# CHARACTERISTIC CURRENTS OF SINGULAR CONNECTIONS

LIVIU I. NICOLAESCU

ABSTRACT. I am trying to understand the work of Harvey and Lawson.

## CONTENTS



## 1. A VERY FAST INTRODUCTION TO GEOMETRIC MEASURE THEORY

We survey some basic facts about currents. Suppose $E$ is a $N$-dimensional real Euclidean vector space. We denote the Euclidean metric on $E$ by $(\bullet, \bullet)$, by $\mathcal{H}^m = \mathcal{H}^m_E$ the $m$-dimensional Hausdorff measure on Borel subsets of $E$ and by $dv_E$ the Lebesgue measure on $E$.

For every smooth map between Euclidean spaces $F : E_0 \to E_1$, every positive integer $k \leq \min(\dim E_0, \dim E_1)$ and every $x \in E_0$ we get a linear map

$$\Lambda^k_x F^* : \Lambda^k T^*_{F(x)} E_1 \to \Lambda^k T^*_x E_0$$

$\Lambda^k T^*_{F(x)} E_1$ and $\Lambda^k T^*_x E_0$ are equipped with natural Euclidean metrics and we denote by $|J_k F|(x)$ the norm of the above linear map.

Suppose $E$ is an *oriented* Euclidean space. Denote by $\Omega^m_{cpt}(E)$ the space of smooth, compactly supported $m$-forms on the Euclidean space $E$ and by $\Omega^m(E)$ the space of smooth $m$-forms. They are naturally equipped with locally convex linear topologies defined by the uniform convergence on compacts of forms and their partial derivatives of any order. The space of $m$-dimensional currents, denoted by $\mathcal{D}_m$ is the topological dual of $\Omega^m_{cpt}$. The space of compactly supported $m$-dimensional currents, denoted by $\mathcal{E}_m$, is the topological dual of $\mathcal{E}^m$. We define

$$\partial : \mathcal{D}_m \to \mathcal{D}_{m-1}$$

by the equality

$$\langle \partial T, \alpha \rangle = \langle T, d\alpha \rangle, \;\; \forall \alpha \in \Omega^{m-1}_{cpt}(E), \;\; T \in \mathcal{D}_m.$$

**Example 1.1.** (a) Let $N = \dim E$. Then any orientation of $E$ determines a natural current

$$[E] \in \mathcal{D}_N, \;\; \langle [E], \alpha \rangle = \int_E \alpha, \;\; \forall \alpha \in \Omega^N_{cpt}.$$

Observe that

$$\partial [E] = 0.$$

*Date*: July, 2005.

For every $\alpha \in \Omega^k(E)$, $T \in \mathcal{D}_m$, $k \leq m$ define $\alpha \cap T \in \mathcal{D}_{m-k}$ by

$$\langle \alpha \cap T, \beta \rangle = \langle T, \alpha \wedge \beta \rangle, \quad \forall \beta \in \Omega_{cpt}^{N-m+k}.$$

We have

$$\langle \partial(\alpha \cap T), \beta \rangle = \langle\, (\alpha \cap T), d\beta \rangle = \langle T, \alpha \wedge d\beta \rangle$$
$$= (-1)^k \langle T, d(\alpha \wedge \beta) - d\alpha \wedge \beta \rangle = (-1)^k \langle \alpha \cap \partial T, \beta \rangle + (-1)^{k+1} \langle d\alpha \cap T, \beta \rangle$$

which yields the *homotopy formula*

$$\partial(\alpha \cap T) = (-1)^{\deg \alpha} \big( \alpha \cap \partial T - (d\alpha) \cap T \big). \tag{1.1}$$

In particular, we obtain an embedding

$$\Omega^{N-m}(E) \hookrightarrow \mathcal{D}_m(E), \ \ \alpha \mapsto [\alpha] = \alpha \cap [E],$$

defined by

$$\langle [\alpha], \beta \rangle = \int_E \alpha \wedge \beta, \ \ \forall \beta \in \Omega_{cpt}^m(E).$$

Observe that

$$\partial[\alpha] = (-1)^{N-m+1} [d\alpha], \ \ \forall \alpha \in \Omega^{N-m}(E). \tag{1.2}$$

(b) If we denote by $\Omega_m(E)$ the space of smooth sections of $\Lambda^m TE$ then we have a natural inclusion

$$\Omega_m(E) \hookrightarrow \mathcal{D}_m(E), \ \ \xi \mapsto dv_E \wedge \xi,$$

where

$$\langle\, dv_E \wedge \xi, \beta \rangle = \int_E (\xi \lrcorner \beta)\, dv_E, \ \ \forall \beta \in \Omega_{cpt}^m(E).$$

If we denote by $^\dagger : \Omega_m(E) \to \Omega^m(E)$ the natural metric duality and by $* : \Omega^m(E) \to \Omega^{N-m}(E)$ the Hodge $*$-operator then

$$dv_E \wedge \xi = (-1)^{m(N-m)} (*\xi^\dagger) \cap [E], \ \ \forall \xi \in \Omega_m(E).$$

(c) Suppose $M$ is an orientable $m$-dimensional submanifold of $E$ and *or* is an orientation on $M$. Then we obtain a current $[M] = [M, or] \in \mathcal{D}_m$ defined by

$$\langle [M], \beta \rangle = \int_M \beta, \ \ \forall \beta \in \Omega_{cpt}^m(E)$$

If $M$ has boundary $\partial M$ then Stokes formula implies

$$\partial[M, or] = [\partial M, \partial or].$$

If $p$ is a point in $E$ we denote by $[p]$ the 0-current determined by the inclusion $\{p\} \hookrightarrow E$.
(d) Consider the 1-form $\varphi = d\theta \in \Omega^1(\mathbb{R}^2 \setminus 0)$. In cartesian coordinates it has the form

$$\varphi = \frac{1}{x^2 + y^2}(xdy - ydx).$$

Then $\varphi$ is locally integrable and thus defines $[\varphi] \in \mathcal{D}_1(\mathbb{R}^2)$. We want to compute its boundary. For every compactly supported smooth function $\beta$ we have

$$\langle \partial[\varphi], \beta \rangle = \int_{\mathbb{R}^2} \varphi \wedge d\beta = -\lim_{\varepsilon \searrow 0} \int_{|z| \geq \varepsilon} d(\beta\varphi) = \lim_{\varepsilon \searrow 0} \int_{|z| = \varepsilon} \beta\varphi = 2\pi\beta(0)$$

so that

$$\partial[d\theta] = 2\pi[0].$$

$\square$

If $f : E \to F$ is a smooth map and $T \in \mathcal{D}_m(E)$ is such that the restriction of $f$ to the support of $T$ is proper then we define $f_*T \in \mathcal{D}_m(F)$ by

$$\langle f_*T, \alpha\rangle = \langle T, f^*\alpha\rangle,\ \ \forall \alpha \in \Omega^m_{cpt}(F).$$

This operation is called the *push-forward* of currents defined by a smooth map. It commutes with the boundary operator $\partial$.

We would like to discuss a few topologies on $\mathcal{D}_m$. Given a current $T \in \mathcal{D}_m$ and a precompact open subset $W \subset E$ we define the *mass* of $T$ in $W$ to be

$$\boldsymbol{m}_W(T) := \sup\bigl\{\langle T, \alpha\rangle : \|\alpha\| \leq 1,\ \ \alpha \in \Omega^m(E),\ \ \operatorname{supp}\alpha \subset W\bigr\}$$

where

$$\|\alpha\| = \sup_{x\in E} \|\alpha(x)\|,$$

$$\|\alpha(x)\| = \max\bigl\{\, (e_1 \wedge \cdots \wedge e_m) \lrcorner\, \alpha(x);\ \ e_1, \cdots, e_m \in E\ \text{ are orthonormal}\,\bigr\}.$$

$\|\alpha\|$ is called the *comass* of $\alpha$. We set that $T$ has *locally finite mass* if

$$\boldsymbol{m}_W(T) < \infty,\ \ \forall W \Subset E.$$

**Example 1.2.** (a) Suppose $M \hookrightarrow E$ is an embedded, oriented $m$-dimensional submanifold. Then

$$\boldsymbol{m}_W(M) = \mathrm{vol}_m(M \cap W),$$

where $\mathrm{vol}_m$ denotes the $m$-dimensional volume induced on $M$ by the Euclidean metric.
(b) Suppose we are given a Radon measure $\mu$ on $E$, a $\mu$-measurable $m$-vector field $\xi \in \Gamma(\Lambda^m TE)$ such that $|\xi(x)| = 1$ $\mu$-a.e. Then we can form the current

$$\mu \wedge \xi \in \mathcal{D}_m$$

by setting

$$\langle \mu \wedge \xi, \alpha\rangle = \int_E \xi \lrcorner\, \alpha d\mu,\ \ \forall \alpha \in \Omega^m_{cpt}.$$

Then $\mu \wedge \xi$ has locally finite mass

$$\boldsymbol{m}_W(\mu \wedge \xi) = \int_W d|\mu|.$$

We say that $\mu \wedge \xi$ is an *integral representation* current. Conversely, every current with locally finite mass $T$ admits a unique integral representation $\mu \wedge \xi$. We set $\mu := \mu_T$, $\xi := \vec{T}$. □

We want to discuss a generalization of the above example. A subset $M \subset E$ is called countably $m$-rectifiable if there exists $Z \subset M$, and a sequence of embedded $C^1$-submanifolds $N_1, \cdots, N_k, \cdots \subset E$ each of dimension $m$ such that

$$\mathcal{H}^m(Z) = 0,$$

and

$$M \setminus Z \subset \bigcup_{k\geq 1} N_k$$

It is called rectifiable if $\mathcal{H}^m(M) < \infty$. If $M \subset E$ is a countably $m$-rectifiable set, $x_0 \in M$ and $V \subset E$ is a $m$-dimensional vector subspace of $E$ then $V$ is called an *approximate tangent space* to $M$ at $x_0$ if for every $f \in C^\infty_{cpt}(E, \mathbb{R})$ we have

$$\lim_{\varepsilon\searrow 0} \int_{\delta_{\varepsilon,x_0}(M)} f d\mathcal{H}^m = \int_V f d\mathcal{H}^m,$$

where $\delta_{\varepsilon,x_0}$ is the dilation of center $x_0$ and factor $\varepsilon^{-1}$

$$\delta_{\varepsilon,x_0} : E \to E, \;\; x \mapsto \frac{1}{\varepsilon}(x - x_0).$$

We have the following characterization of rectifiable sets, [7, Thm. 3.3.5].

**Theorem 1.3.** *A subset $M$ is countably $m$-rectifiable if there exists $Z \subset M$ such that $\mathcal{H}^m(Z) = 0$ and every $x \in M \setminus Z$ has an approximate $m$-dimensional tangent space $T_xM$.* □

Given a countably $m$-rectifiable set $M$ we denote by $M_{reg}$ the subset consisting of points admitting an approximate tangent space. A *measurable orientation* of $M$ is an equivalence class of measurable field of $m$-vectors $\vec{\omega}_M \in \Gamma(\Lambda^m TE)$ on $E$ such that $\vec{\omega}(x) = 0$ if $x \in E \setminus M_{reg}$ while

$$\vec{\omega}(x) = e_1 \wedge \cdots \wedge e_m,$$

where $e_1, \cdots, e_m$ is an orthonormal basis of $T_xM$. Two orientations are called equivalent if they agree $\mathcal{H}^m$-a.e. A pair $(M, \omega_M)$ as above is called an oriented, countably rectifiable set. To such a pair we associated the current

$$[M, \vec{\omega}] := \mathcal{H}^m \wedge \vec{\omega}_M.$$

More generally, if $\nu : E \to \mathbb{Z}$ is a locally $\mathcal{H}^m$-integrable function then we can define $\nu[M, \omega]$ by

$$\langle \nu[M, \omega], \alpha\rangle = \int_M \nu(\vec{\omega} \lrcorner \alpha) d\mathcal{H}^m\rangle.$$

We now want to introduce several important classes of currents.

A current $T \in \mathcal{D}_m$ of the form $\nu[M, \vec{\omega}]$, where $\nu : E \to \mathbb{Z}$ is locally $\mathcal{H}^m$-integrable and $[M, \vec{\omega}]$ is a countably $m$-rectifiable subset is called *locally rectifiable*. It is called *rectifiable* if it has compact support, and hence finite total mass. We denote by $\mathcal{R}_m$ the Abelian group of rectifiable $m$-currents. A rectifiable current $T$ is called *integral* if $\partial T$ is rectifiable. We denote by $\mathcal{I}_m$ the Abelian group of integral $m$-currents.

A current $T$ is called *normal* if it has compact support and

$$\boldsymbol{m}(T) + \boldsymbol{m}(\partial T) < \infty.$$

An $m$-simplex in $E$ is a linearly embedded $m$-simplex. It defines in a natural way an integral current and we denote by $\mathcal{P}_m$ the Abelian group generated by these simplices. We will refer to the elements of $\mathcal{P}_m$ as polyhedral chains. We denote by $\mathbf{P}_m$ the real vector space generated by polyhedral chains.

For every current $T \in \mathcal{D}_m$ we set

$$\mathbf{F}(T) := \sup\big\{ \langle T, \alpha\rangle; \;\; \|\alpha\|, \|d\alpha\| \leq 1 \big\} = \inf\big\{ \boldsymbol{m}(T - \partial S) + \boldsymbol{m}(S); \;\; S \in \mathcal{D}_{m+1} \big\}.$$

We define the *flat metric* to be

$$d(T, S) := \mathbf{F}(S - T).$$

For example, the two oriented segments $T$, $S$ define two rectifiable 1-currents. Observe that

$$T - S = \partial R + (U - V) \Longrightarrow \mathbf{F}(T - S) \leq \boldsymbol{m}(R) + \boldsymbol{m}(U - V) = ld + 2d.$$

A current $T$ is called *flat* if its support is contained in a compact set $K$ and it is a limit in the flat metric of normal currents supported in $K$. We denote by $\mathbf{F}_m$ the *vector space* of flat currents.

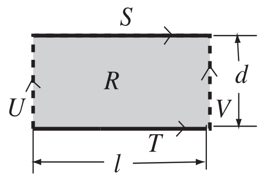


FIGURE 1. *Two parallel segments of length $l$ at distance $d$ apart.*

A current $T \in \mathcal{D}_m$ is called *integrally flat* if $T \in \mathcal{R}_m + \partial\mathcal{R}_{m+1}$. We denote by $\mathcal{F}_m$ the *Abelian group* of integrally flat currents. For such a current we define

$$\mathcal{F}(T) := \sup\big\{\, \boldsymbol{m}(R) + \boldsymbol{m}(S);\ \ T = R + \partial S,\ \ R \in \mathcal{R}_m,\ \ S \in \mathcal{R}_{m+1} \,\big\}.$$

The pushforward operation maps flat (integral) currents to flat (resp. integral currents.)

**Example 1.4.** Any flat $m$-current $T$ can be written in the form

$$T = \xi \wedge dv_E + \partial(\eta \wedge dv_E)$$

where $\xi \in \Gamma(\Lambda^m TE)$ and $\eta \in \Gamma(\Lambda^{m+1}TE)$ are compactly supported and Lebesgue integrable. In particular, if $\dim m = E$ then any top dimensional current can be written in the form $\xi \lrcorner\, dv_E$, where $\xi$ is a Lebesgue integrable section of $\det TE$. For a proof we refer to [2, §4.1.18]. The above equality is equivalent to

$$\langle T, \alpha\rangle = \int_E (\xi \lrcorner\, \alpha) dv_E = \int_E (\eta \lrcorner\, d\alpha) dv_E,\ \ \forall \alpha \in \Omega^m_{cpt}.$$

Consider for example the rectifiable current defined by the segment

$$S = \big\{\, (t, 0) \in \mathbb{R}^2;\ \ t \in [0, 1] \,\big\}.$$

We denote by $(x^1, x^2)$ the Euclidean coordinates on $\mathbb{R}^2$. Then for every $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 \in \Omega^1_{cpt}$ we have

$$\langle S, \alpha\rangle = \int_S \alpha = \int_0^1 \alpha_1(x^1, 0) dx^1.$$

We seek compactly supported, integrable vector fields $\xi = \xi^1\partial_1 + \xi^2\partial_2 \in \Gamma(\Lambda^1 T\mathbb{R}^2)$, $\eta = \rho\partial_1 \wedge \partial_2 \in \Gamma(\Lambda^2 T\mathbb{R}^2)$ such that

$$[S] = dv_E \wedge \xi + \partial(dv_E \wedge \eta),$$

i.e.

$$\int_{\mathbb{R}^2} (\xi^1\alpha_1 + \xi^2\alpha_2) dx^1 dx^2 + \int_{\mathbb{R}^2} (\frac{\partial \alpha_2}{\partial x^1} - \frac{\partial \alpha_1}{\partial x^2}) \rho dx^1 dx^2 = \int_0^1 \alpha_1(x^1, 0) dx^1.$$

If $\alpha$ is supported in a ball disjoint from $S$ then we deduce

$$\xi^1 dx^1 + \xi^2 dx^2 = -d^*(\rho dx^1 \wedge dx^2).$$

Let

$$\delta = \frac{x^2}{d(x)} dv,\ \ d(x) = \mathrm{dist}\,(x, S).$$

We have $\delta(x) \in L^{1,1}_{loc}(\mathbb{R}^2 \setminus S)$ we set

$$\beta = -d^*\delta$$

and

$$Z_\varepsilon = \{d(x) \geq \varepsilon\}.$$

Using the integration by parts formula in [9, Prop. 4.1.40] we deduce that for every compactly supported 1-form $\alpha$ we have

$$\int_{Z_\varepsilon}\langle d\alpha,\delta\rangle dv+\int_{Z_\varepsilon}\langle\beta,\alpha\rangle dv=\int_{Z_\varepsilon}\langle d\alpha,\delta\rangle dv-\int_{Z_\varepsilon}\langle\alpha,d^*\delta\rangle=\int_{\partial Z_\varepsilon}\langle\alpha,\vec{n}\lrcorner\,\delta\rangle ds_\varepsilon,$$

where $\vec{n}$ is the outer unit normal vector field along $\partial Z_\varepsilon$ and $ds_\varepsilon$ denotes the arclength along $\partial Z_\varepsilon$. If we let $\varepsilon\to 0$ we deduce

$$\int_{\mathbb{R}^2}\langle d\alpha,\delta\rangle dv+\int_{\mathbb{R}^2}\langle\beta,\alpha\rangle dv=2\int_S\alpha$$

so that

$$2[S]=dv\wedge\beta_\dagger+\partial(dv\wedge\delta_\dagger),$$

where for $_\dagger:\Omega^p(E)\to\Omega_p(E)$ denotes the metric duality. Unfortunately, $\beta_\dagger$ and $\delta_\dagger$ are not compactly supported. To obtain compactly supported vector fields we choose a compactly supported smooth function $\zeta$ such that

$$\zeta(x)=1,\ \ \forall d(x)\leq 1.$$

Then

$$2[S]=\partial\big(\,dv\wedge(\zeta\delta)_\dagger\,\big)-\big(\,dv\wedge(d^*(\zeta\delta)\,)_\dagger\,\big).$$

□

The supports of flat $m$-dimensional currents have remarkable properties, [2, Thm. 4.1.20].

**Theorem 1.5.** *Suppose $T\in\mathbf{F}_m(E)$. Denote by $G_m(E)$ the Grassmannian of $m$-dimensional vector subspaces of $E$ and by $d\gamma_m$ the invariant measure on $G_m$ of total volume 1. For every $L\in G_m(E)$ we denote by $P_L$ the orthogonal projection onto $L$. Then*

$$\int_{G_m}\mathcal{H}^m(P_L\operatorname{supp}T)d\gamma_m(L)=0\Longleftrightarrow T=0.$$

*In particular, the support of a nontrivial flat $m$-dimensional current cannot be contained in a submanifold of $E$ of dimension $<m$.*

□

The next result explains the importance of integrally flat currents in topology.

**Theorem 1.6.** *Suppose $B\subset A\subset E$ are two "reasonable" subsets, e.g subanalytic. Let*

$$Z_m(A,B)=\big\{\,T\in\mathcal{F}_m;\ \ \operatorname{supp}T\subset A,\ \ \operatorname{supp}\partial T\subset B\,\big\}$$

$$B_m(A,B)=\big\{\,T+\partial S;\ \ T\in\mathcal{F}_m,\ \ S\in\mathcal{F}_{m+1};\ \ \operatorname{supp}T\in B,\ \ \operatorname{supp}\partial S\subset A\,\big\}.$$

*Then the quotient*

$$Z_m(A,B)/B_m(A,B)$$

*is naturally isomorphic to the singular homology $H_m(A,B;\mathbb{Z})$. A similar result holds if we replace $\mathcal{F}$ with $\mathcal{I}$ in the above definitions.*

□

For a proof we refer to [2, §4.4].

A central result in the theory of currents is the *compactness theorem* of Federer-Fleming, [2, Thm. 4.2.17].

**Theorem 1.7.** *Fix a compact subset $K \subset E$ and a positive constant $c$. Then the set*

$$\big\{T \in \mathcal{I}_m;\ \operatorname{supp}(T) \subset K,\ \ \boldsymbol{m}(T) + \boldsymbol{m}(\partial T) \leq c\big\}$$

*is closed with respect to the $\mathcal{F}$-metric while the set*

$$\big\{T \in \mathbf{F}_m;\ \operatorname{supp}(T) \subset K,\ \ \boldsymbol{m}(T) + \boldsymbol{m}(\partial T) \leq c\big\}$$

*is closed with respect to the $\mathbf{F}$ metric.* □

## 2. Singular connections

Suppose $E \to X$ is a smooth vector bundle over an oriented smooth manifold $X$, $\alpha$ is a section of $E$, and then $Z(\alpha)$ is the zero locus of $\alpha$. The Gauss-Bonnet-Chern theorem shows that if $\alpha$ vanishes nondegenerately along $Z(\alpha)$, and if $\nabla$ is a connection on $E$ then the $r$-th Chern-Weil form $c_r(\nabla)$ satisfies an equality of currents

$$c_r(\nabla) - [Z(\alpha)] = \partial T,$$

where $T$ is some current on $X$ of dimension $(N - 2r + 1)$.

We want to associate to the pair $(\nabla, \alpha)$ as above a connection $\overrightarrow{\nabla}^{\alpha}$ with the following properties.

**A**. $\lim_{s\searrow 0} \overrightarrow{\nabla}^{s\alpha} = \nabla$ and the forms $c_r(\overrightarrow{\nabla}^{s\alpha})$ converge in the sense of currents to as $s \nearrow \infty$. We denote this limit current by $c_r(\overrightarrow{\nabla}^{\infty,\alpha})$. Moreover, if $\alpha$ has a nondegenerate zero set then $c_r(\overrightarrow{\nabla}^{\infty}, \alpha) = [Z(\alpha)]$.

**B**. As $s \nearrow \infty$ the family of connections $\overrightarrow{\nabla}^{s}$ converges uniformly on the compacts of $X \setminus Z(\alpha)$ to a connection $\overrightarrow{\nabla}^{\infty} = \overrightarrow{\nabla}^{\infty,\alpha}$ and $c_r(\overrightarrow{\nabla}^{\infty}) = 0 \in \Omega^{2r}(X \setminus Z(\alpha)\,)$.

**C**. The Chern-Weil transgression $T_s \in \Omega^{2r-1}(X)$ satisfying

$$c_r(\nabla^s) = c_r(\nabla) + dT_s,\ \ s > 0$$

has a limit in the sense of currents as $s \nearrow \infty$.

To produce such connections we use a technique introduced by Harvey and Lawson in [5, I.2]. We describe it in a general situation.

Consider two smooth complex vector bundles $E_0, E_1$ a over the smooth, oriented manifold $X$. We denote by $r_i$ the rank of $E_i$, $i = 0, 1$. For a generic section $\alpha \in \operatorname{Hom}(E_0, E_1)$ we have $\operatorname{rank} \alpha_x = r$, for almost all $x \in X$. We set

$$\mathbb{D}(\alpha) := \big\{\, x \in X;\ \ \operatorname{rank} \alpha_x < \min(r_0, r_1)\,\big\}.$$

Suppose we are given two bundle morphisms $\alpha_{10} \in \operatorname{Hom}(E_0, E_1)$ and $\alpha_{01} \in \operatorname{Hom}(E_1, E_0)$ and connections $\nabla^i$ in $E_i$. We obtain connections $\nabla^{ji}$ on $\operatorname{Hom}(E_i, E_j)$, $i, j = 0, 1$. We can then define

$$\overrightarrow{\nabla}^1 = \nabla^1 - (\nabla^{10}\alpha_{10})\alpha_{01},\ \ \overleftarrow{\nabla}^0 = \nabla^0 + \alpha_{01}(\nabla^{10}\alpha_{10}).$$

Note that the definitions are not symmetric. Set for simplicity $\alpha = \alpha_{10}$. In applications we would like $\alpha_{01}$ to be a sort of inverse for $\alpha$. Note that if $\alpha_{10}\alpha_{01} = \mathbb{1}_{E_1}$ then if we set $\alpha^{-1} = \alpha_{01}$ then

$$\overrightarrow{\nabla}^1 = \alpha \nabla^0 \alpha^{-1}$$

In general, on $X \setminus \mathbb{D}(\alpha)$ we have two vector bundles

$$K = \ker \alpha,\ \ R = \operatorname{im} \alpha$$

If we choose hermitian metrics $h_i$ on $E_i$ we would like to have

$$\alpha_{01} = \alpha^{-1} P_R,$$

where $P_R$ denotes the orthogonal projection onto $R$ and $\alpha^{-1}$ denotes the inverse of the map $\alpha_{10} : K^{\perp} \to R$.

If $r_0 \leq r_1$ then on $X \setminus \mathbb{D}(\alpha)$ we have

$$\alpha^{-1} P_R = (\alpha^* \alpha)^{-1} \alpha^*$$

while if $r_0 > r_1$ then on $X \setminus \mathbb{D}_1(\alpha)$ we have

$$\alpha^{-1} P_R = \alpha^* (\alpha \alpha^*)^{-1}$$

Clearly this definition does not make sense over the degeneracy locus $\mathbb{D}(\alpha)$. One way out of this problem is to take an approximate inverse in the above definition. There is no unique way of doing this, and every choice will be called an *approximation mode*.

We choose an *approximate* 1 which is a smooth, increasing function

$$\chi : [0, \infty) \to [0, 1]$$

such that $\chi(0) = 0$ and $\chi(\infty) = 1$. Then the family $\chi_s(t) = \chi(s^2 t)$[1] converges to 1 uniformly on the compacts of $(0, \infty)$ as $s \nearrow \infty$. In the sequel, a very special role will be played by the *algebraic approximation mode* defined by

$$\chi = \frac{t}{1+t}.$$

Now define

$$\beta_s(t) := t^{-1} \chi_s(t).$$

Note that $\beta_s(t)$ converges to $t^{-1}$ uniformly on the compacts of $(0, \infty)$ as $s \nearrow \infty$. For every hermitian, nonnegative operator $A$ the operator $\beta_s(A)$ is an approximate inverse of $A$. We set

$$\alpha_s^{-1} = \left\{ \begin{array}{ll} \beta_s(\alpha^*\alpha)\alpha^* & \text{if} \quad r_0 \leq r_1 \\ \alpha^* \beta_s(\alpha\alpha^*) & \text{if} \quad r_0 > r_1 \end{array} \right. .$$

Note that for every continuous function $f : \mathbb{R} \to \mathbb{R}$ we have the equality[2]

$$f(\alpha^*\alpha)\alpha^* = \alpha^* f(\alpha\alpha^*),$$

so that

$$\alpha_s^{-1} = \beta_s(\alpha^*\alpha)\alpha^* = \alpha^* \beta_s(\alpha\alpha^*).$$

$\alpha_s^{-1}$ is a well defined morphism $E_1 \to E_0$. It vanishes on $R^{\perp}$ and we have

$$\alpha_s^{-1} \alpha = \chi_s(\alpha^*\alpha), \;\; \alpha \alpha_s^{-1} = \chi_s(\alpha\alpha^*).$$

We define

$$\overrightarrow{\nabla}^s = \overrightarrow{\nabla}^{s,\alpha} = \nabla^1 - (\nabla\alpha)\alpha_s^{-1}, \;\; \overleftarrow{\nabla}^s = \nabla^0 + \alpha_s^{-1}(\nabla\alpha).$$

As $s \nearrow \infty$ the connection $\overrightarrow{\nabla}^s$ converges on $X \setminus \mathbb{D}(\alpha)$ to a connection $\overrightarrow{\nabla}^{\infty} = \overrightarrow{\nabla}^{\infty,\alpha}$ called the *singular pushforward* of $\nabla$. Note that in the algebraic approximation mode we have

$$\overrightarrow{\nabla}^{s,\alpha} = \overrightarrow{\nabla}^{1,s\alpha}.$$

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[1] The curios choice of $s^2$ in the definition of $\chi_s$ will be justified a bit later.

[2] To prove this equality note first that is trivially true for $f(x) = x^n$ and thus by linearity it is true for polynomials. To prove it for any continuous function $f$ we choose a sequence of polynomials converging uniformly to $f$ on a compact interval containing the spectra of both $\alpha\alpha^*$ and $\alpha^*\alpha$.

The construction of $\overrightarrow{\nabla}^{s,\alpha}$ is natural in the following sense. If $f: Y \to X$ is a smooth map then

$$f^*(\overrightarrow{\nabla}^{s,\alpha}) = \overrightarrow{(f^*\nabla)}^{s,f^*\alpha}. \tag{2.1}$$

Consider now a special case of the above construction when $E_0 = \underline{\mathbb{C}}$. The morphism $\alpha$ can be identified with a smooth section $E = E_1$. Equip $\underline{\mathbb{C}}$ with the trivial connection and metric and $E$ with a hermitian metric $h = (\bullet, \bullet)_E$ and compatible connection $\nabla = \nabla^E$. Then

$$|\alpha|^2 = \alpha^*\alpha, \;\; \alpha^*(u) = (u,\alpha)_E, \;\; \forall u \in C^\infty(E).$$

Set $\chi_s = \chi_s(|\alpha|^2)$ and $\beta_s = \beta_s(|\alpha|^2)$. We deduce

$$\alpha_s^{-1} = \frac{\chi_s}{|\alpha|^2}\alpha^*, \;\; \overrightarrow{\nabla}^s u = \nabla^{s,\alpha}u = \nabla^E u - \frac{\chi_s \cdot (u,\alpha)_E}{|\alpha|^2}(\nabla^E\alpha).$$

Note that

$$\overrightarrow{\nabla}^s u = (1-\chi_s)\nabla^E u \Longrightarrow \overrightarrow{\nabla}^\infty u = 0.$$

We write

$$\overrightarrow{\nabla}^s = \nabla^E + A_s, \;\; A_s = -\frac{\chi_s}{|\alpha|^2}(\nabla^E\alpha)\alpha^* \tag{2.2}$$

Let us find the curvature $\Omega_s$ of $\overrightarrow{\nabla}^s$. We have

$$\Omega_s = \Omega_E + d^\nabla A_s + A_s \wedge A_s = \Omega_E + d^\nabla A_s + \frac{\chi_s^2}{|\alpha|^4}(\nabla^E\alpha)\alpha^* \wedge (\nabla^E\alpha)\alpha^*$$

and

$$d^\nabla A_s = -d\chi_s \wedge (\nabla^E\alpha)\alpha^* + \frac{\chi_s d|\alpha|^2}{|\alpha|^4}(\nabla^E\alpha)\alpha^* - \frac{\chi_s}{|\alpha|^2}(\Omega_E\alpha)\alpha^* + \frac{\chi_s}{|\alpha|^2}(\nabla^E\alpha) \wedge (\nabla^E\alpha^*).$$

Using the equality

$$d|\alpha|^2 = (\nabla^E\alpha^*)\alpha + \alpha^*\nabla^E\alpha$$

we deduce

$$\frac{\chi_s d|\alpha|^2}{|\alpha|^4} \wedge (\nabla^E\alpha)\alpha^* = -\frac{\chi_s}{|\alpha|^4}(\nabla^E\alpha) \wedge d|\alpha|^2 \wedge \alpha^*$$
$$= -\frac{\chi_s}{|\alpha|^4}(\nabla^E\alpha) \wedge (\nabla^E\alpha^*)\alpha\alpha^* - \frac{\chi_s}{|\alpha|^4}(\nabla^E\alpha)\alpha^* \wedge (\nabla^E\alpha)\alpha^*$$

Putting all of the above together we deduce

$$\Omega_s = \Omega_E\Big(1 - \frac{\chi_s\alpha\alpha^*}{|\alpha|^2}\Big) + \frac{\chi_s}{|\alpha|^2}(\nabla^E\alpha) \wedge (\nabla^E\alpha^*)\Big(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\Big)$$
$$-d\chi_s \wedge (\nabla^E\alpha)\alpha^* - \frac{\chi_s(1-\chi_s)}{|\alpha|^4}(\nabla^E\alpha)\alpha^* \wedge (\nabla^E\alpha)\alpha^*$$
$$= (1-\chi_s)\Omega_E + \chi_s\Big\{\Omega_E + \frac{1}{|\alpha|^2}(\nabla^E\alpha) \wedge (\nabla^E\alpha^*)\Big\}\Big(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\Big)$$
$$-d\chi_s \wedge (\nabla^E\alpha)\alpha^* - \frac{\chi_s(1-\chi_s)}{|\alpha|^4}(\nabla^E\alpha)\alpha^* \wedge (\nabla^E\alpha)\alpha^*$$

As $s \nearrow \infty$ we have $\chi_s(t) \to 1$ uniformly on the compacts of $[0,\infty)$ and we deduce that on compact subsets of the open set $X \setminus \alpha^{-1}(0)$ the curvature $\Omega_s$ converges uniformly to

$$\Omega_0 = \Big(\Omega_E + \frac{1}{|\alpha|^2}(\nabla^E\alpha) \wedge (\nabla^E\alpha^*)\Big)\Big(1 - \frac{\alpha\alpha^*}{|\alpha|^2}\Big).$$

Observe that if $E$ is a line bundle then

$$\alpha\alpha^* = |\alpha|^2 \cdot \mathbb{1}_E$$

so that in this case $\Omega_0 = 0$.

**Example 2.1 (Fundamental universal computation).** Suppose $X = \mathbb{C}$, $E$ is the trivial line bundle $\underline{\mathbb{C}}$, $\nabla^E$ and $h_E$ are the trivial metric and connection on $\underline{\mathbb{C}}$ and $\alpha$ is the tautological section $\alpha(z) = z$. Then if we use the notation $d = \nabla^E$ we deduce

$$\overrightarrow{\nabla}^s = d - \chi_s(|z|^2)\frac{dz}{z}, \quad \overrightarrow{\nabla}^\infty = d - \frac{dz}{z}, \quad \Omega_s = -\frac{d\chi_s \wedge dz}{z}.$$

If $X = \mathbb{C}^r$, $E = \underline{\mathbb{C}}^r$, and $\alpha$ is the tautological section

$$\alpha(z) = z, \quad z = \begin{bmatrix} z^1 \\ \vdots \\ z^r \end{bmatrix}.$$

Then

$$\overrightarrow{\nabla}^s = d - \frac{\chi_s}{|z|^2} dz \otimes z^*, \quad z^* = [\bar{z}^1, \cdots, \bar{z}^r]. \tag{2.3}$$

Observe that $A_s = -\frac{\chi_s}{|z|^2} dz \otimes z^*$ is represented by the $r \times r$ matrix with entries 1-forms

$$A_s = [a_{ij}]_{1 \leq i,j \leq r}, \quad a_{ij} = -\frac{\chi_s}{|z|^2} dz^i \bar{z}^j.$$

The $(i,j)$ entry of $A_s \wedge A_s$ is

$$\frac{\chi_s^2}{|z|^4} \sum_k dz^i \bar{z}^k dz^k \bar{z}^j = \frac{\chi_s^2}{|z|^4} dz^i \bar{z}^j \partial |z|^2.$$

The $(i,j)$-entry of $dA_s$ is

$$\frac{\chi_s}{|z|^2} dz^j \wedge d\bar{z}^j - \chi_s d(|z|^{-2}) dz^i \bar{z}^j - |z|^{-2} d\chi_s dz^i \bar{z}^j.$$

The $(i,j)$ entry of $\Omega_s = dA_s + A_s \wedge A_s$ is then

$$\Omega_{ij} = \underbrace{\frac{\chi_s}{|z|^2} dz^j \wedge d\bar{z}^j}_{:=U_{ij}} + \underbrace{dz^i \bar{z}^j \Big( \frac{\chi_s^2}{|z|^4} \partial |z|^2 + \chi_s d(|z|^{-2}) + |z|^{-2} d\chi_s \Big)}_{:=V_{ij}}$$

We want to compute the top Chern form associated to $\overrightarrow{\nabla}_s$)

$$c_r(\overrightarrow{\nabla}^s) = \left(\frac{\mathbf{i}}{2\pi}\right)^r \det \Omega_s.$$

If we denote by $S_r$ the group of permutations of $r$-objects and by $\epsilon(\varphi)$ the signature of a permutation $\varphi \in S_r$ we deduce

$$\det \Omega_s = \sum_{\varphi \in S_r} \epsilon(\varphi) \prod_{i=1}^r \Omega_{i\varphi(i)} = \sum_{\varphi \in S_r} \epsilon(\varphi) \prod_{i=1}^r (U_{i\varphi(i)} + V_{i\varphi(i)}).$$

Observing that

$$V_{i\varphi(i)} V_{j\varphi(j)} = 0, \ \ \forall i, j$$

we conclude

$$\prod_{i=1}^{r}(U_{i\varphi(i)} + V_{i\varphi(i)}) = \prod_{i=1}^{r} U_{i\varphi(i)} + \sum_{k=1}^{r} V_{k\varphi(k)} \prod_{i\neq k} U_{i\varphi(i)}.$$

We analyze the above two terms separately. We have

$$\prod_{i=1}^{r} U_{i\varphi(i)} = \Big(\chi_s|z|^{-2}\Big)^r \prod_{i=1}^{r} dz^i \wedge d\bar{z}^{\varphi(i)} = (-1)^{r(r-1)/2}\Big(\chi_s|z|^{-2}\Big)^r \prod_{i=1}^{r} dz^i \wedge \prod_{i=1}^{r} d\bar{z}^{\varphi(i)}$$

$$= \epsilon(\varphi)(-1)^{r(r-1)/2}\Big(\chi_s|z|^{-2}\Big)^r \prod_{i=1}^{r} dz^i \wedge \prod_{i=1}^{r} d\bar{z}^i = \epsilon(\varphi)\Big(\chi_s|z|^{-2}\Big)^r \prod_{i=1}^{r} dz^i \wedge d\bar{z}^i.$$

Next,

$$V_{k\varphi(k)} \prod_{i\neq k} U_{i\varphi(i)} = dz^k \bar{z}^{\varphi(k)}\Big( \chi_s d\bar{z}^{\varphi(k)} \frac{\partial |z|^{-2}}{\partial \bar{z}^{\varphi(k)}} + |z|^{-2} d\chi_s \Big) \prod_{i\neq k} U_{i\varphi(i)}$$

$$= dz^k \bar{z}^{\varphi(k)}\Big( -\chi_s \frac{z^{\varphi(k)}}{|z|^4} d\bar{z}^{\varphi(k)} + |z|^{-2} d\chi_s \Big) \prod_{i\neq k} U_{i\varphi(i)}$$

$$= \Big( -\frac{\chi_s |z^{\varphi(k)}|^2}{|z|^4} dz^k \wedge d\bar{z}^{\varphi(k)} + |z|^{-2} dz^k \bar{z}^{\varphi(k)} d\chi_s \Big) \prod_{i\neq k} U_{i\varphi(i)}$$

$$= -\frac{|z^{\varphi(k)}|^2}{|z|^2} \prod_{i} U_{i\varphi(i)} + |z|^{-2} dz^k \bar{z}^{\varphi(k)} d\chi_s \prod_{i\neq k} U_{i\varphi(i)}.$$

Now observe that

$$d\chi_s = \chi_s' d|z|^2, \ \ \chi_s'(t) = \frac{d\chi_s}{dt} = s^2\chi'(s^2 t)$$

so that

$$|z|^{-2} dz^k \bar{z}^{\varphi(k)} d\chi_s \prod_{i\neq k} U_{i\varphi(i)} = |z|^{-2} |z^{\varphi(k)}|^2 dz^k \wedge d\bar{z}^{\varphi(k)} = \frac{\chi_s' |z^{\varphi(k)}|^2}{\chi_s} \prod_{i} U_{i\varphi(i)}$$

Putting all the above together we deduce

$$\sum_{k=1}^{r} V_{k\varphi(k)} \prod_{i\neq k} U_{i\varphi(i)} = -\prod_{i=1}^{r} U_{i\varphi(i)} + \frac{\chi_s'|z|^2}{\chi_s} \prod_{i=1}^{r} U_{i\varphi(i)}$$

and

$$\prod_{i=1}^{r} U_{i\varphi(i)} + \sum_{k=1}^{r} V_{k\varphi(k)} \prod_{i\neq k} U_{i\varphi(i)} = \epsilon(\varphi)\left(\frac{\chi_s}{|z|^2}\right)^{r-1} \chi_s' \prod_{i=1}^{r} dz^i \wedge d\bar{z}^i.$$

Hence

$$\det \Omega_s = r!\left(\frac{\chi_s}{|z|^2}\right)^{r-1} \chi_s' \prod_{i=1}^{r} dz^i \wedge d\bar{z}^i.$$

At this point we use the elementary identity

$$\prod_{i=1}^{r} dz^i \wedge d\bar{z}^i = (-2i)^r dV_r,$$

where $dV_r$ denotes the Euclidean volume form on $\mathbb{C}^r$. Hence

$$c_1(\overrightarrow{\nabla}^s) = \frac{r!}{\pi^r}\left(\frac{\chi_s}{|z|^2}\right)^{r-1}\chi_s' dV_r = \frac{2r}{\sigma_{2r-1}}\chi_s^{r-1}\chi_s'\frac{dV_r}{|z|^{2r-2}},$$

where $\sigma_{2r-1}$ denotes the volume of the unit sphere in $\mathbb{C}^r$, $\sigma_r = \frac{2\pi^r}{(r-1)!}$. In spherical coordinates we have

$$dV_r = \rho^{2r-1}d\rho d\sigma, \ \ \rho = |z|.$$

We deduce

$$c_1(\overrightarrow{\nabla}^s) = 2r\chi_s^{r-1}\chi_s'\rho d\rho\frac{d\sigma}{\sigma_{2r-1}} = d(\chi_s(\rho^2))^r\frac{d\sigma}{\sigma_{2r-1}}. \tag{2.4}$$

Recall that we have a transgression formula

$$c_r(\overrightarrow{\nabla}^{s_0}) - c_r(\overrightarrow{\nabla}^0) = dT_{s_0}$$

where the transgression terms $T_s$ is defined by

$$T_{s_0} = \left(\frac{\mathbf{i}}{2\pi}\right)^r\int_0^{s_0}\det(\dot{A}_s, \Omega_s)ds,$$

with

$$\dot{A}_s = \frac{\partial}{\partial s}A_s, \quad \det(A, \Omega_s) = \frac{d}{dt}\,|_{t=0}\det(tA + \Omega_s).$$

Observe that

$$\dot{A}_s = -\dot{\chi}_s(|z|^2)|z|^{-2}dz\otimes z^*,$$

where

$$\dot{\chi}_s(t) = \frac{d\chi_s}{ds}(t) = \frac{d}{ds}\chi(s^2t) = 2ts\chi'(s^2t).$$

On the other hand

$$\frac{d}{dt}\chi_s(t) = s^2\chi'(s^2t) \Longrightarrow \dot{\chi}_s = \frac{2t}{s}\chi_s'(t).$$

Let $\Xi_s = \Xi_s(t)$ be the matrix representing $\Omega_s + t\dot{A}_s$. Its $(i,j)$ entry is

$$\Xi_{ij}(t) = \Omega_{ij} - t\dot{\chi}_s|z|^{-2}dz^i\bar{z}^j = U_{ij} + W_{ij},$$

$$W_{ij} = V_{ij} + t\frac{2\chi_s'}{s}dz^i\bar{z}^j = dz^i\bar{z}^j\Big(\frac{\chi_s^2}{|z|^4}\partial|z|^2 + \chi_s d(|z|^{-2}) + |z|^{-2}d\chi_s - t\dot{\chi}_s|z|^{-2}\Big).$$

$\det\Xi_s(t)$ is a polynomial of degree $\leq r$ in $t$ and we would like to compute its degree 1 part. We have

$$\det\Xi_s = \sum_{\varphi\in S_r}\epsilon(\varphi)\prod_{i=1}^r\Xi_{i\varphi(i)}(t)$$

and the degree 1-term is obtained by computing the differential of this polynomial at $t = 0$. We have

$$\frac{d}{dt}\,|_{t=0}\det\Xi_s(t) = -\dot{\chi}_s|z|^{-2}\sum_{\varphi\in S_r}\epsilon(\varphi)\left(\sum_{k=1}^r dz^k\bar{z}^{\varphi(k)}\prod_{i\neq k}\Omega_{i\varphi(i)}\right).$$

We analyze each summand separately. We have

$$dz^k\bar{z}^{\varphi(k)}\prod_{i\neq k}\Omega_{i\varphi(i)} = dz^k\bar{z}^{\varphi(k)}\prod_{i\neq k}\left(\frac{\chi_s}{|z|^2}dz^i\wedge d\bar{z}^{\varphi(i)} + dz^i\bar{z}^{\varphi(i)}\Big(\frac{\chi_s^2}{|z|^4}\partial|z|^2 + d\beta_s\Big)\right)$$

$$= dz^k \bar{z}^{\varphi(k)} \prod_{i\neq k} \Bigg( \beta_s dz^i \wedge d\bar{z}^{\varphi(i)} + dz^i \bar{z}^{\varphi(i)} d\beta_s \Bigg)$$

$$= dz^k \bar{z}^{\varphi(k)} \prod_{i\neq k} \Bigg( -\bar{z}^{\varphi(i)} d\beta_s \wedge dz^i - \beta_s d\bar{z}^{\varphi(i)} \wedge dz^i \Bigg) = dz^k \bar{z}^{\varphi(k)} \prod_{i\neq k} dz^i \wedge d(\beta_s \bar{z}^{\varphi(i)})$$

$$= \frac{1}{\beta_s} dz^k \beta_s \bar{z}^{\varphi(k)} \prod_{i\neq k} dz^i \wedge d(\beta_s \bar{z}^{\varphi(i)}).$$

Set

$$w_s^k = \beta_s \bar{z}^k = \chi_s \frac{\bar{z}^k}{|z|^2},$$

to conclude

$$dz^k \bar{z}^{\varphi(k)} \prod_{i\neq k} \Omega_{i\varphi(i)} = \frac{1}{\beta_s} dz^k w_s^{\varphi(k)} \prod_{i\neq k} dz^i \wedge dw_s^{\varphi(i)} = \epsilon(\varphi) \frac{1}{\beta_s} dz^k w_s^k \prod_{i\neq k} dz^i \wedge dw_s^i.$$

Hence

$$\frac{d}{dt}\,|_{t=0} \det \Xi_s(t) = -r! \frac{\dot{\chi}_s}{|z|^2} \sum_{k=1}^r dz^k \bar{z}^k \prod_{i\neq k} dz^i \wedge dw_s^i.$$

Integrating with respect to $s$ and making the change in variables $s \mapsto \chi_s$ so that $d_s\chi_s = \dot{\chi}_s ds$ we deduce

$$T_{s_0} = -r! \left( \frac{\mathbf{i}}{2\pi} \right)^r |z|^{-2} \sum_{k=1}^r dz^k \bar{z}^k \prod_{i\neq k} dz^i \wedge dw_{s_0}^i. \tag{2.5}$$

Letting $s_0 \to \infty$ we deduce

$$T_\infty = -r! \left( \frac{\mathbf{i}}{2\pi} \right)^r \sum_{k=1}^r dz^k (\bar{z}^k |z|^{-2}) \prod_{i\neq k} dz^i \wedge d(\bar{z}^i |z|^{-2}).$$

Observe that the transgression $T_\infty$ is *independent* of the approximation mode. A standard argument now shows that $c_1(\overrightarrow{\nabla}^s)$ converges in the sense of currents to the current $[0] \in \mathbb{C}^r$, i.e.

$$\lim_{s\to\infty} \int_{\mathbb{C}^r} c_r(\overrightarrow{\nabla}_s) u(x) dV = u(0), \quad \forall u \in \Omega^0_{cpt}(\mathbb{C}^r). \tag{2.6}$$

Moreover we have an equality of currents

$$[0] = [0] - c_r(\nabla) = \lim_{s\to\infty} c_r(\overrightarrow{\nabla}_s) - c_r(\nabla) = \partial T_\infty. \tag{2.7}$$

The convergence of the transgression forms $T_s$ to $T_\infty$ in the sense of currents follows from the dominated convergence theorems and the fact that the coefficients of $T_\infty$ are integrable. □

Let us put the above computations in a different light. Consider the trivial vector bundle $\underline{\mathbb{C}}^r_{pt}$ over a point. We denote by $\pi$ the natural projection. Then the trivial bundle of $\underline{\mathbb{C}}^r$ over $\mathbb{C}^r$ is naturally isomorphic to $\pi^*\underline{\mathbb{C}}^r_{pt}$. The section $\alpha(z) = z$ is then the tautological section of $\pi^*\underline{\mathbb{C}}^r_{pt}$.

Suppose now that we start with the trivial vector bundle $\pi : \underline{\mathbb{C}}^r_X \to X$ over the smooth manifold $X$. We equip it with the trivial metric $h_0$ and trivial connection $d$. Now look at the pullback bundle

$$E = \pi^*\underline{\mathbb{C}}^r_X \to \underline{\mathbb{C}}^r_X.$$

We continue to denote by $d$ the pullback connection $\pi^*d$. Given local coordinates $(x^a)$ on $X$ we obtain local coordinates on the base of $E$, $(z^i, x^a)$. Denote by $\alpha$ the tautological section of $E$,

$$\alpha(z,x) = z.$$

Then $\vec{d}^s$ is given by the same equation (2.2)

$$\vec{d}^s = d - \frac{\chi_s}{|z|^2} dz \otimes z^*$$

and the computations in Example 2.1 extend word for word to this more general case. The zero set of the tautological section is naturally identified with the submanifold $X \hookrightarrow \underline{\mathbb{C}}^r_X$. Using the equality (2.6) and the argument in [9, Lemma 7.3.12] based on the integration along the fibers of $\underline{\mathbb{C}}^r_X \to X$ we deduce that $c_r(\vec{d}^s)$ converges in the sense of currents as $s \nearrow \infty$ to $[X]$. More precisely if $\dim_X = n$ and $\eta \in \Omega^n_{cpt}(\underline{\mathbb{C}}^r_X)$ then

$$\int_{\underline{\mathbb{C}}^r_X} c_1(\vec{d}^s) \wedge \eta = \int_X \pi_*(c_r(\vec{d}^s) \wedge \eta,$$

where $\pi_*$ denotes the integration along the fibers of $\pi$. The equation (2.6) then implies

$$\lim_{s \nearrow \infty} \int_X \pi_*(c_r(\vec{d}^s) \wedge \eta = \int_X \eta.$$

If now we start with an arbitrary metric connection $\nabla = d + B$ on $\underline{\mathbb{C}}^r_X$ and we continue to denote by $\nabla$ its pullback to $E$ then using (2.2) we deduce

$$\vec{\nabla}_s = \vec{d}^s - \frac{\chi_s}{|z|^2} Bz \otimes z^*. \tag{2.8}$$

Denote by $\vec{\nabla}^{s,t}$ the connection $\vec{d}^s - t\frac{\chi_s}{|z|^2} Bz \otimes z^*$ and denote by $\Omega_{s,t}$ its curvature. The transgression formula implies

$$c_r(\vec{\nabla}^s) - c_r(\vec{d}^s) = dT_B, \;\; T_B = (-2\pi\mathbf{i})^{-r} \int_0^1 \det\big(\frac{\chi_s}{|z|^2} Bz \otimes z^*, \Omega_{s,t}\big).$$

Observe that the space $\Omega^\bullet(\underline{\mathbb{C}}^r_X)$ of differential forms on the total space of $\underline{\mathbb{C}}^r_X$ admits an increasing filtration $F^k\Omega^\bullet$ by the degree in the fiber variables. More rigorously

$$F^{-1}\Omega^\bullet = (0), \;\; \eta \in F^k\Omega^\bullet \Longleftrightarrow V \lrcorner\, \eta \in F^{k-1}\Omega^\bullet$$

for every vertical tangent vector $V$. We will use the notation $\deg_v \eta \leq k$ for $\eta \in F^k\Omega^\bullet$. Observe that

$$\deg T_B = 2r - 1, \;\; \deg_v T_B \leq 2r - 2, \;\; \deg_v dT_B \leq 2r - 1.$$

We deduce that for every $\eta \in \Omega^n_{cpt}(\underline{\mathbb{C}}^r_X)$ we have

$$\pi_*(\, c_r(\vec{\nabla}_s) \wedge \eta\,) = \pi_*(\, c_r(\vec{d}^s) \wedge \eta\,)$$

and we conclude that

$$\lim_{s \nearrow \infty} c_r(\vec{\nabla}_s) = [X], \;\; \text{in the sense of currents.}$$

Since the Chern-Weil forms are gauge invariant we can reduce the case of an arbitrary metric on $\underline{\mathbb{C}}_X$ to the case of the trivial metric and arbitrary metric connection via a gauge transformation of $\underline{\mathbb{C}}_X$.

The transgression $T_s = Tc_r(\overrightarrow{\nabla}^s, \overrightarrow{\nabla}_0)$ converges in $L^1$ as $s \nearrow \infty$. This can be seen using the transgression formula which implies that $T_s$ has the form

$$T_s = \sum_\ell P_\ell(\chi_s)\omega_\ell,$$

where $P_\ell(x)$ is a universal polynomial (independent of $s$), and $\omega_\ell$ is a form (independent of $\chi$, of degree $2r-1$, of vertical degree $\leq 2r-1$ which is homogeneous of degree 0 with respect to the action of the multiplicative group $(0,\infty)$ along the fibers of $\underline{\mathbb{C}}^r_X$. Putting together these local considerations we obtain the following result.

**Theorem 2.2.** *Suppose $\pi : E \to X$ is a complex vector bundle of rank $r$ over the smooth manifold $X$ equipped with a hermitian metric $h$ and compatible connection $\nabla$. Continue to denote by $\nabla$ its pullback to $\pi^*E \to E$. If we denote by $\alpha$ the tautological section of $\pi^*E \to E$ and we set $\overrightarrow{\nabla}^s = \overrightarrow{\nabla}^{s,\alpha}$ then*

$$c_r(\overrightarrow{\nabla}^0) = c_r(\nabla) = 0 \in \Omega^r(E),$$

$$\lim_{s\nearrow\infty} c_r(\overrightarrow{\nabla}^s) = [X], \quad \textit{as currents},$$

*and the transgressions $T_s = Tc_r(\overrightarrow{\nabla}^s, \overrightarrow{\nabla}^0)$ converge as currents to a current $T_\infty = T_\infty(h,\nabla)$ represented by a $L^1_{loc}$ form of degree $2r-1$ on $E$, smooth outside the zero section and satisfying the current equation*

$$[X] = \partial T_\infty.$$

## 3. Universal compactification

Suppose $X$ is a real analytic manifold, $E \to X$ is a real analytic vector bundle over $X$ equipped with a (real analytic) hermitian metric and compatible (real analytic) connection $\nabla$. For a real analytic section $\alpha$ we form the approximate pushforward connection $\overrightarrow{\nabla}^s$ on $E$ using the *algebraic approximation mode*. In this case we have

$$\chi_s(t) = \frac{s^2 t}{s^2 t + 1}, \quad \beta_s = \frac{s^2}{s^2 t + 1}$$

$$\overrightarrow{\nabla}^s = \overrightarrow{\nabla}^{s,\alpha} = \nabla - (\nabla\alpha)\alpha_s^{-1} = \nabla - s^2(\nabla\alpha)\alpha^*(s^2\alpha\alpha^* + 1)^{-1}.$$

Observe that if we define

$$\overrightarrow{\nabla}^\alpha := \overrightarrow{\nabla}^{1,\alpha} = \nabla - (\nabla\alpha)\alpha^*(\alpha\alpha^* + 1)^{-1}$$

then

$$\overrightarrow{\nabla}^s = \overrightarrow{\nabla}^{s\alpha}.$$

We want to prove that the Chern forms $c_r(\overrightarrow{\nabla}^s)$ converge in the sense of currents as $s \nearrow \infty$ to a current $c_r(\overrightarrow{\nabla}^{\infty,\alpha})$, and the transgression forms $T_s = Tc_r(\overrightarrow{\nabla}^s, \overrightarrow{\nabla})$ converge in the sense of currents to a current $T_\infty$ satisfying the current equation

$$c_r(\overrightarrow{\nabla}^{\infty,\alpha}) - c_r(\nabla) = \partial T_\infty.$$

To achieve this it is convenient to regard $\alpha$ as section of $\mathrm{Hom}(\underline{\mathbb{C}}, E)$. For each $x \in X$ the graph of $\alpha_x$

$$\gamma_\alpha(x) = \big\{ (\lambda, \lambda\alpha(x)) \in \underline{\mathbb{C}}_x \oplus E_x; \;\; \lambda \in \underline{\mathbb{C}}_x \big\}$$

is a line in $\underline{\mathbb{C}}_x \oplus E_x$ which we regard as a point in the projective space $\mathbb{P}(\underline{\mathbb{C}}_x \oplus E_x)$. The morphism $\alpha$ thus determines a section $\gamma_\alpha$ of $\mathbb{P}(\underline{\mathbb{C}} \oplus E)$ and this section completely determines $\alpha$.

A section $\ell$ of $\mathbb{P}(\underline{\mathbb{C}}\oplus E)$ is the graph of a section $\alpha\in \mathrm{Hom}(\underline{\mathbb{C}}, E)$ if and only if

$$\ell_x\subset E_x=0,\;\;\forall x\in X.$$

Equivalently, we can identify the total space of $\mathrm{Hom}(\underline{\mathbb{C}}, E)$ with the open subset $\mathbb{P}(\underline{\mathbb{C}}\oplus E)^0$, which is the complement of the divisor $\mathbb{P}(E)\subset \mathbb{P}(\underline{\mathbb{C}}\oplus E)$.

Denote by $\mathbf{E}$ the pullback of $E$ to $\mathbb{P}(\underline{\mathbb{C}}\oplus E)$ and by $\mathbf{D}$ the pullback of $\nabla$ to $\mathbf{E}$. Denote by $\mathbf{a}$ the tautological section of $\mathbf{E}$ over $\mathrm{Hom}(\underline{\mathbb{C}}, E)\subset \mathbb{P}(\underline{\mathbb{C}}\oplus E)$. We set $\overrightarrow{\mathbf{D}}^s=\overrightarrow{\mathbf{D}}^{s,\mathbf{a}}$. Then

$$\overrightarrow{\mathbf{D}}^s=\mathbf{D}-s^2(\mathbf{D}\mathbf{a})\mathbf{a}^*(s^2\mathbf{a}\mathbf{a}^*+1)^{-1}.$$

Denote by $\mathbb{L}$ the tautological line bundle over $\mathbb{P}(\underline{\mathbb{C}}\oplus E)$ and by $\mathbb{L}^\perp$ its orthogonal complement in $\underline{\mathbb{C}}\oplus\mathbf{E}$. Denote by $P$ (resp. $Q$) the orthogonal projection onto $\mathbb{L}$ (resp. $\mathbb{L}^\perp$. Let us describe $P$ and $Q$ over $\mathrm{Hom}(\underline{\mathbb{C}}, E)\subset \mathbb{P}(\underline{\mathbb{C}}\oplus E)$. Given $a\in \mathrm{Hom}(\underline{\mathbb{C}}_x, E_x)$ the associated line $\gamma_a$ is

$$\gamma_a=\{(\lambda, a\lambda);\;\;\lambda\in\underline{\mathbb{C}}\}.$$

The orthogonal complement of $\gamma_a$ is the graph of $-a^*:E\to\underline{\mathbb{C}}$. Given $(t,v)\in\underline{\mathbb{C}}_x\oplus E_x$ we need to find $u\in E_x$ and $\lambda\in\underline{\mathbb{C}}_x$ such that

$$(\lambda, a\lambda)+(-a^*u,u)=(t,v)\Longleftrightarrow\left[\begin{array}{cc}1&-a^*\\ a&1\end{array}\right]\cdot\left[\begin{array}{c}\lambda\\ u\end{array}\right]=\left[\begin{array}{c}t\\ v\end{array}\right].$$

Set

$$U_a=\left[\begin{array}{cc}1&-a^*\\ a&1\end{array}\right],\;\;V_a=\left[\begin{array}{cc}1&a^*\\ -a&1\end{array}\right].$$

Observe that $V_aU_a=\mathrm{Diag}(1+a^*a, 1+aa^*)$ so that

$$\left[\begin{array}{c}(1+a^*a)\lambda\\ (1+aa^*)u\end{array}\right]=V_a\left[\begin{array}{c}t\\ v\end{array}\right]=\left[\begin{array}{c}t+a^*v\\ -at+v\end{array}\right]\Longrightarrow\lambda=\frac{1}{1+|a|^2}(t+a^*v),\;\;v=\frac{a}{1+|a|^2}t,$$

so that we obtain the block decomposition

$$P=\left[\begin{array}{cc}(1+|\mathbf{a}|^2)^{-1}&(1+|\mathbf{a}|^2)^{-1}\mathbf{a}^*\\ \mathbf{a}(1+|\mathbf{a}|^2)^{-1}&(1+|\mathbf{a}|^2)^{-1}\mathbf{a}\mathbf{a}^*\end{array}\right].\tag{3.1}$$

To find $Q(t,v)$ we need to find $u\in E_x$ and $\lambda\in\underline{\mathbb{C}}_x$ such that

$$(-a^*u,u)+(\lambda,a\lambda)=(t,v)\Longleftrightarrow U_a\left[\begin{array}{c}\lambda\\ u\end{array}\right]=\left[\begin{array}{c}t\\ v\end{array}\right].$$

Hence

$$\left[\begin{array}{c}(1+a^*a)\lambda\\ (1+aa^*)u\end{array}\right]=V_a\left[\begin{array}{c}t\\ v\end{array}\right]\Longrightarrow u=(1+aa^*)^{-1}(-at+v),$$

$$Q=\left[\begin{array}{cc}\mathbf{a}^*(1+\mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}&-\mathbf{a}^*(1+\mathbf{a}\mathbf{a}^*)^{-1}\\ -(1+\mathbf{a}\mathbf{a}^*)^{-1}\mathbf{a}&(1+\mathbf{a}\mathbf{a}^*)^{-1}\end{array}\right].\tag{3.2}$$

The multiplicative group $\mathbb{R}_{>0}$ acts on $\underline{\mathbb{C}}\oplus E$ by $s\cdot(\lambda,u)=(\lambda,su)$. We obtain in this fashion a flow $\Phi_s:\mathbb{P}(\underline{\mathbb{C}}\oplus E)\to\mathbb{P}(\underline{\mathbb{C}}\oplus E)$ and we set $\mathbb{L}_s=\Phi_s^*\mathbb{L}$, $\mathbb{L}_s^\perp=\Phi_s^*\mathbb{L}^\perp$. Over $\mathbb{P}(\underline{\mathbb{C}}\oplus E)^0$ the orthogonal projection $Q_s$ onto $\mathbb{L}_s^\perp$ is obtain from (3.2) by replacing $\boldsymbol{a}$ by $s\boldsymbol{a}$. We obtain the block description

$$Q_s=\left[\begin{array}{cc}s^2\mathbf{a}^*M_s\mathbf{a}&-s\mathbf{a}^*M_s\\ -sM_s\mathbf{a}&M_s\end{array}\right],\;\;M_s=(1+s^2\boldsymbol{a}\boldsymbol{a}^*)^{-1}.$$

This projection restricts to a vector bundle morphism $R_s:\mathbf{E}\to\mathbb{L}_s^\perp$ which is an isomorphism of vector bundles over $\mathbb{P}(\underline{\mathbb{C}}\oplus E)^0$. Consider the connection $D^s$ on $\mathbf{E}$ defined by

$$D^s:=R_s^{-1}(Q_s\mathbf{D})R_s.$$

More precisely, given a section $u$ of $\mathbf{E}$ we have

$$R_s u = \left[ \begin{array}{c} -s\boldsymbol{a}^* M_s u \\ M_s u \end{array} \right].$$

Hence

$$\mathbf{D}DR_s u = \left[ \begin{array}{c} -sd(\boldsymbol{a}^* M_s u) \\ \mathbf{D}(M_s u) \end{array} \right], \;\; Q_s \mathbf{D} R_s = \left[ \begin{array}{c} * \\ s^2 M_s \mathbf{a} d(\mathbf{a}^* M_s u) + M_s \mathbf{D}(M_s u), \;\; R_s^{-1} \end{array} \right]$$

We deduce

$$\begin{aligned} R_s^{-1} Q_s \mathbf{D} R_s u &= s^2 \mathbf{a} d(\mathbf{a}^* M_s u) + \mathbf{D}(M_s u) = s^2 \mathbf{a}(d\mathbf{a}^*) M_s u + s^2 \mathbf{a}\mathbf{a}^* \mathbf{D}(M_s u) + \mathbf{D}(M_s u) \\ &= s^2 \mathbf{a}(\mathbf{D}\mathbf{a}^*) M_s u + \underbrace{(1 + s^2 \mathbf{a}\mathbf{a}^*) M_s}_{\mathbb{1}_{\mathbf{E}}} \mathbf{D} u + (1 + s^2 \mathbf{a}\mathbf{a}^*)(\mathbf{D} M_s) u \\ &= \mathbf{D} u + s^2 \mathbf{a}(d\mathbf{a}^*) M_s u + (M_s^{-1} \mathbf{D} M_s) u \end{aligned}$$

Differentiating the equality $M_s^{-1} M_s = \mathbb{1}_{\mathbf{E}}$ we deduce

$$(M_s^{-1} \mathbf{D} M_s) = -(\mathbf{D} M_s^{-1}) M_s = -s^2 (\mathbf{D}(\mathbf{a}\mathbf{a}^*)\,) M_s = -s^2 (\mathbf{D}\mathbf{a}) \mathbf{a}^* M_s - s^* \mathbf{a}(\mathbf{D}\mathbf{a}^*) M_s$$

Putting all the above together we obtain the identity

$$D^s = R_s^{-1} Q_s \mathbf{D} R_s u = \mathbf{D} u - s^2 (\mathbf{D}\mathbf{a}) \mathbf{a}^* M_s u = \overrightarrow{\mathbf{D}}^s u. \tag{3.3}$$

If we denote by $D_{\mathbb{L}^\perp} = Q\mathbf{D}$ the natural connection on $\mathbb{L}^\perp$ induced by $D$ then

$$Q_s \mathbf{D} = \Phi_s^* D_{\mathbb{L}^\perp}$$

and we thus we can rewrite (3.3) as

$$\overrightarrow{\mathbf{D}}^s = R_s^{-1} \Phi_s^* D_{\mathbb{L}^\perp} R_s, \;\; \forall s \geq 0. \tag{3.4}$$

We have thus obtained the following result.

**Lemma 3.1** (*Universal Desingularization*)**.**

$$c_r(\overrightarrow{\mathbf{D}}^s) = c_r(\Phi_s^* D_{\mathbb{L}^\perp}) = \Phi_s^* c_r(D_{\mathbb{L}^\perp}). \tag{3.5}$$

□

We want to investigate the existence of the limit $\lim_{s\to\infty} \Phi_s^* \omega$ where $\omega$ is a *fixed* differential form on $\mathbb{P}(\underline{\mathbb{C}} \otimes E)$. We will follow the very elegant approach developed by Harvey and Lawson in [6]. This requires a small digression.

Suppose $F : X_0^{n_0} \to X_1^{n_1}$ is a smooth map between smooth oriented manifolds. The pullback by $F$ is a linear map

$$F^* : \Omega^k(X_1) \to \Omega^k(X_0).$$

It is convenient to regard it as a linear map

$$F^\# : \Omega^k(X_1) \to \mathcal{D}_{n_0-k}(X_0), \;\; \alpha \mapsto (F^*\alpha) \cap [X_0].$$

We can rewrite this map as follows. Consider the graph of $F^{-1}$

$$\Gamma_F^* = \big\{ (x_1, x_0) \in X_1 \times X_0; \;\; x_1 = F(x_0) \big\}.$$

Then $\Gamma_F^*$ defines an integral current $[\Gamma_F] \in \mathcal{D}_{n_0}(X_1 \times X_0)$. For $i = 0, 1$ we denote by $\pi_i : X_0 \times X_1 \to X_i$ the canonical projection. We have the following result.

**Lemma 3.2.**

$$F^{\#}\alpha = (\pi_0)_*\big( (\pi_1^*\alpha) \cap [\Gamma_F^*] \big), \;\; \forall \alpha \in \Omega^k(X_1).$$

**Proof** For $\beta \in \Omega^{n_0-k}(X_0)$ we have

$$\big\langle (\pi_0)_*\big( (\pi_1^*\alpha) \cap [\Gamma_F^*] \big), \beta \big\rangle = \big\langle (\pi_1^*\alpha) \cap [\Gamma_F^*], \pi_0^*\beta \big\rangle$$
$$= \big\langle [\Gamma_F^*], \pi_1^*\alpha \wedge \pi_0^*\beta \big\rangle = \int_{X_0} (F \times \mathbb{1}_{X_0})^*(\pi_1^*\alpha \wedge \pi_0^*\beta$$
$$= \int_{X_0} F^*\alpha \wedge \beta = \big\langle (F^*\alpha) \cap [X_0], \beta \big\rangle.$$

□

This suggest the following more general construction. Suppose that we are given a roof, i.e a diagram of the form

$$\begin{tikzcd} & Y \arrow[dl, "f_0"'] \arrow[dr, "f_1"] & \\ X_1 & & X_0 \end{tikzcd},$$

where $X_0, X_1, Y$ are oriented smooth manifolds, and $f_0, f_1$ are smooth maps. Assume $K$ is a $k$-dimensional *kernel* for this roof, i.e. a $k$-dimensional current in $Y$ such that $f_0$ is proper over $\operatorname{supp} K$. Then we obtain a linear map

$$K_{\#} : \Omega^m(X_1) \to \mathcal{D}_{k-m}(X_0), \;\; K_{\#}\alpha = (f_0)_*\big( (f_1^*\alpha) \cap K \big).$$

The result in Lemma 3.2 can be rephrased as the equality

$$F^{\#} = [\Gamma_F^*]_{\#}.$$

We have the following homotopy formula

$$(\partial K)_{\#}\alpha = K_{\#}(d\alpha) + (-1)^m \partial K_{\sharp}\alpha, \;\; \forall \alpha \in \Omega^m(X_1). \tag{3.6}$$

Indeed,

$$K_{\#}(d\alpha) = (f_0)_*\big( d(f_1^*\alpha) \cap K \big) \stackrel{(1.1)}{=} (f_0)_*\big( f_1^*\alpha \cap \partial K - (-1)^m \partial (f_1^*\alpha \cap K) \big)$$
$$= (\partial K)_{\#}\alpha - (-1)^m \partial K_{\#}\alpha.$$

We can rewrite this in operator form

$$(\partial K)_{\#} = K_{\#} \circ d + (-1)^m \partial \circ K_{\#}. \tag{3.7}$$

Observe that if we have a family of smooth maps $F_s : X_0 \to X_1$, $s \in (0, \infty)$ such that $\Gamma_{F_s}^*$ converges as $s \to \infty$ to a current $\Gamma_\infty^*$ in $X_1 \times X_0$ such that $\pi_0$ is proper over $\operatorname{supp} \Gamma_\infty^*$ then for every $k$ form $\alpha$ on $X_1$ the pullbacks $F_s^*\alpha$ converge to the current $(\Gamma_\infty^*)_{\#}\alpha$.

Suppose we are given a smooth map

$$F : [0, \infty) \times X_0 \to X_1, \;\; (s, x_0) \mapsto F_s(x_0).$$

For every $s > 0$ consider the current

$$T_s := \big\{ (x_1, t, x_0) \in X_1 \times \mathbb{R} \times X_0; \;\; x_1 = F_t(x_0), \;\; 0 \leq t \leq s \big\}.$$

We set

$$\partial_s T_s := [s] \times \Gamma_{F_s}^*, \;\; \partial_0 T_s := [0] \times \Gamma_{F_0}^*$$

so that

$$\partial T_s = \partial_s T_s - \partial_0 T_s.$$

We obtain a roof

$$\begin{array}{ccc} & Y = X_1 \times \mathbb{R} \times X_0 & \\ & {\scriptstyle \pi_1} \swarrow \qquad \searrow {\scriptstyle \pi_0} & \\ X_1 & & X_0 \end{array}$$

where $\pi_i$ denote the natural projections. Using $T_s$ as kernel we deduce that for every $k$-form $\alpha$ on $X_1$ we have

$$F_s^{\#}\alpha - F_0^{\#}\alpha = (\partial T_s)_{\#}\alpha \overset{(3.6)}{=} (T_s)_{\#}(d\alpha) + (-1)^k \partial (T_s)_{\#}\alpha.$$

The operator $(T_s)_{\#} : \Omega^k(X_1) \to \mathcal{D}_{n_0+1-k}(X_0)$ is the pullback by $\pi_1$ followed by the integration along the fibers of $T_s \xrightarrow{\pi_1} X_1$. In particular, we deduce that $(T_s)_{\#}$ maps smooth $k$-forms on $X_1$ to smooth $(k-1)$-forms on $X_0$. Since on $(k-1)$-forms (viewed as $(n_0 - k + 1)$-currents) we have $\partial[\alpha] = (-1)^k[d\alpha]$ (see (1.2)) we deduce

$$F_s^*\alpha - F_0^*\alpha = (T_s)_{\#}d\alpha + d(T_s)_{\#}\alpha. \tag{3.8}$$

Denote by $\bar{\mathbb{R}} \cong S^1$ the one-point compactification of $\mathbb{R}$. We set $\bar{Y} = X_1 \times \bar{\mathbb{R}} \times X_0$ and thus we can view the currents $T_s$ as currents in $\bar{Y}$.

**Definition 3.3.** We say that $F : [0,\infty) \times X_0 \to X_1$ has *finite volume* if there exists a metric $\tau$ on $\mathbb{R}$, a metric $g_0$ on $X_0$ and a metric $g_1$ on $X_1$ such that the volume of

$$T_\infty := \Gamma_F^* \subset X_1 \times \bar{\mathbb{R}} \times X_0$$

with respect to the metric $g_1 \oplus \tau \oplus g_0$ is finite.

□

Along $[0,\infty) \subset \bar{\mathbb{R}}$ the metric $\tau$ can be expressed as $\tau = w^2(s)ds^2$. The above volume of $\Gamma_F^*$ is described by the improper integral

$$\int_0^\infty \left( \int_{X_0} (w^2 + |F_s'|_{g_1}^2)^{1/2} \det\big( \mathbb{1} + (D_{x_0}F_s)^* DF_s \big)^{1/2} dV_{g_0}(x_0) \right) ds,$$

where for every $x_0 \in X_0$ we denoted by $(D_{x_0}F_s)^*$ the adjoint of $D_{x_0}F : T_{x_0}X_0 \to T_{F_s(x_0)}X_1$ with respect to the metrics $g_0$, $g_1$. Note that the finiteness of this integral is independent of the metric $\tau$ on $\bar{\mathbb{R}}$.

If we identify $S^1$ with $[-1,1]$ and $[0,\infty) \subset \bar{\mathbb{R}}$ with the subset $[0,1)$ via the map

$$[0,\infty) \ni s \mapsto \chi = \frac{s}{1+s},$$

then the finite volume condition translates to

$$\int_0^\infty \left( \int_{X_0} ((1+s)^{-2} + |F_s'|_{g_1}^2)^{1/2} \det\big( \mathbb{1} + (D_{x_0}F_s)^* DF_s \big)^{1/2} dV_{g_0}(x_0) \right) ds < \infty$$

for some metrics $g_0$, $g_1$ on $X_0$ and $X_1$. Using a smooth proper function $\rho : X_0 \to \mathbb{R}$ we deduce the following.

**Lemma 3.4.** *The finite volume condition is equivalent to*

$$\int_0^\infty \left( \int_{U_0} \big( (1+s)^{-2} + |F_s'|^2_{g_1} \big)^{1/2} \det\big( \mathbb{1} + (D_{x_0}F_s)^* DF_s \big)^{1/2} dV_{g_0}(x_0) \right) ds < \infty, \tag{3.9}$$

*for* any *precompact open subset* $U_0 \subset X_0$ *and* any *metrics* $g_i$ *on* $X_i$.

□

The currents $T_s$ in $Y$ induce currents $\bar{T}_s$ in $\bar{Y}$. The finite volume condition implies that $\Gamma^*_F$ defines a current in $\bar{T}_\infty$ in $\bar{Y}$.

Since $F$ has finite volume then $\bar{T}_s \to \bar{T}_\infty$ in the mass norm. In particular $\bar{T}_s$ converges weakly to $\bar{T}_\infty$ and

$$\partial_s T_s = \bar{T}_s + \partial_0 \bar{T}_s \rightharpoonup \partial_\infty \bar{T}_\infty = \partial_0 \bar{T}_\infty + \bar{T}_\infty.$$

If $\pi_0 : \bar{Y} \to X_0$ is proper over $\operatorname{supp} \bar{T}_\infty$ then for every smooth $k$-form $\alpha$ on $X_1$ the pullbacks $F_s^*\alpha$ converge as currents as $s \nearrow \infty$ to the current $F_\infty^*\alpha := (\partial_\infty T_\infty)_\# \alpha$. Additionally, we have the equality of currents

$$F_\infty^*\alpha - F_0^*\alpha = (\bar{T}_\infty)_\# d\alpha + (-1)^k \partial (\bar{T}_\infty)_\# \alpha.$$

**Proposition 3.5.** *If $F$ is a real analytic map and it is uniformly locally bounded, i.e. for every compact subset $K_0 \subset X_0$ there exists a compact set $K_1 \subset X_1$ such that*

$$F([0,\infty) \times K_0) \subset K_1,$$

*then $F$ has finite volume and the natural projection $\pi_0 : \bar{Y} \to X_0$ is proper over the closure of $\Gamma_F$ in $\bar{Y}$.*

**Proof** The closure of $\Gamma^*_F$ in $X_1 \times \bar{\mathbb{R}} \times X_0$ is a semianalytic set, and thus according to [2, §3.4] or [4, §2] it has locally finite volume. The uniform boundedness condition now implies the finite volume condition on $F$.

□

*Remark* 3.6. The above considerations admit the following immediate generalizations. Suppose $F \to E \stackrel{\pi}{\twoheadrightarrow} B$ is a real analytic, locally trivial fiber bundle and $u_s : B \to E$, $s \in [0\infty)$ is a smooth family of sections. The graph of a section $u : B \to E$ is then the submanifold

$$\Gamma^*_u = u(B) \subset E.$$

The pullback by $u$ is defined by the roof

$$\begin{tikzcd} & E \arrow[dl, "\mathbb{1}_E"'] \arrow[dr, "\pi"] & \\ E & & B \end{tikzcd},$$

with kernel $\Gamma^*_u$. The formulation of the finite volume condition is similar and is left to the reader.

□

In our original situation we have

$$X_0 = X_1 = \mathbb{P}(\underline{\mathbb{C}} \oplus E) \text{ and } F_s(\lambda, u) = \Phi_s[\lambda, u] = [\lambda, su].$$

The map $\Phi : [0,\infty) \times \mathbb{P}(\underline{\mathbb{C}} \oplus E) \to \mathbb{P}(\underline{\mathbb{C}} \oplus E)$ is clearly real analytic. Since the fibers of $\mathbb{P}(\underline{\mathbb{C}} \oplus E) \to X$ are compact we deduce that this map is also locally uniformly bounded. As explained by Fulton in [3, Rem.5.1.1, Ex.18.1.6], the asymptotics of the flow $\Phi_s$ as $s\to\infty$ are intimately related to the deformation to the normal cone construction in algebraic geometry.

We study first the special case when the base $X$ of $E$ is a point. Thus we need to analyze the closure of

$$\Big\{ (\,[\lambda, su], s, [\lambda, u]\,);\;\; s\in[0,\infty), [\lambda,u]\in\mathbb{P}(\mathbb{C}\oplus E)\Big\}$$

in $\mathbb{P}(\mathbb{C}\oplus E)\times\bar{\mathbb{R}}\times\mathbb{P}(\mathbb{C}\oplus E)$. Near $\infty\in\bar{\mathbb{R}}$ we choose a local coordinate $t=1/s$ so that in this coordinate we have

$$\Phi_s[\lambda,u]=[\lambda,t^{-1}u].$$

We need to study the closure of

$$\Gamma^*_\Phi=\Big\{ (\,[t\lambda, u], t, [\lambda, u]\,);\;\; t\in(0,1], [\lambda,u]\in\mathbb{P}(\mathbb{C}\oplus E)\Big\}$$

in $\mathbb{P}(\mathbb{C}\oplus E)\times[0,1]\times\mathbb{P}(\mathbb{C}\oplus E)$. A point $P=([\mu_0,v_0],0,[\lambda_0,u_0])$, $|\lambda_0|+|u_0|=1$ is in this closure if and only if there exist real analytic maps

$$t:[0,1]\to[0,1],\;\; t(x)=t_0x^a,\;\; t_0\neq 0,$$

$$\lambda:[0,1]\to\mathbb{C},\;\; \lambda(x)=\lambda_0+\dot\lambda_0x^b,$$

$$u:[0,1]\to E,\;\; u(x)=u_0+\dot u_0x^c\;\; a,b,c\in\mathbb{Z}_{\geq 0},\;\; a\neq 0,\;\; t_0,\dot\lambda_0,\dot u_0\neq 0,$$

such that,

$$\lim_{x\searrow 0}[t_0\lambda_0x^a+t_0\dot\lambda_0x^{a+b}, u_0+\dot u_0x^c]\to[\mu_0,v_0].$$

We distinguish several cases.

- $u_0,\lambda_0\neq 0$. In this case $\mu_0=0$, $v_0=u_0$.
- $\lambda_0=0$, $u_0\neq 0$. In this case we also have $\mu_0=0$, $v_0=u_0$.
- $\lambda_0\neq 0$, $u_0=0$. In this case we have

$$[\mu_0,v_0]=\lim_{x\searrow 0}[t_0\lambda_0x^a+t_0\dot\lambda_0x^{a+b}, \dot u_0x^c].$$

If $c=a$ then

$$[\mu_0,v_0]=[t_0\lambda_0,\dot u_0],$$

if $c<a$, then

$$[\mu_0,v_0]=[t_0\lambda_0,\dot u_0]=[0,\dot u_0]$$

if $c>a$, then

$$[\mu_0,v_0]=[t_0\lambda_0,0]=[1,0].$$

Hence

$$\big\{([0,u_0],[\lambda_0,u_0]),\;\; u_0\neq 0\big\}\cup\big\{([s_0,\dot u_0],[1,0])\big\}\subset\partial_\infty\Gamma^*_\Phi.$$

We deduce that $\partial_\infty\Gamma^*_\Phi$ is supported by the a variety $\Gamma^*_{\Phi_\infty}=\mathbb{P}(\mathbb{C}\oplus E)\times\mathbb{P}(\mathbb{C}\oplus E)$ which has two irreducible components $H$, $B$, where

$$H=\mathbb{P}(\mathbb{C}\oplus E)\times\{[1,0]\},$$

and $B$ is the closure in $\mathbb{P}(\mathbb{C}\oplus E)\times\mathbb{P}(\mathbb{C}\oplus E)$ of $\Gamma^*_P$, where

$$P:\mathbb{P}(\mathbb{C}\oplus E)\setminus\{[1,0]\}\to\mathbb{P}(E)\subset\mathbb{P}(\mathbb{C}\oplus E)$$

denotes the projection of center $[1,0]$ onto $\mathbb{P}(E)$. Note that $B$ is biholomorphic to the blowup of $\mathbb{P}(\mathbb{C}\oplus E)$ at $[1,0]$. The two components intersect along the common divisor $\mathbb{P}(E)\times\{[1,0]\}$. The deformation $\Gamma^*_{\Phi_s}\to\Gamma^*_{\Phi_\infty}$ is similar in spirit with the deformation depicted in Figure 2.

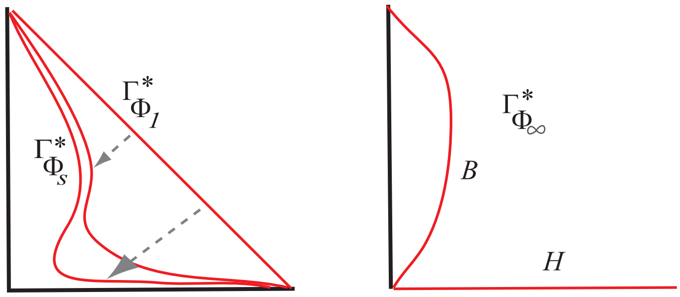


FIGURE 2. *Deforming the graph of the identity map* $\Phi_1 : \mathbb{P}(\underline{\mathbb{C}} \oplus E) \to \mathbb{P}(\underline{\mathbb{C}} \oplus E)$

When $X$ is arbitrary, we set $Y = \mathbb{P}(\underline{\mathbb{C}} \oplus E)$ and consider complex blowup $M = Bl_X(Y \times \mathbb{C})$ of $Y \times \mathbb{C}$ along the zero section[3]

$$X \hookrightarrow \mathbb{P}(\underline{\mathbb{C}} \oplus E) \times \mathbb{C}, \;\; x \longmapsto ([1,0], 0) \in \mathbb{P}(\underline{\mathbb{C}} \oplus E) \times \mathbb{C}.$$

The projection $Y \times \mathbb{C} \to \mathbb{C}$ defines a real analytic function $t : Y \times \mathbb{C} \to \mathbb{C}$ and thus induces a real analytic function $\tilde{t} : M \to \mathbb{C}$. Then $\Gamma^*_{\Phi_{1/t}}$ converge as $t \to 0$ to a current supported by the "divisor" $\{\tilde{t} = 0\}$. This divisor has two irreducible components, $H$ and $B$.

The component $H \cong \mathbb{P}(\underline{\mathbb{C}} \oplus E)$ is the exceptional divisor of the blowup of $\underline{\mathbb{C}} \oplus E$ along $\{0\} \times X$. The component $B$ is the blow-up $Bl_X Y$ of $Y$ along $X \hookrightarrow \mathbb{P}(\underline{\mathbb{C}} \oplus E)$, embedded as the zero section. These two components share in common a $\mathbb{P}(E)$. Let us point out that the total space of the deformation of the zero section embedding $X \hookrightarrow Y$ to the normal cone is precisely $M^0 = M \setminus B$, (see [3, §5.1]).

If now $\alpha$ is a real analytic section of $E$ it induces a real analytic section $\alpha$ of $\mathbb{P}(\underline{\mathbb{C}} \oplus E)$ and we have the tautological equalities

$$E = \alpha^* \mathbb{L}^\perp, \;\; \alpha = \alpha^*(\mathbf{a}), \;\; \overrightarrow{\nabla}^s = \alpha^*(\overrightarrow{\mathbf{D}}^s).$$

Then

$$c_r(\overrightarrow{\nabla}^s) = \alpha^* c_r(\overrightarrow{\mathbf{D}}^s) \stackrel{(3.5)}{=} \alpha^* \Phi_s^* c_r(\mathbf{D}_{\mathbb{L}^\perp}) = \alpha_s^* c_r(\mathbf{D}_{\mathbb{L}^\perp}),$$

where $\alpha_s = \Phi_s \circ \alpha = s\alpha$. Since the family of sections $s\alpha$ is real analytic and the fibers of $\mathbb{P}(\underline{\mathbb{C}} \oplus E) \to X$ are compact, the conditions of Proposition 3.5 are satisfied and we deduce that $c_r(\overrightarrow{\nabla}^s)$ converge as currents as $s \nearrow \infty$.

Imitating [1, Prop.IV. 22], we will refer to the closure of the set $\{(t, \alpha(x));\;\; x \in X\} \subset \mathbb{C}^* \times E$ in $\mathbb{P}(\underline{\mathbb{C}} \oplus E)$ as the *blowup of* $\mathbb{C} \times X$ *along* $Z = \{0\} \times \alpha^{-1}(0)$.[4] We denote this blowup by $Bl_Z(\mathbb{C} \times X)$. The part of this blow-up sitting above $\{0\} \times X$ determines the limit $c_r(\overrightarrow{\nabla}^{\infty,\alpha})$. The part of the blowup above $\{0\} \times X$ is the total transform $\hat{X}$ of $\{0\} \times X$ in $Bl_Z(\mathbb{C} \times X)$. This consists of two components: the exceptional divisor $\mathcal{E}_Z$ of the blowup $Bl_Z(\mathbb{C} \times X)$, and the strict transform $\bar{X}$ of $\{0\} \times X$. If $\alpha$ is a nondegenerate section then $\mathcal{E}_Z = \mathbb{P}(\underline{\mathbb{C}} \oplus E)\,|_Z$ and $\bar{X} = Bl_Z(\{0\} \times X)$.

To prove that the Chern-Weil transgressions $Tc_r(\overrightarrow{\nabla}^s, \overrightarrow{\nabla}^0)$ converge as currents as $s \nearrow \infty$ we need to reformulate the construction of this object in the language of kernels.

Suppose $A_t$, $s \in [0, s]$ is a smooth family of connections on the vector bundle $E \to X$. They determine a connection $\hat{A}$ on the pullback $\hat{E}$ of $E$ to $[0, s] \to X$. The Chern-Weil transgression $Tc_r(A_s, A_0)$ is defined as the form on $X$ obtained by integrating the form $c_r(\hat{A})$

[3]The normal bundle of this zero section is a complex bundle.

[4]If all objects are holomorphic this is indeed the blowup as defined in [1, IV.24].

along the fibers of $[0,s]\times X\to X$. Suppose now that $\Phi_t$ is a smooth 1-parameter family of smooth maps $X\to X$ such that $\Phi_0=\mathbb{1}_X$, and $A$ is connection on $E$. We obtain a 1-parameter family of connections $A_t=\Phi_t^*A$ on $A$ and the associated connection $\hat{A}$ coincides with $\hat{\Phi}^*A$, where

$$\hat{\Phi}:[0,s]\times X\to X,\ \ \hat{\Phi}(t,x)=\Phi_t(x).$$

If we consider the roof

$$\begin{tikzcd} & {[0,s]\times X\times X} \arrow[dl, "\ell"'] \arrow[dr, "r"] & \\ X & & X \end{tikzcd}\ ,\ \ \ell(t,x_1,x_2)=x_1,\ \ r(t,x_1,x_2)=x_2$$

with kernel $T_s=\Gamma^*_{\hat{\Phi}}$ then we deduce that

$$Tc_r(A_s,A_0)=(T_s)_\# c_r(A_0).$$

The convergence of the transgression forms $Tc_r(\overrightarrow{\nabla}^s,\overrightarrow{\nabla}^0)$ now follows again from the convergence of the kernels $T_s$. The current $c_r(\overrightarrow{\nabla}^\infty)$ is defined by

$$\langle c_r(\overrightarrow{\nabla}^\infty),\eta\rangle=\int_{\hat{X}}c_r(\mathbf{D}_{\mathbb{L}^\perp})\wedge\pi^*\eta,\ \ \forall\eta\in\Omega^{2r}_{cpt}(X).$$

When $\alpha$ is a nondegenerate section

$$\int_{\hat{X}}c_r(\mathbf{D}_{\mathbb{L}^\perp})\wedge\pi^*\eta=\underbrace{\int_{\mathbb{P}(\mathbb{C}\oplus E)|_Z}c_r(\mathbf{D}_{\mathbb{L}^\perp})\wedge\pi^*\eta}_{\mathbf{A}}+\underbrace{\int_{\bar{X}}c_r(\mathbf{D}_{\mathbb{L}^\perp})\wedge\pi^*\eta}_{\mathbf{B}}.$$

Over $X\setminus Z$ the strict transform can be identified with the graph of the section

$$X\setminus Z\mapsto[0,\alpha(x)]\subset\mathbb{P}(E)\subset\mathbb{P}(\mathbb{C}\oplus E).$$

Note also that over $X\setminus Z$ we have

$$\mathbb{L}^\perp\mid_{\mathbb{P}(E)}=\pi^*(E)\ \text{ and }\ (\pi^*\nabla)\mid_{\mathbb{P}(E)}\sim\mathbf{D}_{\mathbb{L}^\perp},$$

where $\sim$ denotes gauge equivalence. Hence $c_r(\mathbf{D}_{\mathbb{L}^\perp})\mid_{\bar{X}}=0$ so that $\mathbf{B}=0$. On the other hand

$$\mathbf{A}=\int_{\mathbb{P}(\mathbb{C}_Z\oplus E|_Z)}c_r(\mathbf{D}_{\mathbb{L}^\perp})\wedge\pi^*\eta=\int_Z\pi_*\big(c_r(\mathbf{D}_{\mathbb{L}^\perp})\wedge\pi^*\eta\big)$$

(use projection formula)

$$=\int_Z\big(\pi_*c_r(\mathbf{D}_{\mathbb{L}^\perp})\big)\wedge\eta=\int_Z\eta.$$

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Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618.
*E-mail address*: `nicolaescu.1@nd.edu`
*URL*: `http://www.nd.edu/~lnicolae/`