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Contents

Abstract

We prove an index theorem for families of elliptic boundary value problems and a glueing formula for the index of a family of Dirac operators on a closed manifold. In the process we also obtain a very general result about the cobordism invariance of the index of a family.

To achieve these goals we develop techniques (inspired from symplectic geometry) for computing the index of a family of Fredholm operators.

Key words and phrases: Dirac operators, boundary value problems, Caldèron projectors, index of families, Clifford algebras, Karoubi's $K^{p,q}$-theory, generalized symplectic spaces, generalized symplectic reduction, generalized Maslov index.

Introduction

Consider a closed, compact oriented Riemann manifold (M, g) which is decomposed into two manifolds-with-boundary by an oriented hypersurface Σ: $M = M_1 \cup_\Sigma M_2$. Assume moreover that it is given a continuous family $(D_y)_{y\in Y}$ of Dirac type operators on M. Classically, this family has an index in some K-group. The problem we address in this paper is the following:

Describe the index of the family in terms of its behavior on the two pieces of the decomposition

i.e. we are looking for a splitting formula for the index of a family. If for example the operators have some symmetries (e.g. they are skew or selfadjoint) then the index lies in in a higher K-group (e.g. if all the operators are selfadjoint the index is in $K^1(Y)$). Thus it is very important to take their symmetries into account. Also, it makes a difference whether the operators are complex or real. In this paper we will consider only real operators since they are homotopically more complicated. However all the techniques extend to the complex case. The natural context which coherently takes into account all these aspects is that of Fredholm operators with Clifford symmetries introduced in [AS] and [Ka2].

In a previous paper [N1] we dealt with a special case of the above splitting problem. There we considered a path of selfadjoint Dirac operators $(D^t)_{t\in[0,1]}$ on a bundle $\mathcal{E}$ of Clifford modules with a **fixed** Clifford structure. To any such operator D there is an associated pair of Cauchy data spaces (CD spaces for brevity) Λ_i $(i = 1, 2)$. These are closed subspaces in $L^2(\mathcal{E}\,|_\Sigma)$ defined roughly as follows:

$$\Lambda_i^t = \Lambda_i(D^t) = \{U\,|_\Sigma\,;\; U \in C^\infty(M_i),\;\; D^tU = 0 \;\text{ on }\; M_i\} \quad i = 1, 2.$$

It turns out that $L^2(\mathcal{E}\,|_\Sigma)$ has a natural symplectic structure and the CD spaces form Fredholm pairs (cf. Sec.3) of lagrangian subspaces. The space of Fredholm pairs of lagrangians classifies K^1 and an *explicit* isomorphism $K^1(S^1) \to \mathbb{Z}$ can be constructed, called *the Maslov index.* Then one shows that the index of the original path of Dirac operators (also called *the spectral flow*) equals the Maslov index of the associated path of CD paces.

An equivalent way of looking at this result is to consider the family of boundary value problems

$$B^t = \begin{cases} D_1^t U = V & \text{on } M_1 \\ U\,|_{\partial M_1} \in \Lambda_2^t \end{cases}$$

where $D_1^t = D^t\,|_{M_1}$. Since $(\Lambda_1^t, \Lambda_2^t)$ is a Fredholm pair the operator B^t is Fredholm and because Λ_2^t is lagrangian B^t is selfadjoint. Moreover $\ker B^t = (\ker D^t)\,|_{M_1}$ which

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suggests that the family of boundary value problems may have the same index as the original family (D^t) and thus the splitting formula describes the index of the family (B^t) in terms of boundary data. This is the approach we take in our search for a general splitting formula: replace the original family with a family of boundary problems which has the same index and then describe the index of the family of boundary problems in terms of "interactions along the boundary".

First, we look at Fredholm operators with (Clifford) symmetries (see Prop. 2.9 for their definition). These were shown (cf.[AS], [Ka2]) to form classifying spaces $Fred^{p,q}$ for all K-groups. Thus to any continuous family $y \mapsto T_y \in Fred^{p,q}$ there is an associated index $ind_{p,q}(T_y) \in K^{p,q}(Y)$. Unfortunately the problem of deciding whether two families have the same index can be very delicate.

The other side of the story is recovered by generalizing the notion of lagrangian which is the key notion in this paper. A symplectic space can be viewed as a module over the algebra $C^{1,0} \cong \mathbb{C}$ and one can define lagrangians in terms of this structure: any lagrangian subspace defines a $\mathbb{Z}_2$-grading of the (1,0)-structure. The generalization is now evident. One considers $C^{p,q}$-modules and (p,q)-lagrangians (Sec.3) which can be viewed as defining "super" structures. The space $\mathcal{FL}^{p,q}$ of Fredholm pairs of **infinite dimensional** (p,q)-lagrangians is a classifying space for the $K^{p,q}$-groups of Karoubi (this is also proved in [KGLZ] in a disguised form). Thus to any continuous family $y \mapsto (\Lambda_1^y, \Lambda_2^y) \in \mathcal{FL}^{p,q}$ ($y \in Y$-compact CW complex) one can associate an element $\mu_{p,q}(\Lambda_1^y, \Lambda_2^y) \in K^{p,q}(Y)$ called *the generalized Maslov index.*

The space $\mathcal{FL}^{p,q}$ is very abstract and a natural question imposes itself: given two continuous families in $\mathcal{FL}^{p,q}$ (parameterized by the same compact CW-complex Y) decide whether they have the same generalized Maslov index.

The solution to this problem is the key theoretical result of this paper. The Clifford modules are formally very similar to the usual symplectic spaces. Standard symplectic operations have natural correspondents to Clifford modules (which we called *generalized symplectic spaces*). In particular, the symplectic reduction process generalizes to arbitrary Clifford modules and more important to infinite dimensions. The reduction gives a very efficient way of transforming an infinite dimensional problem to a finite dimensional one. We proved that two continuous families of Fredholm pairs of lagrangians are homotopic iff we can symplectically reduce them to homotopic families of **finite dimensional** lagrangians (see Thm. 4.14 for details).

We next look at families of Clifford symmetric Fredholm operators (see Sec.5 for details) and we ask ourselves the same effectivity question: given two continuous families of Clifford symmetric Fredholm operators decide whether they have the same index. The answer to this question is the second main theoretical contribution of this paper. To approach this problem it is more convenient to think of linear operators in terms of their graphs. Given

$$T : Dom\,(T) \subset H \to H$$

a closed, densely defined, Clifford symmetric Fredholm operator in an Hilbert $C^{p,q+1}$-

module H, its graph

$$\Gamma_T = \{(x, Tx) \in H \times H;\ x \in D(T)\}$$

is a $(p+1, q+1)$-lagrangian with respect to a natural $(p+1, q+1)$-structure in $H \times H$. Moreover $(H \times \{0\}, \Gamma_T)$ is a Fredholm pair. Thus the map $\Gamma : T \mapsto \Gamma_T$ embeds $Fred^{p,q}$ in $\mathcal{FL}^{p+1,q+1}$ and in Theorem 5.5 we prove that Γ is a homotopy equivalence. One major advantage is that we included in our consideration **unbounded operators as well** and this makes Theorem 5.5 a very versatile result. A crucial step in the proof is the construction of some Clifford symmetric operators which are "generators" in K-theory. We called them **Floer operators** since it seems it was Floer who for the first time in [F] emphasized their K-theoretic relevance in the context of symplectic homology. Now using the symplectic reduction trick for abstract lagrangians we can reduce the computation of the index of a family of Fredholm operators to a finite dimensional problem.

Finally we study boundary problems for Dirac operators on a manifold with boundary. If D is a Clifford symmetric Dirac operator on a bundle $\mathcal{E} \to M$ with Clifford symmetries, then $L^2(\mathcal{E}\mid_{\partial M})$ inherits a natural Clifford module structure and the CD space $\Lambda(D)$ of D is a generalized lagrangian subspace. A boundary condition for D corresponds to a choice of a closed subspace $V \subset L^2(\mathcal{E}\mid_{\partial M})$. This boundary condition is elliptic iff $(\Lambda(D), V)$ is a Fredholm pair. The boundary problem thus obtained displays Clifford symmetries iff V is a generalized lagrangian. Therefore we look at a family of boundary value problems

$$B_y = \begin{cases} D_y U = W & \text{on } M \\ U\mid_{\partial M} \in L_y & \end{cases} \qquad y \in Y.$$

Here the parameter space is a compact CW-complex and for each $y \in Y$, L_y is a generalized lagrangian in $L^2(\mathcal{E}\mid_{\partial M})$ -depending continuously upon y- such that the pair $(\Lambda(D_y), L_y)$ is Fredholm. The main application of the previous theoretical considerations is Theorem 6.2. A family of boundary problems as above determines two elements in $K^{p,q}(Y)$. The first one is an index $\mathbf{ind}_{p,q}(D_y, L_y)$ if we think of these problems as defining Fredholm operators. The second element is a measure of the "interaction along the boundary" and is the generalized Maslov index $\mu_{p+1,q+1}(\Lambda(D_y), L_y)$. Theorem 6.2 states a very natural fact:

$$\mathbf{ind}_{p,q}(D_y, L_y) = \mu_{p+1,q+1}(\Lambda(D_y), L_y).$$

The logical structure of the proof is simple. One first shows that the general case is equivalent with a special one, namely when M is in fact a cylinder. The cylinder case is symplectically reduced to a Floer family for which we have already described the index. The main reason this approach works is the extreme rigidity of Dirac operators: they satisfy the unique continuation property. As a corollary we deduce the splitting formula for arbitrary families of Dirac operators (Thm. 6.10) using the approach outlined in the beginning.

Both theorems 6.2 and 6.10 where stated for families of Dirac operators with *constant* symbol. However in Subsection 6.3 we explain how the same proof extends to the more general case of varying symbols. Similar results were recently proved by [DZ] in the complex case, using entirely different methods.

The key ingredient which makes this extension possible is the notion of spectral section defined in [MP], adapted in the obvious manner to a (p, q) setting. Roughly speaking a spectral section is a continuous family of Atiyah-Patodi-Singer family. Using the adiabatic analysis of [N2] we show that the restriction to the boundary of any family of Dirac operators is a family which admits a spectral section. In particular this provides a short proof of a very general result stating the cobordism invariance of arbitrary families of Dirac operators. This arbitrariness is two fold: the space of parameters is any compact CW complex and the index can live in higher K-theory. As explained in Subsection 6.3 the CD spaces induce an "excess of symmetry" on the boundary operators which leads to the vanishing of the index.

We are very pleased of a technical byproduct contained in Appendix B. There we deal with the continuity of families of Dirac operators with varying boundary conditions (thus varying domains). This was one delicate point dealt with in the paper [FO1] when studying paths of *ordinary differential operators* (what we called Floer operators). The authors used a conjugation trick which reduces the problem to families with **constant domain**. Unfortunately this trick does not generalize well to partial differential operators.

Although we were interested only in real case (which is richer from a topological point of view) the methods we develop extend almost verbatim to the complex case and we considered it was not worth lengthening the presentation by dealing with both the real and the complex case.

The paper is divided in six sections and we collected the analytical technicalities in four appendices.

For the readers convenience we present a very short survey of some basic facts concerning the not so popular but extremely versatile $K^{p,q}$ theory of Karoubi. In Section 3 we introduce the notion of (p, q) lagrangians and show how these can be organized to produce a classifying space of the $K^{p,q}$-theory .

The theoretical heart of this work is contained in the sections 4 and 5 which describe in detail the symplectic skeleton of K theory. We gathered the applications to index theory in Sec.6.

We want to mention that (most of) the results of this work were announced in [N2].

Acknowledgments I want to thank B. Boss and K. Wojciechowski for their interest in this work. Their results on the boundary value problems for Dirac operators made me aware of the K-theoretic relevance of the CD spaces.

Also I want to thank W. Zhang for the preprint [DZ]. It was while reading their work that I got the idea for the new proof of the cobordism invariance of the index of families described in Proposition 6.7.

1 Algebraic preliminaries

We gather in this section some standard facts about Clifford algebras and their representations. For details and proofs we refer to [LM] or [Ka1].

§1.1 Clifford algebras Let Q be a quadratic form on V. The Clifford algebra generated by V and Q, denoted by $C(V,Q)$ is the associative unital algebra generated by V with the relations

$$u \cdot v + v \cdot u = -Q(u,v) \cdot 1 \quad \forall\, u, v \in V.$$

If $e_1, \cdots, e_n$ ($n = \dim V$) is a basis of V in which Q is diagonal then $C(V,Q)$ can be alternatively characterized as the associative unital algebra generated by $e_1, \cdots, e_n$ modulo the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2Q(e_i, e_j) \quad \forall\, i, j. \tag{1.1}$$

For any nonnegative integers p, q such that $p + q > 0$ we denote by $\mathbb{R}^{p,q}$ the space $\mathbb{R}^p \oplus \mathbb{R}^q$ endowed with the quadratic form

$$Q(x \oplus y) = |x|^2 - |y|^2 \quad x \in \mathbb{R}^p\,,\; y \in \mathbb{R}^q$$

where $|\cdot|$ denotes the standard euclidian metric. Then $C^{p,q}$ denotes the Clifford algebra generated by $\mathbb{R}^{p,q}$. When $p = q = 0$ we set $C^{0,0} = \mathbb{R}$.

Let $e_1, \cdots e_p; \epsilon_1, \cdots \epsilon_q$ denote the standard basis of $\mathbb{R}^{p,q}$. $C^{p,q}$ decomposes as a $\mathbb{Z}_2$-graded algebra ("superalgebra")

$$C^{p,q} \cong C_+^{p,q} \oplus C_-^{p,q} \tag{1.2}$$

where $C_\pm^{p,q}$ are the vector subspaces generated by the even/odd degree monomials in the basis elements $\{e_i; \epsilon_j\}$.

$C^{p,q}$ can be naturally equipped with a scalar product making $\{e_I \cdot \epsilon_J\}$ an orthonormal basis. Denote the corresponding norm by $\|\cdot\|$. $(C^{p,q}, \|\cdot\|)$ is a $\mathbb{Z}_2$-graded real C^*-algebra with the anti-involution "$*$" uniquely defined by its action on the generators:

$$e_i^* = -e_i\;,\; \epsilon_j^* = \epsilon_j \quad \forall\, i, j.$$

It will be useful to introduce some "super" notions.

Definition 1.1 *(a) A* **superspace** *is a vector space H together with a distinguished direct sum decomposition*

$$H \cong H_0 \oplus H_1.$$

R=diag$(1_{H_0}, -1_{H_1})$ is called the grading of the superstructure. If H is a Hilbert space and the decomposition above is orthogonal H is called a **Hilbert superspace**.
(b) A superalgebra is a $\mathbb{Z}_2$-graded algebra $A \cong A_0 \oplus A_1$. The elements in A_0 (resp.A_1) are called even (resp.odd).
(c) The supercommutator in a superalgebra is the bilinear map $[\cdot,\cdot]_s : A \times A \to A$ defined on homogeneous elements by the formula

$$[x, y]_s = xy - (-1)^{|x|\cdot|y|}yx$$

where $|\cdot| \in \{0, 1\}$ denotes the degree of a homogeneous element.

Example 1.2 If $V = V_0 \oplus V_1$ is a superspace then the algebra of endomorphisms of V has a natural $\mathbb{Z}_2$-grading. The even operators preserve the grading while the odd ones switch it. We will write $\widehat{\mathrm{End}}\,(V)$ to emphasize this super structure of $\mathrm{End}\,(V)$.

§1.2 Clifford modules The notion of Clifford module plays a crucial role in the definition of Karoubi's $K^{p,q}$-theory. We collect here some facts and definitions concerning these objects.

Definition 1.3 *(a) A* **Hilbert (p,q)- module** *is a Hilbert space H together with a morphism of C^*-algebras*

$$\rho : C^{p,q} \to B(H).$$

($B(H)$= bounded linear operators on H).
(b) A $\mathbf{(p,q)\,s-module}$ *is a superspace $V = V_0 \oplus V_1$ together with a morphism of* **superalgebras**

$$\rho : C^{p,q} \to \widehat{\mathrm{End}}\,(V).$$

A **Hilbert (p,q) s-module** *is defined in the obvious way.*

Example 1.4 A Hilbert (p, q) module (H, ρ) is uniquely defined by a choice of operators $J_i = \rho(e_i)$ and $C_j = \rho(\epsilon_j)$ satisfying

$$J_i^* = -J_i \ , \ C_j^* = C_j \ \ \forall\, i, j$$

$$J_i^2 = -I \ , \ C_j^2 = I \ \ \forall\, i, j$$

$$\{J_i, C_j\} = 0 \ \ \forall\, i, j$$

$$\{J_{i_1}, J_{i_2}\} = 0 = \{C_{j_1}, C_{j_2}\} \ \ \forall\, i_1 \neq i_2\; j_1 \neq j_2.$$

where for any linear operators A, B the bracket $\{A, B\}$ denotes their anticommutator $AB + BA$. Note that any finite dimensional $C^{p,q}$-module can be given a structure of Hilbert module constructing a metric using the averaging trick (average with respect to the action of the finite group generated by the e's and ϵ's).

Remark 1.5 *$C^{p,q}$ has a volume element $\omega = e_1 \cdots e_p \cdot \epsilon_1 \cdots \epsilon_q$. It satisfies*

$$\omega^2 = (-1)^{\frac{\delta(\delta+1)}{2}} \ , \ \omega^* = (-1)^{\frac{\delta(\delta+1)}{2}} \omega \tag{1.3}$$

where $\delta = p - q$. Thus ω is a selfadjoint involution if $\delta \equiv 0, -1 (mod\, 4)$. Moreover when $\delta \equiv -1 (mod\, 4)$ ω lies in the center of $C^{p,q}$ since

$$\omega e_i + (-1)^{\delta} e_i \omega = \omega \epsilon_j + (-1)^{\delta} \epsilon_j \omega = 0 \quad \forall\, i, j. \tag{1.4}$$

This implies that multiplication by $\frac{1}{2}(1 - \omega)$ is an idempotent endomorphism of $C^{p,q}$ i.e. $C^{p,q}$ is not simple. Thus in this case every $C^{p,q}$ module H has decomposition

$$H \cong H_+ \oplus H_- \tag{1.5}$$

into the ± 1 eigenspaces of $\Omega = J_1 \cdots J_p \cdot C_1 \cdots C_q$.

Definition 1.6 *A Fredholm selfadjoint operator T in an infinite dimensional Hilbert space is called* **essentially indefinite** *iff its essential spectrum contains both positive and negative elements. In particular a selfadjoint involution C on an infinite dimensional Hilbert space is called* **essential** *if both $\ker(I - C)$ and $\ker(I + C)$ are infinite dimensional. An* **infinite dimensional** *Hilbert (p,q)-module (H, ρ) is called* **essential** *if*
(i) either $\delta \not\equiv -1 (mod\, 4)$;
(ii) or $\delta \equiv -1 (mod\, 4)$ and the subspaces $H_\pm$ in the decomposition (1.5) are both infinite dimensional. Equivalently, this means that the involution $\Omega = J_1 \cdots J_p \cdot C_1 \cdots C_q$ is essential. In particular a (p,q) s-module with grading R is called essential if
(i) either $p - q \not\equiv 0 (mod\, 4)$;
(ii) or $p - q \equiv 0 (mod\, 4)$ and the involution $RJ_1 \cdots J_p C_1 \cdots C_q$ is essential.
A **finite dimensional** *(p,q)- module is called essential if either (p-q) $\not\equiv -1$ (mod 4) or if (p-q) $\equiv -1$ (mod 4) the involution Ω is nontrivial i.e. $\Omega \neq 1$. One defines essential finite dimensional gradings in a similar fashion.*

The topological meaning of essentiality will be revealed in Lemma 3.5

Denote by $M^{p,q}$ the set of isomorphism classes of (p,q)-modules. The direct sum of modules induces on $M^{p,q}$ a structure of abelian monoid. The Grothendieck group associated to $M^{p,q}$ (cf [Ka1]) is denoted by $R^{p,q}$. Analyzing the algebraic structure of the Clifford algebras one deduces the following periodicity result (see [Ka1]).

Proposition 1.7 *If $p - q \equiv p' - q'$ (mod 8) then*

$$R^{p,q} \cong R^{p'-q'}.$$

We define in a similar way the monoid $\hat{M}^{p,q}$ of isomorphism classes of finite dimensional (p,q) s-modules. Let $\hat{R}^{p,q}$ denote its associated Grothendieck group. If $H \cong H_+ \oplus H_-$ is a $C^{p,q}$ supermodule than $C_{q+1} = Proj_{H_+} - Proj_{H_-}$ is a selfadjoint involution on H anticommuting with the generators of the $C^{p,q}$-action (briefly a **(p,q)-grading**) and thus extends the $C^{p,q}$ structure on H to a $C^{p,q+1}$ structure. Conversely, a (p,q)-grading on a (p,q)-module induces a super $C^{p,q}$ structure so that we have

Lemma 1.8

$$\hat{R}^{p,q} \cong R^{p,q+1}.$$

Using the periodicity result of Proposition1.7 one can replace the notation $R^{p,q}$ by R^{p-q} and similarly for the $\hat{R}$'s. The inclusion $C^{p,q} \hookrightarrow C^{p,q+1}$ induces a map

$$i_{p,q} : \hat{R}^{p,q+1} \longrightarrow \hat{R}^{p,q}.$$

Following [ABS] we introduce

$$A_{p-q} = A_{p,q} = \operatorname{coker} i_{p,q} = \hat{R}^{p,q}/i_{p,q}\hat{R}.$$

For any finite dimensional (p,q) s-module M with grading η we denote by $[M,\eta]$ its image in $A_{p,q}$. If (M_k, η_k) $(k = 1,2)$ are two (p,q) s-modules then $[M_1,\eta_1] = [M_2,\eta_2]$ iff there exist $(p,q+1)$ s-modules N_k $(k = 1,2)$ such that

$$M_1 \oplus N_1 \cong M_2 \oplus N_2 \quad \text{as } (p,q) \text{ s} - \text{modules}.$$

The groups $A_{p,q}$ will play an important role in this paper. We list bellow the groups$R^{0,q}$ and $A_{0,q}$ for $q = 0,\ldots,7$.

Table 1.1

q	0	1	2	3	4	5	6	7
$R^{0,q}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$A_{0,q}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

2 Topological preliminaries

This section is a brief survey on the bigraded (real) K-theory introduced by Karoubi. For more details we refer to the efficient presentation in [KGLZ] Exposés II and III or the more comprehensive one in [Ka1].

§2.1 **Karoubi's $K^{p,q}$-theory** Let X be a compact CW-complex and let $E \to X$ a real (metric) bundle of $C^{p,q}$-modules. A **(p,q)-grading** of E is a selfadjoint involution $\eta \in \mathrm{Aut}\,(E)$ anticommuting with the generators of the $C^{p,q}$ structure.The space of all (p,q)-gradings of E will be denoted by $Grad^{p,q}(E)$. The abelian group $KO^{p,q}(X)$ can be described as follows.

1.The generators These are the triples $[E;\eta_0,\eta_1]$ where E is a finite dimensional bundle of $C^{p,q}$-modules over X and η_0,η_1 are (p,q)-gradings of E.

2.The addition

$$[E;\eta_0,\eta_1] + [F;\mu_0,\mu_1] = [E\oplus F;\eta_0\oplus\mu_0,\eta_1\oplus\mu_1].$$

3. The relations First we define the **acyclic** triples to be the triples of the form $[E;\eta,\eta]$. These represent the trivial element in $KO^{p,q}(X)$.

Two triples $[E;\eta_0,\eta_1]$ and $[F;\mu_0,\mu_1]$ define the same element in $KO^{p,q}(X)$ if there exist acyclic triples $[E';\eta,\eta]$ and $[F';\mu,\mu]$ such that

(a) $E\oplus E' \cong F\oplus F'$ as bundles of $C^{p,q}$-modules ;

(b) the gradings $\eta_i\oplus\eta$ and $\mu_i\oplus\mu$ $(i=0,1)$ are homotopic as (p,q)-gradings in $E\oplus E'\cong F\oplus F'$.

A triple $[E;\eta_0,\eta_1]$ as above will be called a **standard grading representation** of the element it represents in $K^{p,q}$. Its opposite in $K^{p,q}$ admits the grading representation $[E;\eta_1,\eta_0]$

The set of generators can be considerably diminished. To this aim it convenient to consider a **cofinal family** of real $C^{p,q}$- **supermodules** i.e. an increasing family of $C^{p,q}$-supermodules $\{(E_n,\eta_n)\}_{n\geq 0}$ such that any other real $C^{p,q}$ -supermodule is a factor of some (E_n,η_n) (here η_n denotes the (p,q)-grading of E_n coming from the "super" structure). As generators for $KO^{p,q}(X)$ it suffices to take the triples

$$[E_n\times X;\eta_n,\eta]$$

where η is a continuous map $\eta : X \to Grad_{p,q}(E_n)$ ([Kal] Prop.III.4.26).

If (X,Y) $(Y \subset X)$ is a pair of compact CW-complexes then the relative group $KO^{p,q}(X,Y)$ is defined following a similar pattern.

1. The generators These are as before triples $[E;\eta_0,\eta_1]$ satisfying the suplimentary condition : $\eta_0|_Y = \eta_1|_Y$.

2. The addition Identical to the one defined above.

3. The relations The acyclic triples are the same. The equivalence relation is similar. The only difference is what we mean by homotopy in 3(b) above. It should be understood as being a homotopy through gradings that agree over Y.

Using the mod 8 periodicity of Clifford algebras one can show the following **algebraic periodicity** result.

Proposition 2.1 *If $p-q \equiv p'-q'$ (mod 8) then*

$$KO^{p,q}(X,Y) \cong KO^{p',q'}(X,Y)$$

for any compact CW-complexes $Y \subset X$.

Let $KO(X,Y)$ be the usual relative K-group for real vector bundles as defined in [A]. Then one can show (see [Kal]).

Proposition 2.2

$$K^{0,0}(X,Y) \cong KO(X,Y).$$

To describe the relationship with the higher KO-groups we need to introduce the periodicity morphism

$$t : KO^{p,q+1}(X) \to KO^{p,q}(X \times D^1, X \times S^0).$$

Identify D^1 with the upper semi-circle $e^{i\theta} : 0 \le \theta \le \pi$ and denote by

$$\pi : X \times D^1 \to X$$

the natural projection. Consider a triple $[E;\eta_1,\eta_2]$ where E is a $C^{p,q+1}$ vector bundle over X and η_1 , $\eta_2 \in Grad^{p,q+1}(E)$. Denote by E' the bundle E viewed as a $C^{p,q}$ bundle. E' has a distinguished grading ϵ_{q+1} coming from the $C^{p,q+1}$ module structure of E. Let $\overline{E} = \pi^*(E') \cong E' \times D^1$. $\bar{E}$ is a $C^{p,q}$-bundle over $X \times D^1$. $\overline{E}$ comes with two gradings

$$\bar{\eta}_i(x,\theta) = \epsilon_{q+1}(x)\cos\theta + \eta_i(x)\sin\theta \quad i = 1,2.$$

$[\bar{E};\bar{\eta}_1,\bar{\eta}_2]$ defines an element in $KO^{p,q}(X \times D^1, X \times S^0)$.

The map $[E;\eta_1,\eta_2] \mapsto [\bar{E};\bar{\eta}_1,\bar{\eta}_2]$ will induce a homomorphism

$$t = KO^{p,q+1}(X) \to KO^{p,q}(X \times D^1, X \times S^0).$$

We can now state the fundamental result in K-theory ([Kal]).

Theorem 2.3 *The homomorphism t defines an isomorphism*

$$KO^{p,q+1}(X) \cong KO^{p,q}(X \times D^1, X \times S^0).$$

Corollary 2.4 *(Bott Periodicity) If $n \cong p - q (mod 8)$*

$$KO^n(X) \cong KO^{p,q}(X).$$

If M is a finite dimensional $C^{p,q}$-supermodule with grading η then $[M;\eta,-\eta]$ defines an element in $KO^{p,q}(pt)$. If moreover M is a $C^{p,q+1}$-supermodule with a grading μ then μ and $-\mu$ are homotopic as $\mathbf{C}^{p,q}$ **– gradings** via the homotopy

$$\mu(\theta) = \cos\theta\mu + \sin\theta\epsilon_{q+1} \quad \theta \in [0,\pi].$$

($\mu(\theta)$ are $C^{p,q}$-gradings since the gradings μ and ϵ_{q+1} anticommute). We thus have a map

$$\alpha : A_{p,q} \to KO^{p,q}(pt)$$

defined by

$$[M,\eta] \mapsto [M;\eta,-\eta].$$

The groups $A_{p,q}$ are defined in purely algebraic terms while the groups $KO^{p,q}(pt)$ have a topological nature. This is why the next result is somewhat surprising.

Corollary 2.5 *(Atiyah-Bott-Shapiro [ABS]) The map*

$$\alpha : A_{p,q} \to KO^{p,q}(pt)$$

is an isomorphism.

Sketch of proof One can show that the map α is surjective (see Remark 3.14 to come). From Corollary 2.4 one deduces easily that $KO^{p,q}(pt) = A_{p,q}$. Looking at the list of $A'_{p,q}s$ given in Table 1.1 we see that the surjective endomorphisms are necessarily automorphisms. □

§2.2 Some elementary examples We interrupt here the flow of presentation to describe *explicit* isomorphisms $KO^{0,0}(pt) \cong \mathbb{Z}$ and $KO^{1,1}(pt) \cong \mathbb{Z}$. Using the periodicity result we will then *explicitly* describe a generator of $K^{1,0}(I, \partial I) \cong \mathbb{Z}$. This will play an important role later when we will identify the spectral flow with a Maslov index.

The elements of $KO^{0,0}(pt)$ are represented by pairs (E, η_0, η_1) where η_i are $\mathbb{Z}_2$ gradings of the finite dimensional Euclidian space E. The map

$$(E, \eta_0, \eta_1) \mapsto \frac{1}{2}\mathrm{tr}\,(\eta_0 - \eta_1) \in \mathbb{Z}$$

defines an isomorphism

$$ind_{0,0} : K^{0,0}(pt) \cong \mathbb{Z}.$$

We will refer to it as the *canonical isomorphism* . The element

$$\gamma_{0,0} = (\mathbb{R}, 1_{\mathbb{R}}, -1_{\mathbb{R}})$$

satisfies $ind_{0,0}\gamma_{0,0} = 1$ and will be called the *canonical generator.*

The algebraic periodicity isomorphism $KO^{0,0}(pt) \cong KO^{1,1}(pt)$ can be explicitly characterized as follows. Start with $u = (E, \eta_0, \eta_1) \in KO^{0,0}(pt)$. The direct sum $\hat{E} = E \oplus E$ has a natural $(1,1)$ structure defined by

$$J = \begin{bmatrix} 0 & 1_E \\ -1_E & 1 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 1_E \\ 1_E & 0 \end{bmatrix}.$$

Now form

$$\hat{u} = (\hat{E}, \eta_0 \oplus -\eta_0, \eta_1 \oplus (-\eta_1)) \in KO^{1,1}(pt).$$

The correspondence $u \mapsto \hat{u}$ defines an isomorphism in K-theory. Denote as usual $\Omega = JR$. For each $v = (F, \nu_0, \nu_1) \in KO^{1,1}(pt)$ set

$$ind_{1,1}(v) \stackrel{def}{=} \frac{1}{4}\mathrm{tr}\,(\Omega\nu_0 - \Omega\nu_1) \in \mathbb{Z}.$$

Note that $ind_{1,1}(\hat{u}) = ind_{0,0}(u)$ for all $u \in KO^{0,0}$. In particular $\gamma_{1,1} = \hat{\gamma}_{0,0}$ will be a generator of $KO^{1,1}(pt)$ which will be called the *canonical generator.* More explicitly

$$\gamma_{1,1} = (\mathbb{R}^2, 1_{\mathbb{R}} \oplus (-1_{\mathbb{R}}), (-1_{\mathbb{R}}) \oplus 1_{\mathbb{R}}).$$

Using the topological periodicity isomorphism $t : KO^{1,1}(pt) \to K^{1,0}(I, \partial I)$ we can now explicitly describe a generator of $KO^{1,0}(I, \partial I)$ this is $\gamma_{1,0} = t(\gamma_{1,1})$ and is defined by the triple

$$\gamma_{1,0} \equiv (\mathbb{R}^2, R_0(\theta), R_1(\theta)), \quad \theta \in [0, \pi]$$

where $\mathbb{R}^2$ is equipped with the $(1,0)$ structure defined by

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the $(1,0)$-gradings $R_0(\theta)$, $R_1(\theta)$ are defined by

$$R_0(\theta) = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \quad R_1(\theta) = \begin{bmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}.$$

Set $\ell_0(\theta) = \ker(1 - R_0(\theta)$ and $\ell_1(\theta) = \ker(1 + R_1(\theta))$. $\ell_0(\theta)$ is the line spanned by the vector

$$v_0(\theta) = (sin\theta + \cos\theta)\mathbf{i} + (\cos\theta - \sin\theta)\mathbf{j}$$

while $\ell_1(\theta)$ is the line spanned by the vector

$$v_1(\theta) = (\sin\theta - \cos\theta)\mathbf{i} - (\sin\theta + \cos\theta)\mathbf{j}.$$

As θ runs from 0 to π the slope of $\ell_0(\theta)$ changes decreasingly from $\tan\pi/4$ to $-\tan\pi/4$ while the slope of $\ell_1(\theta)$ performs the opposite variation.

§2.3 Classifying spaces There are presently many proofs of Theorem 2.3. One approach is to reduce it to a topological result (the original Bott periodicity). This is done via classifying spaces.

To describe one such classifying space consider $M = C^{p,q+1}$. This is tautologically a $C^{p,q}$ module. Any $C^{p,q}$ supermodule is a summand of M^n for n large enough. In other words, $\{(M^n, \epsilon_{q+1})_{n\geq 0}$ is a cofinal family of $C^{p,q}$-supermodules. Let $Grad^{p,q}(n)$ denote the set of (p,q)-gradings of M^n and set

$$Grad^{p,q}(\infty) = \varinjlim Grad^{p,q}(n).$$

Then we have the following result (see [Ka1]).

Proposition 2.6 *$Grad^{p,q}(\infty)$ is a classifying space for $KO^{p,q}$.*

$Grad^{p,q}(\infty)$ can be alternatively described as a homogeneous space. Let $O^{p,q}(n)$ denote the group of orthogonal $C^{p,q}$-automorphisms of M^n and $O_0^{p,q}(n)$ the identity component. $O^{p,q+1}(n)$ is the subgroup of orthogonal $C^{p,q+1}$-automorphisms of M^n and $O_0^{p,q}$ its identity component. Finally note that ϵ_{q+1} is a grading for the $C^{p,q}$-module M^n. Denote by $Grad_0^{p,q}(n)$ the component of ϵ_{q+1} in $Grad^{p,q}(n)$. Any $g \in O_0^{p,q}(n)$ acts on $\eta \in Grad_0^{p,q}(n)$ by $\eta \mapsto g\eta g{-}1$. This action is transitive and the stabilizer of ϵ_{q+1} is clearly $O^{p,q+1}$. Thus

$$Grad_0^{p,q}(n) \cong O_0^{p,q}(n) \,/\, O^{p,q+1}(n).$$

Let

$$O_0^{p,q}(\infty) = \varinjlim O_0^{p,q}(n)$$
$$O^{p,q+1}(\infty) = \varinjlim O^{p,q+1}(n).$$

Then :

Proposition 2.7 *$A_{p-q} \times O_0^{p,q}(\infty) \,/\, O^{p,q+1}(\infty)$ is a classifying space for $KO^{p,q}$. Here A_{p-q} is endowed with the discrete topology.*

Let H be an infinite dimensional $C^{p,q+1}$ module. Denote by $O^{p,q}(H)$ the group of orthogonal operators commuting with the subjacent $C^{p,q}$ action. Similarly define $O^{p,q+1}(H)$. If $\mathcal{K}$ is the ideal of compact operators, denote by $\hat{O}_{\mathcal{K}}^{p,q}$ the identity component of $O^{p,q}(H) \cap (I + \mathcal{K})$. Similarly $O_{\mathcal{K}}^{p,q+1} = O^{p,q+1}(H) \cap (I + \mathcal{K})$.

The techniques of [P] apply and show that $\hat{O}_{\mathcal{K}}^{p,q}$ (resp $O_{\mathcal{K}}^{p,q+1}$) and $O_0^{p,q}(\infty)$ (resp $O^{p,q+1}(\infty)$) are homotopically equivalent. We thus have the following consequence.

Corollary 2.8 *$A_{p,q} \times \hat{O}_{\mathcal{K}}^{p,q}/O_{\mathcal{K}}^{p,q+1}$ is a classifying space for $K^{p,q}$.*

This classifying space reflects the topological character of K-theory. The functional analytical aspect is seen in the following homotopically equivalent description of the classifying space due to Karoubi (see [Ka2] and [KGLZ]; compare with [AS]).

Let H be an infinite dimensional $C^{p,q+2}$ module. Denote by $\mathcal{BF}^{p,q}$ the set of all bounded selfadjoint Fredholm operators D on H such that

$$De_i + e_iD = D\epsilon_j + \epsilon_jD = 0 \;\; \forall i = 1, \ldots, p \;,\;\; j = 1, \ldots, q+1.$$

Denote by $\mathcal{BF}_0^{p,q}$ the component of ϵ_{q+2} in $\mathcal{BF}^{p,q}$. In [Ka2] it is proved the following result.

Proposition 2.9 *$A_{p-q} \times \mathcal{BF}_0^{p,q}$ is a classifying space for $KO^{p,q}$.*

The previous result can be slightly reformulated as follows. Let H be a Hilbert $(p, q+1)$ module and define $\mathcal{BF}_{ess}^{p,q}$ as
$\mathcal{BF}_{ess}^{p,q} = \mathcal{BF}^{p,q}$ if $p - q \not\equiv 1 \pmod 4$.
$\mathcal{BF}_{ess}^{p,q} = \{T \in \mathcal{BF}^{p,q};\ TJ_1 \cdots J_pC_1 \cdots C_{q+1} \text{ is essential}\}$ if $p - q \equiv 1 \pmod 4$.
Notice that when $p - q \equiv 1 \pmod 4$ $TJ_1 \cdots J_pC_1 \cdots C_{q+1}$ is selfadjoint.

Example 2.10 Consider the case $p = 1$ and $q = 0$. Any $(1,1)$ module H_1 can be described as a direct sum of two copies of a Hilbert space H, $H_1 = H \times H$ and the Clifford action is given by

$$J_1 = \begin{bmatrix} 0 & 1_H \\ -1_H & 0 \end{bmatrix} \quad C_1 = \begin{bmatrix} 1_H & 0 \\ 0 & -1_H \end{bmatrix}$$

so that $T \in \mathcal{BF}^{1,0}$ iff T has a block decomposition

$$T = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}$$

where $D : H \to H$ is a Fredholm selfadjoint operator. Hence we can identify $\mathcal{BF}^{1,0}$ with the space of bounded, selfadjoint, Fredholm operators. A simple computation shows that

$$TJ_1C_1 = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix}$$

so that $T \in \mathcal{BF}^{p,q}_{ess}$ iff D is essential. In [AS] it is shown that the space of Fredholm selfadjoint operators has three connected components two of which are contractible and a third consisting precisely of the essential ones, which carries all the important homotopical properties.

For any $T \in \mathcal{BF}^{p,q}_{ess}$, the subspace $\ker T$ is naturally a finite dimensional (p, q) s-module so it defines an element in A_{p-q}. In this way we have an index map

$$\iota : \mathcal{BF}^{p,q}_{ess} \to A_{p-q}, \quad T \mapsto [\ker T] \in A_{p-q}.$$

The following result is proved in Proposition 5.1 of [AS] (see also our Proposition 3.13 to come).

Proposition 2.11 *(i) The map $\iota : \mathcal{BF}^{p,q}_{ess} \to A_{p-q}$ is continuous and moreover: $\iota(T_1T_2) = \iota(T_1) + \iota(T_2)$ so that induces a morphism $\iota_* : \pi_0(\mathcal{BF}^{p,q}_{ess}) \to A_{p,q}$.*
(ii) ι_* is a bijection and moreover all the connected components are homeomorphic to $\mathcal{BF}^{p,q}_0 = \iota^{-1}(0)$.

Propositions 2.9 and 2.11 immediately imply the following result.

Corollary 2.12 *$\mathcal{BF}^{p,q}_{ess}$ is a classifying space for $KO^{p,q}$.*

This generalizes the classical result of Atiyah and Janich ([A]) that the space of Fredholm operators is a classifying space for $KO^{0,0}$.

3 (p,q)-lagrangians and classifying spaces for K-theory

In this section we will present a new description of the classifying space for $KO^{p,q}$. This description is hidden in the definition of the group $KO^{p,q+1}((\check{\mathcal{H}}(X))$ of [KGLZ] Exposé III. However we adopt a different point of view which will emphasize some aspects not discussed there.

§3.1 (p,q)-lagrangian subspaces Let H be a Hilbert $C^{p,q}$-module (throughout this section all modules will be a Hilbert and all subspaces will be assumed closed).

$$\rho : C^{p,q} \to B(H)$$

Set as in Example 1.4 $J_i = \rho(e_i)$ and $C_j = \rho(\epsilon_j)$. A grading of (H, ρ) is then a selfadjoint involution R (reflection) such that

$$\{J_i, R\} = \{C_j, R\} = 0. \tag{3.1}$$

Set $L = \ker(I - R)$. If P is the orthogonal projection onto L then clearly

$$R = 2P - 1$$

and because of (3.1) the subspace L has the following properties

$$J_i L = C_j L = L_1^{\perp} \quad \forall i, j. \tag{3.2}$$

This justifies the following definition.

Definition 3.1 *A subspace L in a $C^{p,q}$-module is called a* $(\mathbf{p}, \mathbf{q}) - \mathbf{lagrangian}$ *if it satisfies condition (3.2).*

It is easily seen that if L is a (p,q)-lagrangian and P (resp. Q) is the orthogonal projection onto L (resp. $L^{\perp}$) then the associated reflection (through L) defined by $R = P - Q$ is a grading for the $C^{p,q}$-module. Hence there is a bijective correspondence between gradings and (p,q)-lagrangians. However, as we shall see in the sequel, the lagrangian description has a more natural occurrence in applications.

Example 3.2 Let H be a $C^{1,0}$-module. $C^{1,0} \cong \mathbb{C}$. Set $J = \rho(e_1)$. H can be interpreted as a Hilbert space with a complex structure given by J. Associated to J is the non-degenerate skew-symmetric bilinear form

$$\omega(x,y) = (Jx,y) \quad \forall x,y \in H.$$

(H,ω) becomes a symplectic vector space. The $(1,0)$-lagrangians are the maximal ω-isotropic subspaces (or lagrangians); see [N] for details).

Let H be an infinite dimensional $C^{p,q}$-module, L a (p,q)-lagrangian and R its associated reflection. L is called an **essential** lagrangian if
(i) either $p-q \not\equiv 0(mod\,4)$
(ii) or $p-q \equiv 0(mod\,4)$ and the $C^{p,q+1}$-structure it induces on H is essential.
Denote the set of essential (p,q)-lagrangians by $\mathcal{L}^{p,q}$.

Example 3.3 A (0,0)-lagrangian is a closed subspace L in H. It is essential iff both L and $L^{\perp}$ are infinite dimensional. In general, note that if $p-q \equiv 0(\mathrm{mod}\,4)$ then $\Omega = J_1 \cdots J_p C_1 \cdots C_q$ is an involution. If L is a (p,q)-lagrangian with associated reflection R then $R\Omega = \Omega R$ and L is essential iff both $\ker(R \pm \Omega)$ are infinite dimensional.

Remark 3.4 *Note that $\mathcal{L}^{p,q}$ is nonempty. Indeed let T be an isometry $T : H \to H \oplus H$. We transport the $C^{p,q}$-module structure from H to $H \oplus H$ as follows*

$$\overline{J_i} = \begin{pmatrix} 0 & TJ_iT^{-1} \\ TJ_iT^{-1} & 0 \end{pmatrix} \quad \overline{C_j} = \begin{pmatrix} 0 & TC_jT^{-1} \\ TC_jT^{-1} & 0 \end{pmatrix}$$

$\overline{J_i}$ and $\overline{C_j}$ define a $C^{p,q}$-module structure on $H \oplus H$ and T is an isomorphism of $C^{p,q}$-modules. Now let $\overline{R} \in B(H \oplus H)$ be the involution

$$\overline{R} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$R = T^{-1}\overline{R}T$ is a (p,q)-grading of H and $\ker(I-R) \in \mathcal{L}^{p,q}$.

$\mathcal{L}^{p,q}$ has a natural topology viewed as a space of operators. To describe its global structure we will need an extension of the results in [Ku] to $C^{p,q}$-modules. At this point the essentiality assumption plays a crucial role.

Lemma 3.5 *Let H be an* **essential** *Hilbert $C^{p,q}$-module and denote by $O^{p,q}(H)$ the group of orthogonal $C^{p,q}$-automorphisms of H. Then:*
(i) $H \cong C^{p,q} \otimes_{\mathbb{R}} H$ as $C^{p,q}$-modules. In particular any two **essential** *(p,q)-modules are isomorphic.*
(ii)$O^{p,q}(H)$ is path connected and contractible.

Proof The proof of (i) will be carried out in several steps.
Step 1 Let $\mathbf{F} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\rho : \mathbf{F} \to B(H)$ an infinite dimensional $\mathbf{F}$-module. Then

$$H \cong \mathbf{F} \otimes_{\mathbb{R}} H \quad \text{as } \mathbf{F}-\text{modules}.$$

We consider only the case $\mathbf{F} = \mathbb{H}$. The other situations are completely similar (and even simpler). Note first that $\mathbb{H} \cong C^{2,0}$. Let $L \subset H$ be a $(2,0)$ lagrangian i.e.

$$iL = jL = L^{\perp} \quad (i, j \in \mathbb{H}).$$

Since $k = i \cdot j$ we deduce that L is k-invariant and thus becomes a $C^{1,0}$-module ($k^2 = -1$). Choose L_1 an (1,0)-lagrangian in L. A simple computation shows that

$$H \cong \mathbf{H} \otimes_{\mathbb{R}} L_1 \quad \text{as } \mathbb{H}-\text{modules}.$$

On the other hand L_1 is infinite dimensional and thus isomorphic with H. Step 1 is completed.

Step 2 Consider

$$\rho : M_n(\mathbf{F}) \to B(H) \quad (\mathbf{F} = \mathbb{R},\ \mathbb{C},\ \mathbb{H})$$

an isomorphism of involutory unital Banach algebras where H is an infinite dimensional Hilbert space and the involution in $M_n(\mathbb{H})$ is induced by the obvious conjugation in $\mathbf{F}$ and the transposition of matrices. Then we have an isomorphism of $M_n(\mathbf{F})$-modules

$$H \cong M_n(\mathbf{F}) \otimes_{\mathbb{R}} H.$$

We follow the idea in the proof of Thm.III 4.4 of [Ka1]. For $i = 1, \dots, n$ consider the diagonal matrix

$$E_i = \operatorname{diag}(\delta_{1i}, \dots, \delta_{ni}).$$

where δ_{ij} is Kronecker's delta. Set $P_i = \rho(E_i)$. One sees that the P_i's are selfadjoint, pairwise commuting idempotents satisfying

$$\sum_i P_i = Id_H.$$

We define $H_i = \text{Range}\,(P_i)$. Since the projections P_i commute with the action of $\mathbf{F}$ each H_i becomes an $\mathbf{F}$-module. Clearly we have a direct sum decomposition

$$H \cong H_1 \oplus \dots \oplus H_n.$$

For $i \neq j$ consider the matrix E_{ij} with entries

$$x_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta} & , \text{ if } \{\alpha, \beta\} \not\subseteq \{i, j\} \\ 1 - \delta_{\alpha\beta} & , \text{ if } \{\alpha, \beta\} \subseteq \{i, j\} \end{cases}$$

and set $T_{ij} = \rho(E_{ij})$. Then $T_{ij}P_i = P_jT_{ij}$ and therefore T_{ij} defines an $\mathbf{F}$-isomorphism between H_i and H_j. In particular we deduce that the summands H_i are infinite dimensional and by Step 1 we have isomorphisms of $\mathbf{F}$-modules

$$H_i \cong \mathbf{F} \otimes_{\mathbb{R}} H_i \cong \mathbf{F} \otimes_{\mathbb{R}} H.$$

The decomposition

$$H \cong \bigoplus_1^n \mathbf{F} \otimes_{\mathbb{R}} H$$

realizes the desired isomorphism of $M_n(\mathbf{F})$-modules.

Step 3 Let $A = C^{3,0} \cong \mathbb{H} \oplus \mathbb{H}$ or $A = C^{0,1} \cong \mathbb{R} \oplus \mathbb{R}$ and consider H an **essential** A-module. Then we have an isomorphism of A-modules

$$H \cong A \otimes_{\mathbb{R}} H.$$

We consider only the first case. Then

$$H \cong H_0 \oplus H_1$$

where H_0, H_1 are $\mathbf{H}$-modules. Since H is essential both H_0 and H_1 are infinite dimensional so they are isomorphic to H as $\mathbb{H}$-modules. This proves the desired isomorphism of A-modules.

Step 4 If H is an essential $C^{p,q}$-module then

$$H \cong C^{p,q} \otimes_{\mathbb{R}} H \quad \text{as } C^{p,q} - \text{modules.} \tag{3.3}$$

The algebraic structure of the Clifford algebras $C^{p,q}$ is known (cf. [Ka1]). They can have one of the following forms

$$M_n(\mathbf{F})\,(\mathbf{F} = \mathbb{R},\ \mathbb{C},\ \mathbb{H}) \text{ or } M_n(\mathbf{F}) \oplus M_n(\mathbf{F}) \oplus M_n(\mathbf{F}),\ (\mathbf{F} = \mathbb{R},\ \mathbb{C}).$$

If $C^{p,q} \cong M_n(\mathbf{F})$ then (3.3) follows from Step 2. If $C^{p,q} \cong M_n(\mathbf{F}) \oplus M_n)\mathbf{F}) \oplus M_n(\mathbf{F})$ the isomorphism (3.3) follows combining Step 2 and 3. (i) is proved.

To prove (ii) note that the isomorphism (3.3) implies

$$O^{p,q}(H) \cong O^{p,q}(C^{p,q} \otimes H) \cong O(H)$$

while O(H) is path connected and contractible by Kuiper's theorem (cf. [Ku]) □

Lemma 3.5 implies the following result.

Corollary 3.6 *The space $\mathcal{L}^{p,q}(H)$ is path connected and contractible.*

Proof The group $O^{p,q}(H)$ acts on $\mathcal{L}^{p,q}(H)$ by

$$L \mapsto TL \ , \ L \in \mathcal{L}^{p,q} \ , \ T \in O^{p,q}(H).$$

If R_L is the reflection associated to L then

$$R_{TL} = TR_L T^*.$$

Given L_1 and L_2 two essential (p,q)-lagrangians they define two essential $C^{p,q+1}$ structures on H. By Lemma 3.5 any two such structures are isomorphic and thus there exists $T \in O^{p,q}$ such that $L_2 = TL_1$. Hence the action of $O^{p,q}$ is transitive. The stabilizer of this action is $O^{p,q+1}$ and therefore

$$\mathcal{L}^{p,q} \cong O^{p,q}/O^{p,q+1}.$$

It is routine to check we have a fibration

$$O^{p,q+1} \hookrightarrow O^{p,q} \rightarrow \mathcal{L}^{p,q}.$$

with contractible fiber and total space. In particular the base $\mathcal{L}^{p,q}$ is weakly contractible. Since by [Mi] $\mathcal{L}^{p,q}$ has the homotopy type of a CW-complex we deduce that it is in fact contractible. □

Definition 3.7 *Let L_1 and L_2 be two subspaces of infinite dimension and codimension in a Hilbert space H. The pair (L_1, L_2) is called* **Fredholm** *if*
(a) $L_1 + L_2$ is closed.
(b) $\dim(L_1 \cap L_2) + \operatorname{codim}(L_1 + L_2) < \infty$.
The integer

$$i(L_1, L_2) = \dim(L_1 \cap L_2) - \operatorname{codim}(L_1 + L_2)$$

is called the (Fredholm) **index** *of the pair.*

Set

$$\mathcal{FL}^{p,q} = \{(L_1, L_2) \in \mathcal{L}^{p,q} \times \mathcal{L}^{p,q} \ / \ (L_1, L_2) \text{ is Fredholm}\}.$$

Remark 3.8 *Let $\mathcal{K}$ denote the ideal of compact operators in H. The Fredholm condition for a pair (L_1, L_2) can alternatively be described as follows. Let P_j be the orthonormal projections onto L_j ($j = 1,2$). Then (L_1, L_2) is Fredholm if and only if one of the following conditions is fullfiled (cf. [C])*
(i) $P_1 + P_2$ and $P_1^\perp + P_2^\perp$ are Fredholm.
(ii)$P_1^\perp - P_2 \in \mathcal{K}$. Moreover if $(L_1, L_2) \in \mathcal{FL}^{p,q}$ with $p + q > 0$ (say $p \geq 1$) then we have

$$(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp = J_1(L_1 \cap L_2)$$

so that in this case

$$i(L_1, L_2) = 0. \tag{3.4}$$

Remark 3.9 *If L_1, L_2 are two (p,q)-lagrangians and R_1 resp. R_2 are the corresponding reflections then we deduce from Remark 3.8 that (L_1, L_2) is Fredholm iff one of the following holds:*
(i) $R_1 - R_2$ is Fredholm.
(ii)$R_1 + R_2 \in \mathcal{K}$.
Since $(R_1 + R_2)^2$ is compact and since $4I = (R_1 - R_2)^2 + (R_1 + R_2)^2$ we deduce that

$$D(R_1, R_2) = 1/4(R_1 - R_2)^2 \in I + \mathcal{K}.$$

§3.2 Graphs of linear operators and lagrangians It is very easy to construct (p, q) lagrangians. The following example illustrates a very important method of creating lagrangians. Most of the lagrangians we will encounter will be of this form, namely as graphs of operators with Clifford symmetries.

Example 3.10 Let (H, ρ) be a Hilbert $C^{p,q+1}$-module ($J_i = \rho(e_i)$, $C_j = \rho(\epsilon_j)$)

$$T : Dom\,(T) \subset H \to H$$

a closed, selfadjoint, densely defined operator satisfying
(a) $\ker T$ is finite dimensional.
(b) Range (T) is closed.
(c) $C^{p,q+1}(Dom\,(T)) \subset Dom\,(T)$
(d)$\{J_i, T\} = \{C_j, T\} = 0 \quad \forall i, j$.
Consider

$$\Gamma_T = \{(x, Tx) \in H \times H \; ; \; x \in Dom\,(T)\}$$

and

$$\Gamma_0 = H \times \{0\} \subset H \times H.$$

Both are closed subspaces of $H \times H$. Since $\ker T$ is finite dimensional $\Gamma_0 \cap \Gamma_T$ is finite dimensional. Since Range (T) is closed we deduce from standard results in functional analysis (see [K] Chap.IV) that $\Gamma_0 + \Gamma_T = H \times$ Range (T) is closed and has finite codimension. Thus (Γ_0, Γ_T) is a Fredholm pair of subspaces in $H \times H$. On the other hand $H \times H$ has a natural structure of $C^{p',q'}$-module ($p' = p + 1$, $q' = q + 1$) given by

$$\rho(e_i) = \begin{bmatrix} 0 & J_i \\ J_i & 0 \end{bmatrix} \; (1 \le i \le p) \; , \; \rho(\epsilon_j) = \begin{bmatrix} 0 & C_j \\ C_j & 0 \end{bmatrix} \; (1 \le j \le q')$$

and

$$\rho(e_{p'}) = \begin{bmatrix} 0 & 1_H \\ -1_H & 0 \end{bmatrix}.$$

The orthogonal complement of Γ_T in $H \times H$ is (cf. [K])

$$\Gamma_T^{\perp} = \{(-Tx, x) \in H \times H \; / \; x \in Dom\,(T)\}.$$

Now a simple computation shows that

$$\rho(e_i)\Gamma_T = \rho(\epsilon_j)\Gamma_T = \Gamma_T^{\perp}, \rho(e_i)\Gamma_0 = \rho(\epsilon_j)\Gamma_0^{\perp} \quad \forall\, 1 \le i \le p',\, 1 \le j \le q'.$$

Thus (Γ_0, Γ_T) is a Fredholm pair of (p', q')-lagrangians in $H \times H$.

The example above has a very important converse.

Example 3.11 Let (H, ρ) be a Hilbert $C^{p',q'}$-module and L_0, L two (p', q')-lagrangians. Denote by R_0 the reflection through L_0. Then $L_0 = \ker(I - R_0)$, $L_0^{\perp} = \ker(I + R_0)$. Assume that L is such that
(a) (L_0, L) is a Fredholm pair.
(b) $L \cap L_0^{\perp} = \{0\}$.
Then $L_0 \cong L_0^{\perp}$ since $J_{p'}L_0 = L_0^{\perp}$. Thus we can identify L_0 with $L_0^{\perp}$ via $-J_{p'}$. Then $H = L_0 \oplus L_0$ and we have the block decompositions

$$R_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ J = J_{p'} = \begin{bmatrix} 0 & 1_{L_0} \\ -1_{L_0} & 0 \end{bmatrix}.$$

Since J_i and C_j anticommute with R_0 and J we deduce

$$J_i = \begin{bmatrix} 0 & \tilde{J}_i \\ \tilde{J}_i & 0 \end{bmatrix} \ ,\ C_j = \begin{bmatrix} 0 & \tilde{C}_j \\ \tilde{C}_j & 0 \end{bmatrix} \tag{3.5}$$

where $\tilde{J}_i$, $\tilde{C}_j : L_0 \to L_0$ are orthogonal maps satisfying

$$-\tilde{J}_i^{\,2} = \tilde{C}_j^{\,2} = 1_{L_0}\ ,\ 1 \le i \le p,\ 1 \le j \le q'.$$

From the condition (b) we see that L can be represented as the graph of a (not necessarily continuous) linear map

$$T : Dom\,(T) \subset L_0 \oplus 0 \to 0 \oplus L_0$$

where $Dom\,(T)$ is the orthogonal projection of $L \subset L_0 \oplus L_0$ onto $L_0 \oplus 0$. One checks easily that

$$Dom\,(T) + L_0^{\perp} = L + L_0^{\perp} = C_1(L^{\perp} + L_0) \quad \text{in H}$$

Since $\overline{(L^{\perp} + L_0)} = (L \cap L_0^{\perp})^{\perp} = H$ and C_1 is an isomorphism we deduce that

$$\overline{Dom\,(T) + L_0^{\perp}} = H$$

i.e. $Dom\,(T)$ is dense in L_0. The pair (L, L_0) is Fredholm so that T is Fredholm i.e it has finite dimensional kernel and a closed finite codimensional range (cf. [K] Chap.IV). The condition that L is lagrangian translates into

$$T\tilde{J}_i + \tilde{J}_i T = T\tilde{C}_j + \tilde{C}_j T = 0 \quad \forall\, i, j$$

i.e. T is an operator as in the Example 3.10.

The ideas involved in Example 3.11 have interesting applications. Let L, L_0 be two transverse (p,q)-lagrangians $(p+q>0)$ i.e

$$L_0 \cap L = 0 \tag{3.6}$$

$$L_0 + L = H \tag{3.7}$$

From (3.6) and (3.7) we deduce that $\forall\, u \in L_0^{\perp}$ there exists a unique $v = V(u) \in L_0$ such that $u + V(u) \in L$. This can be rephrased by saying that L is the graph of a linear map $V : L_0^{\perp} \to L_0$ satisfying some anticommutativity relations that can be described as follows. Decompose as before $H = L_0 \oplus L_0^{\perp}$ and then use a block decomposition as in (3.5)). Then

$$J_i = \begin{bmatrix} 0 & \tilde{J}_i \\ -\tilde{J}_i^* & 0 \end{bmatrix},\ C_j = \begin{bmatrix} 0 & \tilde{C}_j \\ \tilde{C}_j^* & 0 \end{bmatrix}$$

where $\tilde{J}_i, \tilde{C}_j : L_0 \to L_0^{\perp}$ are unitary operators. Then since the graph of V is lagrangian the operator V must satisfy

$$V\tilde{J}_i - \tilde{J}_i^* V^* = V\tilde{C}_j + \tilde{C}_j^* V^*. \tag{3.8}$$

Because L is closed the map V has to be continuous (closed graph theorem). For $0 \le t \le 1$ set $V_t = (1-t)V$ and let Λ_t denote the graph of V_t. Obviously V_t satisfies (3.8) so that (L_0, Λ_t) is a pair of lagrangians satisfying (3.6-7), $\Lambda_0 = L$ and $\Lambda_1 = L_0^{\perp}$. Moreover since $t \mapsto V_t$ is a continuous family of bounded operators $t \mapsto \Lambda_t$ is a continuous family of lagrangians in the gap topology (see [K] Chap.IV). This shows that L is homotopic through lagrangians to $L_0^{\perp}$. In fact we can state a parameterized version of this fact.

Proposition 3.12 *Let X be a compact CW-complex and L_0 a (p,q)-lagrangian in a Hilbert $C^{p,q}$-module H $(p+q>0)$. Set*

$$\mathcal{L}_*^{p,q} = \{L \in \mathcal{L}^{p,q} \ /\ (L_0, L) \text{satisfy (3.6), 3.7}\}$$

and consider a continuous family

$$L : X \to \mathcal{L}_*^{p,q}\ ,\ x \mapsto L_x$$

Then the family L is homotopic in $\mathcal{L}_*^{p,q}$ to the constant family $x \mapsto L_0^{\perp}$.

We refer to Appendix C where we give a different proof this result using a more formal approach.

§3.3 The components of $\mathcal{FL}^{p,q}$ The space of Fredholm pairs of (p,q) lagrangians is in general disconnected. In this subsection we describe the components consisting of pairs of essential ones. The groups $A_{p,q}$ will play a key role.

Let $(L_1, L_2) \in \mathcal{FL}^{p,q}$ and denote by R_i the reflection through L_i. We can form a finite dimensional $C^{p,q}$-supermodule

$$I(L_1, L_2) = (L_1 \cap L_2) \oplus (L_1^\perp \cap L_2^\perp).$$

Its (p,q)-grading is $R_1 = R_2$ (restricted to $I(L_1, L_2)$). Via the Atiyah-Bott-Shapiro map α we obtain an element

$$\mu_{p,q}(L_1, L_2) = [I(L_1, L_2); R_1, -R_2] \in K^{p,q}(pt)$$

Proposition 3.13 *The map*

$$\mu_{p,q} : \mathcal{FL}^{p,q} \to KO^{p,q}(pt)$$

is locally constant and surjective. Moreover

$$\mu_{p,q}(L_1, L_2) = \mu_{p,q}(M_1, M_2)$$

if and only if (L_1, L_2) and (M_1, M_2) lie in the same connected component of $\mathcal{FL}^{p,q}$. Thus $\mu_{p,q}$ induces a bijection

$$ind_{p,q} : [pt, \mathcal{FL}^{p,q}] \to KO^{p,q}(pt).$$

(Compare with Proposition 5.1 of [AS] and Proposition 2.11).

Proof Let $(L_1, L_2) \in \mathcal{FL}^{p,q}$ and denote by R_i the corresponding reflections. Set

$$D = D(L_1, L_2) = 1/4(R_1 - R_2)^2.$$

D is Fredholm, selfadjoint, commutes with the $C^{p,q}$-action and with both R_1 and R_2. Moreover we can naturally identify $I(L_1, L_2)$ as a $C^{p,q}$-supermodule with $\ker D = \ker(R_1 - R_2)$. We will show that if $(L_1', L_2') \in \mathcal{FL}^{p,q}$ (with reflections R_1', R_2') is close to (L_1, L_2) then $\ker D'$ has the same image in $A_{p,q}$ as $\ker D$ (here $D' = D(L_1', L_2') = 1/4(R_1' - R_2')^2$).

By Remark 3.9 we have $D \in I + \mathcal{K}$ so 0 is an isolated point of the spectrum. Let $\varepsilon > 0$ such that

$$\operatorname{spectrum} D \cap [-\varepsilon, \varepsilon] = \{0\}.$$

Choose $\delta = \delta(\varepsilon)$ such that for any selfadjoint operator $T \in I + \mathcal{K}$ with $\|T - D\| < \delta$ we have

$$\pm\varepsilon \notin \operatorname{spectrum} T$$

Now consider a continuous path $(L_1(t), L_2(t)) \in \mathcal{FL}^{p,q}$, $t \in [0,1]$ such that $(L_1(0), L_2(0)) = (L_1, L_2)$ and

$$\|D_t - D\| < \delta \ , \ \forall t \in [0,1] \tag{3.9}$$

where $D_t = D(L_1(t), L_2(t))$. Let P_t denote the spectral projection of D_t corresponding to the spectral interval $[-\varepsilon, \varepsilon]$ and let E_t denote its range. D_t commutes with the $C^{p,q}$-action so E_t is a $C^{p,q}$-submodule. Also since

$$[D_t, R_1(t)] = [D_t, R_2(t)] = 0$$

we conclude that E_t is both $R_1(t)$ and $R_2(t)$-invariant. From (3.9) we deduce that P_t varies continuously with t so that $(E_t)_{t\in[0,1]}$ form a bundle of $C^{p,q}$-modules over $[0,1]$. As in Lemma III.4.21 of [Ka1] we deduce that the image of the triple $(E_t, R_1(t), -R_2(t))$ in $KO^{p,q}(pt)$ is independent of $t \in [0,1]$.

For notational simplicity we let $E' = E_1$, $L_1' = L_1(1)$, $L_2' = L_2(1)$ etc. Denote by U the orthogonal complement of $\ker D'$ in E'. We have the following equality in $KO^{p,q}(pt)$

$$(E'; R_1', -R_2') = (I(L_1', L_2'); R_1', -R_2') \oplus (U; R_1', -R_2') = (I(L_1, L_2), R_1, R_2).$$

We will show that $(U; R_1', -R_2') = 0$ in $KO^{p,q}(pt)$ by proving that $-R_1'$ and R_2' are homotopic as $C^{p,q}$-gradings when restricted to U. Let $T = 1/2(R_2' - R_1')$. Since $D' = T^2$ is invertible so is T. Set

$$T_s = -R_1' + s(T + R_1') = -R_1' + s/2(R_1' + R_2')$$

$$= -R_1'\{I - s\frac{1}{2}R_1'(R_1' + R_2')\}.$$

From $\|R_1'(R_1' + R_2')\| \leq 2$ we deduce that for all $s \in [0,1)$ the operator T_s is invertible. For $s = 1$ we have the equality $T_s = T$ which is invertible). Now set

$$R_s = T_s(T_s^2)^{-1/2} = T_s|T_s|^{-1}.$$

Then clearly $R_s^2 = I$ and since T_s anticommutes with the J_i's and the C_j's so does R_s. Thus R_s is a $C^{p,q}$-grading of U. In particular this shows that $-R_1'$ is homotopic as $C^{p,q}$-gradings with $T|T|^{-1}$. Similarly one shows that R_2' is homotopic with $T|T|^{-1}$ so that

$$(U; R_1', -R_2') = 0 \quad \text{in } KO^{p,q}(pt)$$

i.e.

$$(\ker D; R_1, -R_2) = (\ker D'; R_1', -R_2') \quad \text{in } KO^{p,q}(pt). \tag{3.10}$$

Hence the map $\mu_{p,q} : \mathcal{FL}^{p,q} \to KO^{p,q}(pt)$ is continuous.

Remark 3.14 *The same argument we used to prove (3.10) can be used to show that any triple $[M; \eta_0, \eta_1]$ representing some element in $KO^{p,q}(pt)$ is equivalent (in $KO^{p,q}(pt)$) to a triple of the form $[N; \eta, -\eta]$. We may take $N = \ker(\eta_0 + \eta_1)$ and $\eta = \eta_0 \mid_N$.*

To prove that $\mu_{p,q}$ is surjective consider $[M;\eta,-\eta]$ an element in $KO^{p,q}(pt)$. Let U be an infinite dimensional $C^{p,q}$-supermodule, $U = U_0 \oplus U_1$ with both U_0 and U_1 infinite dimensional. Denote by R the reflection through U_0 and then set $H = M \oplus U$. In Hwe have two (p,q)-gradings $R_0 = \eta \oplus -R$ and $R_1 = \eta \oplus R$. Then R_0, R_1 define a Fredholm pair of lagrangians (L_0, L_1) with $I(L_0, L_1) = M$.

A pair of gradings as above of the form $(\eta \oplus -R\,,\, \eta \oplus R)$ on a direct sum of $C^{p,q}$-modules $H = M \oplus U$ will be called **standard**. The injectivity of $\mu_{p,q}$ will be established in two steps.

Step 1 Every Fredholm pair is homotopic to a standard pair.

Let (R_0, R_1) be a Fredholm pair of gradings (i.e. $R_0 + R_1 \in \mathcal{K}$). Set

$$M = \ker(R_1 - R_0), \quad U = M^{\perp}\ , \quad \eta = R_0\,|_M\,.$$

As in the proof of (3.10) one shows that $R_0\,|_U$ is homotopic through $C^{p,q}$-gradings to $-R_1$. Set $R = R_1\,|_U$. Then we showed that (R_0, R_1) is homotopic (in $\mathcal{FL}^{p,q}$) to the standard pair $(\eta \oplus -R\,,\, \eta \oplus R)$.

Step 2 Any two standard pairs that have the same index are homotopic.

At this point the essentiality assumption is crucial. The proof is a routine exercise using the contractibility of the group $O^{p,q}$ established in Lemma 3.5. □

§3.4 The homotopy type of $\mathcal{FL}^{p,q}$ In the previous subsection we provided a first glance into the K-theoretic nature of the space of Fredholm pairs of lagrangians. We can now state and prove the main result of this section which completely describes the homotopy type of this space.

Theorem 3.15 *Let H be an essential Hilbert $C^{p,q}$-module. Then $\mathcal{FL}^{p,q}$ is a classifying space for $KO^{p,q}$.*

Proof In view of proposition 3.13 it suffices to show that any connected component of $\mathcal{FL}^{p,q}$ is homotopically equivalent to some component of $Grad^{p,q}(\infty)$. To this aim we will closely follow the method proposed in the proof of Thm2.2 of [N] so we will only sketch the main steps.

Henceforth we will identify lagrangians with their associated reflections. First note that we have a projection

$$\pi : \mathcal{FL}^{p,q} \to \mathcal{L}^{p,q} \quad (R_0, R_1) \mapsto R_0.$$

If $R \in \mathcal{L}^{p,q}$ then set

$$\mathcal{L}^{p,q}_R = \pi^{-1}(R) = (-R_0 + \mathcal{K}) \cap \mathcal{L}^{p,q}.$$

One can check easily that we have a fibration. The base is contractible so the total space will have the weak homotopy type of the fiber. According to [Mi] all the above spaces have the homotopy type of CW-complexes so the above weak homotopy is a genuine homotopy equivalence.

Step 1 $\hat{O}_{\mathcal{K}}^{p,q}$ (see Corollary 2.8) acts transitively on the connected components of $\mathcal{L}_R^{p,q}$ by

$$R_1 \mapsto TR_1T^{-1} \ , \ (R, R_1) \in \mathcal{FL}^{p,q}.$$

The proof is word by word the proof of Steps 2-4 in Thm 2.2 of [N1].

Step 2 Conclusion. The stabilizer of the above action is $O_{\mathcal{K}}^{p,q+1}$. Thus

$$\mathcal{L}_R^{p,q} \cong \frac{\hat{O}_{\mathcal{K}}^{p,q}}{O_{\mathcal{K}}^{p,q+1}}$$

The theorem follows using Corollary 2.8 and Proposition 3.13. □

Remark 3.16 *In the case $p = 0$, $q = 0$ this result is proved in [BW1]. The case $p = 1$, $q = 0$ is treated in [N1]. The general case also follows from the very abstract considerations in [KGLZ].*

4 Symplectic reductions

In the previous section we have seen that the space $\mathcal{FL}^{p,q}$ of Fredholm pairs of (p,q)-lagrangians classifies $KO^{p,q}$. In this section we address an effectivity question. Given $f : X \to \mathcal{FL}^{p,q}$ a continuous family of pairs of lagrangians describe the element it induces in $KO^{p,q}$ in a **finite dimensional** language.

§4.1 Generalized symplectic geometries All the results and techniques discussed in this section have their origin in a very simple observation which suggests that Clifford modules are in many respects similar to symplectic spaces. Namely on any Hilbert $C^{p,q}$ -module (H, ρ) (finite or infinite dimensional) one can naturally define a bilinear continuous map

$$\chi : H \times H \to (C^{p,q})^*$$

(where $(C^{p,q})^*$ is the dual of $C^{p,q}$ as a real vector space) defined by

$$\langle \chi(x,y), c\rangle = (\rho(c)x, y) \quad \forall\, x, y \in H\ ,\ c \in C^{p,q}.$$

Here $\langle \cdot,\cdot \rangle : (C^{p,q})^* \times C^{p,q} \to \mathbb{R}$ is the natural pairing and $(\cdot,\cdot)$ is the scalar product in H. Using the natural scalar product on $C^{p,q}$ we identify this Clifford algebra with is dual.

Example 4.1 Let (H, ρ) be a $C^{1,0}$-module. $C^{1,0} \cong \mathbb{C}$, H is a complex Hilbert space and $\chi : H \times H \to \mathbb{C}$ is then

$$\chi(x, y) = (x, y) + i(Jx, y)$$

so that χ is the usual Hermitian induced by the euclidian structure together with the complex structure. Similarly, a $C^{2,0}$ structure on an euclidian space is equivalent with a hyperkähler structure on that space.

Let $C^{p,q}_+$ (resp. $C^{p,q}_-$) denote the subspace of even (resp. odd) elements in $C^{p,q}$. Fix H a Hilbert $C^{p,q}$-module.

Definition 4.2 *Let $W \subset H$ be a closed subspace. Define the* **even** *(resp.* **odd***)* **annihilator** *of W by*

$$W_{+}^{0} \text{ (resp } W_{-}^{0}) = \{x \in H \; ; \; \chi(x,w) \in C_{+}^{p,q} \text{ (resp. } \chi(x,w) \in C_{-}^{p,q} \;\; \forall w \in W\}$$

A closed subspace $W \subset H$ is called **(p,q)-isotropic** *iff $W \subset W_{+}^{0}$ and $W^{\perp} \subset W_{-}^{0}$.*

Remark 4.3 *One can deduce easily from the definition that*

$$W_{+}^{0} = (C_{-}^{p,q}W)^{\perp} \; , \; W_{-}^{0} = (C_{+}^{p,q}W)^{\perp}. \tag{4.1}$$

Thus W is (p,q)-isotropic iff

$$C_{-}^{p,q}W \subset W^{\perp} \; , \; C_{+}^{p,q}W = W.$$

In this case the space $C_{-}^{p,q}W$ will play an important role in the future. It will be called the **dual** *of W and we will denote it by $W^{\#}$(Fig.1). Note that $L \subset H$ is a (p,q)-lagrangian iff L is maximal (p,q)-isotropic.*

Example 4.4 If H is a Hilbert $C^{1,0}$-module then it is naturally a symplectic space. A $(1,0)$-isotropic subspace is then an isotropic subspace in the usual sense of symplectic geometry. In this case the even annihilator of a closed subspace is the standard symplectic annihilator while the odd annihilator is $W^{\perp}$.

The importance of (p,q)-isotropic subspaces stems from the fact that they provide a very efficient method of producing (p,q)-lagrangians.

Definition 4.5 *Let L be a (p,q)-lagrangian and W a (p,q)-isotropic subspace. We say that* **L is clean mod W** *if $L \cap W = \{0\}$ and $L + W$ is closed.*

The next result describes the key technique we will use in this paper in dealing with indices of families of elliptic problems.

Proposition 4.6 Generalized symplectic reduction *Let (H,ρ) be Hilbert $C^{p,q}$-module $(p+q>0)$ and consider L a (p,q)-lagrangian which is clean mod a (p,q)-isotropic subspace W. Denote by H_W the orthogonal complement of W in W_{+}^{0} and by P the orthogonal projection onto H_W. Then:*
(i) H_W is a $C^{p,q}$-module ;
(ii) $L^{W} = P(L \cap W_{+}^{0}) = (L \cap W_{+}^{0})/W$ is a (p,q)-lagrangian in H_W.
H_W and L^W are called the **symplectic reductions** *of H (resp.L) mod W. (Fig. 1)*

In the proof we will need the following technical results.

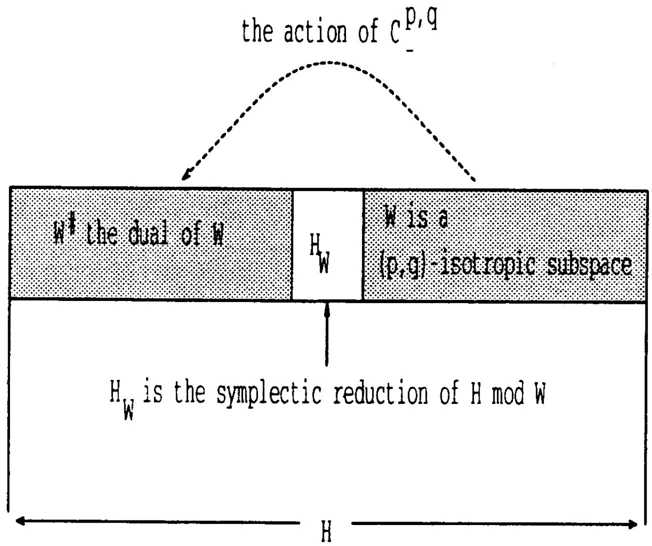

Figure 1: *A (p,q)-isotropic subspace*

Lemma 4.7 *Let H, W be as in Proposition 4.1 and let T denote generically one of the generators of the Clifford action on H i.e. $T = \rho(e_i)$ or $T = \rho(\epsilon_j)$ $(1 \leq i \leq p,\ 1 \leq j \leq q)$. Then*
(i) $W_+^0 = (TW)^\perp = W^\#$.
(ii) H has an orthogonal decomposition as

$$H = TW \oplus H_W \oplus W.$$

(iii) $L^W = (L + W) \cap H_W$.

Proof of Lemma 4.7 (i) Since $W_+^0 = (C_-^{p,q}W)^\perp$ it suffices to show that $TW = C_-^{p,q}W = W^\#$. Clearly $TW \subset C_-^{p,q}W$. Conversely given $S \in C_-^{p,q}$ we have $T^{-1}S \in C_+^{p,q}$ so that by Remark 4.3 $T^{-1}SW \subset W$ since W is (p,q)-isotropic. In particular we deduce $SW \subset TW$ which concludes the proof of (i).
(ii) Follows from (i) and the orthogonal direct sum $W_+^0 = W \oplus H_W$.
(iii) Clearly $L^W \subset (L + W) \cap H_W$. Conversely let $u \in (L + W) \cap H_W$. Thus there exists $x \in L$ and $w \in W$ such that $x + w \in H_W$ i.e. $x \in -u + H_W \subset W + H_W$ Hence $x \in W_+^0 \cap L$. Since $u = Px$ we deduce $u \in P(L \cap W_+^0)$. □

Proof of Proposition 4.6 (i) Let $u \in W_+^0 \cap W^\perp = H_W$. Then

$$Tu \in TW_+^0 \cap TW^\perp = TW_+^0 \cap (TW)^\perp \quad (T \text{ orthogonal}$$

$$= (\text{Lemma 4.7}) T(TW)^\perp \cap W_+^0 = W^\perp \cap W_+^0 = H_W$$

and thus H_W is a $C^{p,q}$-module.

(ii) We have to show that if $T = \rho(e_i)$ or $\rho(\epsilon_j)$ then $TL^W = (L^W)^\perp$. Since $L \cap W = \{0\}$ we deduce that $\forall u \in L^W$ there exists a **unique** $\overline{u} \in L \cap W_+^0$ such that $P\overline{u} = u$. Set $\overline{w} = \overline{u} - u$. Consider u_1, $u_2 \in L^W$. There exists as above elements $\overline{u}_i \in L \cap W_+^0$ such that $P\overline{u}_i = u_i$ $(i = 1,2)$. Then $T\overline{u}_1 \in L^\perp$ since L is lagrangian. Thus

$$(T\overline{u}_1, \overline{u}_2) = 0$$

so that

$$(Tu_1 + \overline{w}_1, u_2 + \overline{w}_2) = 0.$$

Using Lemma 4.7 (ii) we deduce $(Tu_1, u_2) = 0$. Since u_1, u_2 were chosen arbitrarily we conclude that

$$TL^W \subset (L^W)^\perp.$$

Conversely, let $v \in (L^W)^\perp \cap H_W$. This is equivalent to

$$v \in (L \cap W_+^0)^\perp \cap H_W.$$

Since $L + W$ is closed and T is an isomorphism we deduce that $TL + TW = L^\perp + (W_+^0)^\perp$ is closed. Therefore we have

$$v \in (L^\perp + (W_+^0)^\perp) \cap H_W = (TL + TW) \cap H_w = T(L + W) \cap H_W.$$

Since T is a bijection and $TH_W = H_W$ we deduce

$$T^{-1}v \in (L + W) \cap H_W = L_W \quad \text{(by Lemma 4.7 (iii))}$$

and hence $T^{-1}(L^W)^\perp \subset L^W$. Proposition 4.6 is proved. □

Definition 4.8 *A (p,q)-isotropic space W is called* **cofinite** *if the symplectic reduction H_W is finite dimensional.*

When W is cofinite there is an alternate way of viewing the process of symplectic reduction which is often very useful. Thus consider Λ a (p,q)-lagrangian which is clean mod W. Set $L = \Lambda^W$. Choose a lagrangian $L_0 \subset H_W$ such that $L \cap L_0^\perp = \{0\}$ (take for example $L_0 = L$) and consider $\Lambda_0 = L_0 + W^\#$. This is a (p,q)-lagrangian in H and because Λ is clean mod W we deduce

$$\Lambda \cap \Lambda_0^\perp = 0.$$

Since $\Lambda + W$ is closed and L_0 is finite dimensional (W is **cofinite**) we deduce that $\Lambda + \Lambda_0$ is closed. The assumption $p + q > 0$ implies in a standard fashion (Remark 3.8)

$$\Lambda + \Lambda_0^\perp = H.$$

Thus as in Proposition 3.12 we can represent Λ as the graph of a (bounded) linear operator

$$T : \Lambda_0 \to \Lambda_0^\perp.$$

Using the direct sum decompositions

$$\Lambda_0 = L_0 \oplus W^{\#} \ , \ \Lambda_0^{\perp} = L_0^{\perp} \oplus W$$

we can write T in a block form

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A : L_0 \to L_0^{\perp}$. Then L, as a subset of $L_0 \times L_0^{\perp}$ is the graph of the operator A. Now if Λ' is a lagrangian clean mod W which is close to Λ it stays transversal to $\Lambda_0^{\perp}$ and as before it is the graph of an operator $T' : \Lambda_0 \to \Lambda_0^{\perp}$ (close to T) which has the block decomposition

$$T' = \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix}.$$

In particular A' is close to A.

We can rephrase these in a more formal manner. Consider the space $\mathcal{L}_W^{p,q}$ of (p,q)-lagrangians which are clean mod W. The generalized symplectic reduction process defines a map

$$\mathcal{R}_W : \mathcal{L}_W^{p,q}(H) \to \mathcal{L}^{p,q}(H_W). \tag{4.2}$$

The above observations show that as in the usual symplectic case (see [GS]) we have

Proposition 4.9 *If W is cofinite then the reduction map $\mathcal{R}_W$ is continuous.*

§4.2 Homotopic properties of the symplectic reduction process The reduction map constructed above is also surjective and in fact it has a natural section (**extension map**)

$$\mathcal{E}_{W} : \mathcal{L}^{p,q}(H_W) \to \mathcal{L}_W^{p,q}(H) \tag{4.3}$$

defined by

$$\mathcal{E}_W \ : \ L \mapsto L \oplus W^{\#}.$$

Obviously $\mathcal{E}_W$ is continuous.

Proposition 4.10 *Let W be a cofinite (p,q)-isotropic subspace in a Hilbert $C^{p,q}$-module H, where $p+q>0$. Then the reduction map*

$$\mathcal{R}_W : \mathcal{L}_W^{p,q}(H) \to \mathcal{L}^{p,q}(H_W)$$

is a weak homotopy equivalence and $\mathcal{E}_W$ is a homotopy inverse. More precisely if X is a compact CW-complex and

$$\Lambda : X \to \mathcal{L}_W^{p,q}$$

is a continuous family of lagrangians clean mod W then the family

$$\{\Lambda(x)\ ;\ x \in X\}$$

is homotopic (inside $\mathcal{L}^{p,q}_W$) to the family

$$\{\Lambda^W(x) \oplus W^\# \; ; \; x \in X\}.$$

Proof Set $L(x) = \Lambda^W(x)$ and $\Lambda'(x) = \mathcal{E}_W(L(x)) = L(x) \oplus W^\#$. We want to use Proposition 3.12 to show that $\{\Lambda(x)\}$ is homotopic to $\{\Lambda'(x)\}$ Thus we have to check the following

$$\Lambda(x) \cap (\Lambda'(x))^\perp = 0 \quad \forall x \in X. \tag{4.4}$$

$$\Lambda(x) \oplus (\Lambda'(x))^\perp = H \quad \forall x \in X. \tag{4.5}$$

Proof of (4.4) Let $L^\perp(x)$ denote the orthogonal complement of $L(x)$ in H_W. Note that

$$(\Lambda'(x))^\perp = L^\perp(x) \oplus W.$$

Since $\Lambda(x)$ is clean mod W we deduce

$$\Lambda(x) \cap (L^\perp(x) + W) = 0$$

and (4.4) is proved.
Proof of (4.5) Since $\Lambda(x)$ is clean mod W the subspace $\Lambda(x) + W$ is closed. On the other hand, W is cofinite so that $L^\perp(x)$ is finite dimensional. These two observations imply that $\Lambda(x) + W + L^\perp(x)$ is closed. Using condition $p + q > 0$ as in Remark 3.8 we get (4.5). Proposition 4.10 follows from Proposition 3.12. □

Let $W_2 \subset W_1$ be two cofinite isotropic subspaces. Set $H_i = H_{W_i}$ $(i = 1, 2)$. Then obviously $\mathcal{L}^{p,q}_{W_1} \subset \mathcal{L}^{p,q}_{W_2}$ and $H_1 \subset H_2$. If we denote by W_{12} the orthogonal complement of W_2 in W_1 then W_{12} is a (p, q)-isotropic subspace in H_{W_2} and one can check that H_{W_1} is the reduction of H_{W_2} mod W_{12}. Let $\mathcal{L}^{p,q}_{W_{12}}$ denote the family of lagrangians in H_2 which are clean mod W_{12}. It is a routine exercise in linear algebra to show that the reduction mod W_2 of lagrangians clean mod W_1 produces lagrangians in in H_2 clean mod W_{12} and moreover the "tower" bellow is commutative.

$$\begin{tikzcd}
& \mathcal{L}^{p,q}_{W_{12}}(H_2) \arrow[dd, "\mathcal{R}_{W_{12}}"] \\
\mathcal{L}^{p,q}_{W_1}(H) \arrow[ur, "\mathcal{R}_{W_2}"] \arrow[dr, "\mathcal{R}_{W_1}"'] & \\
& \mathcal{L}^{p,q}(H_1)
\end{tikzcd}$$

Diagram 1: *The reduction maps $\mathcal{R}_\bullet$ are compatible with inclusions*

§4.3 The generalized Maslov index The result established above opens new avenues. It provides a simple way of deciding when two families of Fredholm pairs of lagrangians are homotopic by reducing the problem to a finite dimensional situation. In this subsection we will provide a concrete description of the element in K theory defined by such a continuous family. We named this description *the generalized Maslov index* for reasons that will be explained later in this section.

Definition 4.11 *Let Λ be an essential (p,q)-lagrangian in an essential Hilbert $C^{p,q}$-module H. A* **filtration** *of Λ is a family $\mathcal{F}$ of cofinite isotropic subspaces $W \subset \Lambda$ such that*
(a) $\Lambda \in \mathcal{F}$.
(b) $\forall W_1, W_2 \in \mathcal{F}$ there exists $W_3 \in \mathcal{F}$ such that $W_3 \subset W_1 \cap W_2$.
(c)

$$\bigcap_{W \in \mathcal{F}} W = 0.$$

(d) the family $(H_W)_{W\in\mathcal{F}}$ is **cofinal** *i.e. any finite dimensional $C^{p,q}$-module is a direct summand of H_W.*

A lagrangian with a distinguished filtration will be called **filtered**.

Remark 4.12 *Any essential (p,q)-lagrangian admits a filtration. To show this note first that given H_i (i=1,2) two essential $C^{p,q}$-modules and $\Lambda_i \subset H_i$ essential lagrangians then from Corollary 3.6 we deduce that there exists a $C^{p,q}$-isomorphism $T : H_1 \to H_2$ such that $T(\Lambda_1) = \Lambda_2$. Thus the problem reduces to constructing an example of filtered lagrangian.*

Let $(M_n)_{n\geq 1}$ be a sequence of $C^{p,q}$-supermodules such that the sequence of $C^{p,q}$-modules

$$H_k = \bigoplus_{n=1}^{k} M_n$$

is cofinal in the sense of [Ka1]. Denote the grading of M_n by η_n and set

$$L_n = \ker(1 - \eta_n).$$

Now form the Hilbert sum

$$H = \bigoplus_{n=1}^{\infty} M_n.$$

Then H is an essential $C^{p,q}$-module and

$$\Lambda = \bigoplus_{n=1}^{\infty} L_n$$

is an essential (p,q)-lagrangian. Set

$$W_k = \bigoplus_{n>k} L_n \quad k \geq 0.$$

W_k is a cofinite (p,q)-isotropic subspace and in fact

$$H_k = H_{W_k}.$$

The family $(W_k)_{k\geq 0}$ is a filtration of Λ.

Let $\Lambda_0 \subset H$ be a filtered (p,q)-lagrangian and denote its filtration by $\mathcal{F}$. Consider

$$\mathcal{F}\mathcal{L}_0^{p,q} = \{\Lambda \in \mathcal{L}^{p,q} \ / \ (\Lambda_0, \Lambda) \in \mathcal{F}\mathcal{L}^{p,q}\}.$$

We have seen in the proof of Theorem 3.15 that $\mathcal{F}\mathcal{L}_0^{p,q}$ is a classifying space for $KO^{p,q}$. Thus to any continuous map $\Lambda : X \to \mathcal{F}\mathcal{L}_0^{p,q}$ (X- compact CW-complex) there corresponds an element in $KO^{p,q}(X)$ which we provisorily denote by $ind_{p,q}(\Lambda_0, \Lambda)$.

Recall that any element u of $KO^{p,q}(X)$ has a (nonunique) standard grading representation as a triple $[E; \eta_0, \eta_1]$ where E is a **finite dimensional** $C^{p,q}$ -bundle over X and η_0, η_1 are (p,q)-gradings of E. We can then form the lagrangian subbundles

$$L_0 = \ker(1 - \eta_0) \ , \ \ L_1 = \ker(1 + \eta_1)$$

so that η_0 is the reflection through L_0 and $-\eta_1$ is the reflection through L_1. The triple $(E; L_0, L_1)$ will be referred to as a **standard lagrangian representation** of $u \in KO^{p,q}(X)$ or simply **lagrangian representation**. Note that any triple of the form $(E; L, L^{\#})$ represents the trivial element in $KO^{p,q}(X)$. We will look for a standard representation of the element $ind_{p,q}(\Lambda_0, \Lambda)$ introduced above.

We first need the following technical result.

Lemma 4.13 *Let X be a compact CW-complex and*

$$\Lambda : X \to \mathcal{F}\mathcal{L}_0^{p,q} \quad x \mapsto \Lambda_x$$

a continuous map. Then there exists $W \in \mathcal{F}$ such that $\Lambda_x$is clean mod W $\forall x \in X$.

Proof For any $x \in X$ and $W \in \mathcal{F}$ the intersection $\Lambda_x \cap W$ is finite dimensional since (Λ_0, Λ_x) is Fredholm and $W \subset \Lambda_0$. Obviously

$$\bigcap_{W \in \mathcal{F}} (\Lambda_x \cap W) = 0$$

so that $(\Lambda_x \cap W)$ is a filtered family of finite dimensional vector spaces. Hence there exists $W_x \in \mathcal{F}$ such that $\Lambda_x \cap W_x = 0$. Since W_x has finite codimension in Λ_0 the pair (Λ_x, W_x) is in fact Fredholm and in particular $\Lambda_x + W_x$ is closed. It is now easy to show that there exists a neighborhood U_x of x in X such that Λ_y is clean mod W_x for all $y \in U_x$. Now cover X with finitely many such neighborhoods

$$X = U_{X_1} \cup \cdots \cup U_{x_m}$$

and choose $W \in \mathcal{F}$ so that

$$W \subset W_{x_1} \cap \cdots \cap W_{x_m}.$$

Then Λ_x is clean mod W for all $x \in X$. □

For each $W \in \mathcal{F}$ let M_W denote the orthogonal complement of W in Λ_0. M_W is a (p,q)-lagrangian in H_W and in fact is the symplectic reduction of Λ_0 mod $W^\#$.

Denote by $\Lambda : X \to \mathcal{FL}_0^{p,q}$ a continuous family of lagrangians and choose W as in Lemma 4.13 The reduction mod W of the family Λ is a continuous family $x \mapsto L_x^W$ of lagrangians in H_W and thus we obtained a triple $(H_W; M_W, (L_x^W))$ which is a lagrangian representation of some element in $KO^{p,q}(X)$. We claim that this element is independent of the choice of W. Indeed, a standard filtration argument shows that it suffices to consider only the case when $W_2 \subset W_1$ are two isotropic spaces in $\mathcal{F}$ such that the family Λ is clean mod both W_1 and W_2. Set

$$H_i = H_{W_i}, \; L_i = L^{W_i}, \; M_i = M_{W_i} \; (i = 1, 2)$$

and $U = W_{12}$ the orthogonal complement of W_2 in W_1. U is a finite dimensional isotropic subspace in H_2. Then as in Diagram 1

$$L_1 = \mathcal{R}_U L_2 \; , \; M_1 = \mathcal{R}_{U^\#} M_1. \tag{4.6}$$

Set $H_0 = U \oplus U^\#$. From Proposition 4.10 and (4.6) we deduce that L_2 is homotopic to $L_1 \oplus U^\#$ so that we have an equality

$$(H_2; M_2, L_2) = (H_1; M_1, L_1) \oplus (H_0; U, U^\#) \quad \text{in } KO^{p,q}(X)$$

so that $(H_2; M_2, L_2) = (H_1; M_1, L_1)$ in $KO^{p,q}(X)$.

If $\mathcal{N}_0, \mathcal{N}_1 : X \to \mathcal{FL}_0^{p,q}$ are two homotopic families related by a homotopy

$$\mathcal{N} : X \times I \to \mathcal{FL}_0^{p,q} \quad (x,t) \mapsto \mathcal{N}_t(x)$$

then we can pick $W \in \mathcal{F}$ such that $\mathcal{N}_t(x)$ is clean mod W for all x and t. Correspondingly we get an element $(H_W; M_w, \mathcal{N}^W) \in KO^{p,q}(X \times I)$ which restricts to $(H_W, M_W, \mathcal{N}_i^W)$ over the slice $X \times \{i\}$, $i = 0, 1$. We conclude that the symplectic reduction process induces a well defined map

$$\mu_{p,q} : [X, \mathcal{FL}_0^{p,q}] \to KO^{p,q}(X) \quad \Lambda \mapsto \mu_{p,q}(\Lambda_0, \Lambda)$$

which we will call the **generalized Maslov index.**

We can now state the main result of this section which generalizes Proposition 3.13

Theorem 4.14 *The generalized Maslov index*

$$\mu_{p,q} : [X, \mathcal{FL}_0^{p,q}] \to KO^{p,q}(X)$$

is a bijection for any compact CW-complex X.

Proof Surjectivity is almost immediate. Indeed, since $(H_W)_{W\in\mathcal{F}}$ is a cofinal family of $C^{p,q}$-supermodules any element u in $KO^{p,q}(X)$ can be represented as a triple $(H_M; M_W, L)$ where L is a lagrangian subbundle of H_W (see [Ka1]). Now set

$$\Lambda_x = \mathcal{E}_W(L_x) = L_x \oplus W^\#.$$

Then the family $(\Lambda_x)_{x\in X}$ lies in $\mathcal{FL}_0^{p,q}$ and obviously

$$\mu_{p,q}(\Lambda_0, \Lambda) = (H_M; M_W, L).$$

To prove the injectivity consider

$$\Lambda_1\,,\,\Lambda_2 : X \to \mathcal{FL}_0^{p,q}$$

and $W_1\,,\,W_2 \in \mathcal{F}$ such that Λ_i is clean mod W_i $(i = 1, 2)$ and

$$(H_1; M_1, L_1) = (H_2; M_2, L_2) \quad \text{in } KO^{p,q}(X) \tag{4.7}$$

where as before $M_i = M_{W_i}$, $H_i = H_{W_i}$ etc. We may as well assume that $W_2 \subset W_1$ so that $H_1 \subset H_2$. Again since the family $\mathcal{F}$ is cofinal we deduce from (4.7) (see [Ka1] Prop. III.4.26) that there exists $W \in \mathcal{F}$, $W \subset W_2$ and lagrangian $N_i = W_i/W$ in $H_i' = H_W/H_i$ such that

$$L_1 \oplus N_1^\# \cong L_2 \oplus N_2^\# \quad (\text{" } \cong \text{ " : homotopic}) \tag{4.8}$$

and

$$M_1 \oplus N_1 = M_2 \oplus N_2.$$

From (4.8) we deduce immediately that

$$\mathcal{E}_{W_1}(L_1) = \mathcal{E}_W(L_1 \oplus N_1^\#) \cong \mathcal{E}_W(L_2 \oplus N_2^\#) = \mathcal{E}_{W_2}(L_2)$$

so that form Proposition 4.10 we deduce that $\Lambda_1 \cong \Lambda_2$ This shows the map $\mu_{p,q}$ is injective and thus Theorem 4.14 is proved. □

Remark 4.15 *Theorem 4.14 can be slightly extended as follows. Consider* $(\Lambda_0, \Lambda_1) : X \to \mathcal{FL}^{p,q}$ *and assume* $W_x \subset \Lambda_0(x)$ *is a continuous family of cofinite isotropic subspaces such that*
(a) (H_{W_x}) *defines a bundle of* $p,q)$*-modules over* X.
(b) $\Lambda_1(x)$ *is clean mod* W_x *for all* $x \in X$.

Let $L_0^W(x)$ *resp.* $L_1^W(x)$ *denote the reduction of* $\Lambda_0(x)$ *(resp.* $\Lambda_1(x)$*) mod* $W^\#$ *(resp. mod* W*). We thus get an element* $(H_W; L_1^W, L_2^W) \in KO^{p,q}(X)$ *which as before can be shown to be independent of the various choices and coincides with the generalized Maslov index constructed above. Finally to describe the relative* $KO^{p,q}$*-groups we consider the family*

$$\mathcal{FL}_*^{p,q} = \{\Lambda_1 \in \mathcal{FL}_0^{p,q} \;/\; \Lambda_0 \cap \Lambda_1 = 0\}.$$

Then

$$KO^{p,q}(X, Y) \cong [(X, Y); (\mathcal{FL}^{p,q}, \mathcal{FL}_*^{p,q})]$$

and an isomorphism can be explicitly described as in Theorem 4.14.

§4.4 Comparison with the traditional Maslov index In this very brief subsection we explain the motivation behind our terminology. We consider the case $P = 1$, $q = 0$. As we have pointed out in this case the generalized symplectic geometry coincides with what is usually know as symplectic geometry.

In the traditional case one can associate to each path of lagrangians

$$(L_0(t), L_1(t)), \quad t \in [a, b]$$

in a symplectic space H (such that the extremities are transversal) an integer $\mu(L_0(t), L_1(t)$ called the Maslov index (we refer to [CLM1] or [N1] for details). Its homotopic properties immediately imply that it defines an isomorphism

$$\mu : KO^{1,0}(I, \partial I) \to \mathbb{Z}.$$

On the other hand using the lagrangian representations of the elements in K-theory we can define another isomorphism

$$\mu_{1,0} : KO^{1,0}(I, \partial I) \to \mathbb{Z}$$

defined by

$$\mu_{1,0}(\gamma_{1,0}) = 1$$

where $\gamma_{1,0}$ is the canonical generator defined in Subsection 2.2. We claim that $\mu_{1,0} \equiv \mu$.

To verify this claim it suffices to show that

$$\mu(\gamma_{1,0}) = 1.$$

If $(L_0(t), L_1(t))$ is a paths of lines in $\mathbb{R}^2$ equipped with the symplectic structure

$$\omega(u, v) = (Ju, v), \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(Note that $\omega = dy \wedge dx$ rather than the traditional $dx \wedge dy$.) Then its usual Maslov index can be computed as follows (see [CLM1] or [N1]). First decompose the path into little parts so that inside these subintervals neither of the lines is vertical and at extremities the lines intersect transversally. To compute the Maslov index carried by each of the above subintervals denote by $m_i(t)$ the slope of $L_i(t)$, $i = 0, 1$ and form the paths $\Gamma_i(t) = (t, m_i(t))$. Then the local Maslov indices are given by the intersection numbers $\#(\Gamma_0(t) \cap \Gamma_1(t))$. Using the explicit description of $\gamma_{1,0}$ given in Subsection 2.2 it is now very easy to check that its (traditional) Maslov index is 1. Hence the generalized Maslov index coincides with the traditional one.

5 Clifford symmetric Fredholm operators

We are primarily concerned with the indices of families of Fredholm operators as in Proposition 2.9. This section parallels Section 4 in that it addresses a related effectivity question: describe in finite dimensional terms the index of a family of Fredholm operators (**possibly unbounded !**). We adopt a new point of view and we feel it is one of the important contributions of this paper. Namely we realized it is more convenient to think of operators in terms of their graphs. In this way they naturally become lagrangians and the techniques developed in Section 4 will provide a satisfactory answer to the effectivity question above.

§5.1 The space $\mathcal{F}^{p,q}$ Let H be an essential infinite dimensional Hilbert $C^{p,q+1}$-module module. For brevity we will write q' (resp. p') instead of $q+1$ (resp $p+1$). Following Examples 3.10 and 3.11 consider $\mathcal{F}^{p,q}$ the set of densely defined, closed, selfadjoint Fredholm operators

$$T : D(T) \subset H \to H$$

satisfying :
(i) $C^{p,q'}(D(T) = D(T)$.
(ii)$\{T, J_i\} = \{T, C_j\} = 0 \ , \ \forall 1 \leq i \leq p \ 1 \leq j \leq q'$.

If we think of H as a (p, q) s-module with grading given by $C_{q'}$ then the conditions (i) and (ii) above can be rephrased by saying that D is an odd degree linear map supercommuting with the action of $C^{p,q}$ and thus it (almost) defines an element in a Kasparov KK group (see [B]). We will not pursuit this point of view in the present paper but we believe it is deserves some attention.

The set $\mathcal{F}^{p,q}$ can be topologized using the gap topology (Appendix A). $\xrightarrow{g}$ will denote the gap convergence. As in Proposition 2.11 we select a subset of $\overline{\mathcal{F}}$ which we call $\mathcal{F}^{p,q}_{ess}$ defined as follows:
(i) if $(p-q) \not\equiv -1 \pmod 4$ then $\mathcal{F}^{p,q}_{ess} = \mathcal{F}$.
(ii) if $(p-q) \equiv -1 \pmod 4$

$$\mathcal{F}^{p,q}_{ess} = \{T \in \mathcal{F}^{p,q};\ T(1+T^2)^{-1/2} \in \mathcal{B}\mathcal{F}^{p,q}_{ess}\}$$

Proposition 5.1 *The space $\mathcal{F}^{p,q}$ is homotopically equivalent with the space $\mathcal{B}\mathcal{F}^{p,q}$.*

Proof : For $0 < s \leq 1/2$ consider $w_s : \mathbb{R} \to \mathbb{R}$ defined by

$$w_s(\lambda) = \begin{cases} \lambda & \text{if} \quad |\lambda| \leq (1-s)/s \\ (1-s)/s & \text{if} \quad \lambda > (1-s)/s \\ -(1-s)/s & \text{if} \quad \lambda < -(1-s)/s \end{cases}$$

and set $w_0(\lambda) \equiv \lambda$. Using the functional calculus for selfadjoint operators we get for each a map

$$\Psi_s : \mathcal{F}_0^{p,q} \to \mathcal{BF}_0^{p,q}$$

by $T \mapsto w_s(T)$ (since w_s is odd and bounded $w_s(T)$ is in $\mathcal{BF}^{p,q}$). It is easily seen that if $T_n \xrightarrow{g} T$ in $\mathcal{F}^{p,q}$ (or equivalently in the norm resolvent sense of [RS]) then $\Psi_s(T_n) \xrightarrow{g} \Psi_s(T)$ and since $\Psi_s(T_n)$ and their gap limit are bounded operators we deduce by the results of [K] that $\Psi_s(T_n) \to \Psi_s(T)$ in norm and thus the map Ψ_s is continuous. It is a simple exercise in functional analysis to show that if $0 < s_0 \leq 1/2$ and and $s_n \to s_0$ then $\Psi_{s_n}(T) \to \Psi_{s_0}(T)$ in norm, uniformly for T in compacts of $\mathcal{F}^{p,q}$. The continuity of the family Ψ_s at $s = 0$ follows from the uniform gap estimate (see Lemma A.2)

$$\delta(T, \Psi_s(T)) \leq 2^{1/2}s.$$

The map $\Psi_{1/2}$ is a weak homotopy equivalence between $\mathcal{F}_0^{p,q}$ and $\mathcal{BF}_0^{p,q}$. Proposition 5.1 follows from the results in [Mi]. □

Denote by $\mathcal{F}_0^{p,q}$ the connected component of $\mathcal{F}^{p,q}$ containing the the essential $(p, q+1)$-gradings. From the propositions 2.9, 2.11 and 5.1 we deduce that

$$\mathcal{F}_{ess}^{p,q} \cong A_{p-q} \times \mathcal{F}_0^{p,q}$$

and all invertible operators in $\mathcal{F}_{ess}^{p,q}$ lie in $\mathcal{F}_0 p, q$.

§**5.2 Hamiltonian (p,q)-modules** This brief subsection is very technical in content . It describes a few simple methods of detecting *essential* objects (like e.g. lagrangians, Fredholm operators etc.)

Definition 5.2 *A* **Hamiltonian** *(p,q) module is a pair (H,J) where H is a Hilbert (p,q)-module and J is a skewadjoint operator satisfying $J^2 = -1$ and anticommuting with the standard generators of the $C^{p,q}$ action. Notice that J induces a (p+1,q) structure on H. We will call J the* **suspension** *of the hamilton structure.*
(b) A Hamiltonian (p,q) module is called **good** *if its essential both as a (p,q)-module and as a (p+1,q)-module.*
(c) Let (H, J) be a Hamiltonian (p,q)-module. A (p,q)-lagrangian subspace L in H is called **good** *if $JL = L^{\perp}$ (i.e. L is also a $(p+1,q)$ lagrangian) and L is both (p,q) and (p+1,q) essential.*

Let H be a (p,q')-module, $q' = q+1$. $H \times H$ becomes a Hamiltonian $C^{p,q'}$-module with the Clifford action given by

$$e_i \mapsto \begin{bmatrix} 0 & J_i \\ J_i & 0 \end{bmatrix} \; 1 \leq i \leq p \;,\; \epsilon_j \mapsto \begin{bmatrix} 0 & C_j \\ C_j & 0 \end{bmatrix} \; 1 \leq j \leq q'$$

and suspension

$$J = \begin{bmatrix} 0 & 1_H \\ -1_H & 0 \end{bmatrix}.$$

We let the reader check that $(H \times H, J)$ is a good hamiltonian space iff H is $(p, q+1)$ essential.

Given H an essential, infinite dimensional (p,q')-module and $T \in \mathcal{F}^{p,q}$ we can form the graph of T

$$\Gamma_T = \{(x, Tx) \in H \times H \; / \; x \in D(T)\}$$

and

$$\Gamma_0 = \{(x, 0) \in H \times H \; / \; x \in H\}.$$

Both Γ_0 and Γ_T are (p', q')-lagrangians and (Γ_0, Γ_T) is a Fredholm pair. Tautologically, the map $T \mapsto \Gamma_T$ is continuous.

Definition 5.3 *An operator $T \in \mathcal{F}^{p,q}$ is called* **good** *if Γ_T is a good lagrangian in the Hamiltonian space $(H \times H, J)$ constructed above.*

Lemma 5.4 *If H is an infinite dimensional, essential (p,q+1) module then any operator $T \in \mathcal{F}_0^{p,q}(H)$ is good.*

Proof Notice first that the subspace of good operators in $\mathcal{F}_0^{p,q}$ is connected. Indeed it is open since essentiality is an open condition. To see it is closed consider $T_n \xrightarrow{g} T$ in $\mathcal{F}_0^{p,q}$ with T_n good. Denote the reflection associated to Γ_{T_n} by R_n and the reflection associated to Γ_T. Then $R_n \to R$ and we have to show that if R_n are both (p,q+1) and (p+1,q+1) essential then so is R. This boils down to showing that the limit of a convergent sequence of essential involutions is also essential. This is an obvious fact if we work "downstairs" in the Calkin algebra (= Bounded operators modulo the compact ones). Thus, to complete the proof we have to show that there exists good operators in $\mathcal{F}_0^{p,q}$. Choose R an essential $(p, q+1)$-grading. Then the reflection C associated to the graph of R is

$$C = \begin{bmatrix} 0 & R \\ R & 0 \end{bmatrix}.$$

It is a simple exercise in linear algebra to show the following equivalences.
(i) R is $(p, q+1)$ essential $\Longleftrightarrow$ C is $(p, q+1)$ essential.
(ii) H is $(p, q+1)$ essential $\Longleftrightarrow$ C is $(p+1, q+1)$ essential.
We leave the details to the reader. □

§5.3 The graph map and generalized Floer operators Let $\mathcal{FL}_0^{p',q'}$ denote the connected component of the pair $(\Gamma_0, \Gamma_0^\#)$ in

$$\{\Lambda \in \mathcal{L}^{p',q'} \ / \ (\Gamma_0, \Lambda) \in \mathcal{FL}^{p',q'}\}.$$

Notice that $\Gamma_0^\# = \Gamma_0^\perp$ is a good lagrangian so that all lagrangians in $\mathcal{FL}_0^{p',q'}$ are good. We get a continuous map

$$\Gamma \ = \ \Gamma^{p,q} : \mathcal{F}_0^{p,q} \to \mathcal{FL}_0^{p',q'}$$

defined by

$$\Gamma \ : \ T \mapsto \Gamma_T.$$

The main result of this section is the following.

Theorem 5.5 *The map $\Gamma^{p,q}$ is a weak homotopy equivalence so that $\mathcal{F}_0^{p,q}$ is a classifying space for the reduced functors $\tilde{KO}^{p',q'}(\bullet) \cong \widetilde{KO}^{p,q}(\bullet) \cong KO^{p,q}(\bullet, pt)$.*

Proof We have to show that for any compact, connected CW complex X the induced map

$$[X, \mathcal{F}^{p,q}] \to [X, \mathcal{FL}_0^{p',q'}]$$

is a bijection and in fact it suffices to consider only the case when X is some sphere $X \cong S^n$.

Step 1: Surjectivity Since $\mathcal{FL}_0^{p',q'}(H \times H)$ is a classifying space for $\tilde{KO}^{p',q'}$ it suffices that for any $u \in \tilde{KO}^{p',q'}$ with a standard representation $(E; L_0, L_1)$ there exists a continuous map

$$T : X \to \mathcal{F}^{p,q}$$

such that $(E; L_0, L_1)$ is a symplectic reduction of the family $(\Gamma_0, \Gamma_{T_x})_{x\in X}$ i.e.

$$\mu_{p',q'}\left((\Gamma_0, \Gamma_{T_x})_{x\in X}\right) \ = \ (E; L_0, L_1) \quad \text{in } KO^{p',q'}.$$

The novelty of our proof consists in the manner in which we construct the family $(T_x)_{x\in X}$. They will be very natural differential operators associated to the element u.

First, we may assume that $E \cong F \times X$ where F is a finite dimensional Hilbert $C^{p',q'}$-module and L_0 is independent of x. Then $(L_1(x))_{x\in X}$ can be viewed as a continuous family of lagrangians in F.

We will identify $L_0 \oplus L_0$ with F via the correspondence

$$L_0 \oplus L_0 \to F, \quad (u, v) \mapsto u - Jv, \quad J = J_{p'}.$$

J can be rewritten as

$$J \ = \ \begin{bmatrix} 0 & 1_{L_0} \\ -1_{L_0} & 0 \end{bmatrix}. \tag{5.1}$$

It will be convenient to think F is a Hamiltonian $(p, q+1)$-module with suspension J. We can choose from the very beginning F to be a good Hamiltonian module and L_0 a good lagrangian. Since J_i and C_j anticommute with J and with the reflection through L_0 they can be rewritten as

$$J_i = \begin{bmatrix} 0 & U_i \\ U_i & 0 \end{bmatrix}, \ (1 \leq i \leq p) \quad C_j = \begin{bmatrix} 0 & O_j \\ O_j & 0 \end{bmatrix}, \ (1 \leq j \leq q') \tag{5.2}$$

where U_i, O_j are orthogonal operators $L_0 \to L_0$ such that

$$O_j^2 = -U_i^2 = 1_{L_0}.$$

For each $x \in X$ consider the unbounded operator (this justifies the hamiltonian terminology)

$$T_x : D(T_x) \subset H = L^2(0,1;F) \to L^2(0,1;F) \quad u \mapsto J\frac{du}{ds}$$

$$D(T_x) = \{u \in L_1^2(0,1;F) \ / \ u(0) \in L_0, \ \ u(1) \in L_1(x)\}$$

Set

$$\overline{J}_i = JJ_i = \begin{bmatrix} U_i & 0 \\ 0 & -U_i \end{bmatrix}, \ \overline{C}_j = CC_j = \begin{bmatrix} O_j & 0 \\ 0 & -O_j \end{bmatrix} \tag{5.3}$$

Obviously $\overline{J}_i^2 = -I$, $\overline{C}_j^2 = I$. $\overline{J}_i$ and $\overline{C}_j$ extend naturally to H defining a structure of Hilbert $C^{p,q'}$-module. Since the L_0 and $L_1(x)$ are (p', q')-lagrangians they are invariant under $\overline{J}_i$ and $\overline{C}_j$ so that the domain of T_x is invariant under the above $C^{p,q'}$-action. T_x is a special Dirac operator and our choice of boundary conditions makes it a Fredholm selfadjoint operator (see [BW3] or [FO2]). Its kernel can be identified with $L_0 \cap L_1(x)$. It obviously anticommutes with $\overline{J}_i$ and $\overline{C}_j$ so that T_x is a family in $\overline{\mathcal{F}}^{p,q}$. Proposition B.1 of Appendix B shows this is also a continuous family (in the gap topology). The operators we have constructed above will be called *generalized Floer operators* since Andreas Floer was the first to realize their K-theoretic relevance while studying the symplectic Floer homology of a pair of lagrangian submanifolds (see[F] for more details).

Since $(E; L_0, L_1)$ belongs to the reduced KO-group we may assume that for some $x_0 \in X$ we have $L_1(x) = L_0^{\#}$ (i.e. $(F; L_0, L_1(x_0)) = 0$ in $KO^{p',q'}(pt)$). This means that T_{x_0} is invertible and in fact we have the following result.

Lemma 5.6 *All the operators $(T_x)_{x \in X}$ are in $\mathcal{F}_0^{p,q}$.*

Sketch of Proof Since X and $\mathcal{F}_0^{p,q}$ are connected it suffices to show that just one of the operators T_x is good. We will do this for $x = x_0$ where the boundary conditions are $u(0) \in L_0$, $u(1) \in L_0^{\perp}$. In this case the operator $T_0 = T_{x_0}$ is invertible. Its spectrum can be explicitly computed

$$\sigma(T_0) = \{(2k+1)\pi/2; \ k \in \mathbf{Z}\}.$$

The corresponding eigenspaces V_λ can be explicitly described as well.

$$V_\Lambda = \{u(t) \in H;\ u(t) = (\cos\lambda t)x + (\sin\lambda t)Jx,\ x \in L_0\},\ \ \lambda \in \sigma(T_0).$$

Note that $\sigma(T_0)$ is symmetric with respect to the origin. For each $\lambda \in \sigma_+(T_0) = \sigma(T_0) \cap (0,\infty)$ set $H_\lambda = V_\lambda \oplus V_{-\lambda}$ and denote by R_Λ the involution on H_λ defined by $\mathrm{diag}(1_{V_\lambda}, -1_{V_{-\lambda}})$. H_λ is an essential $(p, q+1)$ module because it is naturally isomorphic with F which was chosen to be good. R_Λ defines an essential lagrangian because L_0 is a good lagrangian. Now form the involution

$$R = \oplus_{\lambda\in\sigma_+(T_0)} R_\Lambda.$$

Clearly R is a $(p, q+1)$ grading of H. The operator $T_0(1+T_0^2)^{-1/2}$ can be deformed inside $\mathcal{F}^{p,q}$ to R. (The obvious deformation will do the trick). Because each H_λ is essential (as a finite dimensional module) then so will be the infinite dimensional one

$$H = \oplus_{\lambda\in\sigma_+(T_0)} H_\Lambda.$$

Similarly we deduce R is $(p, q+1)$ essential. Thus T_0 lies in the connected component of essential $(p, q+1)$ lagrangians. □

$H \oplus H$ has a structure of Hamiltonian $(p, q+1)$-module structure given by

$$\tilde{J}_i = \begin{bmatrix} 0 & \overline{J}_i \\ \overline{J}_i & 0 \end{bmatrix},\ \ \tilde{C}_j = \begin{bmatrix} 0 & \overline{C}_j \\ \overline{C}_j & 0 \end{bmatrix}\ \ (1 \le i \le p\,,\ 1 \le j \le q') \tag{5.4}$$

and suspension

$$\tilde{J} = \tilde{J}_{p'} = \begin{bmatrix} 0 & 1_H \\ -1_H & 0 \end{bmatrix}. \tag{5.5}$$

We will show that $(\Gamma_0, \Gamma_x)_{x\in X}$ represents the same element in $\widetilde{KO}^{p',q'}(X)$ using a suitable symplectic reduction. Set

$$\overline{L}_0 = \{u \in H\,;\ u(s) \equiv u_0 \in L_0 \ \text{a.e. } s \in [0,1]\}.$$

$\overline{L}_0 \oplus \overline{L}_0$ becomes a $C^{p',q'}$-submodule of $H \oplus H$. In fact, can show

Lemma 5.7 *$\overline{L}_0 \oplus \overline{L}_0$ is isomorphic as a $C^{p',q'}$-module with $F = L_0 \oplus JL_0$ via the map*

$$\Phi : \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} u(0) \\ v(0) \end{bmatrix}.$$

Proof of Lemma 5.7 We have

$$\Phi\left(\tilde{J}_{p'}\begin{bmatrix} u \\ v \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} v \\ -u \end{bmatrix}\right) = \begin{bmatrix} v(0) \\ -u(0) \end{bmatrix}.$$

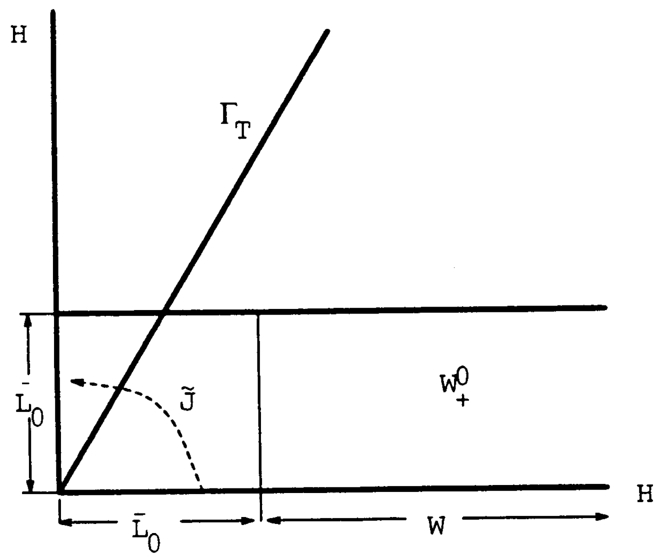

Figure 2: *Constructing the symplectic reduction of a graph*

$$J\Phi\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = J\left(\begin{bmatrix} u(0) \\ v(0) \end{bmatrix}\right) = \begin{bmatrix} v(0) \\ -u(0) \end{bmatrix}.$$

$$\Phi\left(\tilde{J}_i\begin{bmatrix} u \\ v \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} \overline{J}_i v \\ \overline{J}_i u \end{bmatrix}\right) = \begin{bmatrix} \overline{J}_i v(0) \\ \overline{J}_i u(0) \end{bmatrix} = \begin{bmatrix} U_i v(0) \\ U_i u(0) \end{bmatrix}.$$

$$J_i\Phi\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = J_i\left(\begin{bmatrix} u(0) \\ v(0) \end{bmatrix}\right) = \begin{bmatrix} U_i v(0) \\ U_i u(0) \end{bmatrix} \quad \text{by (5.4).}$$

Similarly one shows

$$\Phi\tilde{C}_j = C_j\Phi$$

and Lemma 5.7 is proved. □

Let W denote the orthogonal complement of $\overline{L}_0$ in H. The subspace $W \oplus 0$ is a (p', q')-isotropic subspace of $H \oplus H$ and obviously the symplectic reduction of $H \oplus H$ mod W is $\overline{L}_0 \oplus \overline{L}_0$. We perform the symplectic reduction of Γ_{T_x} mod W. For simplicity we will omit the subscript x.

Note first that Γ_T is clean mod W since $\ker T \subset \overline{L}_0$. It is easily seen that $W_+^0 = H \oplus \overline{L}_0$ (Fig. 2) so that

$$\Gamma_T \cap W_+^0 = \{(u, Tu) \,/\, u \in D(T),\ Tu \in \overline{L}_0\}.$$

Let $v_0 \in \overline{L}_0$. The equation

$$Tu = v_0 \ , \ u \in D(T)$$

is equivalent to the boundary value problem

$$J\dot{u} = v_0 \ , \ u(0) \in L_0 \, , \ u(1) \in L_1.$$

The general solution to $J\dot{u} = v_0 = const.$ is

$$u(s) = u(0) - sJv_0.$$

The boundary condition imposes $u_0 = u(0) \in L_0$ and $u_0 - Ju_0 \in L_1$. Hence we deduce

$$\Gamma_T \cap W_+^0 = \{(u(s), v_0) \ ; \ u(s) = u_0 - sJv_0, \ u_0 \in L_0 \ , \ u_0 - Jv_0 \in L_1\}. \tag{5.6}$$

To find the reduction of Γ_T mod W we have to project $\Gamma_T \cap W_+^0$ orthogonally onto $\overline{L}_0 \oplus \overline{L}_0 = (H \oplus H)_W$. Given $u(s)$ as in (5.6) let $(\tilde{u}, \tilde{v})$ denote the orthogonal projection of $(u(s), J\dot{u}(s))$ onto $\overline{L}_0 \oplus \overline{L}_0$. This means

$$\int_0^1 (u(s) - \tilde{u}, \phi) ds = \int_0^1 (v_0 - \tilde{v}, \psi) ds = 0 \quad \forall \phi \, , \, \psi \in \overline{L}_0.$$

$$\int_0^1 (v_0 - sJv_0 - \tilde{u}, \phi) ds = 0 \ \ \forall \phi \in \overline{L}_0 \ , \ \ \tilde{v} = v_0.$$

$$\int_0^1 (u_0 - \tilde{u}, \phi) ds = 0 \ \ \forall \phi \in \overline{L}_0 \ , \ \ \tilde{v} = v_0.$$

Thus

$$(\Gamma_T)^W = \overline{L}_1 = \{(u_0, v_0) \in \overline{L}_0 \oplus \overline{L}_0 \ / \ u_0 - Jv_0 \in L_1\}.$$

Tautologically, the isomorphism Φ defined in Lemma 5.4 sends $\overline{L}_1$ to L_1. Thus we have a $C^{p',q'}$-isomorphism

$$\Phi : (\overline{L}_0, \overline{L}_1) \mapsto (L_0, L_1)$$

which shows that

$$\mu_{p',q'}((\Gamma_0, \Gamma_{T_x})_{x \in X}) = (F \times X; L_0, L_1(x))$$

and the surjectivity is proved.

Injectivity We have shown so far that the map Γ induces a surjection

$$\Gamma_* \, : \, \pi_n(\mathcal{F}_0^{p,q}) \to \widetilde{KO}^{p'q'}(S^n). \tag{5.7}$$

From Proposition 5.1 we know that we have an isomorphism

$$\pi_n(\mathcal{F}^{p,q}) \cong \widetilde{KO}^{p,q}(S^n) \cong \widetilde{KO}^{p',q'}(S^n).$$

Using Table 1.1 and Bott periodicity (Thm 2.3) we deduce

$$\widetilde{KO}^{p,q}(S^n) \cong 0\,,\, \mathbb{Z}_2\,,\, \mathbb{Z}.$$

For these groups the surjective endomorphisms are also bijective and thus the map of (5.7) is an isomorphism. Theorem 5.5 is proved. □

Definition 5.8 *Let $T : X \to \mathcal{F}^{p,q}$ be a continuous family in $\mathcal{F}^{p,q}$. The element*

$$\mu_{p,q}((\Gamma_0, \Gamma_{T_x})_{x\in X}) \in KO^{p',q'}(X)$$

will be called the index of the family T and will be denoted by $\mathbf{ind}_{p,q}(T)$.

Remark 5.9 *(a)The families of generalized Floer operators we have considered in the proof of Theorem 5.5 were also studied in great detail in the papers [FO1,2] using a different approach. Theorem 5.5 can be rephrased as saying that Floer families generate the KO-groups.*

(b) The same proof as above shows that the connected components $\mathcal{F}^{p,q}$ are in a bijective correspondence with $A_{p,q} = KO^{p,q}(pt.)$.

(c) One can consider more general families of Floer operators of the form

$$T_x = J\frac{d}{ds}$$

where the principal symbol J may also vary *with $x \in X$. This does not affect at all the symplectic reduction proof presented above. The corresponding index of the family will continue to be the generalized Maslov index of the pairs of boundary conditions.*

§5.4 Examples We want to illustrate the above abstract considerations with some examples we believe are very suggestive.

Consider first the simplest of the situations: a single operator $T \in \mathcal{F}^{0,0}$. Thus it can be represented as Fredholm operator $H \oplus H \to H \oplus H$ which has the block decomposition

$$T = \begin{bmatrix} 0 & D^* \\ D & 0 \end{bmatrix}$$

where $D : H \to H$ is a Fredholm operator.

The element defined by T in $KO^{0,0}(pt) \cong \mathbb{Z}$ is the usual index

$$ind\,(T) = ind_{0,0}(T) = \dim\ker D - \dim\ker D^*.$$

Equivalently, $ind_{0,0}T$ can be given the grading representation

$$ind_{0,0}T = (\ker T, R_0, -R_0)$$

where R_0 is the natural involution on $E \stackrel{def}{=} H \oplus H$

$$R_0 = \begin{bmatrix} 1_H & 0 \\ 0 & -1_H \end{bmatrix}.$$

We want to show that the lagrangian approach proposed in the previous subsections produces the same conclusion, albeit in a more roundabout manner.

Look at the graph of T as a subspace $\Gamma_T \subset E \oplus E$. $E \oplus E$ has a natural $(1,1)$ structure defined by

$$J = \begin{bmatrix} 0 & 1_E \\ -1_E & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & R_0 \\ R_0 & 0 \end{bmatrix}.$$

Γ_T is a $(1,1)$ lagrangian with respect to this structure.

Denote by W the orthogonal complement of $\ker T$ in E. Then $W' = W \oplus 0$ is an $(1,1)$ cofinite isotropic subspace of $E \oplus$ and Γ_T is clean mod W'.

The symplectic reduction of $E \oplus E$ mod W' is the subspace $\ker T \oplus \ker T$ equipped with the $(1,1)$ structure induced from $E \oplus E$. The symplectic reduction of Γ_T mod W' is the subspace $\ker T \oplus 0 \subset \ker T \oplus \ker T$. Thus the element defined by T in $K^{1,1}(pt)$ admits the lagrangian representation

$$\mathbf{ind}_{0,0}(T) = \mu_{1,1}(\Gamma_0, \Gamma_T) = (\ker T \oplus \ker T, \ker T \oplus 0, \ker T \oplus 0).$$

If we denote by η the orthogonal reflection in $\ker T \oplus 0$

$$\eta = 1_{\ker T} \oplus (-1_{\ker T})$$

then we obtain a grading representation of this index

$$\mathbf{ind}_{0,0} = (\ker T \oplus \ker T, \eta, -\eta) \in KO^{1,1}(pt) \cong \mathbb{Z}.$$

According to the computations in Subsection 2.2 the above index can be identified with the integer

$$\frac{1}{2}\operatorname{tr}(JR\eta) = \frac{1}{2}\operatorname{tr}\begin{bmatrix} R_0 & 0 \\ 0 & R_0 \end{bmatrix}$$
$$= \operatorname{tr} R_0 = \dim\ker D - \dim\ker D^*.$$

The considerations in the previous subsections can be used to produce yet another proof of the equality

$$\text{Maslov index} = \text{Spectral flow}$$

The spectral flow is an explicit isomorphism

$$SF : K^{1,0}(I, \partial I) \to \mathbb{Z}.$$

Given a continuous path $t \mapsto D_t \in \mathcal{F}^{1,0}$, $t \in [0,1]$ such that D_0 and D_1 are invertible then, roughly speaking, the spectral flow of this path counts with sign the eigenvalues of D_t which change sign as t varies in $[0, 1]$. We refer to [APS], [BF] or [N1] for more details. We want to show that it coincides with the Maslov index $\mu_{1,0}$.

If $(D_t)_{t\in I}$ is a family of selfadjoint Fredholm operators on a Hilbert space H then their graphs Γ_{D_t} are $(1,0)$ lagrangians in $H \oplus H$ equipped with the $(1,0)$ structure defined by

$$J = \begin{bmatrix} 0 & 1_H \\ -1_H & 0 \end{bmatrix}.$$

Its Maslov index is obtained by a symplectic reduction of the family (Γ_0, Γ_{D_t}).

To establish the equality between the Maslov index and the spectral flow it suffices to verify it on a generator of $(KO^{1,0}(I, \partial I)$. It is very easy to select such a generator. Take for example the element with the lagrangian representation

$$\gamma = (\mathbb{R}^2, \mathbb{R} \oplus 0, L_t), \quad |t| \leq \varepsilon$$

where L_t denotes the line with slope t and $\mathbb{R}^2$ is equipped with the symplectic structure $-dx \wedge dy$. The Maslov index of this path is 1.

On the other hand, this element can be obtained by symplectic reduction from the the graphs of the Floer operators

$$D_t : Dom\,(D_t) \subset L^2([0,1], \mathbb{C}) \to L^2([0,1], \mathbb{C}), \quad D_t u(s) = -\mathbf{i}\frac{du}{ds}$$

$$Dom\, D_t = \{u \in L^2_1([0,1], \mathbb{C})\ ;\ u(0) \in \mathbb{R},\ u(1) \in L_t\}.$$

($\mathbb{R}^2 \cong \mathbb{C}$ is equipped with the $(1,0)$ structure defined by multiplication with $-\mathbf{i}$. The induced $(1,0)$-symplectic structure is $-dx \wedge dy$.) It is fairly easy to compute the spectrum of D_t since the equation

$$-\mathbf{i}\frac{du}{ds} = \lambda u$$

has the explicit solutions

$$u(s) = \exp(\mathbf{i}\lambda s).$$

The eigenvalues are

$$\lambda_n(t) = \theta_t + 2\pi n, \quad n \in \mathbb{Z},\ t \in [-\varepsilon, \varepsilon]$$

where $\theta_t \in (-\pi/2, \pi/2)$ is the angle the line L_t makes with the $x$axis. Clearly the spectral flow of this family is 1. This shows the spectral flow coincides with the (generalized) Maslov index.

6 Families of boundary value problems for Dirac operators

In this last section we will use the abstract techniques we developed so far to prove an index theorem for families of boundary value problems for Dirac operators: Thm. 6.2. As an application, we derive a very general splitting formula: Thm. 6.10. In the process we will provide a new very short proof of the cobordism invariance of the index of families using an idea of [MP] coupled with the adiabatic analysis of [N1].The formulations of these results requires the introduction of a suitable language.

Notation L^2_σ will denote the Sobolev space of distributions "σ-times " differentiable with derivatives in L^2. The norm of L^2_σ will be denoted by $|\cdot|_\sigma$ or by $|\cdot|_{\sigma,\Sigma}$ to emphasize the fact we are dealing with distributions defined over a specific manifold Σ. We will denote the L^2 norm by $|\cdot|$.

§6.1 (p,q)-Dirac operators and their Calderon projections Consider an oriented Riemann manifold (M,g). We denote by $C(M)$ the bundle of Clifford algebras generated by T^*M with the induced metric. Denote by $\mathcal{E}$ a bundle of selfadjoint $C(M)$-modules (in the sense of [BGV]). We denote the Clifford multiplication by

$$c : T^*M \to \mathrm{End}(\mathcal{E})$$

and fix a Clifford connection (see [BGV]) $\hat{\nabla}$ in $\mathcal{E}$. On the space $\mathrm{End}(\mathcal{E})$ one can define a norm for each $\sigma \in \mathbb{R}$, thinking of them as bounded maps $L^2_\sigma \to L^2_\sigma$. Correspondingly, we will use the notation $\mathrm{End}_\sigma(\mathcal{E})$ to indicate which topology we use in a specific situation. Furthermore we agree that all Hilbert space notions (like e.g. orthogonality, adjoints etc.) will be understood in the L^2-sense.

Suppose now that $\mathcal{E}$ has an additional $C^{p,q'}$ structure compatible with the $C(M)$-structure and with the connection $\hat{\nabla}$. By this we understand that there exists a morphism of *-algebras

$$\rho(e_i) : C^{p,q'} \to \mathrm{End}(\mathcal{E})$$

such that

$$\{\rho(e_i), c(\alpha)\} = \{\rho(\epsilon_j), c(\alpha)\} = 0 \ , \ \forall \alpha \in \Omega^1(M) \ \forall i,j \tag{6.1}$$

$$\hat{\nabla}\rho(e_i) = \hat{\nabla}\rho(\epsilon_j) = 0 \ , \ \forall i,j. \tag{6.2}$$

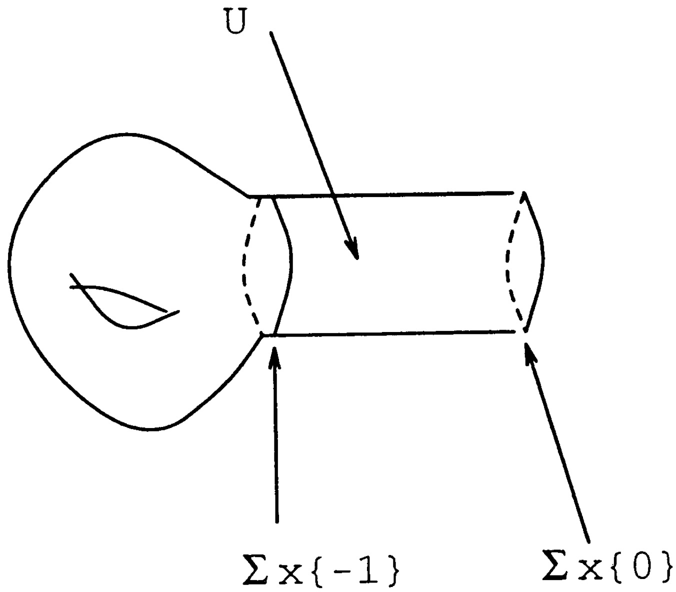

Figure 3: *The boundary has a metrically nice neighborhood*

Moreover, assume that each fiber $\mathcal{E}_x$ is an **essential** finite dimensional (p, q')-module. Set as usual $J_i = \rho(e_i)$, $C_j = \rho(\epsilon_j)$. A bundle with the above properties will be called a $C^{p,q}$**-Dirac bundle**.

It is very easy to construct examples of $C^{p,q}$-Dirac bundles. Start with a $C(M)$-bundle $\mathcal{E}'$ with a Clifford connection ∇'. We can think of $\mathcal{E}'$ as a $\mathbb{Z}_2$-graded $C(M)$-bundle with the trivial grading. Form the trivial bundle $\underline{C}^{p,q'} = C^{p,q'} \times M$ endowed with the trivial connection. We then form the $\mathbb{Z}_2$-graded tensor products $\underline{C}^{p,q'}\hat{\otimes}C(M)$ and $\mathcal{E} = \underline{C}^{p,q'}\hat{\otimes}\mathcal{E}'$. $\mathcal{E}$ with the induced connection has all the required properties. For example the $C(M)$-superbundles of [BGV] are $C^{0,0}$-Dirac bundles.

Let $\mathcal{E}$ be a $C^{p,q}$-Dirac bundle and assume that (M, g) is a compact oriented manifold with oriented boundary $\partial M = \Sigma$ such that the metric g is a product in a tubular neighborhood N of the boundary, $N \cong \Sigma \times (-2, 0]$ (Fig. 3). Set $\mathcal{E}_0 = \mathcal{E}\,|_\Sigma$ and denote by s the longitudinal coordinate along N. Let

$$J = c(ds) : \mathcal{E}_0 \to \mathcal{E}_0$$

denote the endomorphism of $\mathcal{E}_0$ induced by the Clifford multiplication with ds. Any Dirac operator D on $\mathcal{E}$ can be written near Σ as

$$D = J(\hat{\nabla}_s + D_0(s)). \tag{6.3}$$

Denote by $\mathcal{D}^{p,q}$ the space of selfadjoint Dirac operators $D : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ such that

$$\{D, J_i\} = \{D, C_j\} = 0 \ , \ \forall 1 \le i \le p \ 1 \le j \le q'$$

and the operators $D_0(s)$ are **selfadjoint** and **independent** of s. Such operators will be called **cylindrical**. Let $Cyl^{p,q}$ denote the space of selfadjoint endomorphisms A of $\mathcal{E}$ satisfying

$$\{A, J_i\} = \{A, C_j\} = 0 \quad \forall 1 \leq i \leq p \;\; 1 \leq j \leq q' \tag{6.4}$$

$$\{A, J\} = 0 \quad \text{over } N \tag{6.5}$$

$$\hat{\nabla}_s A = 0 \quad \text{over } N. \tag{6.6}$$

$\mathcal{D}^{p,q}$ is an affine space modeled by $Cyl^{p,q}$. It can be topologized using a L^2_σ norm on $Cyl^{p,q}$ where σ is large enough so that L^2_σ embeds continuously into C^3.

J can be used to introduce a structure of Hamiltonian $(p, q+1)$-module in $L^2(\mathcal{E}_0)$ by

$$\overline{J}_i = J_i J \quad 1 \leq i \leq p \tag{6.7}$$

$$\overline{C}_j = C_j J \quad 1 \leq j \leq q' \tag{6.8}$$

and suspension

$$\overline{J}_{p+1} = J.$$

For each $D \in \mathcal{D}^{p,q}$ and $\sigma \in \mathbb{R}$ let

$$\mathcal{K}_\sigma(D) = \{U \in L^2_\sigma(\mathcal{E}) \;\; DU = 0 \;\text{ on } M\}.$$

For any $\sigma \geq 1/2$ there is a well defined trace ([BW3], [S1])

$$R_\sigma : \mathcal{K}_\sigma(D) \to L^2_{\sigma-1/2}(\mathcal{E}_0)$$
$$U \mapsto U|_\Sigma .$$

The **Cauchy data space** (CD space for brevity) of D is defined as

$$\Lambda_M(D) = R_{1/2}\mathcal{K}_1/2(D).$$

$\Lambda_M(D)$ is a closed subspace in $L^2(\mathcal{E}_0)$ and $R_{1/2}$ gives a continuous bijection $\mathcal{K}_1/2(D) \to \Lambda_M(D)$. $\Lambda_M(D)$ consists of the sections $u \in L^2(\mathcal{E}_0)$ which extend to a $L^2_{1/2}$ solution of $DU = 0$ on M. The orthogonal projection

$$\Pi_D : L^2(\mathcal{E}_0) \to L^2(\mathcal{E}_0)$$

onto $\Lambda_M(D)$ is induced by a zeroth order pseudodifferential operator called the **Calderón projection** of D ([BW3], [S1]). Its principal symbol depends upon D only through the $C(M)$-structure in $\mathcal{E}$. Denote this symbol by π_- and set $\pi_+ = 1_{\mathcal{E}_0} - \pi_-$. In $[BW1-3]$ is shown that

$$\Lambda_M(D)^\perp = J\Lambda_M(D).$$

Since D anticommutes with J_i and C_j we deduce

$$J_i\mathcal{K}(D) = C_j\mathcal{K}(D) = \mathcal{K}(D)$$

so that

$$\overline{J}_i\Lambda_D = \overline{C}_j\Lambda_M(D) = \Lambda_M^\perp$$

which shows that $\Lambda_M(D)$ is a (p', q')-lagrangian in $L^2(\mathcal{E}_0)$ with the above $C^{p',q'}$-structure.

§6.2 The index of families of boundary value problems Denote by $\Psi\mathcal{L}^{p',q'} = \Psi\mathcal{L}^{p',q'}_{\pi_+}$ the family of (p',q')-lagrangians $\Lambda \subset L^2(\mathcal{E}_0)$ such that the orthogonal projection onto Λ is a zeroth order pseudodifferential operator whose symbol is π_+. We deduce from the above definition that $\forall D \in \mathcal{D}^{p,q}$ and $\forall \Lambda \in \Psi\mathcal{L}^{p',q'}$ the subspaces $(\Lambda_M(D), \Lambda)$ form a Fredholm pair of lagrangians. We now topologize $\Psi\mathcal{L}^{p',q'}$ using the convergence

$$\Lambda_n \xrightarrow{\psi} \Lambda \quad \text{iff} \quad P_{\Lambda_n} \to P_\Lambda \ \text{as bounded operators } L^2_\sigma \to L^2_\sigma \ \ \sigma = 0\,,\, 1/2.$$

We will call this **ψ-convergence.** Denote by $\Psi\mathcal{L}_0^{p',q'}$ the connected component of $\Psi\mathcal{L}^{p',q'}$ containing $\Lambda_M(D)^\perp$, where $\Lambda_M(D)$ denotes the CD space of some $D \in \mathcal{D}^{p,q}$. Given any $\Lambda \in \Psi\mathcal{L}^{p',q'}$ and $D \in \mathcal{D}^{p,q}$ we can form the operator

$$T = T(D,\Lambda) : Dom(T) \subset L^2(\mathcal{E}) \to L^2(\mathcal{E})$$

$$Dom\,(T) = \{U \in L^2_1(\mathcal{E}) \;/\; U\,|_\Sigma \in \Lambda\}$$

$$TU = DU.$$

The results of [BW3] Chap.18, 19 show that T is a Fredholm selfadjoint operator anticommuting with the generators of the $C^{p,q'}$-action on $L^2(\mathcal{E})$. We thus have a map

$$\mathcal{T} : \mathcal{BP}^{p,q} = \mathcal{D}^{p,q} \times \Psi\mathcal{L}^{p',q'} \to \mathcal{F}^{p,q} \tag{6.9}$$

$$(D,\Lambda) \mapsto T(D,\Lambda).$$

Proposition B.1 from the Appendix shows that this map is actually continuous.

Proposition 6.1 *(a)Any $\Lambda \in \Psi\mathcal{L}_0^{p',q'}$ is an essential (p',q')-lagrangian in $L^2(\mathcal{E}_0)$.*
(b) For any $D \in \mathcal{D}^{p,q}$ and any $\Lambda \in \Psi\mathcal{L}_0^{p',q'}$ the operator $T(D,\Lambda)$ lies in $\mathcal{F}_0^{p,q}$.

For a proof we refer to Appendix D. Part (a) of the above proposition shows that there is a well defined CD-map

$$CD : \mathcal{D}^{p,q} \times \Psi\mathcal{L}_0^{p',q'} \to \mathcal{F}\mathcal{L}_0^{p',q'} \tag{6.10}$$

$$(D,\Lambda) \mapsto (\Lambda_M(D),\Lambda)$$

which is also continuous by Lemma C3 in the Appendix.

Let Y be a compact, connected CW-complex and

$$\beta : Y \to \mathcal{D}^{p,q} \times \Psi\mathcal{L}_0^{p',q'} = \mathcal{BP}^{p,q} \quad y \mapsto (D_y, \Lambda_y)$$

a continuous map. We obtain via $\mathcal{T}$ a continuous family of operators $T(D_y, \Lambda_y)_{y\in Y} \in \mathcal{FL}_0^{p,q}$ which by Theorem 5.5 has a well defined index $\mathbf{ind}_{p,q}(T) \in KO^{p',q'}$. Using the CD map we get a continuous family

$$CD(\beta) : Y \to \mathcal{FL}_0^{p',q'}$$

which by Theorem 4.14 has a generalized Maslov index $\mu_{p',q'}(CD(\beta)) \in KO^{p',q'}(Y)$. The main application of the results we proved so far is the following result.

Theorem 6.2

$$\mathbf{ind}_{p,q}(T(\beta)) = \mu_{p',q'}(CD(\beta)).$$

Before we begin proving Thm. 6.2 it is convenient to describe some special families

$$Y \to \mathcal{D}^{p,q} \times \Psi\mathcal{L}^{p',q'}.$$

Let $D \in \mathcal{D}^{p,q}$. Along N it has the form

$$D = J(\hat{\nabla}_s + D_0)$$

where

$$D_0 : C^\infty(\mathcal{E}_0) \to C^\infty(\mathcal{E}_0)$$

is a selfadjoint Dirac operator on $\mathcal{E}_0$ anticommuting with J. Since D anticommutes with J_i and C_j we deduce

$$0 = \{JD_0,\} = JD_0J_i + J_iJD_0 = -D_0JJ_i + J_iJD_0$$
$$= D_0J_iJ + J_iJD_0 = \{D_0, \overline{J}_i\}.$$

For each $E \geq 0$ denote by $\mathcal{H}_0^E = \mathcal{H}_0^E(D)$ (resp $\mathcal{H}_>^E$, $\mathcal{H}_<^E$) the closed subspace of $L^2(\mathcal{E}_0)$ spanned by the eigenvectors of D_0 corresponding to eigenvalues in the interval $[-E, E]$ (resp. in $[E, \infty)$, $(-\infty, -E]$). Similarly one defines $\mathcal{H}_\geq^E$ and $\mathcal{H}_\leq^E$.

The operator D_0 anticommutes with the generators of the $C^{p',q'}$ action so that $\forall\, E \geq 0$ the subspace $\mathcal{H}_0^E$ is a finite dimensional $C^{p',q'}$-submodule of $L^2(\mathcal{E}_0)$. Moreover $\mathcal{H}_>^E$ and $\mathcal{H}_<^E$ are both (p', q')-isotropic subspaces and in fact $\mathcal{H}_0^E$ is the symplectic reduction of $L^2(\mathcal{E}_0)$ mod either one of them.

It is known that the orthogonal projection of $L^2(\mathcal{E}_0)$ onto $\mathcal{H}_>^E$ is a zeroth order pseudodifferential operator with symbol π_+ (see [APS1],[BW3]). If we choose $L \subset \mathcal{H}_0^E$ a (p', q')-lagrangian then the projection onto L is a smoothing operator (all eigenvalues spanning $\mathcal{H}_0^E$ are smooth) so that the projection onto $\mathcal{H}_>^E \oplus L$ is a zeroth order pseudodifferential operator with principal symbol π_+. Thus

$$\mathcal{H}_>^E \oplus L \in \Psi\mathcal{L}^{p',q'} \tag{6.11}$$

so it is an admissible selfadjoint, elliptic boundary condition for D. We call these special boundary conditions *level E Atiyah-Patodi-Singer conditions* or lagrangians - generalized APS for brevity. We will use the following result proved in Appendix C.

Lemma 6.3 *Let M be a compact manifold with boundary, $D \in \mathcal{D}^{p,q}$ and $\Lambda : Y \to \Psi\mathcal{L}^{p',q'}$ a continuous family of pseudodifferential lagrangians. Then there exists $E > 0$ such that $(\Lambda_y)_{y \in Y}$ is homotopic in the topology of ψ-convergence to a family of level E APS lagrangians*

$$(\Lambda_y) \cong (L_y^E \oplus \mathcal{H}_>^E)$$

Moreover L_y^E can be identified with the symplectic reduction of Λ_y mod $\mathcal{H}_<^E$.

Proof of Theorem 6.2 The proof will be completed in several steps.

Step1: Deforming the operators Any family $\beta : Y \to \mathcal{D}^{p,q} \times \Psi\mathcal{L}^{p',q'}$, $y \mapsto (D_y, \Lambda_y)$ can be deformed to one in which $y \mapsto D_y$ is constant , $D_y \equiv D$. This is trivially true since $\mathcal{D}^{p,q}$ is an affine space.

Step 2: Restricting to the neck. Let $M_{-1} = M \setminus (-1,0] \times \Sigma$ (Fig.3) and set

$$\Lambda_{-1} = \Lambda_{M_{-1}}(D).$$

Consider the family of BVP over the cylinder $C = [-1,0] \times \Sigma$ given by the operators $S_y = S(D, \Lambda_y)$ defined by

$$Dom\,(S_y) = \{U \in L_2^1(\mathcal{E}\,|_C)\,;\; U\,|_{\Sigma\times\{-1\}} \in \Lambda_{-1}, \quad U\,|_{\Sigma\times\{0\}} \in \Lambda_y\}.$$

$$S_y U = DU.$$

$(S_y)_{y\in Y}$ is a continuous family in $\mathcal{F}^{p,q}$ and thus they have an index in $KO^{p',q'}(Y)$. We will show that

$$\mathbf{ind}_{p,q}(T(D,\Lambda_y)) = \mathbf{ind}_{p,q}(S_y).$$

Set $H = L^2(\mathcal{E})$ and $H' = L^2(\mathcal{E}\,|_C)$. There is a natural extension map

$$\mathcal{Z} : H_C \to H \ , \ U \mapsto \overline{U} = \begin{cases} U(x) & , \quad x \in C \\ 0 & , \quad \text{otherwise} \end{cases} . \tag{6.12}$$

This is obviously an embedding of $C^{p,q'}$ modules. We also have an extension map

$$\mathcal{Z}_D : Dom(S_y) \to Dom(T_y)$$

given by

$$U \mapsto \mathcal{Z}_D(U) = \begin{cases} U & , \quad \text{on } C \\ U^D & , \quad \text{on } M_{-1} \end{cases} \tag{6.13}$$

where U^D is the **unique** section in $\mathcal{K}_1(D\,|_{M_{-1}})$ such that

$$U^D\,|_{\Sigma\times\{-1\}} = U\,|_{\Sigma\times\{-1\}} \in \Lambda_{M_{-1}}(D).$$

Now consider the elliptic eigenvalue problem for $S_0 = S(D, \Lambda_M^\#(D))$

$$\left\{\begin{array}{ccll} DU & = & \lambda U, & \text{on } C \\ U\,|_{\Sigma\times\{-1\}} & \in & \Lambda_{-1} & \\ U\,|_{\Sigma\times\{0\}} & \in & \Lambda_M(D)^\# & \end{array}\right. \tag{6.14}$$

S_0 is an elliptic selfadjoint BVP and has real and discrete spectrum.

For any $\nu > 0$ denote by H'_ν the closed subspace of H' generated by the eigenvectors of S_0 corresponding to eigenvalues in the interval $[-\nu, \nu]$ and denote by W'_ν its orthogonal complement in H'. $H' \times H'$ has as usual a $C^{p',q'}$-structure and W_ν are cofinite (p', q')-isotropic subspaces in $H' \times H'$. Obviously $(W'_\nu)_{\nu>0}$ is a filtration of H' and as in Sec.4 we can find $\nu_0 > 0$ such that

$$\Gamma_{S_y} \text{ is clean mod } W'_0 = W'_{\nu_0} \quad \forall\, y. \tag{6.15}$$

We now perform the symplectic reduction of Γ_S mod W'_0. First intersect

$$\Gamma_S \cap (W'_0)^0_+ = \{(U', DU') \in H' \times H' \;/\; U' \in Dom(S_y)\,,\; DU' \in H'_{\nu_0}\}. \tag{6.16}$$

Next, project Γ_S onto $H'_{\nu_0} \times H'_{\nu_0}$

$$(U', DU') \mapsto (\tilde{U}', \tilde{V}')$$

$$\tilde{V}' = DU'\,,\; \int_C (U' - \tilde{U}', \phi') = 0 \quad \forall\, \phi' \in H'_{\nu_0}. \tag{6.17}$$

Denote this symplectic reduction of Γ_S by L_S. Using the extension map $\mathcal{Z}$ of (6.12) we construct

$$H_\nu = \mathcal{Z}(H'_\nu).$$

Let W_ν denote the orthogonal complement of H_ν in H. W_ν is a cofinite (p', q')-isotropic subspace in $H \times H$ and we claim that

$$\Gamma_{T_y} \;\text{ is clean mod } W_0 = W_{\nu_0} \;\; \forall\, y \tag{6.18}$$

where ν_0 **is the same** as in (6.15).

Indeed, suppose the contrary. Then there exists $y \in Y$ such that

$$\Gamma_{T_y} \cap W_0 \neq 0.$$

Hence one can find $V \in Dom\,(T_y) \cap W_0$ such that $DV = 0$, $V \neq 0$. Notice that $V\,|_C \in W'_0$ and $D(V\,|_C) = 0$. Thus

$$V\,|_C \oplus 0 \in \Gamma_{S_y} \cap W'_0 = 0 \oplus 0 \quad \text{(by (6.15))}.$$

By the unique continuation property we deduce $V \equiv 0$ on M and (6.18) is proved.

The symplectic reduction can now be explicitly described.

$$\Gamma_T \cap (W_0)^0_+ = \{(U, DU) \in H \times H \,;\, U \in Dom\,(T),\;\; DU \in H_0\}. \tag{6.19}$$

The projection onto $H_0 \times H_0$

$$(U, DU) \mapsto (\tilde{U}, \tilde{V})$$

can be found by solving the linear system

$$\tilde{V} \;=\; DU, \quad \int_M (U - \tilde{U}, \phi) = 0 \quad \forall\, \phi \in H_0 \tag{6.20}$$

or equivalently

$$\tilde{V} \;=\; DU, \quad \int_C (U - \tilde{U}, \phi') = 0 \quad \forall\, \phi' \in H'_{\nu_0}. \tag{6.21}$$

Denote this symplectic reduction of Γ_T by L_T. There is a natural isomorphism of $C^{p',q'}$-modules

$$|_C : H_{\nu_0} \times H_{\nu_0} \to H'_{\nu_0} \times H'_{\nu_0} \quad (U, V) \mapsto (U\,|_C, V\,|_C). \tag{6.22}$$

We claim that

$$L_T\,|_C = \; L_S. \tag{6.23}$$

It suffices to show that $L_S \subset L_T\,|_C$ since

$$\dim L_s = \dim L_T = \frac{1}{2} \dim H_{\nu_0}.$$

Let $(\tilde{U}', \tilde{V}') \in L_S$. It is the projection of some element (U', DU') as in (6.16) such that (6.17) is satisfied. Thus

$$U' \in Dom(S_y)\,, \quad DU' \in H'_{\nu_0}.$$

Consider $U \; =. \mathcal{Z}_D(U')$ defined as in (6.13). From the construction of U we deduce $U\,|_{\Sigma \times \{0\}} \in \Lambda_y$ (i.e. $U \in Dom\,(T_y)$) and

$$DU \;=\; \begin{cases} DU' & , \quad \text{on } C \\ 0 & , \quad \text{elsewhere} \end{cases}$$

so that $DU \in \mathcal{Z}(H'_{\nu_0}) \;=\; H_{\nu_0}$ i.e.

$$(U, DU) \in \Gamma_{T_y} \cap (W_0)^0_+.$$

Let $(\tilde{U}, \tilde{V})$ denote the orthogonal projection of (U, DU) onto $H_{\nu_0} \times H_{\nu_0}$. By definition

$$(\tilde{U}, \tilde{V}) \in L_T.$$

Obviously $\tilde{V}\,|_C = (DU)\,|_C = DU' = \tilde{V}'$ and from (6.21) we deduce

$$\int_C (U\,|_C - \tilde{U}\,|_C, \phi) = \quad \forall \phi \in H'_{\nu_0}$$

or

$$\int_C (U' - \tilde{U}\,|_C, \phi) = 0 \quad \forall \phi \in H'_{\nu_0}$$

which by (6.17) is equivalent to

$$\tilde{U}\,|_C = \tilde{U}' \;,\; \tilde{V}\,|_C = \tilde{V}'$$

and (6.23) is proved. Now

$$\mathbf{ind}_{p,q}(T) = \mu_{p',q'}(\Gamma_0, \Gamma_T) = \mu_{p',q'}(H_{\nu_0}, L_T)$$

and via the map $|_C$

$$\mu_{p',q'}(H_{\nu_0}, L_T) = \mu_{p',q'}(H'_{\nu_0}, L_S) = \mathbf{ind}_{p,q}(S).$$

Step 2 is completed.

Step 3: Deforming the boundary conditions. We now use Lemma 6.3 for the manifold C with boundary $-\Sigma \times \{-1\} \cup \Sigma \times \{0\}$. Thus in the definition of S_y we can replace Λ_y with $L^E_y \oplus \mathcal{H}^E_>$ on $\Sigma \times \{0\}$ and Λ_{-1} with $L^E_{-1} \oplus \mathcal{H}^E_<$ on $\Sigma \times \{-1\}$, where L^E_{-1} is the symplectic reduction of $\Lambda_M(D)$ mod $\mathcal{H}^E_>$. (One uses $\mathcal{H}^E_<$ on $\Sigma \times \{-1\}$ instead of $\mathcal{H}^E_>$ since $\Sigma \times \{-1\}$ and $\Sigma \times \{0\}$ have different transversal orientations).

Along C the operator D has the form

$$D = J(\frac{d}{ds} + D_0).$$

Note that $\mathcal{H}^E_0$ is D_0 invariant and set $A = D_0\,|_{\mathcal{H}^E_0}$. Finally set $\Lambda_0 = \Lambda_M(D)$ and let L_0 denote the symplectic reduction of Λ_0 mod $\mathcal{H}^E_>$. As in [N1] we have the important relation

$$L^E_{-1} = e^A L^E_0. \tag{6.24}$$

Step 4: Reduction to a Floer family. We use the notations introduced at Step 3. Consider the Floer family

$$\Xi_y : Dom(\Xi_y) \subset L^2([-1,0]; \mathcal{H}^E_0) \to L^2([-1,0]; \mathcal{H}^E_0) = \hat{H}$$

where

$$Dom(\Xi_y) = \{u \in L^2_1([-1,0]; \mathcal{H}^E_0) \;/\; u(-1) \in L^E_{-1} \;,\; u(0) \in L^E_y\}$$

$$\Xi_y(u) = J(\frac{du}{ds} + A).$$

We will show via a suitable symplectic reduction that

$$\mathbf{ind}_{p,q}(\Xi) = \mathbf{ind}_{p,q}(S). \tag{6.25}$$

First note that $\hat{H}$ embeds as a $C^{p,q'}$-module in $H' = L^2(\mathcal{E}\,|_C)$. Let

$$\Theta = \{U \in \hat{H} \;/\; \dot{U} + AU = 0\}.$$

This is a finite dimensional $C^{p,q'}$ submodule in $\hat{H} \subset H'$. Let $\hat{W}$ denote the orthogonal complement of Θ in $\hat{H}$ and W' its orthogonal complement in H'. $\hat{W}$ is a (p',q')-isotropic subspace in $\hat{H} \times \hat{H}$ and W' is a (p',q')-isotropic subspace in $H' \times H'$. Clearly Γ_{Ξ_y} is clean mod $\hat{W}$, $\forall y$ and since $\ker(T_y) \subset \hat{H}$, $\forall y \in Y$ we deduce that Γ_{T_y} is clean mod W', $\forall y$. Using the Fourier decomposition in terms of the eigenvectors of D_0 we deduce

$$D^{-1}(\Theta) \subset \hat{H}. \tag{6.26}$$

(6.26) implies immediately that the reduction of Γ_{T_y} mod W' coincides with the reduction of Γ_{Ξ_y} mod $\hat{W}$ and (6.25) follows from Thm. 4.14, 5.5.

Step 5: Conclusion. Putting together all the informations we have obtained so far we deduce

$$\mathbf{ind}_{p,q}(T) = \mathbf{ind}_{p,q}(\Xi). \tag{6.27}$$

Consider the deformation

$$\Xi^t_y : Dom(\Xi^t_y) \subset \hat{H} \to \hat{H}$$

$$Dom(\Xi^t_y) = \{u \in L^2_1([-1,0];\mathcal{H}^E_0) \;/\; u(-1) \in e^{-tA}L^E_{-1}\ ,\ u(0) \in L^E_y\}$$

$$\Xi^t_y(u) = J(\frac{du}{ds} + (1-t)Au).$$

Since $A = D_0\,|_{\mathcal{H}^E_0}$ is selfadjoint and anticommutes with $\overline{J}_i$ and $\overline{C}_j$ we deduce that $e^{-tA}L^E_{-1}$ is a (p',q') lagrangian in $\mathcal{H}^E_0$. Thus Ξ^t_y is a selfadjoint operator in $\mathcal{F}^{p,q}$ and by Proposition C.1 we deduce that $\Xi^{\cdot}_{\cdot}$ is a continuous family in the gap topology. Thus

$$\mathbf{ind}_{p,q}(\Xi^0_{\cdot}) = \mathbf{ind}_{p,q}(\Xi^1_{\cdot}).$$

Using (6.24) we see that

$$Dom(\Xi^1_{\cdot}) = \{u \in L^2_1([-1,0];\mathcal{H}^E_0 \;/\; u(-1) \in L^E_0\ ,\ u(0) \in L^E_{\cdot}\}$$

$$\Xi^1 = J\frac{du}{ds}.$$

This is precisely the Floer family we used in the proof of Theorem 5.5 where we showed

$$\mathbf{ind}_{p,q}(\Xi^1_{\cdot}) = \mu_{p',q'}(L^E_0, L^E_{\cdot}). \tag{6.28}$$

Recall now that L_0^E is the symplectic reduction of $\Lambda_M(D)$ mod $\mathcal{H}_>^E$ and L^E is the symplectic reduction of Λ_y mod $\mathcal{H}_<^E = \mathcal{H}_>^{E\#}$. Theorem 6.2 follows from (6.27)-(6.28) coupled with Theorem 4.14 (or rather Remark 4.15). □

Remark 6.4 *(a) The deformation we used in the last step has the nice property that if Ξ_y^0 is invertible then Ξ_y^t stays invertible for all t. We will need this in the proof of the splitting formula.*
(b) The result at Step 4 when $p = 1$, $q = 0$ and $Y = S^1$ is discussed from a different perspective in [FO2].
(c) Recently X. Dai and W. Zhang ([DZ]) have proved a complex version of our Theorem 6.2. They consider families of complex Dirac operators parameterized by smooth manifolds which may not have constant symbol as the families in Theorem 6.2 did. However, in the next subsection we will use a result of [MP] to eliminate this condition from the statement of the theorem.
(d) The results of Quillen [Q] provide an explicit construction of the Chern character in a context very similar in spirit with ours: in [Q] as in the present paper the elements in K-theory measure the "homotopical distance" between two lagrangian subbundles in a bundle of Clifford modules as opposed to the more traditional, analytical point of view that K-theory measures the nonexactness of a sequence of bundle morphisms. The "isomorphism" between these two points of view is given by the graph map or the Cayley transform, in Quillen's terminology. This opens possibility of re-obtaining the results of [MP] (and eventually an odd version) from Theorem 6.2.

§6.3 The cobordism invariance of the families index As we have already mentioned the families considered in Theorem 6.2 have constant symbol. In this section we explain how one can eliminate this assumption from the hypotheses of the theorem.

First, a careful analysis of the proof of the theorem shows this assumption enters essentially only at Step 3 when we use Lemma 6.3 which shows that if the operators D_y are *independent* of $y \in Y$ then the boundary value conditions can be deformed to generalized APS conditions. The constant symbol hypothesis is used only to show that the original family can be deformed (via the obvious affine homotopy) to a family of Dirac operators which admit generalized APS conditions. The theorem is true for more general families of operators which admit families of boundary conditions of Atiyah-Patodi-Singer type.

A better way of phrasing this condition is to use the notion of spectral section introduced in [MP]. To define it we first need to introduce slightly more general families of Dirac operators.

Consider a fibration

$$p : Z \to Y$$

over a compact CW-complex Y in which the fibers are *smooth manifolds* diffeomorphic with a compact smooth manifold N with boundary ∂N (possibly empty). We assume the transition maps of this fibration are continuous families of diffeomorphisms of $(M, \partial M)$. Equip Z with Riemann metrics (g_y) along the fibers $p^{-1}(y)$ which vary continuously in the C^∞-topology. If $\partial N \neq \emptyset$ we assume the metrics have the same nice behavior near the boundary as in Subsection 6.1.

Consider a vector bundle $\mathfrak{E} \to Z$ such that its restriction to the fibers $p^{-1}(y)$ is a Dirac bundle $\mathcal{E}_y$. We thus get a bundle

$$\mathfrak{D}^{p,q} \to Y$$

with fibers $\mathcal{D}^{p,q}_y = \mathcal{D}^{p,q}(\mathcal{E}_y)$ topologized using a sufficiently smooth Sobolev topology as in Subsection 6.1. The spaces $L^2(\mathcal{E}_y)$ form a bundle of $(p, q+1)$ Hilbert modules over Y.

A continuous family of Dirac operators is then a continuous section $(D(y))_{y\in Y}$ of $\mathfrak{D}^{p,q}$. Assume now the fibers $p^{-1}(y)$ are manifolds without boundary.

Definition 6.5 *A (p,q)-* **spectral section** *of the family $(D(y))$ is a family (P_y) of selfadjoint projections of $L^2(\mathcal{E}_y)$ satisfying the following conditions.*
(i) Each P_y is a zeroth order pseudodifferential operator.
(ii) The family $y \mapsto P_y$ is ψ-continuous.
(ii) The range of P_y is a $(p, q+1)$ lagrangian in $L^2(\mathcal{E}_y)$.
(iii) There exists a **spectral cut** *i.e. continuous function*

$$R : Y \to [0, \infty)$$

such that for every $y \in Y$

$$D(y)u = \lambda u \Rightarrow \begin{cases} P_y u = u & \text{if} \quad \lambda > R(y) \\ P_y u - 0 & \text{if} \quad \lambda < -R(y) \end{cases} .$$

Exactly as in [MP] one can prove the following result.

Proposition 6.6 *The family $(D(y))_{y\in Y}$ admits a spectral section if and only if*

$$\mathbf{ind}_{p,q}(D(y)) = 0.$$

We want to make two observations.
1. The reflection $R_y = 2P_y - 1$ anticommutes with $D(y)$ up to finite rank operators. We can roughly express this as an "excess of symmetry".
2. The spaces $\mathcal{H}^{R(y)}_{>}$ are $(p, q+1)$ isotropic subspaces in $L^2(\mathcal{E}_y)$ and the symplectic reductions of $L^2(\mathcal{E}_y)$ mod $\mathcal{H}^{R(y)}_{>}$ form a $C^{p,q+1}$-bundle over Y.

We interrupt a little the flow of arguments to illustrate on a simple situation the reason why the "excess of symmetry" in the above proposition implies the vanishing of the index.

Consider the case $p=0$, $q=0$ and $Y=\{pt\}$. Thus we have an odd Dirac operator on a Clifford superbundle $\mathcal{E}=\mathcal{E}_+\oplus\mathcal{E}_-$. Denote by η the $\mathbb{Z}_2$-grading of this bundle and set $H=L^2(\mathcal{E})$. Because D anticommutes with η we deduce that for any $E\geq 0$ η defines an isomorphism

$$\mathcal{H}^E_> \cong \mathcal{H}^{-E}_<.$$

A spectral section with spectral cut $E\geq 0$ defines in this case a $(0,1)$-lagrangian in the space $\mathcal{H}_{[-E,E]}$ spanned by eigenvectors corresponding to eigenvalues in $[-E,E]$. Denote by R the $(0,1)$ grading which corresponds to this lagrangian. Since R anticommutes with $\eta_E=\eta\,|_{\mathcal{H}_{[-E,E]}}$ we deduce

$$\dim\ker(1-\eta_E)\cong\dim\ker(1+\eta_E).$$

In other words

$$ind_{0,0}D=\operatorname{tr}\eta_E=0.$$

Consider a family $\{D(y)\}_{y\in Y}$ of (p,q) Dirac operators on a manifold M with boundary $\Sigma=\partial M$. Assume that for each $y\in Y$ the operator $D(y)$ has the cylindrical form described at the beginning of this section. Denote by $D_0(y)$ the restriction of $D(y)$ to the boundary and set $\mathcal{E}_0=\mathcal{E}_0(y)$ the restriction of $\mathcal{E}_(y)$ to Σ. As we have seen $\mathcal{E}_0(y)$ has a a natural (p',q') structure ($p'=p+1$, $q'=q+1$) and $\{D_0(y)\}$ is a a continuous family of (p',q)-Dirac operators. We want to prove the following result.

Proposition 6.7 *The family $\{D_0(y)\}_{y\in Y}$ admits a (p',q) spectral section.*

Proof Let us first describe the idea of the proof. Attach to $M(y)$ a long cylinder

$$C_r(y)=[0,r]\times\Sigma(y).$$

Denote the resulting manifold by $M_r(y)$ and the obvious extension of $D(y)$ to $M_r(y)$ by $D(Y;r)$. The operator $D(y,r)$ defines a new CD space $\Lambda_r(y)\subset L^2(\mathcal{E}_0(y)$. In [N1] we showed that for any $y\in Y$ the space $\Lambda_r(y)^\perp$ (or rather the corresponding orthogonal projection) converges as $r\to\infty$ to a spectral section of $D_0(y)$ we denote by $\Lambda_\infty(y)^\perp$. We now have a possible candidate for a spectral section of the family $D_0(y)$. Unfortunately the collection $\{\Lambda_\infty(y)^\perp\}$ may not be a continuous family of (p',q')-lagrangians since the above convergence may not be uniform. However, for r sufficiently large the family $\{\Lambda_r(y)^\perp\}$ is continuous and it is very close to being a spectral section (in a sense to be elaborated below).

With this almost spectral section in our hands we will use the same arguments used in the proof of Proposition 1 of [MP] to show that the index of the family $D_0(y)$ in $KO^{p',q}(y)$ is trivial. In other words we establish a general result the *cobordism invariance of the index of families.* At this point we can invoke Proposition 6.6 to conclude that the family $\{D_0(y)\}$ does admit a bona fide spectral section. Now let us fill in the details.

Definition 6.8 *Let $\varepsilon > 0$. A (p', q)-ε-***almost spectral section** *of the family $(D_0(y))$ is a family (P_y) of selfadjoint projections of $L^2(\mathcal{E}_0(y))$ satisfying the following conditions.*
(i) Each P_y is a zeroth order pseudodifferential operator with the same symbol as the orthogonal projection onto $\Lambda(y)$.
(ii) The family $y \mapsto P_y$ is ψ-continuous.
(ii) Range P_y *is a $(p, q+1)$ lagrangian in $L^2(\mathcal{E}_y)$.*
(iii) There exists an **almost spectral cut** *i.e. an upper semicontinuous function*

$$R : Y \to [0, \infty)$$

such that for every $y \in Y$

$$D(y)u = \lambda u \Rightarrow \begin{cases} \|P_y u - u\| \le \varepsilon \|u\| & \text{if} \quad \lambda > R(y) \\ \|P_y u\| \le \varepsilon \|u\| & \text{if} \quad \lambda < -R(y) \end{cases} .$$

The proof of the adiabatic limit result of [N1] also produces the following weaker conclusion.

Lemma 6.9 *There exists $\nu \ge 0$ with the following property: for any $\varepsilon \ge 0$ there exists $r > 0$ (sufficiently large) so that $\Lambda_r(y)^\perp$ is an ε-almost spectral section with an almost spectral cut $R(y)$ satisfying*

$$\max\{R(y)\ ;\ y \in Y\} \le \nu.$$

In the terminology of [N1], the number ν plays the role of a common nonresonance level for the family $D_0(y)$.

Set as in [MP]

$$Q_y = D_0(y)(1 + D_0(y)^2)^{-1/2}.$$

This is a family in $\mathcal{BF}^{p',q}$ with the same index as $\{D_0(y)\}$. Denote by $P_r(y)$ the orthogonal projection onto $\Lambda_r(y)^\perp$. It is not difficult to see that the pseudodifferential operators Q_y and

$$Q_r'(y) = P_r(y) Q_y P_y + (1 - P_r(y)) Q_y (1 - P_r(y))$$

have the same symbol and thus they determine the same element in $KO^{p',q}(Y)$. Now form

$$\hat{Q}_{s,r}(y) = P_r(y)\{Q_y + sG(y)\}P_y + (1 - P_r(y))\{Q_y - sG(y)\}(1 - P_r(y))$$

where $G(y)$ is the compact operator $G(y) = (1 + D_0(y))^{-1/2}$. Hence

$$\mathbf{ind}_{p',q}(D_0(y) = \mathbf{ind}_{p',q}(\hat{Q}_{r,s}(y)), \quad \forall r, s \ge 0.$$

Let ν as in Lemma 6.9 and fix $s \geq \nu$ such that

$$m_+(\nu) = \min\{\frac{s+\lambda}{(1+\lambda^2)^{1/2}} \; ; \; |\lambda| \leq \nu\} > 2$$

and

$$m_-(\nu) = \min\{\frac{s-\lambda}{(1+\lambda^2)^{1/2}} \; ; \; |\lambda| \leq \nu\} > 2.$$

By Lemma 6.9 $P_r(y)$ is an ε-almost spectral section with an almost spectral cut $R(y)$ such that $R(y) \leq \nu$, $\forall y$. One can now elementarily verify that if ε satisfies the inequalities

$$\varepsilon^2 < \frac{1}{2}$$

and

$$\frac{2}{1-\varepsilon^2} < \min\{m_+(\nu), m_-(\nu)\}$$

then $\hat{Q}_{r,s}(y)$ is positive definite on the range of $P_r(y)$ and negative definite on the range of $(1 - P_r(y))$ so that it is invertible for each y. In particular the family can be deformed to a family of (p', q) gradings. Since the space of (p', q)-gradings (i.e. (p', q)-lagrangians is contractible this shows

$$\mathbf{ind}_{p',q}(D_0(s)) = 0$$

so that by Proposition 6.6 the family $\{D_0(y)\}$ admits a spectral section. The proof of Proposition 6.7 is complete. □

As we have mentioned at the beginning of this subsection Theorem 6.2 is valid for more general families of Dirac operators with varying symbols. The structure of the proof in this more general situation is the same. Step 1 no longer applies. Step 2 extend word for word for word to this more general situation. The only change occurs at Step 3 when instead of deforming the boundary conditions to a generalized Atiyah-Patodi-Singer family we deform the to a spectral section whose spectral cut is sufficiently large so we can perform *clean* symplectic reductions. In the process we will obtain families of Floer operators with varying symbols. As we have noted in Section 5 this situation leads to the same conclusion.

§6.4 Gluing formulæ We conclude this section with an application of the ideas involved in the proof of Thm. 6.2.

Consider a closed, compact, oriented Riemann manifold (M, g) and let $\mathcal{E} \to M$ be a $C^{p,q}$-Dirac bundle. Suppose that inside M sits an oriented hypersurface which decomposes M into two manifolds-with-boundary $M = M_1 \cup M_2$ (Fig. 4))

Moreover assume that the metric is a product in a tubular neighborhood $N = (-1, 1) \times \Sigma$. Denote by $\mathcal{D}^{p,q}$ the space of selfadjoint Dirac operators D in $\mathcal{E}$ such that

$$D_i = D\,|_{M_i} \in \mathcal{D}^{p,q}(M_i) \quad i = 1, 2.$$

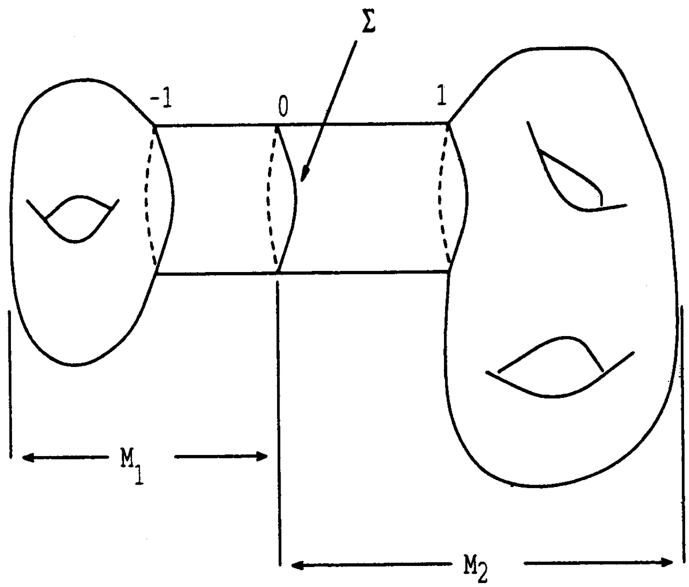

Figure 4: *A metrically nice splitting*

As before $\mathcal{D}^{p,q}(\mathcal{E})$ is an affine space which can be topologized using a L^2_σ metric ($L^2_\sigma \hookrightarrow C^2$). For any $D \in \mathcal{D}^{p,q}$ we get a pair of CD spaces

$$\Lambda_i(D) = \Lambda_{M_i}(D_i) \ , \ i = 1,2.$$

These are (p', q') isotropic spaces in $L^2(\mathcal{E}_0)$ and moreover the pair $(\Lambda_1(D), \Lambda_2(D))$ is Fredholm. In fact, both $\Lambda_i(D)$ are pseudodifferential lagrangians and their (principal) symbols are independent of D. Denote the symbol of Λ_1 by π_- and by the symbol of $\Lambda_2(D)$ by π_+ . Then we have (see [BW3])

$$\pi_- + \pi_+ = 1.$$

Thus we get a map

$$\Lambda^{(2)} : \mathcal{D}^{p,q} \to \Psi\mathcal{L}^{p',q'}_{\pi_-} \times \Psi\mathcal{L}^{p',q'}_{\pi_+} (\subset \mathcal{FL}^{p',q'})$$

$$D :\mapsto (\Lambda_1(D), \Lambda_2(D)) \tag{6.29}$$

which by Lemma C.5 is continuous (in the ψ-convergence topology).

Topologically $\mathcal{D}^{p,q}$ is uninteresting since it is contractible. However, inside $\mathcal{D}^{p,q}$ there is an open set

$$\mathcal{D}^{p,q}_* = \{D \in \mathcal{D}^{p,q} \ / \ D \text{ invertible}\}$$

which we assume is nonempty. The relative topology of the pair becomes interesting. In fact to any map

$$f : (D^n, S^{n-1}) \to (\mathcal{D}^{p,q}, \mathcal{D}^{p,q}_*) \quad y \mapsto D_y.$$

Associated to it is the index $ind_{p,q}(f) = KO^{p',q'}(D^n, S^{n-1})$. Using the map $\Lambda^{(2)}$ of (6.30) we get a generalized Maslov index

$$\mu_{p',q'}(\Lambda^{(2)}(f)) \in KO^{p',q'}(D^n, S^{n-1}).$$

We can now formulate the last result of this paper.

Theorem 6.10 (Splitting formula for the index of families)
For any continuous map $f : (D^n, S^{n-1}) \to (\mathcal{D}^{p,q}, \mathcal{D}^{p,q}_*)$ we have the equality

$$ind_{p,q}(f) = \mu_{p',q'}(\Lambda^{(2)}(f)).$$

Proof Let $f : y \mapsto D(y),\quad y \in D^n$. Consider the family of bôundary value problems

$$T_y = T(D_1(y), \Lambda_2(y)) \quad \Lambda_2(y) = \Lambda_2(D(y))$$

We get a continuous map

$$T : (D^n, S^{n-1}) \to (\mathcal{BP}^{p,q}, \mathcal{BP}^{p,q}_*)$$

where

$$\mathcal{BP}^{p,q}_* = \{(D, \Lambda) \in \mathcal{D}^{p,q} \times \Psi\mathcal{L}^{p',q'}_{\pi_+} \;/\; T(D,\Lambda) \text{ is invertible}\}.$$

Proceeding exactly as in Step 2 of the proof of Thm.6.2 we deduce

$$ind_{p,q}(f) = ind_{p,q}(T) \text{ in } KO^{p',q'}(D^n, S^{n-1}). \tag{6.30}$$

Thus it suffices to show that

$$ind_{p,q}(T) = \mu_{p',q'}(\Lambda^{(2)}(f)) \text{ in } KO^{p',q'}(D^n, S^{n-1}) \tag{6.31}$$

This does not follow directly from Thm. 6.2 which refers to the absolute KO-groups and we have to prove a relative version. The strategy is however completely similar. A little care is needed to make sure that all deformations we perform are relative deformations.

The first problem is at Step 1 when the family

$$y \mapsto T(D_1(y), \Lambda_2(y))$$

is deformed to a family in which $D_1(y)$ is independent of y. The affine deformation we used may not be a relative deformation. This deficiency can be easily removed using Proposition C.7 which shows that $\mathcal{BP}^{p,q}_*$ is contractible. Hence in proving the relative version of Thm. 6.2 we may as well start at the very beginning with a family

$$T : (D^n, S^{n-1}) \to (\mathcal{BP}^{p,q}, \mathcal{BP}^{p,q}_*)$$

such that $T\,|_{S^{n-1}}$ is constant

$$T(D_1(y)\,,\,\Lambda(y)) \;=\; T(\tilde{D}_1,\tilde{\Lambda}) \quad \forall y\in S^{n-1}$$

where $\tilde{\Lambda}$ is an APS lagrangian such that the pair $(\Lambda_{M_1}(\tilde{D}_1)\,\tilde{\Lambda})$ is a transversal Fredholm pair. Now we can linearly deform the whole family $D_1(y)$ to $\tilde{D}_1$.

The deformations we make at Step 3 are now relative deformations. The only problem that might occur is the final deformation at Step 5. However by Remark 6.4(a) these are relative deformations. This proves (6.31). Theorem 6.10 follows using (6.30). □

Remark 6.11 *For p=q=0 and n=0 this result is proved in [BW1-2]. For p=1, q=0 and n=1 this result is proved in different ways in [BF],[Bu],[CLM2] and [N1]. The complex case is discussed in [DZ]. There they deal with arbitrary families (with possibly nonconstant symbols) but they use different approaches depending on the parity of* $p-q$. *Our approach achieves this in "one shot" and extends verbatim to the complex case. Although we stated the above theorem only for families with constant symbol the result is valid for more general families of Dirac operators with varying symbols. In fact the same proof can be slightly altered to cover this more general situation and as in the case of boundary value problems a key ingredient is the result on the existence of spectral sections we established in the previous subsection. We leave the reader perform the necessary modifications.*

A Gap convergence of linear operators

We will prove here some technical results about gap convergence we used in the main body of the paper.

Let H be a real separable Hilbert space. Consider the space $\mathcal{C}(H)$ of densely defined, closed linear operators

$$T : Dom(T) \subset H \to H.$$

For each T we let Γ_T denote its graph

$$\Gamma_T = \{(u, Tu) \in H \times H \; / \; u \in Dom(T)\}.$$

Following [K], Chap.IV we introduce

Definition A.1 *(a) Let $L, M \subset H$ be two closed subspaces of H. Set*

$$\delta(L, M) = \sup\{dist(x, M) \; / \; x \in L \; , \; |x| = 1\}$$

and

$$\hat{\delta}(L, M) = \max(\delta(L, M), \delta(M, L)).$$

$\hat{\delta}(L, M)$ is called the **gap** *between the subspaces L and M.*
(b) A sequence $(T_n) \subset \mathcal{C}(H)$ is said to be **gap convergent** *to T (and we write this as $T_n \xrightarrow{g} T$ if*

$$\hat{\delta}(\Gamma_{T_n}, \Gamma_T) \to 0 \quad \text{as } n \to \infty.$$

By Theorem IV.2.18 of [K]

$$T_n \xrightarrow{g} T \iff \max(\delta(\Gamma_{T_n}, \Gamma_T), \delta(\Gamma_{T_n^*}, \Gamma_{T^*})) \to 0$$

and thus for selfadjoint operators $\delta \equiv \hat{\delta}$. Let us describe explicitly what $\delta(\Gamma_{T_n}, \Gamma_T) \to 0$ really means.

Consider $S, T \in \mathcal{C}(H)$. We first describe $\delta(S, T)$. Let π_T denote the orthogonal projection onto Γ_T. If $(x, Sx) \in \Gamma_S$ then

$$dist\{(x, Sx), \Gamma_T\} = \|(x, Sx) - \pi_T(x, Sx)\|.$$

Note that $(x, Sx) - \pi_T(x, Sx) \in (\Gamma_T)^\perp$ and from [K], III.5.91 one deduces

$$(\Gamma_T)^\perp = \{(-T^*u, u) \; / \; u \in Dom(T^*)\}.$$

Thus there exists $u \in Dom(T^*)$ and $y \in Dom(T)$ such that

$$\pi_T(x, Sx) = (y, Ty) \; , \; (x, Sx) - \pi_T(x, Sx) = (-T^*u, u)$$

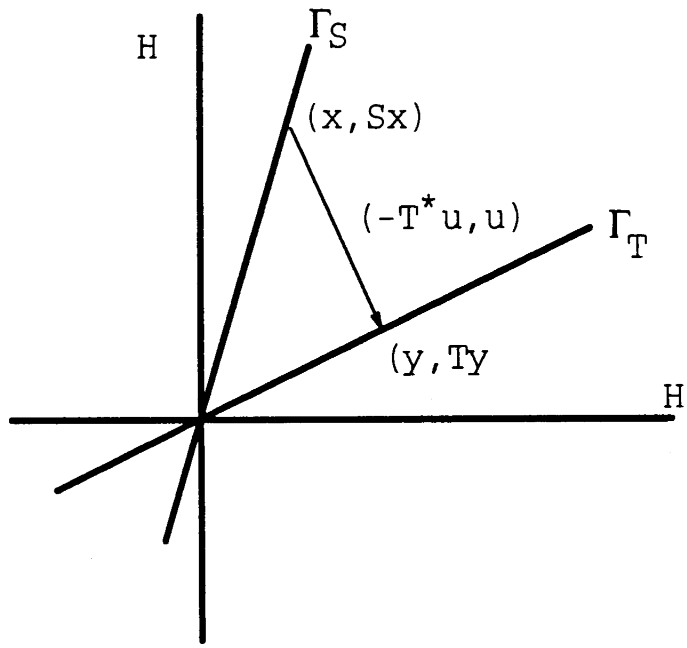

Figure 5: *Computing distances between graphs of linear operators*

so we can decompose (x, Sx) uniquely as (Fig. 5)

$$(x, Sx) = (y, Ty) + (-T^*u, u).$$

The pair (u, y) can be viewed as the unique solutions of the system

$$\left\{ \begin{array}{lllll} T^*u & - & y & = & -x \quad u \in Dom(T^*) \\ u & + & Ty & = & Sx \quad y \in Dom(T) \end{array} \right. . \tag{A.1}$$

Then

$$\delta(S,T)^2 = \sup_{x \in Dom(S)\backslash 0} \{\frac{|u|^2 + |T^*u|^2}{|x|^2 + |Sx|^2}\} \tag{A.2}$$

where u is determined from x via (A.1). Thus $\delta(T_n, T) \to 0$ iff

$$\lim_{n\to\infty} \sup_{x \in Dom(T_n)\backslash 0} \{\frac{|u|^2 + |T^*u|^2}{|x|^2 + |T_n x|^2}\} = 0$$

or equivalently,

$$\lim_{n\to\infty} \sup_{x \in Dom(T_n)\backslash 0} \{\frac{|u|^2 + |T^*u|^2}{|y|^2 + |T^*y|^2}\} = 0. \tag{A.3}$$

Above, u and y are determined from x using (A.1).

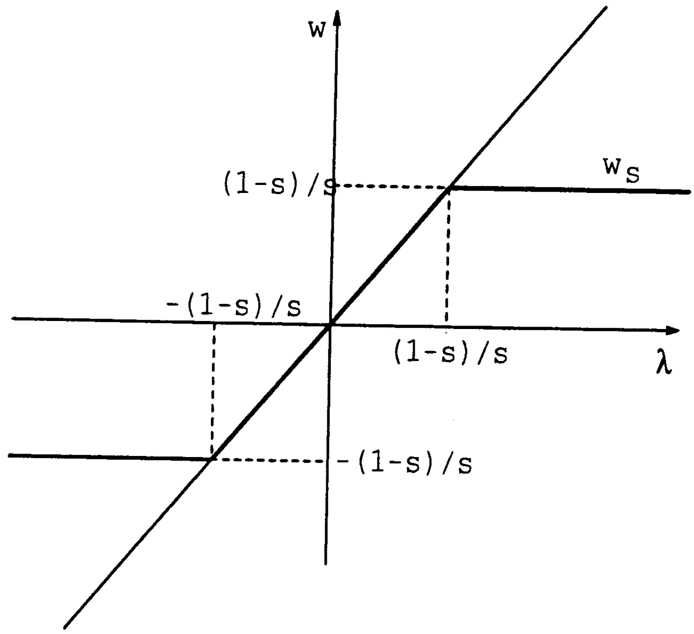

Figure 6: *w_s approximates $1_{\mathbb{R}}$ as $s \to \infty$*

In the remaining part of this section all operators will be assumed Fredholm and selfadjoint. If T is such an operator consider

$$\mathcal{D}(T) = Dom\,(T) \times Dom\,(T) \subset H \times H \to H \times H$$

which has the block form

$$\mathcal{D}(T) = \begin{bmatrix} T & -I \\ I & T \end{bmatrix}.$$

One sees easily that $\mathcal{D}(T)$ is Fredholm and it has a bounded inverse which is

$$\mathcal{D}(T)^{-1} = \begin{bmatrix} T(1+T^2)^{-1} & (1+T^2)^{-1} \\ -(1+T^2)^{-1} & T(1+T^2)^{-1} \end{bmatrix}$$

and obviously

$$\|\mathcal{D}(T)^{-1}\| \leq 1. \tag{A.4}$$

For $0 \leq s \leq 1/2$ consider the continuous function $w_s : \mathbb{R} \to \mathbb{R}$ defined by (Fig. 6)

$$w_s(\lambda) = \begin{cases} \lambda & \text{if} \quad |\lambda| \leq (1-s)/s \\ (1-s)/s & \text{if} \quad \lambda > (1-s)/s \\ -(1-s)/s & \text{if} \quad \lambda < -(1-s)/s \end{cases}.$$

w_s has the following nice properties

$$w_s \text{ is odd and bounded.} \tag{A.5}$$

$$w_s(\lambda) \to \lambda \text{ as } s \to 0 \quad \forall \lambda \in \mathbb{R} \tag{A.6}$$

$$|\Delta_s(\lambda|^2 = \frac{|\lambda - w_s(\lambda|^2}{(1+\lambda^2)(1+w_s(\lambda)^2)} \le 2s^2 \quad \forall s \in [0,1/2]\,,\ \forall \lambda \in \mathbb{R}. \tag{A.7}$$

Proof of (A.7) For $|\lambda| \le (1-s)/s$ this is trivial. For $|\lambda| > (1-s)/s$ we have

$$|\Delta_s(\lambda|^2 = \frac{|\lambda - (1-s)/s|^2}{1+\lambda^2} \cdot \frac{s^2}{s^2 + (1-s)^2} \le 2s^2.$$

For any selfadjoint operator T set $T_s = w_s(T)$. By (A.5) the operators T_s is a bounded.

Lemma A.2 *For any Fredholm selfadjoint operator T we have*

$$\delta(T_s, T) \le \sqrt{2}s.$$

Proof We have to estimate

$$\frac{|u|^2 + |T_s u|^2}{|x|^2 + |Tx|^2}$$

where u is the solution of

$$\mathcal{D}(T_s)\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} -x \\ Tx \end{bmatrix}.$$

Hence

$$\begin{bmatrix} u \\ y \end{bmatrix} = \mathcal{D}^{-1}(T_s)\begin{bmatrix} -x \\ Tx \end{bmatrix}$$

so that

$$u = -T_s(1+T_s^2)^{-1}x + (1+T_s)^{-1}Tx = (1+T_s^2)^{-1}(T-T_s)x.$$

It follows that

$$|u|^2 + |T_s u|^2 = |(1+T_s^2)^{1/2}u|^2 = |(1+T_s^2)^{-1/2}(T-T_s)x|^2.$$

Using the spectral resolution for T we get

$$|x|^2 + |Tx|^2 = |(1+T^2)^{1/2}x|^2.$$

Set $z = (1+T^2)^{1/2}x$. Then

$$\frac{|u|^2 + |T_s u|^2}{|x|^2 + |Tx|^2} = \frac{|(1+T_s^2)^{-1/2}(T-T_s)(1+T^2)^{-1/2}z|^2}{|z|^2}$$

$$= \frac{|\Delta_s(T)z|^2}{|z|^2} \le 2s^2.$$

Lemma A.2 is proved. □

B Gap continuity of families of BVP's for Dirac operators

Let M be a compact oriented Riemann manifold with boundary $\Sigma = \partial M$, $\mathcal{E} \to M$ a bundle of selfadjoint $C(M)$-modules (see [BGV] for more details about the construction of Dirac operators). Set $\mathcal{E}_0 = \mathcal{E}\mid_\Sigma$. The space of Dirac operators on $\mathcal{E}$ is an affine space modelled by End $(\mathcal{E})$ which as usual we topologize using a Sobolev metric L^2_σ such that $L^2_\sigma \hookrightarrow C^3$. Now consider the following data
(i) A sequence of formally selfadjoint Dirac operators D_n converging in the above topology to a (formally selfadjoint) Dirac D.
(ii) A sequence of zeroth order pseudodifferential operators P_n on $\mathcal{E}_0$ which *ψ-converge* to a zeroth order pseudodifferential operator P i.e.

$$P_n \to P \text{ as bounded operators } L^2_\sigma \to L^2_\sigma \quad \sigma = 0, 1/2.$$

Assume
(iii) The operators P_n resp. P are selfadjoint elliptic boundary conditions for D_n and resp. D in the sense described in [BW3].
This means that the operators

$$T_n : Dom(T_n) \subset L^2(\mathcal{E}) \to L^2(\mathcal{E})$$

defined by

$$Dom(T_n) = \{U \in L^2_1(\mathcal{E}) \;/\; P_n U\mid_\Sigma = 0\} \quad T_n U = D_n U$$

are selfadjoint and Fredholm. Define T similarly.

Proposition B.1 *Let $(D_n, P_n)_{n>0}$ and (D,P) satisfy conditions (i)-(iii) above. Then*

$$T_n \xrightarrow{g} T$$

Proof We have to show that $\delta(\Gamma_{T_n}, \Gamma_T) \to 0$. We argue by contradiction. Assume the contrary. Thus there exists a sequence $x_n \in Dom\,(T_n)$

$$|x_n|^2 + |D_n x_n|^2 = 1 \tag{B.1}$$

and

$$\lim_{n\to\infty} \frac{|u_n|^2 + |Du_n|^2}{|y_n|^2 + |Dy_n|^2} = \delta \in (0, \infty] \tag{B.2}$$

$$\left\{ \begin{array}{rcrcl} Du_n & - & y_n & = & -x_n \\ u_n & + & Dy_n & = & D_n x_n \end{array} \right. . \tag{B.3}$$

We will need the following auxilliary result:

Lemma B.2 *There exists* $C > 0$ **independent of n** *such that*

$$|x|_1 \leq C(|x| + |D_n x|) \quad \forall x \in Dom\,(T_n).$$

Lemma B.2 will be proved after we complete the proof of proposition B.1.

The same letter C will be used to denote the various bounds which will be constants independent of n unless otherwise indicated. The relations (A.4), (B.1) and the elliptic estimates for T imply

$$|u_n|_1 + |y_n|_1 < C. \tag{B.4}$$

Lemma B.2 with (B.1) also gives

$$|x_n|_1 < C \tag{B.5}$$

so on a subsequence

$$x_n \rightharpoonup x \quad \text{weakly in } L^2_1(\mathcal{E}) \tag{B.6}$$

$$x_n \rightarrow x \quad \text{strongly in } L^2. \tag{B.7}$$

Using (B.6) and the continuity of the trace map

$$\gamma : L^2_1(\mathcal{E}) \rightarrow L^2_{1/2}(\mathcal{E}_0) \quad u \mapsto u\,|_\Sigma$$

we deduce

$$\gamma x_n \rightarrow \gamma x \quad \text{weakly in } L^2_{1/2} \text{ and strongly in } L^2_\sigma \;\; \sigma < 1/2. \tag{B.8}$$

The ψ-convergence of the operators P_n implies that $P\gamma x = 0$ and in particular

$$x \in Dom\,(T). \tag{B.9}$$

The inequalities (B.4), (B.5) and the elliptic estimates for the first equation of (B.3) yield

$$|u_n|_2 \leq C(|x_n|_1 + |y_n|_1 + |u_n|_1) < C$$

so that on a subsequence

$$u_n \rightarrow u \quad \text{strongly in } L^2_1. \tag{B.10}$$

Because the the graph of T is closed we deduce $u \in Dom(T)$. Using (B.1) and (B.4) and the system (B.3) we conclude that on a subsequence

$$y_n \oplus (D_n x_n - u_n) \rightharpoonup y \oplus (z - u) \;\; \text{weakly in } \; L^2 \times L^2.$$

Here z is the weak limit of Dy_n. The graph of T is also weakly closed so that

$$y \in Dom(T) \;\; , \;\; Dy + u = z. \tag{B.11}$$

Now $D_n \to D$ in C^2 and $x_n \to x$ in L^2. Hence

$$Dx = z \quad \text{in the sense of distributions}$$

Summarizing we see that u, y, x and z satisfy

$$\begin{cases} Du & - & y & = & -x \\ u & + & Dy & = & Dx & = & z \end{cases} \tag{B.12}$$

Equation (B.12) has an obvious solution $y = x$ and $u = 0$. By (A.4) this is the unique solution so that by (B.10)

$$u_n \to 0 \quad \text{strongly in} \quad L^2_1. \tag{B.13}$$

The relation (B.13) implies that the numerator of (B.2) goes to 0. The limit in (B.2) is strictly positive only if

$$|y_n|^2 + |Dy_n|^2 \to 0.$$

Coupling this with elliptic estimates we get

$$y_n \to 0 \quad \text{strongly in} \quad L^2_1. \tag{B.14}$$

The convergences (B.13) and (B.14) can now be used in the second equation of (B.3) and produce

$$D_n x_n \to 0 \ \text{ strongly in } L^2. \tag{B.15}$$

The first equation of (B.3) gives

$$x_n \to 0 \ \text{ strongly in } L^2. \tag{B.16}$$

Obviously (B.15) and (B.16) contradict (B.1). Proposition B.1 is proved. □

Proof of Lemma B.2 Let $x \in Dom(T_n)$. We have an elliptic boundary estimate ([BW3], Chap.19)

$$\begin{aligned} |x|_1 &\le C(|x| + |Dx| + |P\gamma x|_{1/2,\Sigma}) \\ &\le C(|x| + |D_n x| + |P_n \gamma x|_{1/2,\Sigma}) + C(|(D_n - D)x| + |(P_n - P)\gamma x|_{1/2,\Sigma}) \\ &\le C(|x| + |D_n x|) + \epsilon_n(|x|_1 + |\gamma x|_{1/2,\Sigma}) \ \ \text{(we used } (i) \text{ and } (ii)) \\ &\le C\left(|x| + |D_n x| + \epsilon_n |x|_1\right) \ \ \text{by trace inequalities.} \end{aligned}$$

Above, $\epsilon_n \to 0$ as $n \to \infty$. Now the ϵ_n-term can be absorbed in the left hand side and the proof of the lemma is completed. □

Remark B.3 *The above proof of Proposition B.1 can be easily adapted to include the situations when the operators D_n have varying symbols.*

C Pseudodifferential Grassmanians and BVP's for Dirac operators

We gathered here a collection of technical facts we used in Section 6. We will use the terminology introduced in that section

Throughout Sections 1 to 4 we worked in abstract Hilbert spaces neglecting any other analytical structure they may posses. In "real life " the Hilbert spaces one encounters are L^2 spaces of distributions over some smooth Riemannian manifold and as such, they have a natural Sobolev filtration by L^2_σ.

If $\mathcal{E}$ is an euclidian vector bundle over a Riemann manifold M then a closed subspace $W \subset L^2(\mathcal{E})$ is called **pseudodifferential** if the orthogonal projection P_W onto W is induced by a zeroth order pseudodifferential operator. The principal symbol of P_W will be also called **the symbol of W**. This notion of pseudodifferential subspaces was considered in [W] in connection with BVP's for Dirac operators.

Consider now a $C^{p,q-1}$ Dirac bundle $\mathcal{E}_0 \to \Sigma$ over a **closed compact** manifold Σ and choose Λ_0 a pseudodifferential (p,q)-lagrangian. Denote by π the symbol of Λ_0 and consider $\Psi\mathcal{L}^{p,q}_\pi$ the set of all pseudodifferential lagrangians which have the same symbol π as Λ_0. We topologize $\Psi\mathcal{L}^{p,q}_\pi$ with the topology of ψ-convergence as in Sec. 6. Note that if $\Lambda \in \Psi\mathcal{L}^{p,q}_\pi$ the reflection R through Λ is a zeroth order selfadjoint pseudodifferential operator, which moreover is elliptic.

We will need the following technical result

Lemma C.1 *Let Y be a compact CW-complex and*

$$f : Y \to \Psi\mathcal{L}^{p,q}_\pi \quad y \mapsto \Lambda_y$$

a continuous family of pseudodifferential lagrangians such that:
(i)$\Lambda_y \cap \Lambda_0^\perp = 0 \ \ \forall y$.
(ii)$\Lambda_y + \Lambda_0 = L^2(\mathcal{E}_0) \quad \forall y$.
Then f is homotopic in $\Psi\mathcal{L}^{p,q}_\pi$ (in the topology of ψ-convergence) with the constant map $y \mapsto \Lambda_0$.

Proof The proof is based on a simple trick that we used before in the proof of Proposition 3.12. Let $R = R(y) = 2P_{\Lambda_y} - I$ be the reflection through Λ_y and similarly R_0 the reflection through Λ_0. The conditions (i) and (ii) can be rephrased as

$$R(y) + R_0 \ \text{ is invertible } \ \forall y.$$

Set $\Delta = \Delta(y) = \frac{1}{2}(R - R_0)$,

$$S_t = S_t(y) = R_0 + t\Delta \ \text{ where } \ t \in [0,2].$$

Note that $S_1 = \frac{1}{2}(R_0 + R)$ is invertible and it is a zeroth order selfadjoint pseudodifferential operator. For $t \in (0,1)$ rewrite S_t as

$$S_t = R_0(I + tR_0\Delta)$$

where $t\|R_0\Delta\|_0 < 1$. We conclude that S_t is invertible as an operator $L^2 \to L^2$. Since S_t is also elliptic its L^2 inverse is pseudodifferential of zeroth order and is bounded in any L^2_σ.

We proceed similarly for $t \in (1,2]$ rewriting

$$S_t = R - (2-t)\Delta$$

and we conclude that for all t's S_t is invertible with pseudodifferential inverse. Now, using the functional calculus of [S2], form the pseudodifferential operator

$$C_t = S_t|S_t|^{-1/2} \quad t \in [0,2].$$

Note that C_t are (p,q)-gradings of $L^3\mathcal{E}_0$,$\forall t$,$C_0 = R_0$ and $C_2 = C_2(y) = R(y)$. Moreover the $C_t's$ have all the same symbol as R_0 and the R's. The lagrangians determined by C_t gives the desired deformation. Lemma C.1 is proved□

Let W be a cofinite, pseudodifferential (p,q)-isotropic subspace of $L^2(\mathcal{E}_0)$ such that the symplectic reduction H of $L^2(\mathcal{E}_0)$ is a smoothing subspace i.e. it consists of smooth sections of $\mathcal{E}_0$. Let π denote the symbol of $W^\#$. Consider a continuous family

$$(\Lambda_y)_{y\in Y} \in \Psi\mathcal{L}^{p,q}_\pi \tag{C.1}$$

such that

$$\Lambda_y \text{ is clean mod } W \quad \forall y. \tag{C.2}$$

For each y denote by L_y the symplectic reduction of Λ_y mod W and set

$$\tilde{\Lambda}_y = L_y \oplus W^\#.$$

The projection onto H is smoothing we so that the projection onto $\tilde{\Lambda}_y$ is pseudodifferential. The family $(L_y)_{y\in Y}$ is continuous which implies the family $(\tilde{\Lambda}_y)_{y\in Y}$ is continuous in $\Psi\mathcal{L}^{p,q}_\pi$. The reader can verify easily

(i)'$\Lambda_y \cap \tilde{\Lambda}_y = 0 \quad \forall y$.

(ii)' $\Lambda_y + \tilde{\Lambda}_y = L^2(\mathcal{E}_0) \quad \forall y$.

Proceding as in Lemma C.1 we obtain the following result.

Lemma C.2 *The family $(\Lambda_y)_{y\in Y}$ is homotopic in $\Psi\mathcal{L}^{p,q}_\pi$ with the family $(\tilde{\Lambda}_y)_{y\in Y}$.*

Let $D \in \mathcal{D}^{p,q}$. Along the neck it has the form

$$D = J(\hat{\nabla}_s + D_0) \tag{C.3}$$

and consider the spaces $\mathcal{H}^E_0$,$\mathcal{H}^E_<$ and $\mathcal{H}^E_>$ coming from the spectral decomposition of D_0. $\mathcal{H}^E_<$ and $\mathcal{H}^E_>$ are pseudodifferential (p',q')-isotropic subspaces with symbols π_-

and resp. $\pi_+ = 1 - \pi_-$. Moreover $\mathcal{H}_0^E$ is the symplectic reduction of $L^2(\mathcal{E}_0)$ mod $\mathcal{H}_<^E$ (and also mod $\mathcal{H}_>^E$) and is a smoothing subspace.

Consider $(\Lambda_y)_{y\in Y}$ a continuous family in $\Psi\mathcal{L}_{\pi_+}^{p',q'}$. We can find $E > 0$ large enough such that

$$\Lambda_y \text{ is clean mod } \mathcal{H}_<^E \quad \forall y \in Y. \tag{C.4}$$

Let L_y^E denote the symplectic reduction of Λ_y mod $\mathcal{H}_<^E$. Using Lemma C.2 we get

Corollary C.3 *For any continuous family $(\Lambda_y)_{y\in Y}$ in $\Psi\mathcal{L}_{\pi_+}^{p',q'}$ there exists $E > 0$ such that $(\Lambda_y)_{y\in Y}$ is homotopic in $\Psi\mathcal{L}_{\pi_+}^{p',q'}$ to the APS family*

$$(L_y^E \oplus \mathcal{H}_>^E)_{y\in Y}.$$

Corollary C.3 is precisely Lemma 6.3.

Remark C.4 *In the proof of the above corollary instead of working with a* single *operator D we could work with a continuous family of them $(D(y))$ such the restriction to the boundary, $(D_0(y))$ admits a spectral selection in the sense of Definition 6.5. The family of pseudodifferential lagrangians Λ_y wil now possible have varying symbols.*

It is not difficult to see there exist a spectral section whose corresponding spectral cut $R(y)$ is sufficiently large so that Λ_y is clean mod $\mathcal{H}_>^{R(y)}$. Now we can perform the reduction mod $\mathcal{H}_>^{R(y)}$ to conclude the original family of pseudodifferential lagrangians is homotopic to the family

$$L_y \oplus \mathcal{H}_<^{-R(y)}$$

Now consider as in Section 6 a $C^{p,q}$ Dirac bundle over a compact oriented manifold M with boundary $\partial M = \Sigma$. For any $D \in \mathcal{D}^{p,q}$ the CD space $\Lambda_M(D)$ is a pseudodifferential (p',q')-lagrangian in $L^2(\mathcal{E}_0)$ with symbol independent of D which we called π_-. We have a map

$$CD : \mathcal{D}^{p,q} \to \Psi\mathcal{L}_{\pi_-}^{p',q'}$$

As in Proposition 2.4 in [N1] one proves the following result.

Lemma C.5 *The above CD map is continuous.*

Remark C.6 *The proof of Proposition 2.4 in [N1] extends easily to cover the situation when the symbols of the various Dirac operators are not constant.*

Let π_+ denote the symbol of the orthogonal complement of some CD space and set as in Sec. 6

$$\mathcal{BP}_*^{p,q} = \{(D,\Lambda) \in \mathcal{D}^{p,q} \times \Psi\mathcal{L}_{\pi_+}^{p',q'}(\mathcal{E}_0) \;/\; T(D,\Lambda) \text{ is invertible}\}$$

topologized as usually.

Proposition C.7 *The space $\mathcal{BP}^{p,q}_*$ is weakly contractible i.e. any continuous map*

$$\beta : S^n \to \mathcal{BP}^{p,q}_*$$

is homotopic to a constant map.

Proof Let $\beta(y) = (D_y, \Lambda_y)$, $y \in S^n$ and denote by $\Lambda_1(y)$ the CD space of D_y. Lemma C.3 shows this is a ψ-continuous family. Since $(D_y, \Lambda_y) \in \mathcal{BP}^{p,q}_*$ we conclude

$$\Lambda_1(y) \cap \Lambda_y = 0.$$

The pair $(\Lambda_1(y), \Lambda_y)$ is Fredholm. As in Remark 3.8 we deduce that

$$\Lambda_1(y) + \Lambda_y = L^2(\mathcal{E}_0).$$

Lemma C.1 implies there exists a homotopy $(\Lambda^t_y)_{t\in[0,1]}$ in $\Psi\mathcal{L}^{p,'q'}_{\pi_-}$ such that

$$\Lambda^0(y) = \Lambda_y \ , \ \Lambda^1_y = \Lambda_1^\perp(y).$$

Moreover the proof of Lemma C.1 shows that during the deformation $\Lambda^t(y)$ and $\Lambda_1(y)$ stay transversal. Thus

$$(D_y, \Lambda_y) \cong (D_y, \Lambda_1^\perp(y)) \ \text{in} \mathcal{BP}^{p,q}_*.$$

Now fix an operator $\tilde{D} \in \mathcal{D}^{p,q}$ and consider

$$\tilde{D}^t_y = (1-t)D_y + t\tilde{D}.$$

We have a homotopy in $\mathcal{BP}^{p,q}_*$

$$t \mapsto (\tilde{D}^t_y, \Lambda_M(\tilde{D}^t_y)^\perp$$

between $(D_y, \Lambda_1^\perp(y))$ (at $t = 0$) and the constant family $(\tilde{D}, \Lambda_M(\tilde{D})^\perp)$ (at $t = 1$). This concludes the proof of Proposition C.7. □

D The proof of Proposition 6.1

We begin with a simple but useful observation we will frequently use in this section. Let H be an infinite dimensional Hilbert (p,q)-module, $p - q \equiv 0 \pmod 4$ and R a (p,q) grading. The volume operator $\Omega = J_1 \cdots J_p C_1 \cdots C_q$ is in this case an involution which commutes with R. Thus $H_\pm = \ker(1 \mp \Omega)$ are R-invariant subspaces of H. Then, for RΩ to be essential it suffices to show that $R\,|_{H_\pm}$ is essentially indefinite.

Let M be a manifold with boundary $\partial M = \Sigma$ as in Section 6 and $\mathcal{E} \to M$ an essential (p,q)-Dirac bundle.

Proof of (a). The only interesting case is $p - q \equiv 0 \pmod 4$. Fix $D \in \mathcal{D}^{p,q}$ and denote by $\Lambda_0 = \Lambda_M(D)$ the CD-space of D. As in the proof of Proposition 5.1 one

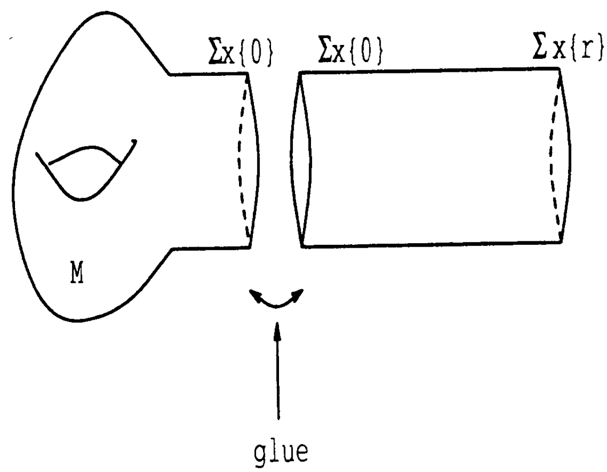

Figure 7: *Adiabatic deformation of the neck*

can show (using the connectivity of $\Psi\mathcal{L}_0^{p',q'}$) that a lagrangian in $\Psi\mathcal{L}_0^{p',q'}$ is essential iff ${\Lambda_0}^{\perp}$ is or equivalently, Λ_0 is.

Define $M(r) = M \cup \Sigma \times [0, r]$. $M(r)$ is usually called an *adiabatic deformation* of M; (see Fig. 7). D has a natural extension $D(r)$ as a neck-compatible Dirac on M(r) and denote by $\Lambda_r \subset L^2(\mathcal{E}\mid_{\partial M(r)})$ the CD space of D(r).

In [N1] we showed that there exists $E > 0$ such that

$$\lim_{r\to\infty} \Lambda_r = L_E \oplus \mathcal{H}_<^E$$

where L_E is a (p', q')-lagrangian in $\mathcal{H}_0^E$. Again a continuity argument shows that Λ_0 is essential iff Λ_∞ is. Denote the grading defined by Λ_∞ by R_∞ and the reflection through $\mathcal{H}_\leq^E$ by R_E. $\mathcal{H}_<^E$ is a (p', q')-isotropic subspace of $L^2(\mathcal{E}_0)$. Since the volume $\overline{\Omega} = \overline{J}_1 \cdots \overline{J}_{p'}\overline{C}_1 \cdots \overline{C}_{q'}$ is even, we deduce $\mathcal{H}_<^E$ is $\overline{\Omega}$-invariant so that R_E commutes with $\overline{\Omega}$. Clearly $R_\infty - R_E$ is compact so that $R_\infty\overline{\Omega}$ is essentially indefinite iff $R_E\overline{\Omega}$ is. Using the opening remark we see that it suffices to show that $R_E\mid_{L^2(\mathcal{E}_0^\pm)}$ is essentially indefinite. Here $\mathcal{E}_0^\pm$ are the ± 1 eigenbundles defined by $\overline{\Omega}$.

$D_0 = D\mid_\Sigma$ anticommutes with the oj_i's and the $\overline{C}_j'$s so that it commutes with $\overline{\Omega}$. In particular $L^2(\mathcal{E}_0^\pm)$ are D_0-invariant. Part (a) is proved if we show that the spectrum of $D_0\mid_{L^2(\mathcal{E}_0^\pm)}$ is unbounded both from below and above. This follows from the observation in [APS1] that the spectral projections defined by the negative and the positive spectrum are both 0th order pseudodifferential operators.

Proof of (b). The interesting case is $p - q \equiv 1 \pmod 4$. Let $D \in \mathcal{D}^{p,q}$ the volume $\Omega = J_1 \cdots J_p C_1 \cdots C_{q'}$ is an involution commuting with D and splits $\mathcal{E}$ into a direct sum of eigenbundles $\mathcal{E}^\pm$. Moreover $L^2(e^\pm)$ are D-invariant. A continuity argument à la Proposition 5.1 shows that $T = T(D, \Lambda)$ is essential for all $\Lambda \in \Psi\mathcal{L}_0^{p',q'}$ iff it is essential for just one boundary condition $\Lambda \in \Psi\mathcal{L}_0^{p',q'}$. Using the observation in the beginning we deduce that for such a T to be essential it suffices that the spectrum of $T\,|_{L^2(\mathcal{E}^\pm)}$ is unbounded both from above and below. Thus part (b) follows if we can show that for any formally selfadjoint Dirac operator D on a manifold with boundary M there exists a selfadjoint boundary condition $\Lambda \in \Psi\mathcal{L}_0$ such that the spectrum of $T = T(D,\Lambda)$ is unbounded both from below and above. Equivalently, it suffices to show the set

$$\mathcal{S} = \{\mathcal{Q}(u) = \frac{(Du,u)}{|u|^2}\,; u \in L^2_1,\ u \neq 0,\ u\,|_\Sigma \in \Lambda\}$$

is unbounded both from above and below.

We pick Λ as a generalized APS condition $\Lambda = L_E \oplus \mathcal{H}^E_>$ such that L_E contains an eigenvector ϕ of D_0 with eigenvalue μ such that $|\phi|_\Sigma = 1$. Set $\psi = J\psi$, where as usual J denotes the Clifford multiplication by ds.

Choose now Lipschitz continuous cutoff functions $\alpha,\ \beta : \mathbb{R} \to \mathbb{R}$ defined by:
(i) $\alpha(s) \equiv 0$ for $s \leq -1$, $\alpha(s) \equiv 1$ for $s \geq -1/2$ and α is linear on [-1/2,0].
(ii)$\beta(s) \equiv -1/2$ for $s \leq -1/2$ and $\beta(s) \equiv s$ for $s \geq -1/2$.
Set

$$A = \int_{-1}^{-1/2} |\alpha(s)|^2 ds > 0,\quad B = \int_{-1}^{-1/2} \alpha(s)ds = 1/4 > 0.$$

For each $n \in \mathbb{Z}$ define $\lambda_n = \operatorname{sign}(n)(\mu^2 + 4\pi^2 n^2)^{1/2}$ and $u_n \in L^2_1(\mathcal{E})$ by

$$u_n = \alpha(s)\left(\cos(2\pi n\beta(s))\phi + \frac{2\pi n}{\lambda_n - \mu}\sin(2\pi n\beta(s))\psi\right).$$

From this definition we immediately deduce that $u_n\,|_\Sigma \in L_E$ and that u_n is supported on the neck $\Sigma \times [-1,0]$. Moreover

$$Du_n = \lambda_n u_n \quad \text{over } (-1/2,0). \tag{D.1}$$

We now evaluate the various quantities which appear in the definition of $\mathcal{Q}$ for our special choices.

The norm $|u_n|^2$. We split the computation into two parts.

$$|u_n|^2 = 1/4\left(1 + \left|\frac{\lambda_n + \mu}{\lambda_n - \mu}\right|^2\right) = 1/2 + O(n^{-1}) \quad \text{over } \Sigma \times (-1/2,0). \tag{D.2}$$

$$|u_n|^2 = A \quad \text{over } \Sigma \times (-1,-1/2). \tag{D.3}$$

(Du_n, u_n). We split the computation as before. Using (D.1) we get

$$(Du_n,u_n) = \lambda_n(1/2 + O(n^{-1})) \quad \text{over } \Sigma \times (-1/2,0) \tag{D.4}$$

$$(Du_n,u_n) = \cos(\pi n)\mu B \quad \text{over } \Sigma \times (-1,-1/2). \tag{D.5}$$

The relations (D.2)-(D.5) and the obvious remark $\lambda_n \sim 2\pi n$ immediately imply that the sequence $\mathcal{Q}(u_n)$ is unbounded both from below and above. Proposition 6.1 is proved. □

Remark D.1 *The argument used in the proof of (b) can be easily adapted to produce a proof of (a) without appealing to the results in [APS1] which rely on deep analytical facts. The functions u_n where found solving an eigenvalue problem for a linear 2-dimensional hamiltonian equation given by the hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ defined by $H(x, y) = \mu(|x|^2 - |y|^2)$. In the boundary-less case it suffices to deform the metric until the manifold Σ looks like a connected sum $\Sigma = \Sigma\#\mathrm{sphere}$ such that the gluing region is cylindrical. Then "graft" on the neck the solutions of the above hamiltonian equations.*

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