^n) \in \mathbb{R}^c;\ \sum_\alpha |t^\alpha|^2 \leq r \bigr\}.$$

Note that we have a diffeomorphism

$$D_r^c \times U \longrightarrow \mathcal{N}_r(U) := \mathcal{N}_r(M)|_U,\ \ (\vec{t}, x) \mapsto (t^\alpha \boldsymbol{e}_\alpha(x), x) \in \mathcal{N}_r(M),$$

and thus we can identify $D_r^c \times U$ with the open subset $\pi^{-1}(U) \subset \mathcal{N}_r(M)$, and we can use $x \in M$ and $\vec{t} \in D_r^c$ as local coordinates on $\pi^{-1}(U)$. Define

$$\mathbb{T}_r(U) := \mathbb{E}_M(\,\mathcal{N}_r(U)\,) \subset \mathbb{R}^n,$$

and

$$\tilde{\boldsymbol{e}}_A : \mathbb{T}_r(U) \to \mathbb{R}^n \ \text{ by } \ \tilde{\boldsymbol{e}}_A(x + t^\alpha \boldsymbol{e}_\alpha) = \boldsymbol{e}_A(x).$$

We have thus extended in a special way the local frame $(\boldsymbol{e}_A)$ of $(T\mathbb{R}^n)|_U$ to a local frame of $(T\mathbb{R}^n)|_{\mathbb{T}_r(U)}$ so that

$$\boldsymbol{D}_{\tilde{\boldsymbol{e}}_\alpha} \tilde{\boldsymbol{e}}_A = 0,\ \ \forall \alpha, A. \tag{3.2}$$

We denote by $(\boldsymbol{\theta}^A)$ the coframe of $\mathbb{T}_r(U)$ dual to $\tilde{\boldsymbol{e}}_A$. We will continue to use the indexing conventions we have used in Section 2.3.

Over $D_r^c \times U$ we have a local frame $(\partial_{t^\alpha}, \boldsymbol{e}_i)$ with dual coframe (ϕ^A) defined by

$$\phi^i = \pi^* \boldsymbol{\theta}^i,\ \ \phi_\alpha = dt^\alpha.$$

Consider the 1-forms $\Theta_B^A \in \Omega^1\bigl(\mathbb{T}_r(U)\bigr)$ associated to the Levi-Civita connection $\boldsymbol{D}$ by the frame $(\tilde{\boldsymbol{e}}_A)$ on $\mathbb{T}_r(U)$, and set

$$\Theta_{CB}^A = \tilde{\boldsymbol{e}}_C \lrcorner\, \Theta_B^A,\ \ \forall i,$$

so that

$$\boldsymbol{D}_{\boldsymbol{e}_C} \boldsymbol{e}_B = \Theta_{CB}^A \boldsymbol{e}_A.$$

Using (3.2) we deduce

$$\Theta_{\alpha B}^A = 0,\ \ \forall \alpha \Longrightarrow \Theta_B^A = \Theta_{iB}^A \boldsymbol{\theta}^i. \tag{3.3}$$

Finally set

$$\Phi_B^A = \pi^*(\Theta_B^A|_M) \in \Omega^1(\mathcal{N}_r(U)),\ \ \Phi_{iB}^A := \pi^*(\Theta_{iB}^A|_U) \in C^\infty(\mathcal{N}_r(U)\,).$$

The equalities (3.3) imply

$$\Phi_B^A = \Phi_{iB}^A \phi^i.$$

On $\mathcal{N}_r(M)|_U$ we use $(\vec{t}, x)$ as coordinates and we have

$$\mathbb{E}_M(t^\alpha \boldsymbol{e}_\alpha(x), x) = x + t^\alpha \boldsymbol{e}_\alpha.$$

We have

$$\begin{aligned} \mathbb{E}_M^* \boldsymbol{\theta}^A &= \sum_i (\boldsymbol{e}_A \bullet \boldsymbol{D}_{\boldsymbol{e}_i} \mathbb{E}_M)\phi^i + \sum_\alpha (\boldsymbol{e}_A \bullet \partial_{t^\alpha} \mathbb{E}_M) dt^\alpha \\ &= \sum_i \boldsymbol{e}_A \bullet (\boldsymbol{e}_i + t^\alpha\, \boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_\alpha)\phi^i + \sum_\alpha \delta_{A\alpha} dt^\alpha \end{aligned}$$

$$= \delta_{Ai}\phi^i + t^\alpha \Phi^A_{i\alpha}\phi^i + \delta_{A\alpha}dt^\alpha$$

Hence

$$\mathbb{E}_M^* \boldsymbol{\theta}^j = \phi^j + t^\alpha \Phi^j_{i\alpha}\phi^i = \phi^j - \sum_\alpha t^\alpha \Phi^\alpha_{ij}\phi^i, \;\; \mathbb{E}_M^* \boldsymbol{\theta}^\beta = dt^\beta.$$

We find it convenient to set

$$\Phi_{ij} = (\Phi^{m+1}_{ij}, \dots, \Phi^n_{ij}) : U \to \mathbb{R}^c,$$

so that

$$\mathbb{E}_M^* \boldsymbol{\theta}^j = \phi^j - \sum_i (\vec{t} \bullet \Phi_{ij})\phi^i.$$

Define the $m \times m$ symmetric matrix

$$S = S(\vec{t}, x) = \big(\, \vec{t} \bullet \Phi_{ij}(x)\,\big)_{1 \le i,j \le m}$$

Note that the volume density on $\mathbb{R}^n$ is

$$|dV_n| = |\theta^{m+1} \wedge \cdots \wedge \theta^n \wedge \theta^1 \wedge \cdots \wedge \theta^m.|$$

$$\mathbb{E}_M^* dV_n = |\det(\mathbb{1} - S(\vec{t}, x)\,)|\,|d\vec{t} \wedge d\phi| = \det(\mathbb{1} - S(\vec{t}, x)\,)\,|d\vec{t} \wedge d\phi|, \tag{3.4}$$

$$d\vec{t} = dt^{m+1} \wedge \cdots \wedge dt^n, \;\; d\phi = d\phi^1 \wedge \cdots \wedge d\phi^m.$$

Recalling that $|d\boldsymbol{\theta}^i|_M|$ is the volume density on M, we deduce

$$\mathbb{E}_M^* |dV_n| = \det(\mathbb{1} - S(\vec{t}, x)\,)|d\vec{t}| \times \pi^*|dV_M|,$$

where $|d\vec{t}|$ denotes the volume density on $\mathbb{R}^c$. For simplicity we write $|dV_M|$ instead of $\pi^*|dV_M|$. Now set

$$\rho := |\vec{t}|, \;\; \omega := \frac{1}{\rho}\vec{t},$$

and denote by $|d\omega|$ the area density on the unit sphere in $\mathbb{R}^c$. Then

$$\mathbb{E}_M^* |dV_n| = \det(\mathbb{1} - \rho S(\omega, x)\,)\rho^{c-1}|d\rho| \times |d\omega| \times |dV_M|. \tag{3.5}$$

Observe that

$$\det(\mathbb{1} - \rho S(\omega, x)\,) = \sum_{\nu=0}^m (-1)^\nu \rho^\nu P_\nu\big(\, \Phi_{ij}(x) \bullet \omega\,\big),$$

where P_ν denotes a homogeneous polynomial of degree ν in the m^2 variables

$$\boldsymbol{u}_{ij} \in \mathbb{R}^c, \;\; 1 \le i, j \le m.$$

We set

$$\overline{P}_\nu\big(\, \Phi_{ij}(x)\,\big) := \int_{S^{c-1}} P_\nu\big(\Phi_{ij}(x) \bullet \omega\,\big) d\omega.$$

Above, $\overline{P}_\nu\big(\, \boldsymbol{u}_{ij}\,\big)$ is an $O(c)$-invariant, homogeneous polynomial of degree ν in the variables $\boldsymbol{u}_{ij} \in \mathbb{R}^c$, $1 \le i, j \le m$. We conclude,

$$\pi_* \mathbb{E}_M^* |dV_n| = \sum_{\nu=0}^m \frac{(-1)^\nu}{c+\nu} r^{c+\nu} \overline{P}_\nu\big(\, \Phi_{ij}(x)\,\big)|dV_M(x)|. \tag{3.6}$$

We would like to determine the invariant polynomials $\overline{P}_\nu\big(\, \boldsymbol{u}_{ij}\,\big)$.

Theorem 3.1.1 implies that $\overline{P}_\nu$ must be a polynomial in the quantities

$$q_{i,j,k,\ell} := \boldsymbol{u}_{ik} \bullet \boldsymbol{u}_{j\ell}.$$

Because these quantities are homogeneous of degree 2 in the variables $\boldsymbol{u}_{ij}$ we deduce $\overline{P}_\nu = 0$ if ν is odd. Assume therefore $\nu = 2h$, $h \in \mathbb{Z}_{\geq 0}$.

For every $\vec{t} \in \mathbb{R}^c = \operatorname{span}(\boldsymbol{e}_{m+1}, \dots, \boldsymbol{e}_{m+c})$ we form the linear operator

$$U(\boldsymbol{u}_{ij}, \vec{t}) : \mathbb{R}^m \to \mathbb{R}^m$$

given by the matrix $(\,\boldsymbol{u}_{ij} \bullet \vec{t}\,)_{1 \leq i,j \leq m}$, we deduce that

$$(-1)^\nu P_\nu(\,\boldsymbol{u}_{ij} \bullet \vec{t}\,) = \operatorname{tr} \Lambda^\nu U(\boldsymbol{u}_{ij}, \vec{t})$$

$=$ the sum of all the $\nu \times \nu$ minors of $U(\vec{t})$ symmetric with respect to the diagonal. These minors are parameterized by the subsets $I \subset \{1, \dots, m\}$ of cardinality $\#I = \nu$. We denote by $\mu_I(\,\boldsymbol{u}_{ij} \bullet \omega\,)$ the corresponding minor, and by $\overline{\mu}_I$ its average,

$$\overline{\mu}_I(\boldsymbol{u}_{ij}) := \int_{S^{c-1}} \mu_I(\,\boldsymbol{u}_{ij} \bullet \omega\,) d\omega.$$

$\overline{\mu}_I$ is an $O(c)$-invariant polynomial in the variables $\{\boldsymbol{u}_{ij}\}_{i,j \in I}$.

Let

$$I = \{1 \leq i_1 < i_2 < \cdots < i_{2h} \leq m\} \subset \{1, \dots, m\},$$

and denote by $\mathcal{S}_I$ the group of permutations of I. For $\sigma \in \mathcal{S}_I$ we set

$$\varphi_j := \varphi(i_j), \;\; \forall j = 1, \dots, 2h.$$

For any $\sigma, \varphi \in \mathcal{S}_I$ we denote by $\epsilon(\sigma, \varphi)$ the signature of the permutation $\sigma \circ \varphi^{-1}$, and by $Q_{\sigma,\varphi}$ the invariant polynomial

$$Q_{I,\sigma,\varphi} = \prod_{j=1}^h q_{\varphi_{2j-1}, \varphi_{2j}, \sigma_{2j-1}, \sigma_{2j}} = \prod_{j=1}^h \boldsymbol{u}_{\varphi_{2j-1}\sigma_{2j-1}} \bullet \boldsymbol{u}_{\varphi_{2j})\sigma_{2j}}.$$

Lemma 3.2.1. *There exists a constant $\xi = \xi_{m,\nu,c}$ depending only on m, ν and c such that*

$$\bar{\mu}_I = \xi Q_I, \;\; Q_I := \sum_{\varphi, \sigma \in \mathcal{S}_I} \epsilon(\sigma, \varphi) Q_{I,\sigma,\varphi}.$$

Proof. We regard $\overline{\mu}_I$ as a function on the vector space of $m \times m$ matrices U with entries in $\mathbb{R}^c$

$$U = [\boldsymbol{u}_{ij}]_{1 \leq i,j \leq m}.$$

We observe that $\overline{\mu}_I$ satisfies the following determinant like properties.

- $\overline{\mu}_I$ changes sign if we switch two rows (or columns).
- $\overline{\mu}_I$ is separately linear in each of the variables $\boldsymbol{u}_{ij}$.
- $\overline{\mu}_I$ is a homogeneous polynomial of degree h in the variables $q_{i,j,k,\ell}$.

We deduce that $\overline{\mu}_I$ is a linear combination of monomials of the form

$$q_{k_1,k_2,\ell_1,\ell_2} \cdots q_{k_{2h-1}k_{2h},\ell_{2h-1},\ell_{2h}},$$

where

$$\{k_1, \dots, k_{2h}\} \;\text{ and }\; \{\ell_1, \dots, \ell_{2h}\}$$

are permutations of I. The skew-symmetry of $\overline{\mu}_I$ with respect to the permutations of rows and columns now implies that $\overline{\mu}_I$ must be a multiple of Q_I.

$\square$

The constant ξ satisfies

$$\xi_{m,\nu,c} = \frac{\overline{\mu}_I(\boldsymbol{u}_{ij})}{Q_I(\boldsymbol{u}_{ij})}, \quad \forall \boldsymbol{u}_{ij} \in \mathbb{R}^c$$

so it suffices to compute the numerator and denominator of the above fraction for some special values of $\boldsymbol{u}_{ij}$. We can assume $I = \{1, 2, \dots, 2h\}$ and we choose

$$\boldsymbol{u}_{ij} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^c.$$

Then, if we set

$$\vec{t} = \begin{bmatrix} t^1 \\ t^2 \\ \vdots \\ t^c \end{bmatrix} \in \mathbb{R}^c, \quad \omega = \frac{1}{|\vec{t}|}\vec{t}$$

we deduce

$$U(\boldsymbol{u}_{ij}, \vec{t}) = \begin{bmatrix} t^1 & 0 & \cdots & 0 \\ 0 & t^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t^1 \end{bmatrix}, \quad \mu_I(\boldsymbol{u}_{ij} \bullet \vec{t}) = |t^1|^\nu.$$

Hence

$$\overline{\mu}_I(\boldsymbol{u}_{ij}) = \int_{S^{c-1}} |\omega^1|^{2h} d\omega. \tag{3.7}$$

On the other hand, we have

$$Q_{I,\sigma,\varphi} = \prod_{j=1}^{h} \boldsymbol{u}_{\varphi_{2j-1}\sigma_{2j-1}} \bullet \boldsymbol{u}_{\varphi_{2j}\sigma_{2j}}$$

which is nonzero if and only if $\sigma = \varphi$. We conclude that for this particular choice of $\boldsymbol{u}_{ij}$ we have

$$Q_I = (2h)!.$$

Hence

$$\xi_{m,\nu,c} = \frac{1}{(2h)!} \int_{S^{c-1}} |\omega^1|^{2h} d\omega.$$

At this point we invoke the following result whose proof is deferred to the end of this section.

Lemma 3.2.2. *For any even, nonnegative integers $2h_1, \dots, 2h_c$ we have*

$$\int_{S^{c-1}} |\omega^1|^{2h_1} \cdots |\omega^c|^{2h_c} d\omega = \frac{2\Gamma(\frac{2h_1+1}{2}) \cdots \Gamma(\frac{2h_c+1}{2})}{\Gamma(\frac{c+2h}{2})},$$

where $h = h_1 + \cdots + h_c$.

We deduce

$$\xi_{m,2h,c} = \frac{2\Gamma(\frac{2h+1}{2})\Gamma(1/2)^{c-1}}{(2h)!\Gamma(\frac{c+2h}{2})}, \tag{3.8}$$

and

$$\overline{P}_\nu(\boldsymbol{u}_{ij}) = \xi_{m,2h,c} \sum_{\#I=\nu} Q_I(\boldsymbol{u}_{ij}). \tag{3.9}$$

We denote by $\mathcal{S}_2$ the group of permutations of a linearly ordered set with two elements. We observe that every element

$$\tau = (\tau_1, \dots, \tau_h) \in G = \underbrace{\mathcal{S}_2 \times \cdots \times \mathcal{S}_2}_{h}$$

defines a permutation of I by regarding τ_1 as a permutation of $\{i_1, i_2\}$, τ_2 as a permutation of $\{i_3, i_4\}$ etc. Thus G is naturally a subgroup of $\mathcal{S}_I$. The space $\mathcal{S}_k/G$ of left cosets of this group can be identified with the subset $\mathcal{S}'_I \subset \mathcal{S}_I$ consisting of bijections $\varphi : I \to I$ satisfying the conditions

$$\varphi_1 < \varphi_2, \;\; \varphi_3 < \varphi_4, \dots, \;\; \varphi_{2h-1} < \varphi_{2h}.$$

We deduce that if $\boldsymbol{u}_{ij} = S_{ij}(x)$ then for every $\sigma \in \mathcal{S}_I$ we have

$$\sum_{\varphi \in \mathcal{S}_I} \epsilon(\sigma, \varphi) Q_{I,\sigma,\varphi} = \sum_{\varphi \in \mathcal{S}'_I, \tau \in G} \epsilon(\sigma, \varphi\tau) Q_{I,\sigma,\varphi\tau}$$

$$= \sum_{\varphi \in \mathcal{S}'_I} \epsilon(\sigma, \varphi) \prod_{j=1}^{h} \bigl(q_{\varphi_{2j-1},\varphi_{2j},\sigma_{2j-1},\sigma_{2j}} - q_{\varphi_{2j},\varphi_{2j-1},\sigma_{2j-1},\sigma_{2j}} \bigr)$$

$$= \sum_{\varphi \in \mathcal{S}'_I} \epsilon(\sigma, \varphi) \prod_{j=1}^{h} R_{\varphi_{2j-1}\varphi_{2j}\sigma_{2j-1}\sigma_{2j}}$$

Using the skew-symmetry $R_{ijk\ell} = -R_{ij\ell k}$ we deduce

$$\sum_{\sigma,\varphi \in \mathcal{S}_I} \epsilon(\sigma, \varphi) Q_{I,\sigma,\varphi} = \sum_{\sigma \in \mathcal{S}_I} \sum_{\varphi \in \mathcal{S}'_I} \epsilon(\sigma, \varphi) \prod_{j=1}^{h} R_{\varphi_{2j-1}\varphi_{2j}\sigma_{2j-1}\sigma_{2j}}$$

$$= \sum_{\sigma \in \mathcal{S}'_I, \tau \in G} \epsilon(\sigma\tau, \varphi) \prod_{j=1}^{h} R_{\varphi_{2j-1}\varphi_{2j}\sigma_{2j-1}\sigma_{2j}}$$

$$= 2^h \underbrace{\sum_{\sigma,\varphi \in \mathcal{S}'_I} \epsilon(\sigma, \varphi) \prod_{j=1}^{h} R_{\varphi_{2j-1}\varphi_{2j}\sigma_{2j-1}\sigma_{2j}}}_{=:\mathcal{Q}_I(R)}$$

We conclude that

$$\overline{P}_\nu(\psi_{ij}) = 2^h \xi_{m,2h,c} \sum_{\#I=2h} \mathcal{Q}_I(R). \tag{3.10}$$

Using (3.1) and (3.6) we deduce that

$$V(M, r) = \operatorname{vol}(\mathbb{T}_r(M)) = \sum_{h=0}^{\lfloor m/2 \rfloor} \boldsymbol{\omega}_{c+2h} r^{c+2h} \frac{2^h \xi_{m,2h,c}}{(c+2h)\boldsymbol{\omega}_{c+2h}} \int_M \mathcal{Q}_h(R) |dV_M|.$$

Let us observe that the constant

$$\frac{2^h \xi_{m,2h,c}}{(c+2h)\boldsymbol{\omega}_{c+2h}}$$

is independent of the codimension c. It *depends only* on h. Indeed, we have

$$\frac{2^h \xi_{m,2h,c}}{(c+2h)\boldsymbol{\omega}_{c+2h}} = \frac{2^h \xi_{m,2h,c}}{\boldsymbol{\sigma}_{c+2h-1}} = \frac{2^h}{\boldsymbol{\sigma}_{c+2h-1}} \frac{2\Gamma(\frac{2h+1}{2})\Gamma(1/2)^{c-1}}{(2h)!\Gamma(\frac{c+2h}{2})}$$

$$= \frac{2^h \Gamma(h+1/2)}{\Gamma(1/2)^{1+2h}(2h)!} = \frac{\boldsymbol{\gamma}(2h)}{\pi^h (2h)!} = \frac{1}{(2\pi)^h h!}.$$

We have thus obtained the following result.

Theorem 3.2.3 (Tube formula)**.** *Suppose M is a closed, compact submanifold of $\mathbb{R}^n$, $\dim M = m$, $c = n - m$. Denote by R the Riemann curvature of the induced metric on M. Then for all $r > 0$ sufficiently small we have*

$$V(M,r) = \mathrm{vol}\,(\mathbb{T}_r(M)) = \sum_{h=0}^{\lfloor m/2 \rfloor} \boldsymbol{\omega}_{c+2h} r^{c+2h} \mu_{m-2h}(M),$$

$$\mu_{m-2h}(M) = \frac{1}{(2\pi)^h h!} \int_M \mathcal{Q}_h(R) |dV_M|$$

where $\mathcal{Q}_h$ is a polynomial of degree h in the curvature. By choosing a local, orthonormal frame $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ of TM we can express the polynomial $\mathcal{Q}_h(R)$ as

$$\mathcal{Q}_h(R) := \sum_{\#I = 2h} \mathcal{Q}_I(R),$$

where for every $I = \{i_1 < i_2 < \cdots < i_{2h}\} \subset \{1, \dots, m\}$ we define

$$\mathcal{Q}_I(R) = \sum_{\sigma,\varphi \in \mathcal{S}'_I} \epsilon(\sigma,\varphi) \prod_{j=1}^{h} R_{\varphi(i_{2j-1})\varphi(i_{2j})\sigma(i_{2j-1})\sigma(i_{2j})}. \qquad \square$$

Example 3.2.4. (a) If $h = 0$ then

$$\mu_m(M) = \mathrm{vol}\,(M).$$

(b) Assume now that m is even, $m = 2h$, and oriented. Then $c + 2h = m + c = n$ and

$$\mu_0(M) = \frac{1}{(2\pi)^h} \int_M \frac{1}{h!} \mathcal{Q}_h(R) dV_M.$$

Comparing the definition of $\mathcal{Q}_h(R)$ with (2.17a) we deduce that the top dimensional form

$$\frac{1}{(2\pi)^h} \frac{1}{h!} \mathcal{Q}_h(R) dV_M \in \Omega^{2h}(M)$$

is precisely the Euler form associated with the orientation of M and the induced metric, so that

$$\mu_0(M,g) = \int_M \boldsymbol{e}(M,g). \tag{3.11}$$

(c) Suppose now that M is a hypersurface. Consider the second fundamental form

$$S = (S_{ij})_{1 \le i,j \le m}, \quad S_{ij} = (\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_j) \bullet \boldsymbol{e}_{m+1},$$

where we recall that $\boldsymbol{e}_{m+1}$ is in fact the oriented unit normal vector field along M. Fix a point $x_0 \in M$ and assume that at this point the frame $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ diagonalizes the second fundamental form so that

$$S_{ij} = \kappa_i \delta_{ij}.$$

The eigenvalues $\kappa_1, \dots, \kappa_m$ are the principal curvatures at the point x_0. We denote by $c_\nu(\kappa)$ the elementary symmetric polynomial of degree ν in the variables κ_i. In this case $c = 1$ and we have

$$\mathbb{E}_M^* dV_{\mathbb{R}^{m+1}} = \det(\mathbb{1} - t_S) dt \wedge dV_M = \sum_{\nu=0}^{m} (-1)^\nu t^\nu c_\nu(\kappa) dt \wedge dV_M$$

$$\pi_* \mathbb{E}_M^* dV_{\mathbb{R}^{m+1}} = \sum_{\nu=0}^{m} \Big(\int_{-r}^{r} t^\nu dt \Big) c_\nu(\kappa) dV_M = 2 \sum_{h=0}^{\lfloor m/2 \rfloor} \frac{r^{2h+1}}{2h+1} c_{2h}(\kappa) dV_M$$

so that

$$\sum_{h=0}^{\lfloor m/2 \rfloor} \boldsymbol{\omega}_{1+2h} r^{1+2h} \mu_{m-2h}(M) = V(M, r) = 2 \sum_{h=0}^{\lfloor m/2 \rfloor} \frac{r^{2h+1}}{2h+1} \int_M c_{2h}(\kappa) dV_M.$$

We conclude that

$$\mu_{m-2h}(M) = \frac{2}{\boldsymbol{\sigma}_{2h}} \int_M c_{2h}(\kappa) dV_M, \;\; \boldsymbol{\sigma}_{2h} = (2h+1)\boldsymbol{\omega}_{1+2h}.$$

If $M = S^m \hookrightarrow \mathbb{R}^{m+1}$ is the unit sphere then $\kappa_i = 1$ and we deduce that

$$\mu_{m-2h}(S^m) = 2 \frac{\boldsymbol{\sigma}_m}{\boldsymbol{\sigma}_{2h}} \binom{m}{2h}. \tag{3.12}$$

(d) Using the definition (2.3) of the scalar curvature we deduce that for any m-dimensional submanifold $M^m \hookrightarrow \mathbb{R}^n$ we have

$$\mu_{m-2}(M, g) = \mathrm{const}_m \int_M s_g |dV_g|,$$

where s denotes the scalar curvature of the induced metric g, and const_m is an universal constant, depending only on m. We see that the map $g \to \mu_{m-2}(M, g)$ is precisely the *Einstein functional*.

To find const_m we compute $\mu_{m-2}(M)$ when $M = S^m$. Using (3.12) we deduce

$$\frac{2\boldsymbol{\sigma}_m}{\boldsymbol{\sigma}_2} \binom{m}{2h} = \mathrm{const}_m \int_M s_{round} |dV_{S^m}|,$$

where s_{round} denotes the scalar curvature of the round metric on the unit sphere. Using the definition (2.3) we deduce

$$s_{round} = \sum_{i,j} R_{ijij} = \sum_{i,j} 1 = 2 \binom{m}{2}.$$

Hence

$$\mathrm{const}_m = \frac{2}{\boldsymbol{\sigma}_2} = \frac{1}{2\pi} \Longrightarrow \mu_{m-2}(M, g) = \frac{1}{2\pi} \int_M s_g |dV_g|.$$

(e) The polynomial $\mathcal{Q}_2$ still has a "reasonable form"

$$\mathcal{Q}_2(R) = \sum_{\#I=4} \mathcal{Q}_I$$

Then $\#\mathcal{S}'_I = 6$ and

$$\mathcal{Q}_I = \sum_{\sigma,\varphi\in\mathcal{S}'_I} \epsilon(\sigma,\varphi) R_{\sigma_1\sigma_2\varphi_1\varphi_2} R_{\sigma_3\sigma_4\varphi_3\varphi_4}$$

$$= \sum_{\sigma\in\mathcal{S}'_I} R_{\sigma_1\sigma_2\sigma_1\sigma_2} R_{\sigma_3\sigma_4\sigma_3\sigma_4} + \sum_{\sigma\neq\varphi\in\mathcal{S}'_I} \epsilon(\sigma,\varphi) R_{\sigma_1\sigma_2\varphi_1\varphi_2} R_{\sigma_3\sigma_4\varphi_3\varphi_4}.$$

The first sum has only three different monomials, each of them appearing twice is them sum. The second sum has $\binom{6}{2}$ different monomials (corresponding to subsets of cardinality 2 of $\mathcal{S}'_I$) and each of them appears twice. □

Definition 3.2.5. If (M,g) is a closed, compact, oriented, Riemann manifold, $m = \dim M$, and w is nonnegative integer. If $m-w$ is odd we set

$$\mu_w(M) = 0.$$

If $m-w$ is an even, nonnegative integer, $m-w = 2h$, then we set

$$\mu_w(M,g) = \frac{1}{(2\pi)^h h!} \int_M \mathcal{Q}_h(R) |dV_M|.$$

We will say that $\mu_w(M,g)$ is the *weight w curvature measure of* (M,g). We set

$$|d\mu_w| := \frac{1}{(2\pi)^h h!} \mathcal{Q}_h(R)|dV_M|,$$

and we will refer to it as *the (weight w) curvature density.* □

Remark 3.2.6. Let us observe that for any Riemann manifold M, orientable or not, the quantities $|d\mu_w|$ are indeed well defined, i.e. independent of the choice of local frames used in their definition. The fastest way to argue this is by invoking Nash embedding theorem which implies that any compact manifold is can be isometrically embedded in an Euclidean space. For submanifolds of $\mathbb{R}^n$, the proof of the tube formula then implies that these densities are indeed well defined.

We can prove this by much elementary means by observing that, for any finite set I, the relative signature $\epsilon(\sigma,\varphi)$ of two permutations $\varphi,\sigma : I \to I$ is defined by choosing a linear ordering on I, but it is *independent of this choice.* □

Proof of Lemma 3.2.2. Consider the integral

$$I(h_1,\dots,h_c) = \int_{\mathbb{R}^c} e^{-|\vec{t}|^2} |t^1|^{2h_1} \cdots |t^c|^{2h_c} dt.$$

We have $e^{-|\vec{t}|^2} = e^{-|t+1|^2} \cdots e^{-|t^c|^2}$ so that

$$I(h_1,\dots,h_c) = \prod_{j=1}^c \Bigl(\int_{-\infty}^{\infty} e^{-s^2} s^{2h_j} ds \Bigr) = 2^c \prod_{j=1}^c \Bigl(\int_0^{\infty} e^{-s^2} s^{2h_j} ds \Bigr)$$

$(u = s^2)$

$$= \prod_{j=1}^c \Bigl(\int_0^{\infty} e^{-u} t^{h_j - 1/2} du \Bigr) = \prod_{j=1}^c \Gamma\Bigl(\frac{2h_j+1}{2} \Bigr).$$

On the other hand, using spherical coordinates, $\rho = |\vec{t}|$, $\omega = \frac{1}{|\vec{t}|}\vec{t}$, and recalling that $h = h_1 + \cdots + h_c$, we deduce that

$$I(h_1, \dots, h_c) = \Bigl(\int_{S^{c-1}} |\omega^1|^{2h_1} \cdots |\omega^c|^{2h_c} d\omega\Bigr)\Bigl(\int_0^\infty e^{-\rho^2}\rho^{2h+c-1}d\rho\Bigr)$$

$(u = \rho^2)$

$$= \frac{1}{2}\Bigl(\int_{S^{c-1}} |\omega^1|^{2h_1} \cdots |\omega^c|^{2h_c} d\omega\Bigr)\int_0^\infty e^{-u}u^{\frac{c+2h}{2}-1}du$$

$$= \frac{1}{2}\Gamma\Bigl(\frac{c+2h}{2}\Bigr)\Bigl(\int_{S^{c-1}} |\omega^1|^{2h_1} \cdots |\omega^c|^{2h_c} d\omega\Bigr). \qquad \square$$

3.3. Gauss-Bonnet formula for arbitrary submanifolds

Suppose $M^m \subset \mathbb{R}^n$ is a closed, compact submanifold of $\mathbb{R}^n$. As usual, set $c = n - m$, and we denote by g the induced metric on M. For every sufficiently small positive real number r we set

$$M_r := \{x \in \mathbb{R}^n;\ \operatorname{dist}(x, M) = r\} = \partial\mathbb{T}_r(M).$$

M_r is a compact hypersurface of $\mathbb{R}^n$ and we denote by g_r the induced metric. Observe that for r and ε sufficiently small we have

$$\mathbb{T}_\varepsilon(M_r) = \mathbb{T}_{r+\varepsilon}(M) - \mathbb{T}_{r-\varepsilon}(M)$$

so that

$$V(M_r, \varepsilon) = V(M, r+\varepsilon) - V(M, r-\varepsilon)$$

which implies that

$$\sum_{h\geq 0} \boldsymbol{\omega}_{1+2h}\varepsilon^{1+2h}\mu_{n-1-2h}(M_r, g_r)$$

$$= \sum_{k\geq 0} \boldsymbol{\omega}_{c+2k}\Bigl\{(r+\varepsilon)^{c+2k} - (r-\varepsilon)^{c+2k}\Bigr\}\mu_{n-c-2k}(M, g).$$

We deduce

$$\mu_{n-1-2h}(M_r, g_r) = \frac{2}{\boldsymbol{\omega}_{1+2h}}\sum_{k\geq 0} \boldsymbol{\omega}_{c+2k}\binom{c+2k}{1+2h}r^{c-1+2k-2h}\mu_{n-c-2k}(M, g).$$

We make a change in variables. We set

$$p := n - 1 - 2h,\ \ w = n - c - 2k = m - 2k.$$

Then $c + 2k = n - w$, $1 + 2h = n - p$, $c + 2k - 1 - 2h = p - w$ so that we can rewrite the above formula as

$$\mu_p(M_r, g_r) = 2\sum_{w=0}^{m} \binom{n-w}{n-p}\frac{\boldsymbol{\omega}_{n-w}}{\boldsymbol{\omega}_{n-p}}r^{p-w}\mu_w(M, g). \tag{3.13}$$

In the above equality it is understood that $\mu_w(M) = 0$ if $m - w$ is odd. In particular, we deduce that *if the codimension of M is odd* then

$$\lim_{r\to 0} \mu_p(M_r, g_r) = 2\mu_p(M, g),\ \ \forall 0 \leq p \leq m = \dim M. \tag{3.14}$$

If in the formula (3.13) we assume that the manifold M is a point, then we deduce that M_r is the $(n-1)$-dimensional sphere of radius r, $M_r = S_r^{n-1}$, and we conclude that

$$\mu_p(S_r^{n-1}) = 2\binom{n}{p}\frac{\boldsymbol{\omega}_n}{\boldsymbol{\omega}_{n-p}}r^p = 2\boldsymbol{\omega}_p\begin{bmatrix} n \\ p \end{bmatrix}r^p, \;\; n-p \equiv 1 \mod 2, \tag{3.15}$$

where $\left[\begin{smallmatrix} n \\ p \end{smallmatrix}\right]$ is defined by (1.17). The last equality agrees with our previous computation (3.12).

If in the formula (3.13) we let $p = 0$ we deduce

$$\mu_0(M_r, g_r) = 2\mu_0(M, g), \;\; \forall 0 < r \ll 1, \;\; \text{if codim } M \text{ is odd.}$$

Observe that the tube $\mathbb{T}_r(M)$ is naturally oriented, even though the manifold M may not be orientable. The Gauss-Bonnet theorem for oriented hypersurfaces implies

$$\chi(M_r) = \mu_0(M_r, g_r)$$

so that

$$\mu_0(M, g) = \frac{1}{2}\chi(M_r), \;\; \forall 0 < r \ll 1. \tag{3.16}$$

Theorem 3.3.1 (Gauss-Bonnet)**.** *Suppose M is a closed, compact submanifold of an Euclidean space $\mathbb{R}^n$. Denote by g the induced metric. Then*

$$\mu_0(M, g) = \chi(M).$$

Proof. If $m = \dim M$ is odd then both $\chi(M)$ and $\mu_0(M)$ are equal to zero and the identity is trivial. Assume therefore that m is even. If the dimension n of the ambient space is odd then the Poincaré duality for the *oriented* n-dimensional manifold with boundary $\mathbb{T}_r(M)$ implies

$$\chi(M_r) = \chi(\,\partial\mathbb{T}_r(M)\,) = 2\chi(\,\mathbb{T}_r(M)\,) = 2\chi(M)$$

and the theorem follows from (3.16).

If n is even, we apply the above argument to the embedding

$$M \hookrightarrow \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1},$$

where we observe that the metric induced by the embedding $M \hookrightarrow \mathbb{R}^{n+1}$ coincides with the metric induced by the original embedding $M \hookrightarrow \mathbb{R}^n$. □

Remark 3.3.2. We want emphasize that in the above theorem *we did not require that M be orientable* which is the traditional assumption in the Gauss-Bonnet theorem. □

Let us record for later use the following corollary of the above proof.

Corollary 3.3.3. *For every closed compact, smooth submanifold M of an Euclidean space V such that* $\dim V - \dim M$ *is odd we have*

$$\chi(M) = 2\chi(\,\partial\mathbb{T}_r(M)\,), \;\; \forall 0 < r \ll 1. \qquad \square$$

3.4. Curvature measures of domains in an Euclidean space

The second fundamental form of a submanifold is in fact a bilinear form with values in the normal bundle. If the submanifold happens to be the *boundary* of a domain, then the normal bundle admits a *canonical trivialization* and the second fundamental form will be a scalar valued form. The next definition formalizaes this observation.

Definition 3.4.1. For any relatively compact open subset D of an Euclidean space V we define the *co-oriented second fundamental form* of D to be the symmetric bilinear map

$$S_D : \mathrm{Vect}(\partial D)\times \mathrm{Vect}(\partial D)\to C^\infty(\partial D),$$

$$S_D(X,Y) = (\boldsymbol{D}_X Y)\bullet \boldsymbol{n},\;\; X,Y\in \mathrm{Vect}(\partial D),$$

where $\boldsymbol{n} : \partial D\to V$ denotes the *outer* unit normal vector field along ∂D. □

Suppose $D\subset \mathbb{R}^{m+1}$ is an open, relatively compact subset with smooth boundary $M := \partial D$. We denote by $\boldsymbol{n}$ the unit outer normal vector field along $M := \partial D$ and by $S = S_D$ the co-oriented second fundamental form of D. For every symmetric bilinear form B on an Euclidean space V we define $\mathrm{tr}_j(B)$ the j-th elementary symmetric polynomial in the eigenvalues of B, i.e.,

$$\sum_{j\geq 0} z^j\,\mathrm{tr}_j(B) = \det(\mathbb{1}_V + zB).$$

Equivalently,

$$\mathrm{tr}_j\,B = \mathrm{tr}\left(\Lambda^k B : \Lambda^k V\to \Lambda^k V\right).$$

We define the tube of radius r around D to be

$$\mathbb{T}_r(D) := \bigl\{ x\in\mathbb{R}^{m+1};\;\; \mathrm{dist}\,(x,D)\leq r\,\bigr\}.$$

We denote by $\mathbb{E}_M$ the exponential map

$$\mathbb{E}_M : (T\mathbb{R}^{m+1})|_M\to \mathbb{R}^{m+1},$$

$$(X,p)\longmapsto \mathbb{E}_M(X,p) = p+X,\;\; p\in M,\;\; X\in T_p\mathbb{R}^{m+1}.$$

For $r>0$ we denote by $\Delta_r\subset (T\mathbb{R}^{m+1})|_M$ the closed set

$$\Delta_r := \bigl\{ (t\boldsymbol{n}(p),p);\;\; p\in M,\;\; t\in[0,r]\,\bigr\}.$$

For sufficiently small r the map $\mathbb{E}_M$ defines a diffeomorphism

$$\mathbb{E}_M : \Delta_r\to \mathbb{T}_r(D)\setminus D,$$

so that

$$\mathrm{vol}\,(\mathbb{T}_r(D)\,) = \mathrm{vol}\,(D) + \int_{\Delta_r}\mathbb{E}_M^* dV_{\mathbb{R}^{m+1}}.$$

Fix $p_0\in M$ and a local, positively oriented local orthonormal frame

$$(\boldsymbol{e}_1,\dots,\boldsymbol{e}_m)$$

of TM defined in a neighborhood U of p_0 in M, such that, for every $p\in U$, the collection

$$(\boldsymbol{n}(p),\boldsymbol{e}_1(p),\dots,\boldsymbol{e}_m(p))$$

is a positively oriented, orthonormal frame of $\mathbb{R}^n$. We obtain a dual coframe $\boldsymbol{\theta},\boldsymbol{\theta}^1,\dots,\boldsymbol{\theta}^n$.

As in the previous section, the pullback of $\mathbb{E}_M^* dV_{\mathbb{R}^{m+1}}$ to Δ_r has the description

$$\begin{aligned}\mathbb{E}_M^* dV_{\mathbb{R}^{m+1}} &= \det\big(\mathbb{1} - tS_M\big) dt \wedge d\boldsymbol{\theta}^1 \wedge \cdots \wedge \boldsymbol{\theta}^m \\ &= \left(\sum_{j=1}^m \operatorname{tr}_j(-S_M) t^j\right) dt \wedge d\boldsymbol{\theta}^1 \wedge \cdots \wedge \boldsymbol{\theta}^m.\end{aligned}$$

We deduce

$$\int_{\Delta_r} \mathbb{E}_M^* dV_{\mathbb{R}^{m+1}} = \sum_{j\geq 0} \frac{r^{j+1}}{j+1}\left(\int_M \operatorname{tr}_j(-S_M) dV_M\right).$$

Define

$$\mu_{m-j}(D) := \frac{1}{\boldsymbol{\sigma}_j}\left(\int_M \operatorname{tr}_j(-S_M) dV_M\right),\ \ 0 \leq j \leq m,$$

and

$$\mu_{m+1}(D) := \operatorname{vol}(D)$$

so that using the equality $\boldsymbol{\sigma}_j = (j+1)\boldsymbol{\omega}_j$ we deduce the *tube formula for domains*,

$$\operatorname{vol}\big(\mathbb{T}_r(D)\big) = \sum_{k=0}^{m+1} \boldsymbol{\omega}_{m+1-k} r^{m+1-k} \mu_k(D). \tag{3.17}$$

Theorem 2.5.1 shows that, just as in the case of submanifolds, we have $\mu_0(D) = \chi(D)$.

Definition 3.4.2. Suppose D is a relatively compact domain with smooth boundary of an Euclidean space V, $\dim V = n$. Then *the curvature densities of* D are the densities $|d\mu_j|$ on ∂D defined by

$$|d\mu_j| := \frac{1}{\boldsymbol{\sigma}_{n-j}} \operatorname{tr}_{n-j}(-S_D)|dV_{\partial D}|,$$

where $|dV_{\partial D}|$ denotes the volume density on ∂D indiced by the Euclidean metric on V. □

We denote by $\mathbb{D}_r^{m+1}$ the ball of radius r in $\mathbb{R}^{m+1}$. Then

$$\mathbb{T}_\varepsilon(\mathbb{D}_r^{m+1}) = \mathbb{D}_{r+\varepsilon}^{m+1}$$

so that

$$\boldsymbol{\omega}_{m+1}(r+\varepsilon)^{m+1} = \sum_{k\geq 0} \boldsymbol{\omega}_{m+1-k} \varepsilon^{m+1-k} \mu_k(\mathbb{D}_r^{m+1}).$$

We conclude

$$\mu_k(\mathbb{D}_r^{m+1}) = \frac{\boldsymbol{\omega}_{m+1}}{\boldsymbol{\omega}_{m+1-k}} \binom{m+1}{k} r^k = \boldsymbol{\omega}_k \begin{bmatrix} m+1 \\ k \end{bmatrix} r^k. \tag{3.18}$$

Suppose $X \hookrightarrow \mathbb{R}^{m+1}$ is a closed, compact smooth submanifold. Then for every sufficiently small $r > 0$, the tube $D_r := \mathbb{T}_r(X)$ is a compact domain with smooth boundary and

$$\mathbb{T}_\varepsilon(D_r) = \mathbb{T}_{r+\varepsilon}(X).$$

The tube formula for X implies that

$$\sum_{j\geq 0} \boldsymbol{\omega}_j \varepsilon^j \mu_{m+1-j}(D_r) = \sum_{k\geq 0} \boldsymbol{\omega}_k \big(r+\varepsilon\big)^k \mu_{m+1-k}(X).$$

We deduce that

$$\mu_{m+1-j}(D_r) = \frac{1}{\boldsymbol{\omega}_j} \sum_{k \geq j} \boldsymbol{\omega}_k \binom{k}{j} r^{k-j} \mu_{m+1-k}(X)$$

We set $n := m+1$ and we make the change in variables

$$p := n - j, \quad w := n - k.$$

Then $k - j = p - w$ and we obtain the following generalization of the tube formula

$$\begin{aligned} \mu_p\bigl(\mathbb{T}_r(X)\bigr) &= \frac{1}{\boldsymbol{\omega}_{n-p}} \sum_w \boldsymbol{\omega}_{n-w} \binom{n-w}{n-p} r^{p-w} \mu_w(X) \\ &= \sum_w \boldsymbol{\omega}_{p-w} \begin{bmatrix} n-w \\ p-w \end{bmatrix} r^{p-w} \mu_w(X). \end{aligned} \tag{3.19}$$

We deduce

$$\lim_{r \to 0} \mu_p\bigl(\mathbb{T}_r(X)\bigr) = \mu_p(X), \quad \forall 0 \leq p \leq \dim X. \tag{3.20}$$

3.5. Crofton Formulæ for domains of an Euclidean space

Suppose D is an open, relatively compact subset of the Euclidean space $\mathbb{R}^{m+1}$ with smooth boundary $M = \partial D$. We denote by g the induced metric on M, by $\mathbf{Gr}^c$ the Grassmannian of linear subspaces of $\mathbb{R}^{m+1}$ of codimension c, and by $\mathbf{Graff}^c$ the affine Grassmannian of codimension c affine subspaces of $\mathbb{R}^{m+1}$.

Recall that on $\mathbf{Gr}^c$ we have a natural metric with volume density $|d\gamma_c|$ and total volume

$$V_c := \frac{\prod_{j=0}^m \boldsymbol{\sigma}_j}{\bigl(\prod_{i=0}^{c-1} \boldsymbol{\sigma}_i\bigr) \cdot \bigl(\prod_{j=0}^{m-c} \boldsymbol{\sigma}_j\bigr)}.$$

We rescale this volume density as in (1.18) to obtain a new volume density $|d\nu_c|$ with total volume

$$\int_{\mathbf{Gr}^c} |d\nu_c| = \begin{bmatrix} m+1 \\ c \end{bmatrix}. \tag{3.21}$$

As explained in Section 1.5 these two densities produce two invariant densities $|d\tilde{\gamma}_c|$ and $|d\tilde{\nu}_c|$ on $\mathbf{Graff}^c$ which differ by a multiplicative constant.

Theorem 3.5.1 (Crofton Formula)**.** *Let $1 \leq p \leq m - c$ and consider the function*

$$f : \mathbf{Graff}^c \to \mathbb{R}, \quad f(L) = \mu_p(L \cap D).$$

If the function f is $|d\tilde{\nu}|$-integrable then

$$\begin{bmatrix} p+c \\ p \end{bmatrix} \mu_{p+c}(D) = \int_{\mathbf{Graff}^c} \mu_p(L \cap D) |d\tilde{\nu}_c(L)|.$$

Proof. For simplicity, we set $V = \mathbb{R}^{m+1}$, $n = m + 1 = \dim V$. We will carry out the proof in several steps.

Step 1. We will prove that there exists a constant $\xi_{c,p}$, depending only on m, c, and p such that

$$\xi_{m,c,p}\mu_{p+c}(D) = \int_{\mathbf{Graff}^c} \mu_p(L \cap D)|d\tilde{\nu}_c(L)|.$$

Step 2. We will show that the constant ξ is equal to $\left[{p+c \atop p}\right]$ by explicitly computing both sides of the above equality in the special case $D = \mathbb{D}^{m+1}$.

Step 1. We will rely on a basic trick in integral geometry. For every $S \in \mathbf{Graff}^c$ we denote by $[S] \in \mathbf{Gr}_c$ the parallel translate of S containing the origin. We introduce the incidence relation

$$\mathcal{I} = \big\{ (v, S) \in V \times \mathbf{Graff}^c;\ \ v \in S \big\} \subset V \times \mathbf{Graff}^c.$$

Observe that we have a diffeomorphism

$$\mathcal{I} \to V \times \mathbf{Gr}^c,\ \ \mathcal{I} \ni (v, S) \longmapsto (v, [S]) \in V \times \mathbf{Gr}^c$$

with inverse

$$V \times \mathbf{Gr}^c(V) \ni (v, L) \longmapsto (v, v + L) \in \mathcal{I}.$$

We obtain a double fibration

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ V & & \mathbf{Graff}^c \end{tikzcd}$$

we set

$$\mathcal{I}(M) := \ell^{-1}(M) = \big\{ (v, S) \in V \times \mathbf{Graff}^c;\ \ v \in S \cap M \big\}.$$

Since $\dim \mathcal{I} = \dim V + \dim \mathbf{Gr}^c = n + c(n - c)$ we deduce

$$\dim \mathcal{I}(M) = n + c(n - c) - \operatorname{codim} M = m + c(n - c) = m + c(m + 1 - c).$$

Again we have a diagram

$$\begin{tikzcd} & \mathcal{I}(M) \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ M & & \mathbf{Graff}^c \end{tikzcd}$$

The map r need not be a submersion. Fortunately, r fails to be a surjection on a rather thin set.

Denote by $\mathbf{Graff}^c(M)$ the set of codimension c affine planes which intersect M transversally. Then Sard's theorem implies that $\mathbf{Graff}^c(M)$ is *open* in $\mathbf{Graff}^c$ and its complement has *measure zero*. We set

$$\mathcal{I}(M)^* := r^{-1}\big(\mathbf{Graff}^c(M) \big).$$

The set $\mathcal{I}(M)^*$ is an open subset of $\mathcal{I}(M)$, and we obtain a double *fibration*

$$\begin{tikzcd} & \mathcal{I}(M)^* \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ M & & \mathbf{Graff}^c(M) \end{tikzcd} \tag{3.22}$$

The fiber of r over $L \in \mathbf{Graff}^c$ is the slice $M_L := L \cap M$ which is the boundary of the domain $D_L := (L \cap D) \subset L$.

The vertical bundle of the fibration $r : \mathcal{I}^*(M) \to \mathbf{Graff}^c(M)$ is equipped with a natural density given along a fiber $L \cap M$ by curvature density $|d\mu_k|$ of the the domain D_L. We will denote this density by $|d\mu_k^L|$. As explained in Section 1.2, using the pullback $r^*|d\tilde{\gamma}_c|$ we obtain a density

$$|d\lambda| = |d\mu_p^L| \times r^*|d\tilde{\gamma}_c|$$

on $\mathcal{I}^*(M)$ satisfying

$$\int_{\mathcal{I}^*(M)} |d\lambda| = \int_{\mathbf{Graff}^c(M)} \left(\int_{L\cap M} |d\mu_p^L| \right) |d\tilde{\gamma}_c(L)| = \int_{\mathbf{Graff}^c} \mu_p(L \cap D) |d\tilde{\gamma}_c(L)|.$$

To complete Step 1 in our strategy it suffices to prove that there exists a constant ξ, depending only on m and c such that

$$\ell_*|d\lambda| = \xi |d\mu_c|,$$

where the curvature density is described in Definition 3.2.5.

Set $h = (m - c)$. The points in $\mathcal{I}(M)$ are pairs (x, L) where $x \in M$, and L is an affine plane of dimension $h + 1$. Suppose $(x_0, L_0) \in \mathcal{I}^*(M)$. Then we can parametrize a small open neighborhood of (x_0, L_0) in $\mathcal{I}^*(M)$ by a family

$$(x, \boldsymbol{e}_0(S), \boldsymbol{e}_1(S), \dots, \boldsymbol{e}_h(S), \boldsymbol{e}_{h+1}(S), \dots, \boldsymbol{e}_m(S)),$$

where x runs in a small neighborhood of $x_0 \in M$, S runs in a small neighborhood $\mathcal{U}_0$ of $[L_0]$ in $\mathbf{Gr}^c$ so that the following hold for every S.

- $$\{\boldsymbol{e}_0(S), \boldsymbol{e}_1(S), \dots, \boldsymbol{e}_h(S), \boldsymbol{e}_{h+1}(S), \dots, \boldsymbol{e}_m(S)\}$$
  is an orthonormal frame of $\mathbb{R}^n$.
- $$S = \operatorname{span}\{\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_h\},$$
- $$T_{x_0}M \cap S = \operatorname{span}\{be_1, \cdots, \boldsymbol{e}_h\}.$$

A neighborhood of (x_0, L_0) in $\mathcal{I}$ is parametrized by the family

$$(\vec{r}, \boldsymbol{e}_0(S), \boldsymbol{e}_1(S), \dots, \boldsymbol{e}_h(S), \boldsymbol{e}_{h+1}(S), \dots, \boldsymbol{e}_m(S)),$$

where $\vec{r}$ runs in a neighborhood of x_0 in the ambient space V.

We denote by S_D, the co-oriented second fundamental form of D and by S_L the co-oriented second fundamental form of $D_L \subset L$, and by $|dV_{L\cap M}|$ the volume density on $L \cap M$. Then, if we set $k = \dim L - p = m - c - p$, we deduce

$$|d\mu_p^L| = \frac{1}{\boldsymbol{\sigma}_k} \operatorname{tr}_k(-S_L)|dV_{L\cap M}|.$$

In the sequel we will use the following conventions.

- i, j, k denote indices running in the set $\{0, \dots, h\}$.
- α, β, γ denote indices running in the set $\{h + 1, \dots, m\}$.
- A, B, C denote indices running in the set $\{0, 1, \dots, m\}$.

We denote by (θ^A) the dual coframe of $(\boldsymbol{e}_A)$, and set

$$\theta_{AB} := (\boldsymbol{D}\, e_A \bullet e_B).$$

Then, the volume density of the natural metric on $\mathbf{Gr}^c$ is

$$|d\gamma_c| = \Big|\bigwedge_{\alpha,i} \theta_{\alpha i}\Big|.$$

Then

$$|d\tilde{\gamma}_c| = \Big|\bigwedge_{\alpha} \boldsymbol{D}\,\vec{r} \bullet \boldsymbol{e}_\alpha\Big| \times |d\gamma_c| = \Big|\bigwedge_{\alpha} \theta^\alpha\Big| \times |d\gamma_c|$$

and

$$|d\lambda| = |d\mu_p^L| \times |d\tilde{\gamma}_c| = \frac{1}{\boldsymbol{\sigma}_k} \det(-S_{L\cap M})|dV_{L\cap M}| \times \Big|\bigwedge_{\alpha} \theta^\alpha\Big| \times |d\gamma_c|. \tag{3.23}$$

The fiber of $\ell : \mathcal{I}(M) \to M$ over x_0 is described by

$$G_{x_0} := \big\{ (\vec{r}, \boldsymbol{e}_A(S)) \in \mathcal{I}(M),\ \ \vec{r} = x_0 \,\big\}.$$

We set

$$G^*_{x_0} := G_{x_0} \cap \mathcal{I}^*(M).$$

$G^*_{x_0}(M)$ can be identified with the space of linear subspaces S of codimension c such that $T_{x_0}M + S = V$, i.e., the affine subspace $x_0 + S$ intersects M transversally at x_0.

Denote by $\boldsymbol{n}$ a smooth unit normal vector field defined in a neighborhood of x_0 in M, i.e.

$$\boldsymbol{n}(x) \perp T_x M,\ \ |\boldsymbol{n}(x)| = 1.$$

Lemma 3.5.2. *Suppose $x_0 + S$ intersects M transversally at x_0. We set $\boldsymbol{e}_A = \boldsymbol{e}_A(S)$. Then at the point $x_0 \in M$ we have*

$$|(\boldsymbol{n} \bullet \boldsymbol{e}_0)| \cdot |dV_M| = |\theta^1 \wedge \cdots \wedge \theta^m|,$$

i.e., for any $X_1, \dots, X_m \in T_{x_0}M$ we have

$$|(\boldsymbol{n} \bullet \boldsymbol{e}_0)| \cdot |dV_M|(X_1, \dots, X_m) = |\theta^1 \wedge \cdots \wedge \theta^m|(X_1, \dots, X_m).$$

Proof. It suffices to verify this for one basis $X_1, \dots, X_m$ of $T_{x_0}M$ which we can choose to consists of the orthogonal projections $\boldsymbol{f}_1, \dots, \boldsymbol{f}_m$ of $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$. These projections form a basis since S intersects $T_{x_0}M$ transversally.

Observe that

$$\boldsymbol{f}_i = \boldsymbol{e}_i,\ \ \forall 1 \le i \le 2h,\ \ \boldsymbol{f}_\alpha = \boldsymbol{e}_\alpha - (\boldsymbol{e}_\alpha \bullet \boldsymbol{n})\boldsymbol{n}.$$

Then

$$|dV_M|(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)^2 = \det(\boldsymbol{f}_A \bullet \boldsymbol{f}_B)_{1 \le A,B \le m}$$

We observe that

$$\boldsymbol{f}_i \bullet \boldsymbol{f}_j = \delta_{ij},\ \ \boldsymbol{f}_i \bullet \boldsymbol{f}_\alpha = 0,\ \ \forall 1 \le i,j \le 2h < \alpha$$
$$\boldsymbol{f}_\alpha \bullet \boldsymbol{f}_\beta = \delta_{\alpha\beta} - n_\alpha n\beta,\ \ n_\alpha := \boldsymbol{n} \bullet \boldsymbol{e}_\alpha.$$

We deduce

$$|dV_M|(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m)^2 = \det(\mathbb{1} - A),$$

where A denotes the $c \times c$ symmetric matrix with entries $n_\alpha n_\beta$, $2h < \alpha, \beta < m$. If we denote by u the vector

$$\vec{u} = \begin{bmatrix} n_{2h+1} \\ \vdots \\ n_{2h_c} \end{bmatrix} \in \mathbb{R}^c$$

which we also regard as a $c \times 1$, matrix then we deduce

$$A = \vec{u}\vec{u}^t.$$

This matrix has a $c-1$ dimensional kernel corresponding to vectors orthogonal to $\vec{u}$. The vector $\vec{u}$ itself is an eigenvector of A and the corresponding eigenvalue λ is obtained from the equality

$$\lambda\vec{u} = |\vec{u}|^2\vec{u} \Longrightarrow \lambda = |\vec{u}|^2 = \sum_\alpha n_\alpha^2 = |\boldsymbol{n}|^2 - |\boldsymbol{n} \bullet \boldsymbol{e}_0|^2 = 1 - |\boldsymbol{n} \bullet \boldsymbol{e}_0|^2.$$

We conclude that

$$\det(\mathbb{1} - A) = |\boldsymbol{n} \bullet \boldsymbol{e}_0|^2 \Longrightarrow |dV_M|(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m) = |\boldsymbol{n} \bullet \boldsymbol{e}_0|.$$

On the other hand

$$|\theta^1 \wedge \dots \wedge \theta^m|(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m) = |\det(\boldsymbol{e}_A \bullet \boldsymbol{f}_B)_{1 \leq A,B \leq m}|$$

We have again

$$\boldsymbol{e}_i \bullet \boldsymbol{f}_j = \delta_{ij}, \quad \boldsymbol{e}_i \bullet \boldsymbol{f}_\alpha = 0, \quad \forall 1 \leq i,j \leq 2h < \alpha$$
$$\boldsymbol{e}_\alpha \bullet \boldsymbol{f}_\beta = \delta_{\alpha\beta} - n_\alpha n_\beta,$$

so that

$$|\theta^1 \wedge \dots \wedge \theta^m|(\boldsymbol{f}_1, \dots, \boldsymbol{f}_m) = |\boldsymbol{n} \bullet \boldsymbol{e}_0|^2.$$

The lemma is now proved. □

Lemma 3.5.3 (Euler-Meusnier). *Suppose $L \in \mathbf{Graff}^c$ intersects M transversally and $x_0 \in L$. If $\boldsymbol{n}$ is a unit vector perpendicular to $T_{x_0}M$, then*

$$S_L = (\boldsymbol{n} \bullet \boldsymbol{e}_0) S_D|_{T_{x_0} \cap [L]},$$

that is,

$$S_L(\boldsymbol{e}_i, \boldsymbol{e}_j) = (\boldsymbol{n} \bullet \boldsymbol{e}_0) S_D(\boldsymbol{e}_i, \boldsymbol{e}_j), \quad \forall 1 \leq i,j \leq 2h.$$

Proof. We have

$$S_L(\boldsymbol{e}_i, \boldsymbol{e}_j) = \boldsymbol{e}_0 \bullet (\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_j)$$

Let us now observe that the vector $(\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_j)$ is parallel with the plane L because the vectors $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ lie in this plane. Thus, $\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_j$ decomposes into two components, one component parallel to $\boldsymbol{e}_0$, and one component $(De_{\boldsymbol{e}_i}\boldsymbol{e}_j)^\tau$ tangent to $L \cap M$. Hence

$$\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_j = S_L(\boldsymbol{e}_i, \boldsymbol{e}_j)\boldsymbol{e}_0 + \sum_{k=1}^h S_{ij}^k \boldsymbol{e}_k.$$

Taking the inner product with $\boldsymbol{n}$ we deduce

$$S_D(\boldsymbol{e}_i, \boldsymbol{e}_j) = (\boldsymbol{D}_{\boldsymbol{e}_i} \boldsymbol{e}_j) \bullet \boldsymbol{n} = \pm S_L(\boldsymbol{e}_i, \boldsymbol{e}_j)(\boldsymbol{e}_0 \bullet \boldsymbol{n}).$$ □

From the above lemma we deduce

$$\operatorname{tr}_k(-S_L|_{x_0}) = |\boldsymbol{n} \bullet \boldsymbol{e}_0|^k \operatorname{tr}_k(-S_D|_{T_{x_0}M\cap[L]}).$$

for any $(x_0, L) \in \mathcal{I}^*(M)$. In a neighborhood of $(x_0, L_0) \in \mathcal{I}^*(M)$ we have

$$|d\lambda|(x, L) = \frac{1}{\boldsymbol{\sigma}_k} \operatorname{tr}_k(-S_L)|dV_{L\cap M}| \times \Big|\bigwedge_{\alpha} \theta^\alpha\Big| \times |d\gamma_c|$$

$$= \frac{1}{\boldsymbol{\sigma}_k}|\boldsymbol{n} \bullet \boldsymbol{e}_0|^k\big(\operatorname{tr}_k(-S_D|_{T_xM\cap[L]})\big)\Big|\bigwedge_{A=1}^m \theta^A\Big| \times |d\gamma_c|$$

(use Lemma 3.5.2)

$$= \frac{1}{\boldsymbol{\sigma}_k}|\boldsymbol{n} \bullet \boldsymbol{e}_0|^{k+1}\big(\operatorname{tr}_k(-S_D|_{T_xM\cap[L]})\big)|dV_M| \times |d\gamma_c|$$

This proves that along the fiber $G^*_{x_0}$ we have

$$|d\lambda|/|dV_M| = \frac{1}{\boldsymbol{\sigma}_k}|\boldsymbol{n} \bullet \boldsymbol{e}_0|^{k+1}\big(\operatorname{tr}_k(-S_D|_{T_{x_0}M\cap[L]})\big)|d\gamma_c|([L])$$

If we denote by $\theta([L], T_{x_0}M)$ the angle between $[L]$ and the hyperplane $T_{x_0}M$ we deduce

$$|d\lambda|/|dV_M| = \frac{1}{\boldsymbol{\sigma}_k} \cdot |\cos\theta([L], T_{x_0}M)|^{k+1}\big(\operatorname{tr}_k(-S_D|_{T_{x_0}M\cap[L]})\big)|d\gamma_c|([L]).$$

The map

$$G^*_{x_0} \ni (x_0, L) \longmapsto [L] \in \mathbf{Gr}^c$$

identifies $G^*_{x_0}$ with an open subset of $\mathbf{Gr}^c$ whose complement has measure zero. We now have the following result.

Lemma 3.5.4. *V is an Euclidean space, $\dim V = m+1$, $H \subset V$ is a hyperplane through the origin, and $B : H \times H \to \mathbb{R}$ a symmetric bilinear map. Denote by $O(H)$ the subgroup of orthogonal transformations of V which map H to itself and suppose*

$$f : \mathbf{Gr}_c \to \mathbb{R}$$

is an $O(H)$ invariant function. Define

$$\mathbf{Gr}^c_H = \big\{S \in \mathbf{Gr}^c;\ \ S \text{ intersects } H \text{ trasversally}\big\}.$$

Then for every $0 \leq k \leq m-c$ there exists a constant $\xi = \xi_{m,c,k}$ depending only on m, c and k such that

$$I_k(f, B) := \int_{\mathbf{Gr}^c_H} f(S)\operatorname{tr}_k(B|_{H\cap S})|d\gamma_c|(S) = \xi_{m,c,k}\operatorname{tr}_k(B)\int_{\mathbf{Gr}^c} f|d\gamma_c|(S).$$

Proof. Observe that for fixed f the map $B \to I_k(f, B)$ is an $O(H)$-invariant homogeneous polynomial of degree k in the entries of B. We can therefore express it as a polynomial

$$I_k(f, B) = P_f\big(\operatorname{tr}_1(B), \ldots, \operatorname{tr}_k(B)\big)$$
$$= \xi_f \operatorname{tr}_k(B) + Q_f\big(\operatorname{tr}_1(B), \ldots, \operatorname{tr}_{k-1}(B)\big).$$

Let us prove that $Q_f \equiv 0$. To do this, we apply the above formula to a symmetric bilinear form B such that

$$\dim\ker B > m - k.$$

Thus, at least $m-k+1$ of the m eigenvalues of B vanish, so that $\mathrm{tr}_k(B)=0$. For such forms we have

$$I_k(f,B)=Q\big(\mathrm{tr}_1(B),\dots,\mathrm{tr}_{k-1}(B)\big).$$

On the other hand, for almost all $S\in\mathbf{Gr}^c_H$ we have

$$\dim S\cap\ker B>m-c-k.$$

The restriction of B to $S\cap H$ has $m-c$ eigenvalues, and from the above inequality we deduce that at least $m-c-k$ of them are trivial. Hence

$$I_k(f,B)=0,\ \ \forall B,\ \ \dim\ker B>m-k\Longrightarrow Q_f=0.$$

Now choose B to be the bilinear form corresponding to the inner product on H. Then

$$\mathrm{tr}_k(B)=\binom{m}{k}\ \text{ and }\ \mathrm{tr}_k(B|_{H\cap S})=\binom{m-c}{k},\ \ \forall S\in\mathbf{Gr}^c_H,$$

and we conclude that

$$\binom{m-c}{k}\int_{\mathbf{Gr}^c}f|d\gamma_c|(S)=\xi_f\binom{m}{k}.$$ □

Now apply the above lemma in the special case

$$H=T_{x_0}M,\ \ B=-S_D,\ \ f(S)=\frac{1}{\boldsymbol{\sigma}_k}|\cos\theta(S,H)|^{k+1}$$

to conclude that

$$\ell_*|d\lambda|=\xi\,\mathrm{tr}_k(-S_D)|dV_M|=\xi|d\mu_{p+c}|$$

so that

$$\begin{aligned}\xi\mu_{p+c}(D)&=\int_M\ell_*|d\lambda|=\int_{\mathcal{I}^*(M)}|d\lambda|\\&=\int_{\mathbf{Graff}^c(M)}r_*|d\lambda|=\int_{\mathbf{Graff}^c(M)}\mu_p(L\cap D)|d\tilde{\gamma}_c|(L).\end{aligned}$$

Thus, rescaling $|d\tilde{\gamma}_c|$ to $|d\tilde{\nu}_c|$, we deduce that there exists a constant ξ depending only on m and c such that

$$\xi\mu_{p+c}(D)=\int_{\mathbf{Graff}^c(M)}\mu_p(L\cap D)|d\tilde{\nu}_c|(L).$$

Step 2. To determine the constant ξ in the above equality we apply it in the special case $M=\mathbb{D}^m$. Using (3.18) we deduce

$$\xi\mu_{p+c}(\mathbb{D}^{m+1})=\xi\boldsymbol{\omega}_{p+c}\begin{bmatrix}m+1\\p+c\end{bmatrix}=\xi\frac{\boldsymbol{\omega}_{m+1}}{\boldsymbol{\omega}_{m+1-c-p}}\binom{m+1}{p+c}.$$

Now observe that for $L\in\mathbf{Graff}^c$ we set $r=r(L)=\mathrm{dist}\,(L,0)$. Then $L\cap\mathbb{D}^{m+1}$ is empty if $r>1$, and it is a disk of dimension $(m+1-c)=\dim L$ and radius $(1-r^2)^{1/2}$ if $r<1$. We conclude that

$$\mu_p(L\cap\mathbb{D}^{m+1})=\mu_p(\mathbb{D}^{m+1-c})\times\begin{cases}(1-r^2)^{p/2} & r<1\\ 0 & p>1.\end{cases}$$

We set

$$\mu_{m,c,p}=\mu_p(\mathbb{D}^{m+1-c})=\boldsymbol{\omega}_p\begin{bmatrix}m+1-c\\p\end{bmatrix}=\frac{\boldsymbol{\omega}_{m+1-c}}{\boldsymbol{\omega}_{m+1-c-p}}\binom{m+1-c}{p}.$$

Using Theorem 1.5.1 we deduce

$$\begin{aligned}
&\int_{\mathbf{Graff}^c} \mu_p(L\cap \mathbb{D}^{m+1})|d\tilde{\nu}_c|(L)\\
&= \int_{\mathbf{Gr}^c}\left(\int_{[L]^\perp} \mu_p\big(\mathbb{D}^{m+1}\cap(x+[L])\big)|dV_{[L]^\perp}|(x)\right)|d\nu_c|([L])\\
&= \mu_{m,c,p}\int_{\mathbf{Gr}^c}\underbrace{\left(\int_{x\in[L]^\perp,\ |x|<1}(1-|x|^2)^{p/2}|dV_{[L]^\perp}|(x)\right)}_{=:I_{c,p}}|d\nu_c|([L])\\
&= \mu_{m,c,p}I_{c,p}\int_{\mathbf{Gr}^c}|d\nu_c(S)| \stackrel{(3.21)}{=} \mu_{m,c,p}I_{c,p}\begin{bmatrix} m+1\\ c\end{bmatrix}.
\end{aligned}$$

Hence

$$\xi\frac{\boldsymbol{\omega}_{m+1}}{\boldsymbol{\omega}_{m+1-c-p}}\binom{m+1}{p+c} = \frac{\boldsymbol{\omega}_{m+1-c}}{\boldsymbol{\omega}_{m+1-c-p}}\binom{m+1-c}{p}I_{c,p}\begin{bmatrix} m+1\\ c\end{bmatrix}.$$

Using spherical coordinates on $\mathbb{R}^c$ we deduce

$$I_{c,p} = \int_{\mathbb{R}^c}(1-|x|^2)^{p/2}dV_{\mathbb{R}^c} = \boldsymbol{\sigma}_{c-1}\int_0^1 r^{c-1}(1-r^2)^{p/2}dr \stackrel{s=r^2}{=} \frac{\boldsymbol{\sigma}_{c-1}}{2}\int_0^1 s^{\frac{c-2}{2}}(1-s)^{p/2}ds$$

$$\stackrel{(1.12)}{=} \frac{\boldsymbol{\sigma}_{c-1}}{2}B\left(\frac{c}{2},\frac{p}{2}+1\right) \stackrel{(1.13)}{=} \frac{\boldsymbol{\sigma}_{c-1}}{2}\frac{\Gamma(\frac{c}{2})\Gamma(1+\frac{p}{2})}{\Gamma(1+\frac{c}{2}+\frac{p}{2})} \stackrel{(1.14)}{=} \Gamma(1/2)^c\frac{\Gamma(1+\frac{p}{2})}{\Gamma(1+\frac{c}{2}+\frac{p}{2})} = \frac{\boldsymbol{\omega}_{p+c}}{\boldsymbol{\omega}_p}.$$

Hence

$$\begin{aligned}
\xi\boldsymbol{\omega}_{m+1}\binom{m+1}{p+c} &= \frac{\boldsymbol{\omega}_{m+1-c}\boldsymbol{\omega}_{p+c}}{\boldsymbol{\omega}_p}\begin{bmatrix} m+1\\ c\end{bmatrix}\binom{m+1-c}{p}\\
&= \frac{\boldsymbol{\omega}_{m+1}\boldsymbol{\omega}_{p+c}}{\boldsymbol{\omega}_p\boldsymbol{\omega}_c}\binom{m+1}{c}\binom{m+1-c}{p}
\end{aligned}$$

We deduce

$$\xi = \frac{\boldsymbol{\omega}_{p+c}}{\boldsymbol{\omega}_p\boldsymbol{\omega}_c}\frac{\binom{m+1}{c}}{\binom{m+1}{p+c}\binom{m+1-c}{p}} = \frac{\boldsymbol{\omega}_{p+c}}{\boldsymbol{\omega}_p\boldsymbol{\omega}_c}\binom{p+c}{p} = \begin{bmatrix} p+c\\ p\end{bmatrix}$$

□

We now describe a simple situation when the function $\mathbf{Graff}^c \ni L \mapsto \mu_0(L\cap M)$ is integrable.

Proposition 3.5.5. *If the domain D with smooth boundary is also semialgebraic (see Appendix B) then the function*

$$\mathbf{Graff}^c \ni L \mapsto \chi(L\cap D)$$

is bounded and semialgebraic. In particular, it is integrable so that

$$\mu_c(D) = \int_{\mathbf{Graff}^c}\mu_0(L\cap D)|d\tilde{\nu}|(L).$$

3.6. Zero order Crofton formulæ for submanifolds in an Euclidean space

Suppose M is a closed, compact smooth submanifold of dimension m of the Euclidean space $\mathbb{R}^n$. We continue to denote by $\mathbf{Graff}^c$ the Grassmannian of affine planes in $\mathbb{R}^n$ of codimension c. We want to prove the following result.

Theorem 3.6.1 (General Crofton Formula)**.** *If the manifold M is also semialgebraic, then the function*

$$\mathbf{Graff}^c \ni L \mapsto \chi(L \cap M) \in \mathbb{Z}$$

is $|d\nu_c|$-integrable and

$$\mu_c(M) = \int_{\mathbf{Graff}^c} \chi(L \cap M)|d\tilde{\nu}_c|(L).$$

Proof. We can assume that $k = \operatorname{codim} M < 1$. For every $x \in \mathbb{R}^n$ set $d(x) := \operatorname{dist}(x, M)$. Fix $R > 0$ such that for $x \in \mathbb{T}_R(M)$ there exists a unique point $\bar{x} \in M$ such that

$$|x - \bar{x}| = d(x).$$

For every $r < R$ consider the tube of radius r, around M,

$$D_r := \mathbb{T}_r(M),$$

and set $M_r = \partial\mathbb{T}_r(M)$. From (3.14) we deduce

$$\mu_c(M) = \lim_{r\to 0} \mu_c(D_r).$$

D_r is a semialgebraic domain with smooth boundary, and Theorem 3.5.1 implies

$$\mu_c(D_r) = \int_{\mathbf{Graff}^c} \chi(L \cap D_r)|d\nu_c|(L).$$

Thus it suffices to show that

$$\mu_c(M) = \lim_{r\to 0} \int_{\mathbf{Graff}^c} \chi(L \cap D_r)|d\nu_c|(L) = \int_{\mathbf{Graff}^c} \mu_0(L \cap M)|d\tilde{\nu}_c|(L).$$

For $r \in (0, R)$ we define

$$f_r : \mathbf{Graff}^c \to \mathbb{R}, \;\; f_r(L) = \chi(L \cap D_r).$$

For uniformity we set $f_0(L) = \chi(L \cap M)$.

Lemma 3.6.2. *There exists $C > 0$ such that*

$$|f_r(L)| \leq C, \;\; \forall L \in \mathbf{Graff}^c, \;\; r \in [0, R).$$

Proof. *Use semialgebraicity. To be included later*

□

Let

$$\mathbf{Graff}^c(M) := \big\{ L \subset \mathbf{Graff}^c; \;\; L \text{ intersects } M \text{ transversally} \big\},$$

and define $\mathbf{Graff}^c(M_r)$ similarly. Observe that $\mathbf{Graff}^c(M)$ is an open subset of $\mathbf{Graff}^c$ with negligible complement. For every $r > 0$ we set

$$\mathcal{X}_r = \big\{ L \in \mathbf{Graff}^c(M);\ \ L \in \mathbf{Graff}^c(M_s);\ \ \chi(L \cap D_s) = \chi(L \cap M),\ \ \forall s \in (0, r] \,\big\}$$

Observe that

$$\mathbf{Graff}^c(M, r_1) \subset \mathbf{Graff}^c(M, r_0),\ \ \forall r_1 \geq r_0.$$

To proceed further we need the following technical result, whose proof will presented at the end of this section.

Lemma 3.6.3. *The sets $\mathcal{X}_r$ are measurable in $\mathbf{Graff}^c$ and*

$$\bigcup_{r>0} \mathcal{X}_r = \mathbf{Graff}^c(M).$$

Set

$$\mathbf{Graff}^c_* = \big\{ L \in \mathbf{Graff}^c;\ \ L \cap D_R \neq \emptyset \big\}.$$

$\mathbf{Graff}^c_*(M)$ is a relatively compact subset of $\mathbf{Graff}^c$, and thus it has finite measure. Define

$$\mathcal{X}^*_r := \mathcal{X}_r \cap \mathbf{Graff}^c_*,\ \ \mathcal{Y}^r_* = \mathbf{Graff}^c_* \setminus \mathcal{X}^*_r.$$

For $0 < r < R$ we have

$$\mu_c(D_r) = \int_{\mathbf{Graff}^c} f_r(L)|d\tilde{\nu}_c|(L) = \int_{\mathbf{Gr}^c_*} f_r(L)|d\tilde{\nu}_c|(L)$$

$$= \int_{\mathcal{X}^*_r} f_r(L)|d\tilde{\nu}_c| + \int_{\mathcal{Y}^*_r} f_r(L)|d\tilde{\nu}_c| = 2\int_{\mathcal{X}^*_r} f_0(L)|d\tilde{\nu}_c| + \int_{\mathcal{Y}^*_r} f_r(L)|d\tilde{\nu}_c|$$

Hence

$$\left| \mu_c(M_r) - \int_{\mathcal{X}^*_r} f_0(L)|d\tilde{\nu}_c| \right| \leq \int_{\mathcal{Y}^*_r} |f_r(L)||d\tilde{\nu}_c| \leq C \mathrm{vol}\,(\mathcal{Y}^*_r).$$

We now let $r \to 0$, and since $\mathrm{vol}\,(\mathcal{Y}^*_r) \to 0$ we conclude that

$$\mu_c(M) = \lim_{r\to 0} \mu_c(D_r) = \lim_{r \to 0} \int_{\mathcal{X}^*_r} f_0(L)|d\tilde{\nu}_c| = \int_{\mathbf{Graff}^c} f_0(L)|d\tilde{\nu}_c|.$$

This concludes the proof of Theorem 3.6.1. □

Proof of Lemma 3.6.3. We will prove that for any given $L_0 \in \mathbf{Graff}^c(M)$ there exists and $\rho = \rho(L_0)$ such that

$$L_0 \in \mathcal{X}_\rho.$$

The measurability[1] follows from the fact that $\mathcal{X}_r$ is described using countably many boolean operations on measurable sets.

Consider the normal bundle

$$N = (TM)^\perp \to M.$$

For x in M we denote by N_x the fiber of N over x.

Let $y \in L_0 \cap M$. We denote by N^0_y the orthogonal complement in L_0 of $T_y(L_0 \cap M)$,

$$N^0_y = L_0 \cap \big(T_y(L_0 \cap M) \big)^\perp.$$

[1]With a little bit of work one can show that the sets $\mathcal{X}_r$ are in fact semi-algebraic, and in particular, measurable.

We think of N_y^0 as an affine subspace of $\mathbb{R}^n$ containing y. Because L_0 intersects M transversally we have

$$\dim N_y = \dim N_y^0 = k = \operatorname{codim} M.$$

For every $r > 0$ we set $N_y^0(r) := N_y^0 \cap D_r$.

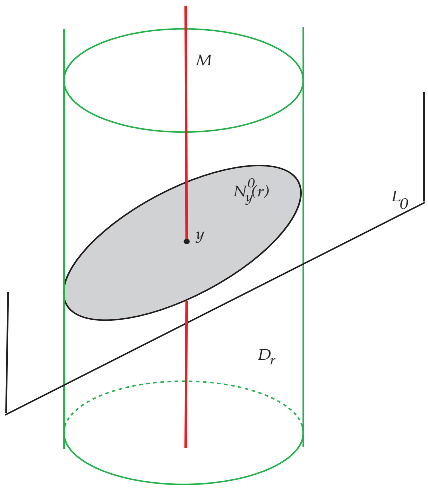

Figure 3.1. *Slicing the tube D_r around the submanifold M by a plane L_0.*

The collection $(N_y^0)_{y\in L_0\cap M}$ forms a vector subbundle $N^0 \to L_0 \cap M$ of $(T\mathbb{R}^n)|_{L_0\cap M}$. We have an exponential map

$$\mathbb{E}_{L_0\cap M} : N^0 \to L_0.$$

Denote by δ the pullback to N^0 of the distance function $x \mapsto d(x) = \operatorname{dist}(x, M)$,

$$\delta_{L_0} = d \circ \mathbb{E}_{L_0\cap M} : N^0 \to \mathbb{R}.$$

The zero section $L_0 \cap M \hookrightarrow N^0$ is a Bott nondegenerate critical submanifold of δ because for every $y \in L_0 \cap M$ the restriction to N_y^0 of the Hessian of d at y is positive definite. Hence there exists $\rho = \rho(L_0)$ sufficiently small such that the map

$$\mathbb{E}_{L_0\cap M} : \{\delta \le \rho\} \to L_0 \cap D_\rho$$

is a diffeomorphism. We deduce that we have a natural projection

$$\pi : L_0 \cap D_\rho \to L_0 \cap M,$$

which is continuous and defines a locally trivial fibration with fibers $N_y^0(\rho)$.

For every $y \in L_0 \cap M$ the fiber $N_y^0(\rho)$ is homeomorphic to a disk of dimension k, because we have a proper Morse function $N_y^0(\rho) \ni x \mapsto d(x)$, with a unique critical point, its minimum y. Thus $L_0 \cap D_\rho$ is homeomorphic to a tube in L_0 around $L_0 \cap M \subset L_0$ so that

$$\chi(L_0 \cap D_\rho) = \chi(L_0 \cap M).$$

The downward gradient flow of the restriction to $L_0 \cap D_\rho$ of the distance function $d(x)$ produces diffeomorphisms of manifolds with boundary

$$L_0 \cap D_\rho \cong L_0 \cap D_r, \ \ \forall r \in (0, \rho).$$

Hence

$$\chi(L_0 \cap D_r) = \chi(L_0 \cap D_\rho) = \chi(L_0 \cap M), \ \ \forall r \in (0, \rho].$$

Since the restriction to $L_0 \cap D_\rho$ of the distance function $d(x)$ has no critical points other than the minima $y \in L_0 \cap M$, we deduce that L_0 is transversal to the level sets

$$\{ d(x) = r \} = M_r, \ \ \forall r \in (0, \rho].$$

This proves $L_0 \in \mathcal{X}_\rho$. □

Corollary 3.6.4. *Suppose $C \subset \mathbb{R}^2$ is a smooth, closed, compact semialgebraic curve. For every line $L \in \mathbf{Gr}_1(\mathbb{R}^2) = \mathbf{Gr}^1(\mathbb{R}^2)$ we set*

$$n_C(L) := \#(L \cap C).$$

Then the function $L \mapsto n_C(L)$ belongs to $L^\infty(\mathbf{Gr}_1(\mathbb{R}^2), |d\tilde{\nu}|_{2,1})$, has compact support and

$$\operatorname{length}(C) = \int_{\mathbf{Gr}_1(\mathbb{R}^2)} n_C(L) |d\tilde{\nu}_{2,1}|(L). \qquad \square$$

More stuff to come...✍

3.7. Higher order Crofton formulæ

More stuff to come...✍

Chapter 4

The symplectic geometry of the cotangent bundle

4.1. Symplectic linear algebra

A *symplectic pairing* on a finite dimensional real vector space V is a skew-symmetric, nondegenerate bilinear map

$$\omega : V \times V \to \mathbb{R}, \;\; (\boldsymbol{v}_1, \boldsymbol{v}_2) \mapsto \omega(\boldsymbol{v}_2, \boldsymbol{v}_1) = -\omega(\boldsymbol{v}_2, \boldsymbol{v}_1), \;\; \forall \boldsymbol{v}_1, \boldsymbol{v}_2 \in V.$$

More precisely, ω is an element $\omega \in \Lambda^2 V^*$ such that the linear map

$$I_\omega : V \to V^*, \;\; V \ni \boldsymbol{v} \longmapsto I_\omega(\boldsymbol{v}) = \boldsymbol{v} \lrcorner\, \omega \in V^*$$

is a linear isomorphism. This map is called the *symplectic duality.* A *symplectic vector space* is a pair (V, ω), V is a finite dimensional real vector space and ω is a symplectic pairing on V.

Example 4.1.1. (a) **(The canonical symplectic pairing.)** Suppose U is a finite dimensional vector space, denote by

$$\langle -, - \rangle : U^* \times U \to \mathbb{R}$$

the canonical pairing, and set $V := U^* \times U = T^*U$. Then the bilinear map

$$\Omega : V \times V \to \mathbb{R}, \;\; \Omega\big((\boldsymbol{\xi}_1, \boldsymbol{u}_1), (\boldsymbol{\xi}_2, \boldsymbol{u}_2) \big) := \langle \boldsymbol{\xi}_1, \boldsymbol{u}_2 \rangle - \langle \boldsymbol{\xi}_2, \boldsymbol{u}_1 \rangle$$

We will say that Ω is the *canonical symplectic pairing* on $U^* \times U$. Observe that $V^* \cong U \times U^*$, and the symplectic duality

$$I_\Omega : U^* \times U \to U \times U^*$$

is given by

$$I_\Omega(\boldsymbol{\xi}, \boldsymbol{u}) = (-\boldsymbol{u}, \boldsymbol{\xi})$$

Indeed,

$$\big\langle\, I_\Omega(\boldsymbol{\xi}_1, \boldsymbol{u}_1), (\boldsymbol{\xi}_2, \boldsymbol{u}_2) \,\big\rangle = \big\langle\, (-\boldsymbol{u}_1, \boldsymbol{\xi}_1), (\boldsymbol{\xi}_2, \boldsymbol{u}_2) \,\big\rangle = \Omega\big((\boldsymbol{\xi}_1, \boldsymbol{u}_1), (\boldsymbol{\xi}_2, \boldsymbol{u}_2) \big).$$

(b) Suppose (V_i, ω_i), $i = 0, 1$ are symplectic vector spaces. Then the bilinear form $\omega_1 \times (-\omega_0)$ on $V_1 \times V_0$ is a symplectic pairing. □

Definition 4.1.2. A *morphism* of symplectic vector spaces (V_i, ω_i), $i = 0, 1$, is a linear map $T : V_0 \to V_1$ such that

$$\omega_1(T\boldsymbol{u}_0, T\boldsymbol{v}_0) = \omega_0(\boldsymbol{u}_0, \boldsymbol{v}_0), \quad \forall \boldsymbol{u}_0, \boldsymbol{v}_0 \in V_0. \qquad \square$$

We will also say that T is a *symplectomorphism*.

Remark 4.1.3. Observe that any symplectomorphism must be an *injective* map. □

Suppose (V, ω) is a symplectic vector space. For every vector subspace $U \subset V$ we set

$$\check{U} := \{ \boldsymbol{\xi} \in V^*; \ \ \langle \boldsymbol{\xi}, \boldsymbol{u} \rangle = 0, \ \ \forall \boldsymbol{u} \in U \},$$

and we define the *symplectic annihilator* to be

$$U^\flat := I_\omega^{-1}(\check{U}) = \{ \boldsymbol{v} \in V; \ \ \omega(\boldsymbol{v}, \boldsymbol{u}) = 0, \ \ \forall \boldsymbol{u} \in U \}.$$

Observe that

$$\dim U^\flat = \dim \check{U} = \dim V - \dim U, \quad (U^\flat)^\flat = 0$$

Definition 4.1.4. Suppose (V, ω) is a symplectic vector space, and U is a subspace of V. Then U is called *isotropic* if $U \subset U^\flat$, *co-isotropic* (or *involutive*) if $U^\flat \subset U$, and *Lagragian*, if U is simultaneously, isotropic and involutive, i.e.,

$$U = U^\flat. \qquad \square$$

Observe that

$$U \text{ isotropic} \Longleftrightarrow U^\flat \text{ involutive}.$$

We denote by $\mathcal{I}_-(V, \omega)$ the set of isotropic subspaces, by $\mathcal{I}_+(V, \omega)$ the set of involutive subspaces, and by

$$\mathrm{Lag}(V, \omega) := \mathcal{I}_-(V, \omega) \cap I_+(V, \omega)$$

the set of Lagrangian subspaces. Observe that $\mathcal{I}_-(V)$ is nonempty because it contains all the one-dimensional subspaces. The set $\mathcal{I}_-(V)$ is ordered by inclusion.

Proposition 4.1.5. $\mathrm{Lag}(V)$ *coincides with the subset of maximal elements of* $\mathcal{I}_-(V)$. *In particular,* $\mathrm{Lag}(V)$ *is non-empty.*

Proof. Clearly, a Lagrangian subspace L is maximal isotropic because any isotropic subspace U satisfies $2\dim U \leq \dim U + \dim U^\flat = \dim V$, and in particular

$$\dim L = \dim L^\flat = \frac{1}{2} \dim V.$$

Conversely, assume L is a maximal isotropic subspace. Then $L = L^\flat$, because for any vector $\boldsymbol{v} \in L^\flat \setminus L$ the subspace $L + v$ is still isotropic. □

Example 4.1.6. (a) Suppose U is a vector space, and $V = U^* \times U$ is equipped with the canonical symplectic structure Ω. Then for any subspace $S \subset U$ the subspace

$$L_S = \check{S} \times S \subset U^* \times U$$

is Lagrangian.

(b) Suppose $A : U \to U^*$ is a linear map. We define its graph as the subspace

$$\Gamma_A = \{ (A\boldsymbol{u}, \boldsymbol{u}) \in U^* \times U; \;\; \boldsymbol{u} \in U \}.$$

Then Γ_A is Lagrangian if and only if A is *symmetric*, i.e. $A = A^*$, where A^* is the adjoint of A

$$A^* : (U^*)^* = U \longrightarrow U^*.$$

To see this observe that

$$\check{\Gamma}_A = \{(-\boldsymbol{v}, A^*\boldsymbol{v}) \in U \times U^* \}$$

so that

$$\Gamma_A^\flat = I_\Omega^{-1}(\check{\Gamma}_A) = \Gamma_{A^*}.$$

(c) Suppose (V_i, ω_i), $i = 0, 1$ are two symplectic spaces. Then a linear map $T : V_0 \to V_1$ is a symplectic morphism, if and only if its graph

$$\Gamma_T := \{ (T\boldsymbol{v}_0, \boldsymbol{v}_0); \;\; \boldsymbol{v}_0 \in V_0 \} \subset V_1 \times V_0$$

is Lagrangian with respect to the symplectic pairing $\omega_1 \times (-\omega_0)$. □

Suppose L is a Lagrangian subspace of the symplectic space (V, ω). Observe that for every $\boldsymbol{v} \in L$ the linear functional $I_\omega \boldsymbol{v}$ vanishes on L so that it induces a linear functional on V/L. In particular, we have a natural map

$$I_\omega(L) \to (V/L)^*,$$

which is an isomorphism since L is Lagrangian.

4.2. Lagrangian submanifolds

4.3. Distributions and their singularities

4.4. Fourier integral operators

Chapter 5

The conormal cycle of a definable set

Definable sets

Appendix B

A brief trip in geometric measure theory

Bibliography

[**ABP**] M.F. Atiyah, R. Bott, V.K. Patodi: *On the heat equation and the index theorem*, Invent. Math., **19**(1973), 279-330.

[**Gr**] A. Gray: *Tubes*, Addison Wessley, 1990.

[**GS**] V. Guillemin, S. Sternberg: *Geometric Asymptotics*, Math. Surveys and Monographs, vol. 14, Amer. Math. Soc. 1990.

[**KR**] D.A. Klain, G.-C. Rota: *Introduction to Geometric Probability*, Cambridge University Press, 1997.

[**N**] L.I. Nicolaescu: *Lectures on the Geometry of Manifolds*, World Sci. Pub. Co. 1996.

[**San**] L. Santaló: *Integral Geometry and Geometric Probability*, Cambridge University Press, 2004.

[**W1**] H. Weyl: *On the volume of tubes*, Amer. J. Math., **61**(1939), 461-472.

[**W2**] H. Weyl: *The Classical Groups. Their Invaraints and Representations*, Princeton University Press, 1939.

Index