

EXISTENCE AND REGULARITY FOR A SINGULAR SEMILINEAR STURM-LIOUVILLE PROBLEM

LIVIU I. NICOLAESCU

Department of Mathematics, University "Al. I. Cuza" Iasi 6600, Romania

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Abstract. In this paper, we consider the following Sturm-Liouville problem

$$-\frac{1}{r^\gamma}(r^\gamma u')' = r^\beta |u|^{p-1}u \quad \text{in } (0, R)$$

$$u(R) = 0, \quad \int_0^R r^\gamma |u'|^2 dr < \infty$$

where $p \geq 1$, $\beta > -2$, $\gamma \in \mathbb{R}$. Using the Mountain Pass Lemma we prove the existence of a weak positive solution under optimal conditions on the parameters. Beyond these conditions a variant of Pohozaev's identity gives non-existence. Using M\"oser's iteration technique we prove the boundedness of this solution which in turn gives the uniqueness. Applying a symmetric version of the Mountain Pass Lemma we proved the existence of infinitely many weak solutions changing sign. The main tool in the proof is the generalized Hardy-Littlewood inequality. We apply these results to semilinear degenerate elliptic equations in \mathbb{R}^N and we get new interesting corollaries.

0. Introduction. We consider the following boundary value problem

$$-\operatorname{div}(r^\alpha \nabla u) = r^\delta |u|^{p-1}u \quad \text{in } B_R(0) \subset \mathbb{R}^N, \quad N \geq 3, \quad r = |x|$$

$$u = 0 \quad \text{on } |x| = R. \tag{0.1}$$

If we restrict our search to radially symmetric solutions then this equation reduces to the following one

$$-\frac{1}{r^\gamma}(r^\gamma u')' = r^\beta u^p \quad \text{in } (0, R), \quad \gamma = N + \alpha - 1, \quad \beta = \delta - \alpha$$

$$u(R) = 0 + \text{a boundedness condition at } r = 0. \tag{0.2}$$

One is tempted to set $u'(0) = 0$ but this is too restrictive since as we shall see later there exist weak solutions of (0.2) which are bounded at 0 and are not differentiable at this point. The natural boundedness condition is

$$\int_0^R r^\gamma |u'|^2 dr < \infty \tag{0.3}$$

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which arises naturally by considering the variational form of (0.2).

First we look for positive solutions of (0.2) + (0.3). We obtain a weak solution under the condition $\gamma > 1$, $\beta > -2$, $1 < p < (\gamma + 3 + 2\beta)/(\gamma - 1)$, making use of the Mountain Pass Lemma.

This is a generalization of a previous result of W. M. Ni [8]. A by-product of the proof is a compactness result for weighted Sobolev spaces which has its own interest. The main ingredient in the proof will be the generalized Hardy-Littlewood inequality.

Since we found only a weak solution an important question is about its regularity. The problem is classical in the interior of $(0, R)$ but raises difficulties when it is considered in a neighborhood of 0. Using a weighted variant of M\"oser's iteration technique we were able to establish the boundedness of the positive solution. Then using the special form of the equation we get further information about the behaviour of the solutions near 0, e.g., the uniqueness of the positive solution.

We also establish multiplicity results using a symmetric Mountain Pass Lemma as in Rabinowitz [12]. When applied to (0.1) these results provide surprising corollaries like the non-existence of a critical exponent as one is accustomed to in the usual case of the laplacian, i.e., $\alpha = 0$.

The paper is divided into five sections.

Section 1: Some results about weighted Sobolev spaces in one dimension.

Section 2: A priori study of the positive solutions of (0.2), (0.3).

Section 3: Existence of a positive solution.

Section 4: Existence of infinitely many solutions changing sign.

Section 5: Applications.

1. Some results about weighted Sobolev spaces in one dimension. Let us consider $S = \{u \in C^1[0, R] \mid u \equiv 0 \text{ in a neighborhood of } R\}$. For $\gamma \in \mathbb{R}$ denote by E_γ the completion of S in the norm

$$\|u\|_\gamma = \left(\int_0^R r^\gamma |u'|^2 dr \right)^{1/2}.$$

We are interested in imbedding E_γ into various weighted Lebesgue spaces. These can be obtained using the following generalized Hardy-Littlewood inequality (see Maz'ja [5] sect. 1.3.)

Theorem 1.1. *Let μ and ν be nonnegative Borel measures on \mathbb{R} and let ν^* be the absolutely continuous part of ν . Let $1 \leq p \leq q \leq \infty$.*

In order that there exist a constant $C > 0$ independent of f such that

$$\left[\int_{-\infty}^{+\infty} \left| \int_x^{+\infty} f(t) dt \right|^q d\mu(x) \right]^{1/q} \leq C \left[\int_{-\infty}^{+\infty} |f(x)|^p d\nu(x) \right]^{1/p} \quad (1.1)$$

it is necessary and sufficient that

$$B = \sup_{r \in (-\infty, +\infty)} \left[\mu((-\infty, r)) \right]^{1/q} \left[\int_r^{+\infty} \left(\frac{d\nu^*}{dx} \right)^{-1/(p-1)} \right]^{(p-1)/p} < \infty. \quad (1.2)$$

Let us consider the following particular case

$$p = 2, \quad d\nu(x) = d\nu^*(x) = \begin{cases} x^\gamma dx, & x \in (0, R) \\ 0, & x \notin (0, R) \end{cases}$$

$$d\mu(x) = d\mu^*(x) = \begin{cases} x^\theta dx, & x \in (0, R) \\ 0, & x \notin (0, R) \end{cases}$$

By Theorem 1.1, we see that E_γ is continuously imbedded in $L^q(0, R; r^\theta dr)$, $q \geq 2$, if and only if $\theta > \max(-1, \gamma - 2)$ and $(\theta + 1)/q \geq (\gamma - 1)/2$. We state this result in the following

Imbedding Lemma. $E_\gamma \hookrightarrow L^q(0, R; r^\theta dr)$ continuously for $q \geq 2$ if and only if

$$\theta > \max(-1, \gamma - 2), \quad \frac{\theta + 1}{q} \geq \frac{\gamma - 1}{2}. \tag{1.3}$$

If (1.3) is satisfied then E_γ is also imbedded in L_θ^q for $q \in [1, 2]$ since $r^\theta dr$ is a finite measure on $[0, R]$.

We use the notation L_θ^q for $L^q(0, R; r^\theta dr)$. Our aim is to prove the compactness of these imbeddings (except possibly a critical one). We need a generalization of Ni's Radial Lemma [7]. This is the purpose of our next result.

Radial Lemma. *There exists $C = C(\gamma) > 0$ such that for $u \in E_\gamma$*

$$|u(r)| \leq (C_\gamma/r^{-\frac{\gamma-1}{2}}) \|u\|_\gamma.$$

Proof: $-u(r) = u(R) - u(r) = \int_r^R u'(s) ds$. Thus,

$$|u(x)| \leq \int_r^R |u'(s)| ds \leq \left(\int_r^R |u'(s)|^2 r ds \right)^{1/2} \left(\int_r^R s^{-\gamma} ds \right)^{1/2} \leq (C_\gamma/r^{-\frac{\gamma-1}{2}}) \|u\|_\gamma.$$

Compactness Lemma. E_γ is compactly imbedded in L_θ^q ($q \geq 2$) if

$$\gamma > 1, \quad \theta > \max(-1, \gamma - 2), \quad \frac{\theta + 1}{q} > \frac{\gamma - 1}{2}. \tag{1.4}$$

Proof: Step 1. E_γ is compactly imbedded in $L_{\gamma/2}^1$. Indeed by the Imbedding Lemma, $E_\gamma \hookrightarrow L_{\gamma/2}^1$.

Let us first observe that given $u \in E_\gamma$, then

$$r^{\gamma/2}u \in H_0^1(0, R) - \text{the usual Sobolev space.}$$

Indeed, since $\frac{\gamma}{2} > -1$ we get by the Imbedding lemma that $u \in L_\gamma^2$, hence, $r^{\gamma/2}u \in L^2(0, R)$,

$$\frac{d}{dr}(r^{\gamma/2}u) = \frac{\gamma}{2}r^{(\gamma-2)/2}u + r^{\gamma/2}u'.$$

Since $\gamma > 1$ we get by the Imbedding Lemma that $u \in L^2_{\gamma-2}$ and therefore,

$$\frac{d}{dr}(r^{\gamma/2}u) \in L^2(0, R).$$

By the Radial Lemma we get that $|r^{\gamma/2}u| \leq C_\gamma r^{1/2}$ and therefore, $r^{\gamma/2}u \in H^1_0(0, R)$. Since $H^1_0(0, R)$ is compactly imbedded in $L^1(0, R)$ we get that E_γ is compactly imbedded in $L^1_{\gamma/2}$ via the sequence of mappings $E_\gamma \xrightarrow{T_1} H^1_0 \xrightarrow{T_2} L^1(0, R)$, where $u \xrightarrow{T_1} r^{\gamma/2}u$ and $v \xrightarrow{T_2} T$.

Step 2. E_γ is compactly imbedded in L^q_θ for every $1 \leq q < \omega = (2(\theta+1))/(\gamma-1)$ (i.e., $(\theta+1)/q > (\gamma-1)/2$).

By the Hölder inequality we have for $a \in (0, 1)$

$$\begin{aligned} \int_0^R |u|^q r^\theta dr &= \int_0^R r^{(a\gamma)/2} |u|^a r^{\theta-(a\gamma)/2} |u|^{q-a} dr \\ &\leq \left(\int_0^R r^{\gamma/2} |u| dr \right)^a \left(\int_0^R r^{(\theta-\frac{a\gamma}{2})/(1-a)} |u|^{(q-a)/(1-a)} dr \right)^{1-a}. \end{aligned}$$

We claim that we can choose a small enough so that

$$E_\gamma \hookrightarrow L^{(q-a)/(1-a)}_{(\theta-\frac{a\gamma}{2})/(1-a)}.$$

By the Imbedding Lemma this is true if and only if

$$f(a) = \frac{(\theta - \frac{a\gamma}{2})/(1-a) + 1}{(q-a)/(1-a)} \leq \frac{\gamma-1}{2}.$$

But $\lim_{a \rightarrow 0} f(a) = (\theta+1)/q > (\gamma-1)/2$ by (1.4) so if a is small enough $f(a) > (\gamma-1)/2$. Then again by the Imbedding Lemma we get

$$\|u\|_{L^q_\theta} \leq \text{const.} \|u\|_{L^1_{\gamma/2}}^a \|u\|_\gamma^{1-a}. \quad (1.5)$$

By Step 1 and (1.5) we get the desired compactness. ■

The range of applicability of the Compactness Lemma can be considerably enlarged by making the following observation. If $\gamma \leq 1$, then $E_\gamma \hookrightarrow E_\lambda \forall \lambda > 1$. If $\theta > -1$ and $q > 1$ are arbitrary, then one can find $\lambda > 1$ such that $(\theta+1)/q > (\lambda-1)/2$ and therefore, by our previous result E_γ is compactly imbedded in L^q_θ . Hence, we have the following general compactness result

$$E_\gamma \xrightarrow[\text{compact}]{} L^q_\theta, \quad \forall \gamma \in \mathbb{R}, \quad q \geq 1, \quad \theta > \max(-1, \gamma-2), \quad \frac{\theta+1}{q} > \frac{\gamma-1}{2}. \quad (1.6)$$

2. A priori study of the positive solutions of (0.2). We begin this section by stating an estimate for the positive solutions near 0.

Lemma 2.1. *Let u be a positive solution of the equation*

$$-1/r^\gamma(r^\gamma u')' = r^\beta u^p \quad \text{in } (0, R), \quad u(R) = 0, \quad (2.1)$$

where $\gamma > 1$, $\beta > -1 - \gamma$, $p > 1$. Then, there exists $C > 0$ such that

$$u(r) < C/(r^{(2+\beta)/(p-1)}). \quad (2.2)$$

Remark 2.1. Estimate (2.2) has a long history. It was first obtained by Gridaspruck [3] for positive solutions of (0.1) with $\alpha = 0$. Subsequent proofs were given by Ni-Saks [9], Ni-Serrin [11]. For a proof we refer the reader to the work [9].

By a weak solution of problem (0.2) and (0.3), we mean a function $u \in E_\gamma$ such that $\forall \phi \in E_\gamma$

$$\int_0^R r^\gamma u' \phi' dr = \int_0^R r^\theta u^p \phi dr, \quad \theta = \beta + \gamma. \quad (2.5)$$

If $\gamma \leq 1$, $\theta > -1$, by the Imbedding Lemma we get that in this case $E_\gamma \hookrightarrow L^\infty$ so any weak solution of (0.2) + (0.3) is bounded.

What can be said when $\gamma > 1$?

We first prove a non-existence result.

Proposition 2.2. *Let $\gamma > 1$. If $(\theta + 1)/(p + 1) < (\gamma - 1)/2$, then there exist no non-trivial positive weak solutions of (0.2) and (0.3).*

Proof: The proof is based on a variant of Pohozaev's identity similar to that used by Ni-Serrin [10]. If u is a weak solution of (0.2) and (0.3), then u satisfies

$$-(r^\gamma u')' = r^\theta u^p \quad \text{in } (0, R), \quad (2.6)$$

classically, by standard interior elliptic regularity. We multiply (2.6) with ru' and we get

$$\begin{aligned} r^{\gamma+1} u' u'' + \gamma r^\gamma |u'|^2 + r^{\theta+1} u^p u' &= 0 \\ \frac{1}{2} r^{\gamma+1} \frac{d}{dr} |u'|^2 + \gamma r^\gamma |u'|^2 + \frac{1}{p+1} r^{\theta+1} \frac{d}{dr} (u^{p+1}) &= 0. \end{aligned}$$

We integrate by parts on (ε, R) the last equality. We obtain

$$\begin{aligned} \frac{1}{2} R^{\gamma+1} |u'(R)|^2 - \frac{1}{2} \varepsilon^{\gamma+1} |u'(\varepsilon)|^2 - \frac{1}{p+1} \varepsilon^{\theta+1} u^{p+1}(\varepsilon) \\ + \frac{\gamma-1}{2} \int_\varepsilon^R r^\gamma |u'(r)|^2 dr - \frac{\theta+1}{p+1} \int_\varepsilon^R r^\theta u^{p+1}(r) dr &= 0. \end{aligned} \quad (2.7)$$

Since $u \in E_\gamma$, we get that on a subsequence $\varepsilon_k \rightarrow 0$ we have

$$\varepsilon_k^{\gamma+1} |u'(\varepsilon_k)|^2 \rightarrow 0. \quad (2.8)$$

Also since $(\theta + 1)/(p + 1) < (\gamma - 1)/2$, we get $\theta + 1 - (p + 1)/(p - 1)(2 + \beta) > 0$ and by Lemma 2.1 we get

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\theta+1} u^{p+1}(\varepsilon) = 0. \quad (2.9)$$

If in (2.7) we let $\varepsilon = \varepsilon_k \rightarrow 0$ we infer

$$\frac{1}{2}R^{\gamma+1}|u'(R)|^2 + \frac{\gamma-1}{2} \int_0^R r^\gamma |u'(r)|^2 dr - \frac{\theta+1}{p+1} \int_0^R r^\theta u^{p+1}(r) dr = 0. \quad (2.10)$$

If now we multiply (2.6) with u and integrate on (ε, R) we obtain

$$- \int_\varepsilon^R r^\gamma |u'|^2 dr + \int_\varepsilon^R r^\theta u^{p+1} - \varepsilon^\gamma u(\varepsilon)u'(\varepsilon) = 0.$$

By the Radial Lemma

$$u(\varepsilon) \leq C\varepsilon^{-(\gamma-1)/2}$$

and according to (2.8) we get that on a subsequence $\varepsilon_k \rightarrow 0$

$$\varepsilon_k^{\gamma+1} u(\varepsilon_k)u'(\varepsilon_k) \rightarrow 0.$$

Now, let $\varepsilon = \varepsilon_k$ and let $k \rightarrow \infty$. We get

$$\int_0^R r^\gamma |u'|^2 dr = \int_0^R r^\theta u^{p+1} dr. \quad (2.11)$$

From (2.10) and (2.11) we infer

$$\frac{1}{2}R^{\gamma+1}|u'(R)|^2 + \left(\frac{\gamma-1}{2} - \frac{\theta+1}{p+1}\right) \int_0^R r^\gamma |u'(r)|^2 dr = 0.$$

So we infer that $u \equiv 0$. ■

Condition $(\theta+1)/(p+1) < (\gamma-1)/2$ can be rewritten as $p > (\gamma+3+2\beta)/(\gamma-1)$. Hence, for $p > (\gamma+3+2\beta)/(\gamma-1)$ there exist no nontrivial positive weak solutions of (0.2) and (0.3). It is therefore natural to study the regularity only for $p \leq (\gamma+3+2\beta)/(\gamma-1)$. We have the following result.

Theorem 2.3. *Let u be a weak positive solution of (0.2) and (0.3) in which $\gamma > 1$, $\beta > -2$, $p > 1$, $(\theta+1)/(p+1) > (\gamma-1)/2$ (i.e., $1 < p < (\gamma+3+2\beta)/(\gamma-1)$). Then, $u \in L^\infty(0, R)$.*

Proof: We shall use a weighted variant of M\"oser's iteration method. We first note that since $(\theta+1)/(p+1) > (\gamma-1)/2$, then

$$\theta - \frac{\gamma-1}{2}(p-1) > \gamma - 2.$$

Hence, there exists $\lambda > 0$ such that

$$\theta - \frac{\gamma-1}{2}(p-1) = \gamma - 2 + \lambda.$$

By the Radial Lemma, we get

$$r^\theta u^p \leq Cr^{\theta - \frac{\gamma-1}{2}(p-1)} u \leq Cr^{\gamma-2+\lambda} u, \quad \text{with } \lambda > 0 \text{ small enough.} \quad (2.12)$$

Let $k > 0$ and $\bar{u} = u + k$. We consider the functions

$$F(\bar{u}) = F_l(\bar{u}) = \begin{cases} \bar{u}^q, & k \leq \bar{u} \leq \ell \\ q\ell^{q-1}\bar{u} - (q-1)\ell^q, & \bar{u} \geq \ell \end{cases} \quad q > 1, \ell > k$$

$$G(u) = F(u) \cdot F'(u) - qk^{2q-1}, \quad u > 0.$$

The above functions satisfy the following inequalities

$$F \leq q\ell\bar{u}, \quad \bar{u}F' \leq qF, \quad G \leq F \cdot |F'|, \quad G' \geq |F'|^2. \quad (2.13)$$

In (2.5) we choose $\phi = \eta^2(r)G(\bar{u})$, where $\eta \in C^\infty[0, R]$ is a cut-off function, $\eta \equiv 1$ in a neighborhood of 0, $\eta \equiv 0$ in a neighborhood of R . We have, from (2.13)

$$\begin{aligned} r^\gamma u' \phi' - r^\theta u^p \phi &= r^\gamma u' \eta^2 G' u' + 2r^\gamma u' \eta \eta' G - r^\theta \eta^2 G(\bar{u}) u^p \\ &\geq r^\gamma |\eta u' F'|^2 - 2r^\gamma \eta' F |\eta F' u'| - r^\theta \eta^2 G(\bar{u}) u^p \end{aligned}$$

According to (2.5) we get

$$\int_0^R r^\gamma |\eta \bar{u}' F'|^2 dr \leq 2 \int_0^R r^\gamma |\eta' F| |\eta F' u'| dr + \int_0^R r^\theta \eta^2 |G(\bar{u})| u^p dr.$$

Let us denote $v = F(\bar{u})$. Then, taking (2.12) into account we get

$$\begin{aligned} \int_0^R r^\gamma |\eta v'|^2 dr &\leq 2 \int_0^R r^\gamma |\eta' v| |\eta v'| dr + C \int_0^R r^{\gamma-2+\lambda} \eta^2 |G(u)| u dr \\ &\leq C \left[\int_0^R r^\gamma |\eta' v| |\eta v'| dr + q \int_0^R r^{\gamma-2+\lambda} |\eta v|^2 dr \right], \end{aligned} \quad (2.14)$$

where we have used (2.13). We estimate the integrals lying on the right hand side of (2.14)

$$\int_0^R r^\gamma |\eta' v| |\eta v'| dr \leq \|\eta' v\|_{L_\gamma^2} \|\eta v'\|_{L_\gamma^2} \leq C \|\eta' v\|_{L_{\gamma-2+\lambda}^2} \|\eta v'\|_{L_\gamma^2} \quad (2.15)$$

since for $\lambda > 0$ small enough $L_{\gamma-2+\lambda}^2 \hookrightarrow L_\gamma^2$

$$\begin{aligned} \int_0^R r^{\gamma-2+\lambda} |\eta v|^2 dr &= \int_0^R r^{\gamma-2+\lambda} |\eta v|^\epsilon |\eta v|^{2-\epsilon} dr \\ &\leq \|\eta v\|_{L_{\gamma-2+\lambda}^2}^\epsilon \|\eta v\|_{L_{\gamma-2+\epsilon}^2}^{2-\epsilon} \\ &\leq C \|\eta v\|_{L_{\gamma-2+\lambda}^2}^\epsilon \|\eta v\|_{L_\gamma^2}^{2-\epsilon} \quad (\text{by the Imbedding Lemma}) \\ &\leq C \|\eta v\|_{L_{\gamma-2+\lambda}^2}^\epsilon (\|\eta' v\|_{L_\gamma^2}^{2-\epsilon} + \|\eta v'\|_{L_\gamma^2}^{2-\epsilon}). \end{aligned} \quad (2.16)$$

Substituting (2.15) and (2.16) in (2.14) we get

$$\|\eta v'\|_{L^2_\gamma}^2 \leq C \left[\|\eta'v\|_{L^2_{\gamma-2+\lambda}} \|\eta v'\|_{L^2_\gamma} + q \|\eta v\|_{L^2_{\gamma-2+\lambda}}^\epsilon \left(\|\eta'v\|_{L^2_\gamma}^{2-\epsilon} + \|\eta v'\|_{L^2_\gamma}^{2-\epsilon} \right) \right]. \quad (2.17)$$

Now, we divide both sides of (2.17) by $\|\eta'v\|_{L^2_{\gamma-2+\lambda}}^2$. If we denote

$$z = \frac{\|\eta v'\|_{L^2_\gamma}}{\|\eta'v\|_{L^2_{\gamma-2+\lambda}}}, \quad \zeta = \frac{\|\eta v\|_{L^2_{\gamma-2+\lambda}}}{\|\eta'v\|_{L^2_{\gamma-2+\lambda}}}$$

then, (2.17) becomes

$$z^2 \leq C[z + q(\zeta^\epsilon + \zeta^\epsilon z^{2-\epsilon})].$$

We proceed further as in Serrin [13]. We obtain that

$$z \leq Cq^{2/\epsilon}(1 + \zeta) \quad (2.18)$$

or equivalently

$$\|\eta v'\|_{L^2_\gamma} \leq Cq^{2/\epsilon}(\|\eta v\|_{L^2_{\gamma-2+\lambda}} + \|\eta'v\|_{L^2_{\gamma-2+\lambda}}). \quad (2.19)$$

Since $((\gamma - 2 + \lambda) + 1)/2 > (\gamma - 1)/2$, we infer by the Imbedding Lemma that there exists $\chi > 1$ such that

$$E_\gamma \hookrightarrow L^{2\chi}_{\gamma-2+\lambda}.$$

Using this fact, (2.19) can be transformed via the Imbedding Lemma into

$$\|\eta v\|_{L^{2\chi}_{\gamma-2+\lambda}} \leq Cq^{2/\epsilon}(\|\eta v\|_{L^2_{\gamma-2+\lambda}} + \|\eta'v\|_{L^2_{\gamma-2+\lambda}}). \quad (2.19)'$$

We lose no generality if we assume $R > 2$. Now, we choose η such that $\eta \equiv 1$ in $(0, h')$, $\eta \equiv 0$ in (h, R) where $0 < h' < h < 2$ and $\max|\eta'| \leq 2/(h - h')$. If we set this function in (2.18) we get

$$\|v\|_{2\chi, h'} \leq \frac{C}{h - h'} q^{2/\epsilon} \|v\|_{2, h} \quad (2.20)$$

where for the sake of simplicity we have denoted by $\|\cdot\|_{q, h}$ the norm in the space $L^q_{\gamma-2+\lambda}(0, h)$.

We recall the definition of v , i.e., $v = F_\ell(\bar{u})$. We let $\ell \rightarrow \infty$ in (2.20). $F_\ell(\bar{u}) \rightarrow \bar{u}^q$. This yields

$$\|\bar{u}^q\|_{2\chi, h'} \leq \frac{C}{h - h'} q^{2/\epsilon} \|\bar{u}^q\|_{2, h}. \quad (2.21)$$

If we set $p = 2q$ (2.21) is transformed into

$$\|\bar{u}\|_{p\chi, h'} \leq [C(\frac{p}{2})^{2/\epsilon}(h - h')^{-1}]^{2/p} \|\bar{u}\|_{p, h}.$$

For $n \in \mathbb{N}$ we denote $p_n = 2\chi^n$, $h_n = 1 + 2^{-n}$, $h'_n = h_{n+1}$. We infer

$$\|\bar{u}\|_{p_{n+1}, h_{n+1}} \leq C^{1/\chi^n} K^{n/\chi^n} \|\bar{u}\|_{p_n, h_n}$$

where $K = 2\chi^{2/\epsilon}$. The iteration of the last inequality yields

$$\|\bar{u}\|_{p_{n+1}, h_{n+1}} \leq C^{\Sigma(1/\chi^k)} K^{\Sigma(k/\chi^k)} \|\bar{u}\|_{2, 2} \leq \text{const.} \|\bar{u}\|_{L^2_{\gamma-2+\lambda}}. \quad (2.22)$$

We let $n \rightarrow \infty$ and we get

$$\|u\|_{\infty, 1} \leq \text{const.} \|\bar{u}\|_{L^2_{\gamma-2+\lambda}},$$

since $r^{\gamma-2+\lambda} dr$ is a finite measure on $(0, R)$. Theorem 2.3 is proved. ■

The purpose of our next result is to establish further regularity for u near zero.

Proposition 2.4. *Let u be a nonnegative weak solution of (0.2) and (0.3), $p > 1$, $\beta > -2$, $\gamma > 1$, $(\theta + 1)/(p + 1) > (\gamma - 1)/2$. Then, $u \in C[0, R] \cap C^2(0, R)$ and moreover*

$$u(r) \sim u(0) + \frac{u^p(0)}{(\beta + \gamma + 1)(\beta + 2)} r^{\beta+2} \quad \text{as } r \searrow 0. \tag{2.23}$$

Proof: We follow the method used by Ni [8]. As we have already seen u satisfies (2.6) in $(0, R)$ and moreover, $\lim_{r \rightarrow 0} u(r)$ exists and it is a positive real number. Also, $(r^\gamma u')' < 0$ and therefore, $\lim_{r \rightarrow 0} r^\gamma u'$ exists. Since, on a subsequence $r_k \rightarrow 0$, $\lim_{r_k \rightarrow 0} r_k^{(\gamma+1)/2} u'(r_k) = 0$ (cf. (2.8)) this yields, because $\gamma > \frac{\gamma+1}{2}$,

$$\lim_{r_k \rightarrow 0} r^\gamma u'(r) = 0. \tag{2.24}$$

$(r^\gamma u')' + r^{\beta+\gamma} u^p = 0$. Integrating from ϵ to r and letting $\epsilon \rightarrow 0$ we get (by (2.24))

$$r^\gamma u'(r) = - \int_0^r t^{\beta+\gamma} u^p(t) dt.$$

Therefore,

$$\left| r^\gamma u_r(r) - \left(- \int_0^r t^{\beta+\gamma} u^p(0) dt \right) \right| \leq \int_0^r t^{\beta+\gamma} |u^p(t) - u^p(0)| dt;$$

i.e.,

$$\begin{aligned} \left| r^\gamma u_r(r) + \frac{u^p(0)}{\beta + \gamma + 1} r^{\beta+\gamma+1} \right| &= r^{\beta+\gamma+1} o(1) = o(r^{\beta+\gamma+1}) \\ \left| u_r(r) + \frac{u^p(0)}{\beta + \gamma + 1} r^{\beta+1} \right| &= o(r^{\beta+1}). \end{aligned}$$

Thus,

$$u_r(r) \sim - \frac{u^p(0)}{\beta + \gamma + 1} r^{\beta+1} \quad \text{near } r = 0$$

and therefore,

$$u(r) \sim u(0) - \frac{u^p(0)}{(\beta + \gamma + 1)(\beta + 2)} r^{\beta+2}. \quad \blacksquare$$

The last result of this section is a uniqueness result.

Theorem 2.5. *Problem (0.2) and (0.3) possesses at most one nontrivial positive weak solution when $\gamma \neq 1$, $\beta > \max\{-2, -\gamma - 1\}$.*

Proof: We consider the initial value problem

$$\frac{1}{r^\gamma} (r^\gamma u')' + r^\beta u^p = 0, \quad u(0) = a > 0. \tag{2.25}$$

The following rescaling relates the first zero of u and the initial value $u(0)$ in a 1-1 correspondence:

$$u_\lambda(x) = \lambda^{(2+\beta)/(p-1)} u(\lambda x).$$

Thus, we have to show that each initial value $u(0)$ only gives rise to one solution of the corresponding initial value problem. We make the change of variables

$$s = r^{-(\gamma-1)}, \quad v(s) = u(r). \quad (2.26)$$

Then, $d/dr = ds/dr d/ds = -(\gamma-1)r^{-\gamma} d/ds$. Hence, $r^\gamma du/dr = -(\gamma-1) dv/ds$. We get

$$v_{ss} + \frac{1}{(\gamma-1)^2} \frac{v^p}{s^{2+(2+\beta)/(\gamma-1)}} = 0, \quad v(\infty) = a > 0. \quad (2.27)$$

Note that since $v > 0$ near $s = \infty$ it must be concave there; thus $v_s(\infty) = 0$. It suffices to show that for each $a > 0$ there is at most one solution v of (2.27) near $s = \infty$. We first observe that if v satisfies (2.27) near $s = \infty$, then $0 \leq v \leq a$ near $s = \infty$ and it is a solution of the following integral equation near $s = \infty$

$$v(s) = a - \int_s^\infty (t-s) \frac{1}{(\gamma-1)^2} \frac{1}{t^{2+(2+\beta)/(\gamma-1)}} v^p(t) dt. \quad (2.28)$$

To see this we set $g(s) = 1/[(\gamma-1)^2 s^{2+(2+\beta)/(\gamma-1)}]$. Then $\int_s^\infty sg(s) ds < \infty$ since $2+\beta > 0$. Let v be a solution of (2.27) near $s = \infty$. Integrating from s to s_1 and then letting $s_1 \rightarrow \infty$ we get

$$v_s(s) = \int_s^\infty g(t)v^p(t) dt.$$

Integrating again from s_2 to s_1 and letting $s_1 \rightarrow \infty$

$$a - v(s_2) = \int_{s_2}^\infty \int_{s_2}^\infty g(t)v^p(t) dt ds.$$

It is easy to see that one can apply Fubini's theorem here to obtain

$$a - v(s_2) = \int_{s_2}^\infty \int_{s_2}^t g(t)v^p(t) ds dt = \int_{s_2}^\infty (t-s_2)g(t)v^p(t) dt,$$

i.e., v satisfies (2.28). Now, we claim that (2.28) has at most one solution with $v \leq a$ near $s = \infty$. To do this we use a contraction mapping argument. Let

$$B_x = \{v \mid 0 \leq v \leq a \text{ in } [x, \infty)\}$$

equipped with the L_∞ -norm. We claim that (2.28) has exactly one solution in B_x for $x \geq x_0$, where x_0 is a positive number such that

$$\lambda \equiv pa^{p-1} \int_{x_0}^\infty (t-x_0)g(t) dt \leq 1.$$

On B_x we define

$$T(v)(s) = a - \int_s^\infty (t-s)g(t)v^p(t) dt.$$

It is easy to see that T is well-defined, $T(v) \leq a$ for all $s \geq x$ and $T(v)(s) \geq 0$ for all $s \geq x$, since $\lambda < 1$ and $p > 1$. Thus, $T(B_x) \subset B_x$.

Next, we show that T is a contraction on B_x . For $v, w \in B_x$ and $s \geq x$

$$\begin{aligned} |T(v)(s) - T(w)(s)| &\leq |v - w|_{B_x} \int_s^\infty (t - s)g(t) \left| \frac{v^p(t) - w^p(t)}{v(t) - w(t)} \right| dt \\ &\leq |v - w|_{B_x} \int_s^\infty (t - s)g(t) p a^{p-1} dt \\ &\leq \lambda |v - w|_{B_x}. \end{aligned}$$

Since $\lambda < 1$, T is a contraction on B_x for all $x \geq x_0$. Thus, there exists a unique fixed point of T on B_x .

Remark 2.2. The idea of the previous proof of uniqueness is due to Ni [8], and it works in more general situations.

3. Existence of a positive solution. In this section we shall prove the existence of a (unique) positive solution of (0.2) + (0.3). This will be achieved using the Mountain Pass Lemma.

Mountain Pass Lemma (Ambrosetti-Rabinowitz [1]). *Let E be a real Banach space and $I \in C^1(E, R)$. Suppose I satisfies (PS), $I(0) = 0$,*

(I.1) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$ and*

(I.2) *there is an $e \in E \setminus B_\rho$ such that $I(e) \leq 0$.*

Then, I possesses a critical value $c \geq x$ which can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g[0,1]} I(u),$$

where

$$\Gamma = \{g \in C([0, 1], E) \mid g(0) = 0, g(1) = e\}.$$

We first note that the following version of the maximum principle is true.

Maximum Principle. *Let $u \in E_\gamma$ such that*

$$\int_0^R r^\gamma u' \varphi' dr \geq 0, \quad \forall \varphi \in E_\gamma, \varphi \geq 0. \quad (3.1)$$

Then $u \geq 0$.

Indeed, this follows from the simple observation that if $u, v \in E_\gamma$, then $\max(u, v), \min(u, v) \in E_\gamma$ and, then choose in (3.1), $\varphi = -\min(u, 0)$. (One should remark that $u \in E_\gamma \Leftrightarrow r^{\gamma/2}u \in H_0^1(0, R)$, $r^{(\gamma-2)/2}u \in L^2(0, R)$ and all the statements above are obtained by translation from the usual unweighted situation.)

As in Ni [7] we shall actually work with the equation

$$-\frac{1}{r^\gamma} (r^\gamma u')' = r^\beta |u|^p, \quad u \in E^\gamma, u(R) = 0, \quad (3.2)$$

whose solutions are positive weak solutions of (0.2) and (0.3).

We consider the functional

$$I : E_\gamma \longrightarrow \mathbf{R} \text{ by } I(u) = \frac{1}{2} \|u\|_\gamma^2 - \frac{1}{p+1} \int_0^R r^\theta |u|^p u \, dr.$$

By the Imbedding Lemma, we see that I is well-defined when

$$\frac{\theta + 1}{p + 1} \geq \frac{\gamma - 1}{2}.$$

Lemma 3.1. *If $(\theta + 1)/(p + 1) > (\gamma - 1)/2$, $p > 1$, $\beta > -2$, $\theta > -1$, then $I \in C^1(E_\gamma, \mathbf{R})$ and*

$$\langle I'(u), \varphi \rangle = \int_0^R r^\gamma u' \varphi' \, dr - \int_0^R r^\theta |u|^p \varphi \, dr, \quad \forall \varphi \in E_\gamma.$$

The proof of this lemma is technical and follows the same lines as the proof in Rabinowitz [12] Proposition B.10, so we omit it.

By Lemma 3.1 we see that the critical points of I are actually weak solutions of (3.2).

Theorem 3.2. *Equation (3.2) possesses at least a nontrivial weak solution when $(\theta + 1)/(p + 1) > (\gamma - 1)/2$, $\beta > -2$, $p > 1$, $\theta > -1$.*

Proof: We shall verify conditions (I.1) and (I.2) in the statement of the Mountain Pass Lemma. (I.2) follows from the simple observation that

$$\lim_{t \rightarrow \infty} I(tu) = -\infty \quad \forall u \neq 0$$

since $p > 1$. To prove (I.1), we note that by the Imbedding Lemma

$$\int_0^R r^\theta |u|^p u \, dr = o(\|u\|_\gamma^2) \quad \text{as } \|u\|_\gamma \rightarrow 0.$$

Therefore, $I(u) = \frac{1}{2} \|u\|_\gamma^2 + o(\|u\|_\gamma^2)$ as $u \rightarrow 0$ and (I.1) holds.

The only thing remaining to be proved is that I satisfies the Palais-Smale condition. Let $\{u_n\}$ be a sequence with $I(u_n)$ bounded and $I'(u_n) \rightarrow 0$. We first prove that $\{u_n\}$ is bounded in E_γ . $I'(u_n) \rightarrow 0$ implies

$$|\langle I'(u_n), u_n \rangle| = \left| \|u_n\|_\gamma^2 - \int_0^R r^\theta |u_n|^p u_n \, dr \right| \leq \|u_n\|_\gamma \quad (3.3)$$

for $n \geq 1$ large enough. $|I(u_n)| \leq C$ is equivalent to

$$\left| \frac{1}{2} \|u_n\|_\gamma^2 - \frac{1}{p+1} \int_0^R r^\theta |u_n|^p u_n \, dr \right| \leq C \quad (3.4)$$

which yields

$$\|u_n\|_\gamma^2 \leq 2C + \frac{2}{p+1} \left| \int_0^R r^\theta |u_n|^p u_n \, dr \right| \leq 2C + \frac{2}{p+1} \|u_n\|_\gamma^2 + \frac{2}{p+1} \|u_n\|_\gamma;$$

i.e.,

$$\left(1 - \frac{2}{p+1}\right) \|u_n\|_\gamma^2 \leq 2C + \frac{2}{p+1} \|u_n\|_\gamma.$$

Therefore, $\{u_n\}$ is a bounded sequence in E_γ .

Let E_γ^* be the dual of E_γ . We denote by $D : E_\gamma^* \rightarrow E_\gamma$ the Riesz-Fréchet duality between E_γ^* and E_γ given by

$$\langle Du, v \rangle = (u, v)_{E_\gamma}.$$

We also consider $T : E_\gamma \rightarrow E_\gamma^*$, given by

$$\langle Tu, v \rangle_{(E_\gamma^*, E_\gamma)} = \int_0^R r^\theta |u|^p v \, dr.$$

With this notation we have

$$I'(u_n) = Du_n - Tu_n$$

and therefore,

$$u_n = D^{-1}Tu_n + D^{-1}I'(u_n).$$

Since $I'(u_n) \rightarrow 0$, u_n will have a convergent subsequence, if and only if Tu_n has such a subsequence. To this effect we prove that T is a compact mapping. Here an important ingredient will be the Compactness Lemma. T can be decomposed in the following chain of mappings

$$E_\gamma \xrightarrow{i} L_\theta^{p+1} \xrightarrow{T_1} L^{(p+1)/p} \xrightarrow{i^*} E_\gamma^*$$

$$T = i^* \circ T_1 \circ i$$

where i is the natural injection $E_\gamma \rightarrow L_\theta^{p+1}$ given by the Imbedding Lemma.

$T_1 : L_\theta^{p+1} \rightarrow L_\theta^{(p+1)/p}$, $T_1 u = |u|^p$ is continuous, $L_\theta^{(p+1)/p}$ is the dual of L_θ^{p+1} and i^* is the adjoint of i and is given by $\langle i^* u, v \rangle_{(E_\gamma^*, E_\gamma)} = \int_0^R r^\theta u v \, dr$ - the duality between L_θ^{p+1} and $L_\theta^{(p+1)/p}$.

By the Compactness Lemma and the hypotheses of the theorem we infer that i is compact (and even i^* is compact by Schauder's theorem).

Therefore T is compact. So I satisfies (PS) and Theorem 3.2 is proved.

Corollary 3.3.

- a) Problem (0.2) and (0.3) possess a unique positive weak solution when $p > 1$, $\beta > \max(-2, -\gamma - 1)$, $(\theta + 1)/(p + 1) > (\gamma - 1)/2$
- b) If $\gamma > 1$, $\beta > -2$, $p > 1$, $(\theta + 1)/(p + 1) < (\gamma - 1)/2$, then (0.2) and (0.3) has no non-trivial positive solution.

We state the above conditions explicitly. We distinguish two situations

- A. $\gamma > 1$, $\beta > -2$. If $p \in [1, (\gamma + 3 + 2\beta)/(\gamma - 1))$ equation (0.2) and (0.3) possesses exactly one positive weak solution. If $p \in ((\gamma + 3 + 2\beta)/(\gamma - 1), \infty)$ there exist no such solutions.
- B. $\gamma \leq 1$, $\beta > -\gamma - 1$. Equation (0.2) and (0.3) possesses exactly one positive weak solution for every $p > 1$.

The next section deals with multiplicity results for (0.2) and (0.3).

4. Existence of infinitely many solutions changing sign. The existence of an infinity of solutions will be obtained as a consequence of a symmetric variant of the Mountain Pass Lemma due to Rabinowitz [12]. To this effect, we need some facts about the spectral properties of the differential operator defining the equation.

Let $\theta > \max(-1, \gamma - 2)$ and $\gamma \in \mathbf{R}$. We consider $H_\theta = L_\theta^2$ and $A : D(A) \subset H_\theta \rightarrow H_\theta$ given by

$$(Au, v)_{H_\theta} = \int_0^R r^\gamma u' v' dr.$$

By the Imbedding Lemma we get that A is a symmetric positive definite linear operator. Its energetic space H_A (cf. Mihlin [6]) is just E_γ . By the Compactness Lemma H_A is compactly imbedded in H_θ . $u \in H_\theta$ is a generalized eigenvector corresponding to the eigenvalue λ if it satisfies

$$(Au, v)_{H_\theta} = \lambda(u, v)_{H_\theta} \quad \forall v \in H_\theta. \quad (4.1)$$

The following fact is known (Mihlin [6]).

Lemma 4.1.

- a) *The generalized eigenvalues of A are positive and form an unbounded sequence*

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow \infty.$$

Each of them has finite multiplicity. (In the sequence above each eigenvalue appears as many times as its multiplicity.)

- b) *For $k \geq 1$ let $v_k \neq 0$ be an eigenvalue corresponding to λ_k and let E_k be the subspace spanned by v_1, \dots, v_k . Then for $v \in E_k$*

$$\|v\|_\gamma \geq \lambda_{k+1} \|v\|_{H_\theta}. \quad (4.2)$$

Moreover, $\{v_k\}_{k \geq 1}$ can be chosen to form an orthonormal basis in H_θ .

As we have already mentioned we shall use a symmetric version of the Mountain Pass Lemma which we state below.

Symmetric Mountain Pass Lemma. *Let E be an infinite dimensional Banach space and let $I \in C^1(E, \mathbf{R})$ be even, satisfy (PS) and $I(0) = 0$. If $E = V \oplus X$ where V is finite dimensional and I satisfies*

(I'.1) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$ and*

(I'.2) *for each finite dimensional subspace $\tilde{E} \subset E$ there is an $R = R(\tilde{E})$ such that $I \leq 0$ on $\tilde{E} \setminus B_{R(E)}$;*

then I possesses an unbounded sequence of critical values.

We can state the main result of this section.

