CRITICAL SETS OF RANDOM SMOOTH FUNCTIONS ON COMPACT MANIFOLDS

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ABSTRACT. Given a compact, connected Riemann manifold without boundary (M, g) of dimension m and a large positive constant L we denote by $\boldsymbol{U}_L$ the subspace of $C^\infty(M)$ spanned by eigenfunctions of the Laplacian corresponding to eigenvalues $\leq L$. We equip $\boldsymbol{U}_L$ with the standard Gaussian probability measure induced by the L^2-metric on $\boldsymbol{U}_L$, and we denote by $\mathcal{N}_L$ the expected number of critical points of a random function in $\boldsymbol{U}_L$. We prove that $\mathcal{N}_L \sim C_m \dim \boldsymbol{U}_L$ as $L \to \infty$, where C_m is an explicit positive constant that depends only on the dimension m of M and satisfying the asymptotic estimate $\log C_m \sim \frac{m}{2} \log m$ as $m \to \infty$.

CONTENTS

1. INTRODUCTION

Suppose that (M, g) is a smooth, compact, connected Riemann manifold of dimension $m > 1$. We denote by $|dV_g|$ the volume density on M induced by g. For any $\boldsymbol{u}, \boldsymbol{v} \in C^\infty(M)$ we denote by

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$(\boldsymbol{u},\boldsymbol{v})_g$ their L^2 inner product,

$$(\boldsymbol{u},\boldsymbol{v})_g := \int_M \boldsymbol{u}(x)\boldsymbol{v}(x)\,|dV_g(x)|.$$

The L^2-norm of a smooth function u is then

$$\|\boldsymbol{u}\| := \sqrt{(\boldsymbol{u},\boldsymbol{u})_g}.$$

Let $\Delta_g : C^\infty(M) \to C^\infty(M)$ denote the scalar Laplacian defined by the metric g. For $L > 0$ we set

$$\boldsymbol{U}_L = \boldsymbol{U}_L(M,g) := \bigoplus_{\lambda\in[0,L]} \ker(\lambda - \Delta_g), \;\; d(L) := \dim \boldsymbol{U}_L.$$

We equip $\boldsymbol{U}_L$ with the Gaussian probability measure.

$$d\boldsymbol{\gamma}_L(\boldsymbol{u}) := (2\pi)^{-\frac{d(L)}{2}} e^{-\frac{\|\boldsymbol{u}\|^2}{2}} |d\boldsymbol{u}|.$$

For any $\boldsymbol{u} \in \boldsymbol{U}_L$ we denote by $\mathcal{N}_L(\boldsymbol{u})$ the number of critical points of $\boldsymbol{u}$. If L is sufficiently large, then $\mathcal{N}_L(\boldsymbol{u})$ is finite with probability 1. We obtain in this fashion a random variable $\mathcal{N}_L = \mathcal{N}_{L,M,g}$, and we denote by $\boldsymbol{E}(\mathcal{N}_L)$ its expectation

$$\boldsymbol{E}(\mathcal{N}_L) := \int_{\boldsymbol{U}_L} \mathcal{N}_L(\boldsymbol{u}) d\boldsymbol{\gamma}_L(\boldsymbol{u}).$$

In this paper we investigate the behavior of $\boldsymbol{E}(\mathcal{N}_L)$ as $L \to \infty$. More precisely we will prove the following result.

Theorem 1.1. *For any $m > 1$ there exists a positive constant $C = C(m)$ such that* ***for any*** *compact, connected, m-dimensional Riemannian manifold M we have*

$$\boldsymbol{E}(\mathcal{N}_{L,M,g}) \sim C(m) \dim \boldsymbol{U}_L(M,g) \;\; \text{as} \;\; L \to \infty. \tag{1.1}$$

The constant $C(m)$ can be expressed in terms of certain statistics on the space $\mathcal{S}_m$ the space of symmetric $m \times m$ matrices . We denote $d\boldsymbol{\gamma}_*$ the Gaussian measure[1]on $\mathcal{S}_m$ given by

$$\begin{gathered} d\boldsymbol{\gamma}_*(X) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}\sqrt{\mu_m}} \cdot e^{-\frac{1}{4}\left(\operatorname{tr} X^2 - \frac{1}{m+2}(\operatorname{tr} X)^2\right)} 2^{\frac{1}{2}\binom{m}{2}} \prod_{i\leq j} dx_{ij}, \\ \mu_m = 2^{\binom{m}{2}+1}(m+2)^{m-1}. \end{gathered} \tag{1.2}$$

Then

$$C(m) = \left(\frac{4\pi}{m+4}\right)^{\frac{m}{2}} \Gamma(1+\frac{m}{2}) \underbrace{\int_{\mathcal{S}_m} |\det X|\, d\boldsymbol{\gamma}_*(X)}_{=:I_m}. \tag{1.3}$$

A similar result holds in the case $m = 1$. In this case $M = S^1$ and $\boldsymbol{U}_L$ is the space of trigonometric polynomials of degree $\leq L$. One can show (see [33])

$$\boldsymbol{E}(\mathcal{N}_{L,S^1}) \sim \sqrt{\frac{3}{5}} \dim \boldsymbol{U}_L \;\; \text{as} \;\; L \to \infty.$$

We can say something about the behavior of $C(m)$ as $m \to \infty$.

[1]We refer to Appendix B for a detailed description of a 3-parameter family Gaussian measures $d\Gamma_{a,b,c}$ on $\mathcal{S}_m$ that includes $d\boldsymbol{\gamma}_*$ as $d\boldsymbol{\gamma}_* = d\Gamma_{3,1,1}$.

Theorem 1.2.
$$\log C(m) \sim \log I_m \sim \frac{m}{2}\log m \;\; as \;\; m\to\infty. \tag{1.4}$$

The proof of (1.1) is achieved in two steps. First we prove an integral formula[2] (2.2) that expresses the expected number of critical points of a function in $\boldsymbol{U}_L$ as an integral
$$\boldsymbol{E}(\mathcal{N}_L) = \int_M \rho_L(\boldsymbol{x})\,|dV_g(\boldsymbol{x})|. \tag{$*$}$$
As explained in [11], the integral in the right-hand side of the above equality is the the total curvature of the immersion given by the evaluation map
$$\mathbf{ev}: M \to \operatorname{Hom}(\boldsymbol{U}_L,\mathbb{R}),\;\; \boldsymbol{p}\mapsto \mathbf{ev}_{\boldsymbol{p}}, \tag{1.5}$$
where $\mathbf{ev}_{\boldsymbol{p}}(\boldsymbol{u}) = \boldsymbol{u}(p)$, $\forall \boldsymbol{u}\in\boldsymbol{U}_L$.

The integrand $\rho_L(\boldsymbol{x})$ has a probabilistic interpretation. To formulate it let us denote by $\operatorname{Hess}_{\boldsymbol{x}}(\boldsymbol{u},g)$ the Hessian at $\boldsymbol{x}$ of the random function $\boldsymbol{u}\in\boldsymbol{U}_L$ computed using the Levi-Civita connection of the metric g. We identify $\operatorname{Hess}_{\boldsymbol{x}}(\boldsymbol{u},g)$ with a symmetric linear operators $T_{\boldsymbol{x}}M\to T_{\boldsymbol{x}}M$. Then
$$\rho_L(\boldsymbol{x}) = \frac{1}{\boldsymbol{E}(|d\boldsymbol{u}(x)|_g^2)}\boldsymbol{E}\Big(\,|\det \operatorname{Hess}_{\boldsymbol{x}}(\boldsymbol{u},g)|\;\Big|\; d\boldsymbol{u}(x)=0\Big). \tag{1.6}$$
Above, $\boldsymbol{E}(|d\boldsymbol{u}(x)|_g^2)$ denotes the expectation of the random variable $\boldsymbol{U}_L\ni\boldsymbol{u}\mapsto |d\boldsymbol{u}(x)|_g^2\in\mathbb{R}$ and the quantity
$$\boldsymbol{E}\Big(\,|\det \operatorname{Hess}_{\boldsymbol{x}}(\boldsymbol{u},g)|\;\Big|\; d\boldsymbol{u}(x)=0\Big)$$
is the conditional expectation of the random variable $\boldsymbol{u}\mapsto|\det\operatorname{Hess}_{\boldsymbol{x}}(\boldsymbol{u},g)|$ given that $d\boldsymbol{u}(x)=0$. A result similar to (1.6) is proved in [8, 14, 15] using the strategy pioneered by Kac, [25] and Rice, [39]. We use a different approach based on the double-fibration trick in integral geometry, [31, §9.1.1]. We refer to Subsection 2.1 for a comparison between these two approaches.

The random vector $\boldsymbol{u}\mapsto d\boldsymbol{u}(\boldsymbol{x})$ is Gaussian, and the expectation of $|d\boldsymbol{u}(\boldsymbol{x})|_g^2$ can be expressed in terms of the covariance matrix of this random vector. Using the *regression formula* (see [4, Prop. 1.2] or (A.2) we express this conditional expectation as the unconditional expectation of a new random variable $|\det A_L(\boldsymbol{x})|$, where $A_L(\boldsymbol{x})$ denotes a random, Gaussian symmetric $m\times m$ matrix whose covariance takes into account the correlations between the Gaussian variables $\boldsymbol{u}\mapsto\operatorname{Hess}_{\boldsymbol{x}}(\boldsymbol{u},g)$ and $\boldsymbol{u}\mapsto d\boldsymbol{u}(x)$.

Next, we reduce reduce the large L asymptotics of the Gaussian random vector $d\boldsymbol{u}(x)$ and matrix $A_L(\boldsymbol{x})$ to questions concerning the asymptotics of the spectral function $\mathcal{E}_L$ of the Laplacian, i.e., the Schwartz kernel of the orthogonal projection onto $\boldsymbol{U}_L$. Fortunately, such questions were addressed in the mathematical literature, [7, 13, 40, 44], by refining the wave kernel method of L. Hörmander, [22]. The clincher is an asymptotic independence result explained in ❷ below.

We actually prove a bit more. We show that
$$\lim_{L\to\infty} L^{-\frac{m}{2}}\rho_L(\boldsymbol{x}) = \frac{C(m)\boldsymbol{\omega}_m}{(2\pi)^m},\;\;\text{uniformly in } \boldsymbol{x}\in M, \tag{1.7}$$
where $\boldsymbol{\omega}_m$ denotes the volume of the unit ball in $\mathbb{R}^m$. Using the classical Weyl estimates (3.2) we see that (1.7) implies (1.1).

The equality (1.7) has an interesting interpretation. We can think of $\rho_L(\boldsymbol{x})|dV_g(\boldsymbol{x})|$ as the expected number of critical points of a random function in $\boldsymbol{U}_L$ inside an infinitesimal region of volume

[2] Formulæ of this type appear in the literature under different names, Chern-Lashof or Kac-Rice, depending on one's mathematical bias. This author usually resides in the Chern-Lashof part of the mathematical world.

$|dV_g(\boldsymbol{x})|$ around the point $\boldsymbol{x}$. From this point of view we see that (1.7) states that *for large L we expect the critical points of a random function in $\boldsymbol{U}_L$ to be approximatively uniformly distributed.*

We are inclined to believe that as $L \to \infty$ the ratio

$$q_L = \frac{\boldsymbol{var}(\mathcal{N}_L)}{\boldsymbol{E}(\mathcal{N}_L)}$$

has a finite limit $q(M, g)$. Such a result would show that $\mathcal{N}_L$ is highly concentrated near its mean value as $L \to \infty$. In [33] we prove that this is the case when $M = S^1$ and moreover, $q(S^1) \approx 0.4518....$ For a holomorphic counterpart of such an estimate we refer to [41]. The Riemannian case is much more challenging since we lack good general off-diagonal estimates for the spectral function.

We obtain the asymptotics of $C(m)$ by relying on a trick used by Y.V. Fyodorov [18] in a related context. This reduces the asymptotics of the integral I_m to known asymptotics of the 1-point correlation function in random matrix theory, more precisely, Wigner's semi-circle law.

Philosophically, the universality result contained in Theorem 1.1 is a consequence of a universal behavior of the spectral function $\mathcal{E}_L$ along the diagonal. Roughly speaking, if we rescale the metric g so that in the limit it becomes flatter, and flatter, then the corresponding spectral function begins to resemble the spectral function of the Laplacian on the Euclidean space $\mathbb{R}^m$. For a precise formulation of this universal rescaling phenomenon we refer to [26, 34].

A related problem was considered by M. Douglas, B. Shiffman, S. Zelditch, [14, 15] where they investigate the number of critical points of a random holomorphic section of a large power N of a positive holomorphic line bundle $\mathcal{L}$ over a Kähler manifold X. In these papers the role of our $\boldsymbol{U}_L$ is played by the space of holomorphic sections $H^0(X, \mathcal{L}^N)$, and the large L asymptotics is replaced by large N asymptotics. The large N asymptotics ultimately follow from the refined asymptotics of the Szegö kernels obtained by S. Zelditch in [43]. These refined asymptotics then lead to a complete asymptotic expansion as $N \to \infty$ for the expected number of critical points of a random holomorphic section of $\mathcal{L}^N$. In our case, the asymptotics of the spectral function $\mathcal{E}_L$ is very sensitive to the background metric g and it is unrealistic to seek a complete asymptotic expansion of $\boldsymbol{E}(\mathcal{N}_L)$ as $L \to \infty$ that is valid for any metric on g. Finally, the papers [14, 15] do not investigate large dimension asymptotics of the type described in our Theorem 1.2.

The proof of Theorem 1.1 reveals several additional interesting universal rescaling phenomena.

❶ We identify $\boldsymbol{U}_L$ with $\boldsymbol{U}_L^\vee = \operatorname{Hom}(\boldsymbol{U}_L, \mathbb{R})$ using the L^2-metric. We can thus view the evaluation map in (1.5) as a map $\mathbf{ev} : M \to \boldsymbol{U}_L$. For large L this map is an embedding, and we denote by $\boldsymbol{\sigma}_L$ the pullback to M via $\mathbf{ev}$ of the L^2-metric on $\boldsymbol{U}_L$. Equivalently, if $(\boldsymbol{\psi}_k)$ is an orthonormal basis of $\boldsymbol{U}_L$, then

$$\boldsymbol{\sigma}_L = \sum_k d\boldsymbol{\psi}_k \otimes d\boldsymbol{\psi}_k.$$

The equality (3.9) in the proof of Theorem 1.1 shows that the rescaled metric $g(L) := L^{-\frac{m+2}{2}}\boldsymbol{\sigma}_L$ converges in the C^0 topology to $K_m g$, where g is the original metric on M and K_m is a certain, explicit constant that depends only on m; see (3.5). This was also observed by S. Zelditch, [44, Prop. 2.3]. A closely related result was proved in [5, Thm.5].

To obtain the convergence of $g(L)$ in stronger topologies we would need bounds on the sectional curvature of $g(L)$. In Subsection 3.3 we show that these bounds are equivalent to some refined asymptotic estimates satisfied by certain linear combinations of fourth order derivatives of the spectral function, (3.20). These estimates hold for homogeneous spaces equipped with invariant metrics.

A related embedding can be constructed in the holomorphic case and S. Zelditch [43] has proved that the resulting sequence of suitably rescaled metrics g_N converges C^∞ to the original Kähler metric. The main reason for such a stronger form of convergence is the better behavior of the Szegö kernels. Such a regular behavior is not to be expected for the spectral function $\mathcal{E}_L$.

❷ Let us denote by $\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g)$ the Hessian at $\boldsymbol{p}$ of a smooth function $\boldsymbol{u}$ defined in terms of the Levi-Civita connection ∇^g; see (2.19). Fix an orthonormal basis of $T_{\boldsymbol{p}}M$ so we can identify $\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g)$ with a symmetric $m \times m$ matrix. The correspondence

$$\boldsymbol{U}_L \ni \boldsymbol{u} \mapsto \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g) \in \mathcal{S}_m$$

defines a Gaussian random symmetric matrix. Similarly, the correspondence

$$\boldsymbol{U}_L \ni \boldsymbol{u} \mapsto d\boldsymbol{u}(\boldsymbol{p}) \in T^*_{\boldsymbol{p}}M$$

defines a Gaussian random vector. The estimates (3.13) and (3.14) imply that as $L \to \infty$ the random vectors

$$L^{-\frac{m+4}{4}} \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g) \;\text{ and }\; L^{-\frac{m+2}{4}} d\boldsymbol{u}(\boldsymbol{p})$$

converge to *independent* random Gaussian vectors. Ultimately, this is an asymptotic manifestation of the surprising independence result [1, Eq. (12.2.11)].

❸ The limit as $L \to \infty$ of the random Gaussian symmetric matrix $L^{-\frac{m+4}{2}} \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g)$ is a random symmetric matrix A_∞ whose entries a_{ij} are centered Gaussian variables satisfying the correlation equalities

$$\boldsymbol{E}(a_{ij}a_{k\ell}) = \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}, \;\; \forall i, j, k, \ell. \tag{1.8}$$

where c_m is a certain, explicit universal constant that depends only on m. (Up to a rescaling, the probability density of A_∞ is given by (1.2)). Observe that the random matrix A_∞, is independent of the point $\boldsymbol{p}$, the choice of metric g and even on the manifold M!] The *universal* gaussian ensemble $\mathcal{U}_m$ described by (1.8) is *not* the GOE_m (Gaussian Orthogonal Ensemble) typically investigated in random matrix theory and given by the correlation equalities

$$\boldsymbol{E}(a_{ij}a_{k\ell}) = \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}, \;\; \forall i, j, k, \ell.$$

However $\mathcal{U}_m$ is closely related to GOE_m. More precisely a random matrix A in the ensemble $\mathcal{U}_m$ can equivalently be described as a sum of random matrices

$$A = B + Y\mathbb{1}_m,$$

where B belongs to GOE_m and Y is a standard scalar gaussian random variable independent of B.

The present paper is structured as follows. Section 2 contains the formulation and the proof of the key integral formula (1.6), including several reformulations in the language of random processes. In this section we also present a simple application of this formula to the number of critical points of random spherical harmonics of large degree on S^2. This sheds additional light on a recent result of Nazarov and Sodin [30] on the number of nodal domains of random spherical harmonics. More precisely, the inequality (2.41) shows that the expected number δ_n of zonal domain on S^2 of a random harmonic polynomial of large degree n satisfies the upper bound $\delta_n < 0.29n^2$.

Section 3 contains the proof of the asymptotic estimate (1.1) and Section 4 contains the proof of the estimate (1.3).

For the reader's convenience we have included in Appendix A a coordinate-free, brief survey of several facts about Gaussian measures and Gaussian processes in a form adapted to the applications in this paper. Appendix B contains a detailed description of a 3-parameter family of Gaussian measures on the space $\mathcal{S}_m$ of real, symmetric $m \times m$ matrices. These measures play a central role in the proof of (1.1) and we could not find an appropriate reference for the mostly elementary facts discussed in this appendix. Appendix C contains the computations of several Gaussian integrals involving random 2×2 matrices.

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NOTATIONS

(i) For any random variable ξ we denote by $\boldsymbol{E}(\xi)$ and respectively $\boldsymbol{var}(\xi)$ its expectation and respectively its variance.
(ii) $\mathcal{S}_m$ denotes the space of symmetric $m \times m$ real matrices.
(iii) For any finite dimensional real vector space $\boldsymbol{V}$ we denote by $\boldsymbol{V}^\vee$ its dual, $\boldsymbol{V}^\vee := \operatorname{Hom}(\boldsymbol{V}, \mathbb{R})$.
(iv) For any Euclidean space $\boldsymbol{V}$, we denote by $S(\boldsymbol{V})$ the unit sphere in $\boldsymbol{V}$ centered at the origin and by $B(\boldsymbol{V})$ the unit ball in $\boldsymbol{V}$ centered at the origin.
(v) We will denote by $\boldsymbol{\sigma}_n$ the "area" of the round n-dimensional sphere S^n of radius 1, and by $\boldsymbol{\omega}_n$ the "volume" of the unit ball in $\mathbb{R}^n$. These quantities are uniquely determined by the equalities (see [31, Ex. 9.1.11])

$$\boldsymbol{\sigma}_{n-1} = n\boldsymbol{\omega}_n = 2\frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{n\pi^{\frac{n}{2}}}{\Gamma\big(1+\frac{n}{2}\big)},\quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{σ}$$

where Γ is Euler's Gamma function.
(vi) If $\boldsymbol{V}_0$ and $\boldsymbol{V}_1$ are two Euclidean spaces of dimensions $n_0, n_1 < \infty$ and $A : \boldsymbol{V}_0 \to \boldsymbol{V}_1$ is a linear map, then the *Jacobian* of A is the nonnegative scalar $J(A)$ defined as the norm of the linear map

$$\Lambda^k A : \Lambda^k \boldsymbol{V}_0 \to \Lambda^k \boldsymbol{V}_1,\;\; k := \min(n_0, n_1).$$

More concretely, if $n_0 \leq n_1$, and $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n_0}\}$ is an orthonormal basis of $\boldsymbol{V}_0$, then

$$J(A) = \big(\det G(A)\big)^{1/2}, \tag{J_-}$$

where $G(A)$ is the $n_0 \times n_0$ Gramm matrix with entries

$$G_{ij} = \big(A\boldsymbol{e}_i, A\boldsymbol{e}_j\big)_{\boldsymbol{V}_1}.$$

If $n_1 \geq n_0$ then

$$J(A) = J(A^\dagger) = \big(\det G(A^\dagger)\big)^{1/2}, \tag{J_+}$$

where $A^\dagger$ denotes the adjoint (transpose) of A. Equivalently, if $d\mathrm{Vol}_i \in \Lambda^{n_i}\boldsymbol{V}_i^*$ denotes the metric volume form on $\boldsymbol{V}_1$, and $d\mathrm{Vol}_A$ denotes the metric volume form on $\ker A$, then $J(A)$ is the positive number such that

$$d\mathrm{Vol}_0 = \pm d\mathrm{Vol}_A \wedge A^* d\mathrm{Vol}_1. \tag{J'_+}$$

2. THE KEY INTEGRAL FORMULA

2.1. **A Chern-Lashof type formula.** As we mentioned in the introduction, a key component in the proof of Theorem 1.1 is an integral formula that describes the expected number of critical points as an integral over the background manifold M. The goal of this section is to state and prove this formula, and then give several probabilistic reformulations.

The literature on random fields contains many formulæ of this type, and their proofs follow the strategy pioneered by M. Kac and S. Rice, [25, 39]. As mentioned in the introduction, the general integral formula [8, Thm. 4.2] or [14, Thm. 4.4] covers the situation of interest to us. We believe that it would greatly benefit a geometrically inclined reader to see an alternate approach that does not

[3]He suddenly and untimely passed away in June 2011. I will miss his generosity and expertise.

follow the usual probabilistic pattern pioneered by Kac and Rice, but instead relies on the ubiquitous double-fibration trick in integral geometry, [2, 20, 31]. As a matter of fact, our main integral formula (2.2) contains as special cases the integral formulæ of Chern-Lashof, [11] and Milnor, [28]. We should perhaps describe the differences between the Kac-Rice approach and the double-fibration approach.

Denote by δ_M the measure on T^*M defined by the integration along the zero section. The Kac-Rice approach identifies the measure $\rho_g|dV_g|$ on M as an average (in the space of measures on M) of the family of measures $\{\mu_{\boldsymbol{u}} := (d\boldsymbol{u})^*\delta_M;\ \ \boldsymbol{u}\in\boldsymbol{U}_L\}$, where we denoted by $(d\boldsymbol{u})^*\delta_M$ the pullback in the sense of distributions of the measure δ_M via the map $d\boldsymbol{u}: M\to T^*M$. As is well known, [23, Thm. 8.4.2], the pullback operation is well defined provided a certain micro-local transversality condition is satisfied. For this reason, the implementation of the Kac-Rice strategy requires two intermediary steps, namely replacing the measure δ_M with a regularization δ_M^ε followed by a passage to limit $\varepsilon\searrow 0$.

The double-fibration approach is in some sense dual to the Kac-Rice approach. It leads immediately to a description of $\rho_g|dV_g|$ as the pushforward of a certain measure defined on the total space of a fibration $\pi: P\to M$, and the bulk of the proof is devoted to explicitly computing this pushforwad.

Suppose that M is a smooth, compact, connected manifold without boundary. Set $m:=\dim M$.

Definition 2.1. (a) For any nonnegative integer k, any point $\boldsymbol{p}\in M$ and any $f\in C^\infty(M)$ we will denote by $j_k(f,\boldsymbol{p})$ the k-th jet of f at $\boldsymbol{p}$.
(b) Suppose that $\boldsymbol{U}\subset C^\infty(M)$ is a linear subspace. If k is nonnegative integer then we say that $\boldsymbol{U}$ is k-*ample* if for any $\boldsymbol{p}\in M$ and any $f\in C^\infty(M)$ there exists $\boldsymbol{u}\in\boldsymbol{U}$ such that

$$j_k(\boldsymbol{u},\boldsymbol{p}) = j_k(f,\boldsymbol{p}). \qquad \square$$

Fix a finite dimensional vector space $\boldsymbol{U}\subset C^\infty(M)$ and set $N:=\dim\boldsymbol{U}$. We have an evaluation map

$$\mathbf{ev}=\mathbf{ev}^{\boldsymbol{U}}: M\to\boldsymbol{U}^\vee:=\operatorname{Hom}(\boldsymbol{U},\mathbb{R}),\ \ \boldsymbol{p}\mapsto\mathbf{ev}_{\boldsymbol{p}},$$

where for any $\boldsymbol{p}\in M$ the linear map $\mathbf{ev}_{\boldsymbol{p}}:\boldsymbol{U}\to\mathbb{R}$ is given by

$$\mathbf{ev}_{\boldsymbol{p}}(\boldsymbol{u})=\boldsymbol{u}(\boldsymbol{p}),\ \ \forall\boldsymbol{u}\in\boldsymbol{U}.$$

For any $\boldsymbol{u}\in C^\infty(M)$ we denote by $\mathcal{N}(\boldsymbol{u})$ the number of critical points of $\boldsymbol{u}$. In the remainder of this section we will assume that $\boldsymbol{U}$ is 1-ample. This implies that the evaluation map $\mathbf{ev}^{\boldsymbol{U}}$ is an immersion. Moreover, as explained in [32, §1.2], the 1-ampleness condition also implies that almost all functions $\boldsymbol{u}\in\boldsymbol{U}$ are Morse functions and thus $\mathcal{N}(\boldsymbol{u})<\infty$ for almost all $\boldsymbol{u}\in\boldsymbol{U}$.

We fix an inner product $h=(-,-)_h$ on $\boldsymbol{U}$ and we denote by $|-|_h$ the resulting Euclidean norm. Using the metric h we identify $\boldsymbol{U}$ with its dual and thus we can regard the evaluation map as a smooth map $\mathbf{ev}: M\to\boldsymbol{U}$. We define the expected number of critical points of a function in $\boldsymbol{U}$ to be the quantity

$$\mathcal{N}(\boldsymbol{U},h):=\frac{1}{\boldsymbol{\sigma}_{N-1}}\int_{S(\boldsymbol{U})}\mathcal{N}(\boldsymbol{u})\,|dA_h(\boldsymbol{u})| = \int_{\boldsymbol{U}}\mathcal{N}(\boldsymbol{u})\,\underbrace{\frac{e^{-\frac{|\boldsymbol{u}|_h^2}{2}}}{(2\pi)^{\frac{N}{2}}}\,|dV_h(u)|}_{=:d\boldsymbol{\gamma}_h(\boldsymbol{u})}, \tag{2.1}$$

where $\boldsymbol{\sigma}_{n-1}$ denotes the "area" of the unit sphere in $\mathbb{R}^n$, $|dA_h|$ denotes the "area" density on the unit sphere $S(\boldsymbol{U})$, and $|dV_h(\boldsymbol{u})|$ denotes the volume density on $\boldsymbol{U}$ determined by the metric h. A priori, the expected number of critical points could be infinite, but in any case, it is independent of any choice of metric on M. The space $\boldsymbol{U}$ equipped with the Gaussian probability measure $d\boldsymbol{\gamma}_h$ is a probability space. We denote by $\mathcal{N}_U$ the random variable $\boldsymbol{U}\ni\nu\mapsto\mathcal{N}(\boldsymbol{u})\in\mathbb{Z}$ so that

$$\mathcal{N}(\boldsymbol{U},h)=\boldsymbol{E}(\mathcal{N}_{\boldsymbol{U}},d\boldsymbol{\gamma}_h),$$

where $\boldsymbol{E}(-, d\boldsymbol{\gamma}_h)$ denotes the expectation computed with respect to the probability measure $d\boldsymbol{\gamma}_h$. We will refer to the pair $(\boldsymbol{U}, h)$ as the *sample space*.

The main goal of this section is to express $\mathcal{N}(\boldsymbol{U}, h)$ as the integral of an explicit density on M. To describe this formula it is convenient to fix a metric g on M. We will express $\mathcal{N}(\boldsymbol{U}, h)$ as an integral

$$\int_M \rho_g(\boldsymbol{p})\, |dV_g(\boldsymbol{p})|.$$

The function ρ_g does *depend on* g, but the density $\rho_g(\boldsymbol{p})\,|dV_g(\boldsymbol{p})|$ *is independent of* g. The concrete description of $\rho_g(\boldsymbol{p})$ relies on several fundamental objects naturally associated to the triplet $(\boldsymbol{U}, h, g)$.

For any $\boldsymbol{p} \in M$ we set

$$\boldsymbol{U}^0_{\boldsymbol{p}} := \big\{ \boldsymbol{u} \in \boldsymbol{U};\;\; d\boldsymbol{u}(\boldsymbol{p}) = 0 \big\}.$$

The 1-ampleness assumption on $\boldsymbol{U}$ implies that for any $\boldsymbol{p} \in M$ the subspace $\boldsymbol{U}^0_{\boldsymbol{p}}$ has codimension m in $\boldsymbol{U}$ so that

$$\dim \boldsymbol{U}^0_{\boldsymbol{p}} = N - m.$$

Denote by $dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}$ the area density along the unit sphere $S(\boldsymbol{U}^0_{\boldsymbol{p}}) \subset \boldsymbol{U}^0$.

The differential of the evaluation map at $\boldsymbol{p}$ is a linear map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$. We will refer to $\mathcal{A}_{\boldsymbol{p}}$ as the *adjunction map* and we will denote by $J_g(\boldsymbol{p}) = J_g(\boldsymbol{p}, \boldsymbol{U})$ its Jacobian. More precisely, if $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m)$ is a g-orthonormal basis of $T_{\boldsymbol{p}}M$, then

$$J_g(\boldsymbol{p})^2 = \det\Big[\big(\mathcal{A}_{\boldsymbol{p}}\boldsymbol{e}_i, \mathcal{A}_{\boldsymbol{p}}\boldsymbol{e}_j \big)_h \Big]_{1 \leq i,j \leq m}.$$

Since $\mathbf{ev}^{\boldsymbol{U}}$ is an immersion we have $J_g(\boldsymbol{p}) \neq 0, \forall \boldsymbol{x} \in M$.

For any $\boldsymbol{p} \in M$ and any $\boldsymbol{u} \in \boldsymbol{U}^0_{\boldsymbol{p}}$, the Hessian of $\boldsymbol{u}$ at $\boldsymbol{p}$ is a well defined symmetric bilinear form on $T_{\boldsymbol{p}}M$ that can be identified via the metric g with a symmetric endomorphism $\mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g)$ of $T_{\boldsymbol{p}}M$. We denote this symmetric endomorphism by $\mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g)$.

Theorem 2.2. *If $(\boldsymbol{U}, h)$ is a 1-ample sample space on M, then*

$$\begin{aligned}
\mathcal{N}(\boldsymbol{U}, h) &= \frac{1}{\boldsymbol{\sigma}_{N-1}} \int_M \frac{1}{J_g(\boldsymbol{p})} \left(\int_{S(\boldsymbol{U}^0_{\boldsymbol{p}})} |\det \mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{v}, g)|\, |dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{v})| \right) |dV_g(\boldsymbol{p})| \\
&= (2\pi)^{-\frac{m}{2}} \int_M \frac{1}{J_g(\boldsymbol{p})} \underbrace{\left(\int_{\boldsymbol{U}^0_{\boldsymbol{p}}} |\det \mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g)| \frac{e^{-\frac{|\boldsymbol{u}|_h^2}{2}}}{(2\pi)^{\frac{N-m}{2}}} |dV_h(\boldsymbol{u})| \right)}_{=:I_{\boldsymbol{p}}} |dV_g(\boldsymbol{p})|.
\end{aligned} \tag{2.2}$$

Proof. Denote by $\underline{\boldsymbol{U}}_M$ the trivial vector bundle over M with fiber $\boldsymbol{U}$, $\underline{\boldsymbol{U}}_M := (\boldsymbol{U} \times M \to M)$. For any $\boldsymbol{p} \in M$ we denote by $\boldsymbol{K}_{\boldsymbol{p}}$ the orthogonal complement of $\boldsymbol{U}^0_{\boldsymbol{p}}$ in $\boldsymbol{U}$.

Lemma 2.3. *The subspace $\boldsymbol{K}_{\boldsymbol{p}}$ coincides with the range of the adjunction map $\mathcal{A}_{\boldsymbol{p}}$.*

Proof. Indeed, if $(\Psi_n)_{1 \leq n \leq N}$ is an orthonormal basis of $(\boldsymbol{U}, h)$, then

$$\mathbf{ev}_{\boldsymbol{p}} = \sum_n \Psi_n(\boldsymbol{p})\Psi_n \in \boldsymbol{U}.$$

and for any vector field X on M we have

$$\mathcal{A}_{\boldsymbol{p}} X_{\boldsymbol{p}} = \sum_n (X\Psi_n)_{\boldsymbol{p}} \Psi_n.$$

Thus, the function $\boldsymbol{u}=\sum_n u_n\Psi_n\in\boldsymbol{U}$, $u_n\in\mathbb{R}$, belongs to $\boldsymbol{K}_{\boldsymbol{p}}^{\perp}$ if and only if for any vector field X on M we have

$$0=\sum_n u_n(X\Psi_n)_{\boldsymbol{p}}=X\cdot\boldsymbol{u}(\boldsymbol{p})\Longleftrightarrow\boldsymbol{u}\in\boldsymbol{U}_{\boldsymbol{p}}^0.$$

□

This proves that the collection $(\boldsymbol{K}_{\boldsymbol{p}})$ defines a subbundle $\boldsymbol{K}$ of $\underline{\boldsymbol{U}}_M$ and the adjunction map induces an isomorphism of vector bundle $\mathcal{A}:TM\to\boldsymbol{K}$. We deduce that the collection of spaces $(\boldsymbol{U}^0_{\boldsymbol{p}})_{\boldsymbol{p}\in M}$ also forms a vector subbundle $\boldsymbol{U}^0$ of the trivial bundle $\underline{\boldsymbol{U}}_M$ and we have an orthogonal direct sum decomposition

$$\underline{\boldsymbol{U}}_M=\boldsymbol{U}^0\oplus\boldsymbol{K}.$$

For any section u of $\underline{\boldsymbol{U}}_M$ we denote by u^0 its $\boldsymbol{U}^0$-component.

The bundle $\underline{\boldsymbol{U}}_M$ is equipped with a canonical trivial connection D. More precisely, if we regard a section of u of $\underline{\boldsymbol{U}}_M$ as a smooth map $u:M\to\boldsymbol{U}$, then for any vector field X on M we define D_Xu as the smooth function $M\to\boldsymbol{U}$ obtained by derivating u along X. The *shape operator* of the subbundle $\boldsymbol{K}$ is the bundle morphism $\boldsymbol{\Xi}:TM\otimes\boldsymbol{K}\to\boldsymbol{U}^0$ defined by the equality

$$\boldsymbol{\Xi}(X,\boldsymbol{u}):=(D_X\boldsymbol{u})^0,\ \ \forall X\in C^\infty(TM),\ \ \boldsymbol{u}\in C^\infty(\boldsymbol{K}).$$

For every $\boldsymbol{p}\in M$, we denote by $\boldsymbol{\Xi}_{\boldsymbol{p}}$ the induced linear map $\boldsymbol{\Xi}_{\boldsymbol{p}}:T_{\boldsymbol{p}}M\otimes\boldsymbol{K}_{\boldsymbol{p}}\to\boldsymbol{U}^0$. If we denote by $\mathbf{Gr}_m(\boldsymbol{U})$ the Grassmannian of m-dimensional subspaces of $\boldsymbol{U}$, then we have a Gauss map

$$M\ni\boldsymbol{p}\overset{\mathcal{G}}{\longmapsto}\mathcal{G}(\boldsymbol{p}):=\boldsymbol{K}_{\boldsymbol{p}}\in\mathbf{Gr}_m(\boldsymbol{U}).$$

The shape operator $\boldsymbol{\Xi}_{\boldsymbol{p}}$ can be viewed as a linear map

$$\boldsymbol{\Xi}_{\boldsymbol{p}}:T_{\boldsymbol{p}}M\to\operatorname{Hom}(\boldsymbol{K}_{\boldsymbol{p}},\boldsymbol{U}^0_{\boldsymbol{p}})=T_{\boldsymbol{K}_{\boldsymbol{p}}}\mathbf{Gr}_m(\boldsymbol{U}),$$

and, as such, it can be identified with the differential of $\mathcal{G}$ at $\boldsymbol{p}$, [31, §9.1.2]. Any $\boldsymbol{v}\in\boldsymbol{U}^0_{\boldsymbol{p}}$ determines a bilinear map

$$\boldsymbol{\Xi}_{\boldsymbol{p}}\cdot\boldsymbol{v}:T_{\boldsymbol{p}}M\otimes\boldsymbol{K}_{\boldsymbol{p}}\to\mathbb{R},\ \ \boldsymbol{\Xi}_{\boldsymbol{p}}\cdot\boldsymbol{v}(\boldsymbol{e},\boldsymbol{u}):=\big(\,\boldsymbol{\Xi}_{\boldsymbol{p}}(\boldsymbol{e},\boldsymbol{u}),\boldsymbol{v}\,\big)_h,$$

By choosing orthonormal bases $(\boldsymbol{e}_i)$ in $T_{\boldsymbol{p}}M$ and $(\boldsymbol{u}_j)$ of $\boldsymbol{K}_{\boldsymbol{p}}$ we can identify this bilinear form with an $m\times m$-matrix. This matrix depends on the choices of bases, but the absolute value of its determinant is independent of these bases. It is thus an invariant of the pair $(\boldsymbol{\Xi}_{\boldsymbol{p}},\boldsymbol{v})$ that we will denote by $|\det\boldsymbol{\Xi}_{\boldsymbol{p}}\cdot\boldsymbol{v}|$.

Lemma 2.4.

$$\mathcal{N}(\boldsymbol{U},h)=\frac{1}{\boldsymbol{\sigma}_{N-1}}\int_M\left(\int_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}|\det\boldsymbol{\Xi}_{\boldsymbol{p}}\cdot\boldsymbol{v}|\,|dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{v})|\right)|dV_g(\boldsymbol{p})|. \tag{2.3}$$

Proof. Consider the incidence variety

$$\mathcal{I}:=\big\{(\boldsymbol{p},\boldsymbol{v})\in M\times S(\boldsymbol{U});\ \ d\boldsymbol{v}(\boldsymbol{p})=0\,\big\}=\big\{(\boldsymbol{x},\boldsymbol{v})\in M\times S(\boldsymbol{U});\ \ \boldsymbol{v}\in S(\boldsymbol{U}^0_{\boldsymbol{p}})\,\big\}.$$

We have a natural double "fibration"

$$M\overset{\boldsymbol{\lambda}}{\longleftarrow}\mathcal{I}\overset{\boldsymbol{\rho}}{\longrightarrow}S(\boldsymbol{U}),$$

where the left/right projections $\boldsymbol{\lambda},\boldsymbol{\rho}$ are the canonical projections. The left projection $\boldsymbol{\lambda}:\mathcal{I}\to M$ describes $\mathcal{I}$ as the unit sphere bundle associated to the metric vector bundle $\boldsymbol{U}^0$. In particular, this shows that $\mathcal{I}$ is a compact, smooth manifold of dimension $(N-1)$. For generic $\boldsymbol{v}\in S(\boldsymbol{U})$ the fiber $\rho^{-1}(\boldsymbol{v})$ is finite and can be identified with the set of critical points of $\boldsymbol{v}: M\to\mathbb{R}$. We deduce

$$\mathcal{N}(\boldsymbol{U},h)=\frac{1}{\operatorname{area}\left(S(\boldsymbol{U})\right)}\int_{S(\boldsymbol{U})}\#\boldsymbol{\rho}^{-1}(\boldsymbol{v})\,|dA_h(\boldsymbol{u})|. \tag{2.4}$$

Denote by $g_{\mathcal{I}}$ the metric on $\mathcal{I}$ induced by the metric on $M\times S(\boldsymbol{U})$ and by $|dV_{\mathcal{I}}|$ the induced volume density. The coarea formula, [10, §13.4], implies that

$$\int_{S(\boldsymbol{U})}\#\boldsymbol{\rho}^{-1}(\boldsymbol{v})|dA_h(\boldsymbol{v})|=\int_{\mathcal{I}}J_{\boldsymbol{\rho}}(\boldsymbol{p},\boldsymbol{v})|dV_{\mathcal{I}}(\boldsymbol{p},\boldsymbol{v})|, \tag{2.5}$$

where the nonnegative function $J_{\boldsymbol{\rho}}$ is the Jacobian of $\boldsymbol{\rho}$ defined by the equality

$$\boldsymbol{\rho}^*|dA_h|=J_{\boldsymbol{\rho}}\cdot|dV_{\mathcal{I}}|.$$

To compute the integral in the right-hand side of (2.5) we need a more explicit description of the geometry of the incidence variety $\mathcal{I}$.

Fix a local orthonormal frame $(\boldsymbol{e}_1,\dots,\boldsymbol{e}_m)$ of TM defined in a neighborhood $\mathcal{O}$ in M of a given point $\boldsymbol{p}_0\in M$. We denote by $(\boldsymbol{e}^1,\dots,\boldsymbol{e}^m)$ the dual co-frame of T^*M. Set

$$\boldsymbol{f}_i(\boldsymbol{p}):=\mathcal{A}_{\boldsymbol{p}}\boldsymbol{e}_i(\boldsymbol{p})\in\boldsymbol{U},\ \ i=1,\dots,m,\ \ \boldsymbol{p}\in\mathcal{O}.$$

More explicitly, $\boldsymbol{f}_i(\boldsymbol{u})$ is defined by the equality

$$\left(\,\boldsymbol{f}_i(\boldsymbol{p}),\boldsymbol{v}\,\right)_h=\partial_{\boldsymbol{e}_i}\boldsymbol{u}(\boldsymbol{p}),\ \ \forall\boldsymbol{u}\in\boldsymbol{U}. \tag{2.6}$$

Fix a neighborhood $\mathcal{U}\subset\lambda^{-1}(\mathcal{O})$ in $M\times S(\boldsymbol{U})$ of the point $(\boldsymbol{p}_0,\boldsymbol{v}_0)$, and a local orthonormal frame $\boldsymbol{u}_1(\boldsymbol{p},\boldsymbol{v}),\dots,\boldsymbol{u}_{N-1}(\boldsymbol{p},\boldsymbol{v})$ over $\mathcal{U}$ of the bundle $\rho^*TS(\boldsymbol{U})\to M\times S(\boldsymbol{U})$ such that the following hold.

- The vectors $\boldsymbol{u}_1(\boldsymbol{p},\boldsymbol{v}),\dots,\boldsymbol{u}_m(\boldsymbol{p},\boldsymbol{v})$ *are independent of the variable* $\boldsymbol{v}$ and form an orthonormal basis of $K_{\boldsymbol{x}}^{\perp}$. (E.g., we can obtain such vectors from the vectors $\boldsymbol{f}_1(\boldsymbol{p}),\dots,\boldsymbol{f}_m(\boldsymbol{p})$ via the Gramm-Schmidt process.)
- For $(\boldsymbol{p},\boldsymbol{v})\in\mathcal{U}$, the space $T_{\boldsymbol{p}}E_{\boldsymbol{x}}$ is spanned by the vectors $\boldsymbol{u}_{m+1}(\boldsymbol{p},\boldsymbol{v}),\dots,\boldsymbol{u}_{N-1}(\boldsymbol{p},\boldsymbol{v})$.

The collection $\boldsymbol{u}_1(\boldsymbol{p}),\dots,\boldsymbol{u}_m(\boldsymbol{p})$ is a collection of smooth sections of $\underline{\boldsymbol{U}}_M$ over $\mathcal{O}$. For any $\boldsymbol{p}\in\mathcal{O}$ and any $\boldsymbol{e}\in T_{\boldsymbol{p}}M$, we obtain the vectors (functions).

$$D_{\boldsymbol{e}}\boldsymbol{u}_1(\boldsymbol{p}),\dots,D_{\boldsymbol{p}}\boldsymbol{u}_m(\boldsymbol{x})\in\boldsymbol{U},$$

where we recall that D denotes the trivial connection on $\underline{\boldsymbol{U}}_M$. Observe that

$$\mathcal{I}\cap\mathcal{U}=\big\{(\boldsymbol{p},\boldsymbol{v})\in\mathcal{U};\ \ U_i(\boldsymbol{p},\boldsymbol{v})=0,\ \ \forall i=1,\dots,m\,\big\}, \tag{2.7}$$

where U_i is the function $U_i:\mathcal{O}\times\boldsymbol{U}\to\mathbb{R}$ given by

$$U_i(\boldsymbol{p},\boldsymbol{v}):=\big(\,\boldsymbol{u}_i(\boldsymbol{p}),\boldsymbol{v}\,\big)_h.$$

Thus, the tangent space of $\mathcal{I}$ at $(\boldsymbol{p},\boldsymbol{v})$ consists of tangent vectors $\dot{\boldsymbol{p}}\oplus\dot{\boldsymbol{v}}\in T_{\boldsymbol{x}}M\oplus T_{\boldsymbol{v}}S(\boldsymbol{V})$ such that

$$dU_i(\dot{\boldsymbol{p}},\dot{\boldsymbol{v}})=0,\ \ \forall i=1,\dots,m.$$

We let ω_U denote the m-form

$$\omega_U:=dU_1\wedge\cdots\wedge dU_m\in\Omega^m(\mathcal{U}),$$

and we denote by $\|\omega_U\|$ its norm with respect to the product metric on $M\times S(\boldsymbol{U})$. Denote by $|\widehat{dV}|$ the volume density on $M\times S(\boldsymbol{U})$ induced by the product metric. The equality (2.7) implies that

$$|\widehat{dV}|=\frac{1}{\|\omega_U\|}\,|\omega_U\wedge dV_E\,|\,.$$

Hence

$$J_{\boldsymbol{\rho}}|\widehat{dV}| = \frac{1}{\|\omega_U\|}|\omega_U \wedge \boldsymbol{\rho}^* dA|.$$

We deduce

$$J_{\boldsymbol{\rho}}(\boldsymbol{p}_0, \boldsymbol{v}_0) = J_{\boldsymbol{\rho}}(\boldsymbol{p}_0, \boldsymbol{v}_0)|\widehat{dV}|(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m, \boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1})$$

$$= \frac{1}{\|\omega_U\|}|\omega_U \wedge \boldsymbol{\rho}^* dS|(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m, \boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1}) = \frac{1}{\|\omega_U\|} \underbrace{\left|\omega_U\big(\,\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\,\big)\right|_{(\boldsymbol{p}_0, \boldsymbol{v}_0)}}_{=:\Delta_U(\boldsymbol{p}_0, \boldsymbol{v}_0)}.$$

Hence,

$$\int_{S(\boldsymbol{U})} \#\boldsymbol{\rho}^{-1}(\boldsymbol{w})|dA_h(\boldsymbol{v})| = \int_{\mathcal{I}} \frac{\Delta_U}{\|\omega_U\|}\,|dV_{\mathcal{I}}(\boldsymbol{p}, \boldsymbol{v})|. \tag{2.8}$$

Sublemma 2.5. *We have the equality*

$$J_{\boldsymbol{\lambda}} = \frac{1}{\|\omega_U\|}, \tag{2.9}$$

where $J_{\boldsymbol{\lambda}}$ denotes the Jacobian of the projection $\boldsymbol{\lambda} : \mathcal{I} \to M$.

Proof. Along $\mathcal{U}$ we have

$$|\widehat{dV}| = \frac{1}{\|\omega_U\|}\,|\omega_U \wedge dV_{\mathcal{I}}\,|\,,$$

while the definition of the Jacobian implies that

$$|dV_{\mathcal{I}}| = \frac{1}{J_{\boldsymbol{\lambda}}}|dV_g \wedge dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}|.$$

Therefore, it suffices to show that along $\mathcal{U}$ we have

$$|\widehat{dV}| = |\omega_U \wedge dV_g \wedge dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}|,$$

i.e.,

$$\Big|\omega_U \wedge dV_g \wedge dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m, \boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1})\Big| = 1.$$

Since $dU_i(\boldsymbol{u}_k) = 0$, $\forall k \geq m+1$ we deduce that

$$\Big|\omega_U \wedge dV_g \wedge dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{e}_1, \dots, \boldsymbol{e}_m, \boldsymbol{u}_1, \dots, \boldsymbol{u}_{N-1})\Big| = |\omega_U(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m)|.$$

Thus, it suffices to show that

$$|\omega_U(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m)| = 1.$$

This follows from the elementary identities

$$dU_i(\boldsymbol{u}_j) = (\boldsymbol{u}_i, \boldsymbol{u}_j)_h = \delta_{ij}, \quad \forall 1 \leq i, j \leq m,$$

where δ_{ij} is the Kronecker symbol. □

Using (2.9) in (2.8) and the coarea formula we deduce

$$\int_{S(\boldsymbol{U})} \#\boldsymbol{\rho}^{-1}(\boldsymbol{w})|dA_h(\boldsymbol{v})| = \int_M \left(\int_{S(\boldsymbol{U}^0_{\boldsymbol{p}})} \Delta_U(\boldsymbol{p}, \boldsymbol{v})\,|dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{v})| \right) |dV_g(\boldsymbol{p})|. \tag{2.10}$$

Observe that at a point $(\boldsymbol{p}, \boldsymbol{v}) \in \lambda^{-1}(\mathcal{O}) \subset \mathcal{I}$ we have

$$dU_i(\boldsymbol{e}_j) = \big(\,D_{\boldsymbol{e}_j}\boldsymbol{u}_i(\boldsymbol{p}), \boldsymbol{v}\,\big)_h.$$

We can rewrite this in terms of the shape operator $\boldsymbol{\Xi}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \otimes \boldsymbol{K}_{\boldsymbol{p}} \to \boldsymbol{U}^0_{\boldsymbol{p}}$. More precisely,

$$dU_i(\boldsymbol{e}_j) = \big(\,\boldsymbol{\Xi}_{\boldsymbol{p}}(\boldsymbol{e}_j, \boldsymbol{u}_i), \boldsymbol{v}\,\big)_h.$$

Hence,

$$\Delta_U(\boldsymbol{x},\boldsymbol{v}) = \Big|\det\big(\,\boldsymbol{\Xi}_{\boldsymbol{p}}(\boldsymbol{e}_j,\boldsymbol{u}_i),\boldsymbol{v}\,\big)_h\,\Big|,$$

We conclude that

$$\int_{S_h(\boldsymbol{U})}\#\boldsymbol{\rho}^{-1}(\boldsymbol{v})|dA_h(\boldsymbol{v})| = \int_M\left(\int_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}|\det\boldsymbol{\Xi}_{\boldsymbol{p}}\cdot\boldsymbol{v}|\,|dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{v})|\right)|dV_M(\boldsymbol{p})|.$$

This proves (2.3) □

To proceed further observe that the left-hand side of (2.3) is plainly independent of the metric g on M. This raises the hope that if we judiciously choose the metric on M, then we can obtain a more manageable expression for $\mu(M,\boldsymbol{V})$. One choice presents itself.

Let $\boldsymbol{\sigma}$ be the pullback to M of the metric on $\boldsymbol{V}$ via the immersion $\mathbf{ev}: M\to\boldsymbol{U}$. More concretely, for any $\boldsymbol{p}\in M$ and any $X,Y\in T_{\boldsymbol{p}}M$, we have

$$\boldsymbol{\sigma}_{\boldsymbol{p}}(X,Y) = \big(\,\mathcal{A}_{\boldsymbol{p}}X,\mathcal{A}_{\boldsymbol{p}}Y\,\big)_h.$$

When $g=\boldsymbol{\sigma}$, the equality (2.3) is precisely the main theorem of Chern and Lashof, [11].

Fix $\boldsymbol{p}\in M$ and a $\boldsymbol{\sigma}$-orthonormal frame$(\boldsymbol{e}_i)_{1\le i\le m}$ of TM defined in a neighborhood $\mathcal{O}$ of $\boldsymbol{p}$. Then the collection $\boldsymbol{u}_j=\mathcal{A}\boldsymbol{e}_j$, $1\le j$, is a local orthonormal frame of $\boldsymbol{K}|_{\mathcal{O}}$. The shape operator has the simple description

$$\boldsymbol{\Xi}_{\boldsymbol{p}}(\boldsymbol{e}_i,\boldsymbol{u}_j) = \big(\,D_{\boldsymbol{e}_i}\mathcal{A}\boldsymbol{e}_j\,\big)^0.$$

Fix an orthonormal basis $(\Psi_n)_{1\le n\le N}$ of $\boldsymbol{U}$ so that every $\boldsymbol{v}\in\boldsymbol{U}$ has a decomposition

$$\boldsymbol{v} = \sum_\alpha v_n\Psi_n,\quad v_n\in\mathbb{R}.$$

Then

$$\mathcal{A}_{\boldsymbol{p}}\boldsymbol{e}_j(\boldsymbol{p}) = \sum_n(\partial_{\boldsymbol{e}_j}\Psi_n)_{\boldsymbol{p}}\Psi_n,\quad D_{\boldsymbol{e}_i}\mathcal{A}^\dagger\boldsymbol{e}_j(\boldsymbol{p}) = \sum_n(\partial^2_{\boldsymbol{e}_i\boldsymbol{e}_j}\Psi_n)_{\boldsymbol{p}}\Psi_n,$$

and

$$\big(\,(D_{\boldsymbol{e}_i}\mathcal{A}\boldsymbol{e}_j)_{\boldsymbol{p}},\boldsymbol{v}\,\big)_h = \sum_\alpha v_n(\partial^2_{\boldsymbol{e}_i\boldsymbol{e}_j}\Psi_\alpha)_{\boldsymbol{p}} = \partial^2_{\boldsymbol{e}_i\boldsymbol{e}_j}\boldsymbol{v}(\boldsymbol{p}).$$

If $\boldsymbol{v}\in\boldsymbol{U}^0_{\boldsymbol{p}}$, then the Hessian of $\boldsymbol{v}$ at $\boldsymbol{p}$ is a well-defined, symmetric bilinear form $\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v})$ on $T_{\boldsymbol{p}}M$ that can be identified via the metric $\boldsymbol{\sigma}$ with a symmetric linear operator

$$\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v},\boldsymbol{\sigma}): T_{\boldsymbol{p}}M\to T_{\boldsymbol{p}}M.$$

If we fix a $\boldsymbol{\sigma}$-orthonormal frame $(\boldsymbol{e}_i)$ of $T_{\boldsymbol{p}}M$, then the operator $\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v},\boldsymbol{\sigma})$ is described by the symmetric $m\times m$ matrix with entries $\partial^2_{\boldsymbol{e}_i\boldsymbol{e}_j}\boldsymbol{v}(\boldsymbol{x})$. We deduce that

$$|\det\boldsymbol{\Xi}_{\boldsymbol{p}}\cdot\boldsymbol{v}| = |\det\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v},\boldsymbol{\sigma})\,|,\quad\forall\boldsymbol{v}\in S(\boldsymbol{U}^0_{\boldsymbol{p}}).$$

In particular, we deduce that

$$\mathcal{N}(\boldsymbol{U},h) = \frac{1}{\boldsymbol{\sigma}_{N-1}}\int_M\left(\int_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}|\det\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v},\boldsymbol{\sigma})|\,|dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{v})\right)|dV_{\boldsymbol{\sigma}}(\boldsymbol{p})|. \tag{2.11}$$

Finally, we want to express (2.11) entirely in terms of the adjunction map $\mathcal{A}$. For any $\boldsymbol{p}\in M$ and any $\boldsymbol{v}\in\boldsymbol{U}_{\boldsymbol{p}}$, we define the density

$$\mu_{\boldsymbol{p},\boldsymbol{v}}:\Lambda^mT_{\boldsymbol{p}}M\to\mathbb{R},$$

$$\begin{aligned}\mu_{\boldsymbol{p},\boldsymbol{v}}(X_1\wedge\cdots\wedge X_m) &= \Big|\det\big(\,\partial^2_{X_iX_j}\boldsymbol{v}(\boldsymbol{p})\,\big)_{1\le i,j\le m}\Big|\cdot\Big(\det\big(\,(\mathcal{A}_{\boldsymbol{p}}X_i,\mathcal{A}_{\boldsymbol{p}}X_j)_h\,\big)_{1\le i,j\le m}\Big)^{-1/2}\\ &= \Big|\det\big(\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v})(X_i,X_j)\,\big)_{1\le i,j\le m}\Big|\cdot\Big(\det\big(\,\boldsymbol{\sigma}(X_i,X_j)\,\big)_{1\le i,j\le m}\Big)^{-1/2},\end{aligned}$$

for any basis $X_1, \dots, X_m$ of $T_{\boldsymbol{p}}M$. Observe that for any $\boldsymbol{\sigma}$-orthonormal frame $\boldsymbol{e}_1, \dots, \boldsymbol{e}_m$ of $T_{\boldsymbol{p}}M$ we have

$$\mu_{\boldsymbol{p},\boldsymbol{v}}(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_m) = |\det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v}, \boldsymbol{\sigma})\,|.$$

If we integrate $\mu_{\boldsymbol{p},\boldsymbol{v}}$ over $\boldsymbol{v} \in S(\boldsymbol{U}^0_{\boldsymbol{p}})$, we obtain a density

$$|d\mu_{\boldsymbol{U}}(\boldsymbol{p})| : \Lambda^m T_{\boldsymbol{p}}M \to \mathbb{R},$$

$$|d\mu_{\boldsymbol{U}}(\boldsymbol{p})|(X_1 \wedge \cdots \wedge X_m) = \int_{S(\boldsymbol{U}_{\boldsymbol{p}})} \mu_{\boldsymbol{p},\boldsymbol{v}}(X_1 \wedge \cdots \wedge X_m)\, |dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{v})|,$$

$\forall X_1, \dots, X_m \in T_{\boldsymbol{p}}M$.

Clearly $|d\mu_{\boldsymbol{U}}(\boldsymbol{p})|$ varies smoothly with $\boldsymbol{p}$, and thus it defines a density $|d\mu_{\boldsymbol{U}}(-)|$ on M. We want to emphasize that this density *depends on the metric on* $\boldsymbol{U}$ but *it is independent* of any metric on M. We will refer to it as *the density of* $\boldsymbol{U}$. By construction

$$\mathcal{N}(\boldsymbol{U}, h) = \frac{1}{\boldsymbol{\sigma}_{n-1}} \int_M |d\mu_{\boldsymbol{U}}(\boldsymbol{p})|.$$

If we now return to our original metric g on M, then we can express $|d\mu_{\boldsymbol{U}}(-)|$ as a product

$$|d\mu_{\boldsymbol{U}}(\boldsymbol{p})| = \delta_g(\boldsymbol{p}) \cdot |dV_g(\boldsymbol{p})|,$$

where $\delta_g = \delta_{g,\boldsymbol{U}} : M \to \mathbb{R}$ is a smooth nonnegative function.

To find a more useful description of ρ_g, we choose local coordinates $(x^1, \dots, x^m)$ near $\boldsymbol{p}$ such that (∂_{x^i}) is a *g-orthonormal* basis of $T_{\boldsymbol{p}}M$. Then

$$\mu_{\boldsymbol{p},\boldsymbol{v}}(\partial_{x_1} \wedge \cdots \wedge \partial_{x_m}) = \left|\det\big(\, \partial^2_{x_i x_j}\boldsymbol{v}(\boldsymbol{p})\,\big)_{1\leq i,j\leq m}\right| \cdot \Big(\det\big(\,(\mathcal{A}_{\boldsymbol{p}}\partial_{x_i}, \mathcal{A}_{\boldsymbol{p}}\partial_{x_j})_h\,\big)_{1\leq i,j\leq m}\Big)^{-1/2}.$$

Observe that the matrix $\big(\,\partial^2_{x_i x_j}\boldsymbol{v}(\boldsymbol{p})\,\big)_{1\leq i,j\leq m}$ describes the Hessian operator

$$\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{v}, g) : T_{\boldsymbol{p}}M \to T_{\boldsymbol{p}}M$$

induced by the Hessian of $\boldsymbol{v}$ at $\boldsymbol{p}$ and the metric g.

The scalar $\Big(\det\big(\,(\mathcal{A}_{\boldsymbol{p}}\partial_{x_i}, \mathcal{A}_{\boldsymbol{p}}\partial_{x_j})_h\,\big)_{1\leq i,j\leq m}\Big)^{1/2}$ is precisely the Jacobian $J_g(\boldsymbol{p})$ of the adjunction map $\mathcal{A}_{\boldsymbol{p}} : T_{\boldsymbol{p}}M \to \boldsymbol{U}$ defined in terms of the metric g on $T_{\boldsymbol{p}}M$ and the metric h on $\boldsymbol{U}$. We set

$$\Delta_{\boldsymbol{x}}(\boldsymbol{V}, g) := \int_{S(\boldsymbol{U}^0_{\boldsymbol{p}})} |\det \operatorname{Hess}_{\boldsymbol{x}}(\boldsymbol{v}, g)|\, |dA_{S(\boldsymbol{U}^0_{\boldsymbol{p}})}(\boldsymbol{v})|.$$

Since $|dV_g(\boldsymbol{p})|(\partial_{x_1} \wedge \cdots \wedge \partial_{x_m}) = 1$, we deduce

$$\delta_{g,\boldsymbol{V}}(\boldsymbol{p}) = \Delta_{\boldsymbol{p}}(\boldsymbol{V}, g) \cdot J_g(\boldsymbol{p})^{-1}. \tag{2.12}$$

This proves the first equality in (2.2). The second equality follows from the first by invoking (A.6) and the explicit formula (σ) for $\boldsymbol{\sigma}_{N-1}$. □

Remark 2.6 (***A Gauss-Bonnet type formula***)**.** With a little care, the above arguments lead to a Gauss-Bonnet type theorem. More precisely, if we assume that M is oriented, then, under appropriate orientation conventions, the Morse inequalities imply that the degree of the map $\rho : \mathcal{I} \to S_h(\boldsymbol{U})$ is equal to the Euler characteristic of M. If instead of working with densities, we work with forms, then we conclude that

$$\chi(M) = (2\pi)^{-\frac{m}{2}} \int_M \frac{1}{J_g(\boldsymbol{p})} \left(\int_{\boldsymbol{U}^0_{\boldsymbol{p}}} \det \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g) \frac{e^{-\frac{|\boldsymbol{u}|_h^2}{2}}}{(2\pi)^{\frac{N-m}{2}}} |dV_h(\boldsymbol{u})| \right) |dV_g(\boldsymbol{p})|. \tag{2.13}$$

When M is a submanifold of the Euclidean space $\boldsymbol{U}$, and $g = \boldsymbol{\sigma}$, then the above argument yields the Gauss-Bonnet theorem for submanifolds of a Euclidean space. $\square$

2.2. **A formula for variance.** The equality (2.2) can be used in some instances to compute the variance of the number of critical points of a random function in $\boldsymbol{U}$. More precisely, to a sample space $(\boldsymbol{U}, h) \subset C^\infty(M)$ we associate a sample space $(\boldsymbol{U}^\Delta, h_\Delta) \subset C^\infty(M \times M)$, the image of $\boldsymbol{U}$ via the diagonal injection

$$\mathcal{D} : C^\infty(M) \to C^\infty(M \times M), \quad \boldsymbol{u} \mapsto \boldsymbol{u}^\Delta,$$

$$\boldsymbol{u}^\Delta(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{u}(x) + \boldsymbol{u}(\boldsymbol{y}), \quad \forall \boldsymbol{u} \in \boldsymbol{U}, \quad \boldsymbol{x}, \boldsymbol{y} \in M.$$

The metric h_Δ is defined by requiring that the map $\mathcal{D} : (\boldsymbol{U}, h) \to (\boldsymbol{U}^\Delta, h_\Delta)$ is an isometry.

There is a slight problem. The sample space $\boldsymbol{U}^\Delta$ is *never* 1-ample, no matter how large we choose $\boldsymbol{U}$. More precisely, the ampleness is always violated along the diagonal $\Delta_M \subset M \times M$. We denote by $\boldsymbol{U}^\Delta_*$ the space of restrictions to $M^2_* := M \times M \setminus \Delta_M$ of the functions in $\boldsymbol{U}^\Delta$. Note that for any $\boldsymbol{u} \in \boldsymbol{U}$ we have

$$\mathcal{N}_{\boldsymbol{U}^\Delta_*}(\boldsymbol{u}) = \mathcal{N}(\boldsymbol{u}^\Delta|_{M^2_*}) = \mathcal{N}(\boldsymbol{u})^2 - \mathcal{N}(\boldsymbol{u}).$$

If $\boldsymbol{U}$ is sufficiently large, then the sample space $\boldsymbol{V}^\Delta_*$ is 1-ample and we deduce that

$$\boldsymbol{E}\big(\, \mathcal{N}^2_{\boldsymbol{U}} - \mathcal{N}_{\boldsymbol{U}}, d\gamma_h \,\big) = \boldsymbol{E}\big(\, \mathcal{N}_{\boldsymbol{U}^\Delta_*}, d\boldsymbol{\gamma}_{h_\Delta} \,\big). \tag{$\boldsymbol{M}_2$}$$

The quantity in the left-hand side of ($\boldsymbol{M}_2$) is the so called *second combinatorial momentum* of the random variable $\mathcal{N}_{\boldsymbol{U}}$. Using (2.2) we can express the right-hand side of ($\boldsymbol{M}_2$) as an integral of a density over M^2_*. Since $\boldsymbol{U}^\Delta$ fails to be 1-ample along the diagonal so that the corresponding Jacobian term vanishes along the diagonal. We deduce that

$$\boldsymbol{E}\big(\, \mathcal{N}^2_{\boldsymbol{U}} - \mathcal{N}_{\boldsymbol{U}}, d\gamma_h \,\big)$$

$$= \frac{1}{(2\pi)^m} \int_{M^2_*} \frac{1}{J^\Delta_g(\boldsymbol{p}, \boldsymbol{q})} \left(\int_{\boldsymbol{U}^0_{\boldsymbol{p},\boldsymbol{q}}} |\det \mathrm{Hess}_{\boldsymbol{p},\boldsymbol{q}}(\boldsymbol{u}^\Delta)| \frac{e^{\frac{-|\boldsymbol{u}|^2_h}{2}}}{(2\pi)^{\frac{N-2m}{2}}} |dV_h(\boldsymbol{u})| \right) |dV_{g\times g}(\boldsymbol{p}, \boldsymbol{q}))|.$$

where $\boldsymbol{U}^0_{\boldsymbol{p},\boldsymbol{q}} := \boldsymbol{U}^0_{\boldsymbol{p}} \cap \boldsymbol{U}^0_{\boldsymbol{q}}$,

$$\det \mathrm{Hess}_{\boldsymbol{p},\boldsymbol{q}}(\boldsymbol{u}^\Delta, g \times g) = \det \mathrm{Hess}_{\boldsymbol{p}}(\boldsymbol{u}, g) \det \mathrm{Hess}_{\boldsymbol{q}}(\boldsymbol{u}, g), \quad \forall \boldsymbol{u} \in \boldsymbol{U}^0_{\boldsymbol{p}} \cap \boldsymbol{U}^0_{\boldsymbol{q}},$$

and J^Δ_g is the Jacobian of the adjunction map $\mathcal{A}^\Delta_{\boldsymbol{p},\boldsymbol{q}} : T_{(\boldsymbol{p},\boldsymbol{q})} M \times M \to \boldsymbol{U}^\Delta$.

2.3. **A Gaussian random field perspective.** The formula (2.2) looks hopeless for two immediately visible reasons.

- The Jacobian $J_g(\boldsymbol{p})$ seems difficult to compute.
- The integral $I_{\boldsymbol{p}}$ in (2.2) may be difficult to compute since the domain of integration $\boldsymbol{U}^0_{\boldsymbol{p}}$ may be impossible to pin down.

We will deal with these difficulties simultaneously by relying on some probabilistic principles inspired from [1]. For the reader's convenience we have gathered in Appendix A the basic probabilistic notions and facts needed in the sequel.

Consider again the metric $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{\boldsymbol{U}}$, the pullback of the metric h on $\boldsymbol{U}$ via the evaluation map. We will refer to it as the *stochastic metric* associated to the sample space $(\boldsymbol{U}, h)$. It is convenient to have a local description of the stochastic metric.

Fix an orthonormal basis $\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_N$ of $\boldsymbol{U}$. The evaluation map $\mathbf{ev}^{\boldsymbol{U}} : M \to \boldsymbol{U}$ is then given by

$$M \ni \boldsymbol{x} \mapsto \sum_n \boldsymbol{\psi}_n(\boldsymbol{x}) \cdot \boldsymbol{\psi}_n \in \boldsymbol{U}.$$

If $\boldsymbol{p} \in M$ and $\mathcal{U}$ is an open coordinate neighborhood of $\boldsymbol{p}$ with coordinates $x = (x^1, \dots, x^m)$, then

$$\boldsymbol{\sigma}_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}) = \sum_n \frac{\partial \boldsymbol{\psi}_n}{\partial x^i}(\boldsymbol{p}) \frac{\partial \boldsymbol{\psi}_n}{\partial x^j}(\boldsymbol{p}), \;\; \forall 1 \leq i, j \leq m. \tag{2.14}$$

Note that if the collection $(\partial_{x^i})_{1 \leq i \leq m}$ forms a g-orthonormal frame of $T_{\boldsymbol{p}}M$, then

$$J_g(\boldsymbol{p})^2 = \det\Big[\, \boldsymbol{\sigma}_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}) \Big]_{1 \leq i,j \leq m}. \tag{2.15}$$

To the sample space $(\boldsymbol{U}, h)$ we associate in a tautological fashion a Gaussian random field on M as follows. The measure $d\boldsymbol{\gamma}_h$ in (2.1) is a probability measure and thus $(\boldsymbol{U}, d\gamma_h)$ is naturally a probability space. We have a natural map

$$\xi : M \times \boldsymbol{U} \to \boldsymbol{R}, \;\; M \times \boldsymbol{U} \ni (\boldsymbol{p}, \boldsymbol{u}) \mapsto \xi_{\boldsymbol{p}}(\boldsymbol{u}) := \boldsymbol{u}(\boldsymbol{p}).$$

The collection of random variables $(\xi_{\boldsymbol{p}})_{\boldsymbol{p} \in M}$ is a Gaussian random field on M.

Using the orthonormal basis $(\boldsymbol{\psi}_k)$ of $\boldsymbol{U}$ we obtain a linear isometry

$$\mathbb{R}^N \ni \boldsymbol{t} = (t_1, \dots, t_n) \mapsto \boldsymbol{u}_{\boldsymbol{t}} = \sum_k t_k \boldsymbol{\psi}_k \in \boldsymbol{U},$$

with inverse $\boldsymbol{u} \mapsto t_k(\boldsymbol{u}) = h(\boldsymbol{u}, \boldsymbol{\psi}_k)$. For any $\boldsymbol{p} \in M$ and any $\boldsymbol{t} \in \mathbb{R}^N$ we have

$$\xi_{\boldsymbol{p}}(\boldsymbol{u}_{\boldsymbol{t}}) = \sum_k t_k \boldsymbol{\psi}_k(\boldsymbol{p}).$$

The covariance kernel of this field is the function $\mathcal{E} = \mathcal{E}_{\boldsymbol{U}} : M \times M \to \mathbb{R}$ given by

$$\begin{aligned} \mathcal{E}(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{E}(\xi_{\boldsymbol{p}}, \xi_{\boldsymbol{q}}) &= \sum_{j,k=1}^{N} \left(\int_{\mathbb{R}^N} t_j t_k \, d\gamma_N(\boldsymbol{t}) \right) \boldsymbol{\psi}_j(\boldsymbol{p}) \boldsymbol{\psi}_k(\boldsymbol{q}) \\ &= \sum_{k=1}^{M} \boldsymbol{\psi}_k(\boldsymbol{p}) \boldsymbol{\psi}_k(\boldsymbol{q}), \end{aligned} \tag{2.16}$$

where $d\gamma_N$ is the canonical Gaussian measure on $\mathbb{R}^N$.

If $\boldsymbol{p} \in M$ and $\mathcal{U}$ is an open coordinate neighborhood of $\boldsymbol{p}$ with coordinates $x = (x^1, \dots, x^m)$ such that $x(\boldsymbol{p}) = 0$, then we can rewrite (2.14) in terms of the covariance kernel alone

$$\boldsymbol{\sigma}_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}) = \frac{\partial^2 \mathcal{E}(x, y)}{\partial x^i \partial y^j}|_{x=y=0}. \tag{2.17}$$

Note that any vector field X determines a new Gaussian random field on M, the derivative of $\boldsymbol{u}$ along X. We obtain the Gaussian random variables

$$\boldsymbol{u} \mapsto (X\boldsymbol{u})_{\boldsymbol{p}}, \;\; \boldsymbol{u} \mapsto (Y\boldsymbol{u})_{\boldsymbol{p}},$$

and we have

$$\boldsymbol{\sigma}_{\boldsymbol{p}}(X, Y) = \boldsymbol{E}\big((X\boldsymbol{u})_{\boldsymbol{p}}, (Y\boldsymbol{u})_{\boldsymbol{p}} \big). \tag{2.18}$$

The last equality justifies the attribute stochastic attached to the metric $\boldsymbol{\sigma}$.

We denote by ∇ the Levi-Civita connection of the metric g. The Hessian of a smooth function $f : M \to \mathbb{R}$ with respect to the metric g is the symmetric $(0, 2)$-tensor $\nabla^2 f$ on M defined by the equality

$$\nabla^2 f(X, Y) := XYf - (\nabla_X Y)f, \;\; \forall X, Y \in \operatorname{Vect}(M). \tag{2.19}$$

If $\boldsymbol{p}$ is a critical point of f then $\nabla^2_{\boldsymbol{p}} f$ is the usual Hessian of f at $\boldsymbol{p}$. More generally, if $(x^1, \dots, x^m)$ are g-*normal* coordinates at $\boldsymbol{p}$, then

$$\nabla^2_{\boldsymbol{p}} f(\partial_{x^i}, \partial_{x^j}) = \partial^2_{x^i x^j} f(\boldsymbol{p}), \;\; \forall 1 \leq i, j \leq m.$$

For any $\boldsymbol{p} \in \boldsymbol{M}$ and any $f \in C^\infty(M)$ we use the metric $g_{\boldsymbol{p}}$ to identify the bilinear form $\nabla^2_{\boldsymbol{p}} f$ on $T_{\boldsymbol{p}} M$ with an element of $\mathcal{S}(T_{\boldsymbol{p}} M)$, the vector space of symmetric endomorphisms of the Euclidean space $(T_{\boldsymbol{p}} M, g_{\boldsymbol{p}})$. For any $\boldsymbol{p} \in M$ we have two random Gaussian vectors

$$\boldsymbol{U} \ni \boldsymbol{u} \mapsto \nabla^2_{\boldsymbol{p}} \boldsymbol{u} \in \mathcal{S}(T_{\boldsymbol{p}} M), \;\; \boldsymbol{U} \ni \boldsymbol{u} \mapsto d\boldsymbol{u}(\boldsymbol{p}) \in T^*_{\boldsymbol{x}} M.$$

Note that the expectation of both random vectors are trivial while (2.17) shows that the covariance form of $d\boldsymbol{u}(\boldsymbol{p})$ is the metric $\boldsymbol{\sigma}_{\boldsymbol{p}}$.

To proceed further we need to make an additional assumption on the sample space $\boldsymbol{U}$. Namely, in the remainder of this section we will assume that it is 2-*ample*. In this case the map

$$\boldsymbol{U} \ni \boldsymbol{u} \mapsto \nabla^2_{\boldsymbol{p}} \boldsymbol{u} \in \mathcal{S}(T_{\boldsymbol{p}} M)$$

is surjective so the Gaussian random vector $\nabla^2_{\boldsymbol{p}} \boldsymbol{u}$ is nondegenerate. A simple application of the co-area formula shows that the integral $I_{\boldsymbol{p}}$ in (2.2) can be expressed as a conditional expectation

$$I_{\boldsymbol{p}} = \boldsymbol{E}\big(\,|\det \nabla^2_{\boldsymbol{p}} \boldsymbol{u}| \,\big|\, d\boldsymbol{u}(\boldsymbol{p}) = 0 \,\big).$$

Observing that

$$J_g(\boldsymbol{p}) = (\det \boldsymbol{S}_{d\boldsymbol{u}(p)})^{\frac{1}{2}}, \tag{2.20}$$

we deduce that

$$\mathcal{N}(\boldsymbol{U}, h) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M (\det \boldsymbol{S}_{d\boldsymbol{u}(p)})^{-\frac{1}{2}} \boldsymbol{E}\big(\,|\det \nabla^2_{\boldsymbol{p}} \boldsymbol{u}| \,\big|\, d\boldsymbol{u}(\boldsymbol{p}) = 0 \,\big)\, |dV_g(\boldsymbol{p})|. \tag{2.21}$$

The last equality is very similar to the main conclusion of the Expectation Metatheorem, [1, Thm. 11.2.1] or the expectation formula in [4, Thm. 6.2]. We can simplify the equality (2.21) even more by taking full advantage of the Gaussian nature of the various random variables involved in this equality.

The covariance form of the pair of random variables $\nabla^2_{\boldsymbol{p}} \boldsymbol{u}$ and $d\boldsymbol{u}(p)$ is the bilinear map

$$\Omega : \mathcal{S}(T_{\boldsymbol{p}} M)^\vee \times T_{\boldsymbol{p}} M \to \mathbb{R},$$

$$\Omega(\xi, \eta) = \boldsymbol{E}\big(\langle \xi, \nabla^2_{\boldsymbol{p}} \boldsymbol{u} \rangle \cdot \langle d\boldsymbol{u}, \eta \rangle \,\big), \;\; \forall \xi \in \mathcal{S}^\vee_m, \;\; \eta \in T_{\boldsymbol{p}} M.$$

Using the natural inner products on $\mathcal{S}(T_{\boldsymbol{p}} M)$ and $T_{\boldsymbol{p}} M$ defined by $g_{\boldsymbol{p}}$ we can regard the covariance form as a linear operator

$$\boldsymbol{\Omega}_{\boldsymbol{p}} : T_{\boldsymbol{p}} M \to \mathcal{S}(T_{\boldsymbol{p}} M).$$

Similarly, we can identify the covariance forms of $\nabla^2_{\boldsymbol{p}} u$ and du with symmetric positive definite operators

$$\boldsymbol{S}_{\nabla^2_{\boldsymbol{p}} \boldsymbol{u}} : \mathcal{S}(T_{\boldsymbol{p}} M) \to \mathcal{S}(T_{\boldsymbol{p}} M)$$

and respectively

$$\boldsymbol{S}_{d\boldsymbol{u}(\boldsymbol{p})} : T_{\boldsymbol{p}} M \to T_{\boldsymbol{p}} M.$$

Using the regression formula (A.4) we deduce that

$$\boldsymbol{E}\big(\,|\det \nabla^2_{\boldsymbol{p}} \boldsymbol{u}| \,\big|\, d\boldsymbol{u}(\boldsymbol{p}) = 0 \,\big) = \boldsymbol{E}(|\det Y_{\boldsymbol{p}}|), \tag{2.22}$$

where $Y_{\boldsymbol{p}} : \boldsymbol{U} \to \mathcal{S}(T_{\boldsymbol{p}} M)$ is a Gaussian random vector with mean value zero and covariance operator

$$\boldsymbol{\Xi}_{\boldsymbol{p}} = \boldsymbol{\Xi}_{Y_p} := \boldsymbol{S}_{\nabla^2_p \boldsymbol{u}} - \Omega \boldsymbol{S}^{-1}_{d\boldsymbol{u}(\boldsymbol{p})} \Omega^\dagger : \mathcal{S}(T_{\boldsymbol{p}} M) \to \mathcal{S}(T_{\boldsymbol{p}} M). \tag{2.23}$$

Since $\boldsymbol{U}$ is 2-ample the operator Ξ_p is invertible and we have

$$\boldsymbol{E}(|\det Y_{\boldsymbol{p}}|) = (2\pi)^{-\frac{\dim \mathcal{S}(T_{\boldsymbol{p}})}{2}} (\det \Xi_{\boldsymbol{p}})^{-\frac{1}{2}} \int_{\mathcal{S}(T_{\boldsymbol{p}}M)} |\det Y|\, e^{-\frac{(\Xi_{\boldsymbol{p}}^{-1}Y,Y)}{2}} dV_g(Y). \tag{2.24}$$

We deduce that *when $\boldsymbol{U}$ is 2-ample* we have

$$\mathcal{N}(\boldsymbol{U},h) = \frac{1}{(2\pi)^{\frac{m}{2}}} \int_M (\det \boldsymbol{S}_{d\boldsymbol{u}(p)})^{-\frac{1}{2}} \boldsymbol{E}(|\det Y_{\boldsymbol{p}}|)\, |dV_g(\boldsymbol{p})|, \tag{2.25}$$

where $Y_{\boldsymbol{p}}$ is a Gaussian random symmetric endomorphism of $T_{\boldsymbol{p}}M$ with expectation 0 and covariance operator $\Xi_{\boldsymbol{p}}$ described by (2.23).

To compute the above integral we choose normal coordinates $(x^1,\dots,x^n)$ near $\boldsymbol{p}$ and thus we can orthogonally identify $T_{\boldsymbol{p}}M$ with $\mathbb{R}^m$. We can view the random variable $\nabla^2_{\boldsymbol{p}}\boldsymbol{u}$ as a random variable

$$H^{\boldsymbol{p}} : \boldsymbol{U} \to \mathcal{S}_m := \mathcal{S}(\mathbb{R}^m),\;\; \boldsymbol{U} \ni \boldsymbol{u} \mapsto H^{\boldsymbol{p}}(\boldsymbol{u}) \in \mathcal{S}_m,\;\; H^{\boldsymbol{p}}_{ij}(\boldsymbol{u}) = \partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{p}),$$

and the random variable $d\boldsymbol{u}(\boldsymbol{p})$ as a random variable

$$D^{\boldsymbol{p}} : \boldsymbol{U} \to \mathbb{R}^m,\;\; \boldsymbol{u} \mapsto D^{\boldsymbol{p}}\boldsymbol{u} \in \mathbb{R}^m,\;\; D^{\boldsymbol{p}}_i\boldsymbol{u} = \partial_{x^i}\boldsymbol{u}(\boldsymbol{p}).$$

The covariance operator $\boldsymbol{S}_{d\boldsymbol{u}(p)}$ of the random variable $D^{\boldsymbol{p}}$ is given by the symmetric $m\times m$ matrix with entries

$$\boldsymbol{\sigma}_{\boldsymbol{p}}(\partial_{x^i},\partial_{x^j}) = \frac{\partial^2\mathcal{E}(x,y)}{\partial x^i\partial y^j}|_{x=y=0}. \tag{2.26}$$

To compute the covariance form $\boldsymbol{\Sigma}_{H^{\boldsymbol{p}}}$ of the random matrix $H^{\boldsymbol{p}}$ we observe first that we have a canonical basis $(\xi_{ij})_{1\leq i\leq j\leq m}$ of $\mathcal{S}_m^\vee$ so that ξ_{ij} associates to a symmetric matrix A the entry a_{ij} located in the position (i,j). Then

$$\begin{aligned}\boldsymbol{\Sigma}_{H^{\boldsymbol{p}}}(\xi_{ij},\xi_{k\ell}) &= \boldsymbol{E}\big(\, H^{\boldsymbol{p}}_{ij}(\boldsymbol{u}), H^{\boldsymbol{p}}_{k\ell}(\boldsymbol{u})\,\big) = \boldsymbol{E}\big(\,\partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{x})\partial^2_{x^kx^\ell}u(\boldsymbol{x})\,\big)\\ &= \sum_{n=1}^N \partial^2_{x^ix^j}\boldsymbol{\psi}_n(\boldsymbol{x})\partial^2_{x^kx^\ell}\boldsymbol{\psi}_n(\boldsymbol{x}) = \frac{\partial^4\mathcal{E}(x,y)}{\partial x^i\partial x^j\partial y^k\partial y^\ell}|_{x=y=0}.\end{aligned} \tag{2.27}$$

Similarly we have

$$\Omega(\xi_{ij},\partial_{x^k}) = \boldsymbol{E}\big(\,\partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{p}),\partial_{x^k}\boldsymbol{u}(\boldsymbol{p})\,\big) = \frac{\partial^3\mathcal{E}(x,y)}{\partial x^i\partial x^j\partial y^k}|_{x=y=0}. \tag{2.28}$$

To identify Ω with an operator it suffices to observe that (∂_{x^k}) is an orthonormal basis of $T_{\boldsymbol{p}}M$, while the collection $\{\,\hat{\xi}_{ij}\,\}_{i\leq j} \subset \mathcal{S}_m^\vee$,

$$\hat{\xi}_{ij} = \begin{cases}\xi_{ij}, & i=j\\ \sqrt{2}\xi_{ij}, & i<j\end{cases}$$

is an orthonormal basis of $\mathcal{S}_m^\vee$. If we denote by $\widehat{E}_{ij}$ the dual orthonormal basis of $\mathcal{S}_m$, then

$$\Omega\partial_{x^k} = \sum_{i\leq j}\Omega(\hat{\xi}_{ij},\partial_{x^k})\widehat{E}_{ij}.$$

Remark 2.7. If the metric g coincides with the stochastic metric $\boldsymbol{\sigma}$, then the covariance operator Ω is trivial. For a proof of this and of many other nice properties of the metric $\boldsymbol{\sigma}$ we refer to [1, §12.2]. □

2.4. **Zonal domains of spherical harmonics of large degree.** In the conclusion of this section we want to discuss an immediate application of the above results to critical sets of random spherical harmonics.

Let (M,g) be the unit round sphere S^2. The spectrum of the Laplacian on S^2 is

$$\lambda_n = n(n+1),\;\; n=0,1,2,\dots,\;\; \dim\ker(\lambda_n-\Delta) = 2n+1 = d_n.$$

The space $\boldsymbol{U}_n = \ker(\lambda_n - \Delta)$ has a well known descrition: it consists of sperical harmonics, i.e., restrictions to S^2 of harmonic polynomials of degree n in three variables. We want to describe the behavior of $\mathcal{N}(\boldsymbol{U}_n)$ as $n\to\infty$, where $\boldsymbol{U}_n$ is equipped with the L^2-metric. In other words we want to find the expected number of critical points of a spherical harmonic of very large degree.

In this case the covariance kernel $\mathcal{E}_n(\boldsymbol{p},\boldsymbol{q})$ of $\boldsymbol{U}_n$ has a very simple description. More precisely, if $(\Psi_k)_{1\leq k\leq 2n+1}$ is an orthonormal basis of $\boldsymbol{U}_n$, then the classical addition theorem, [29, §1.2] shows that

$$\mathcal{E}_n(\boldsymbol{p},\boldsymbol{q}) = \sum_k \Psi_k(\boldsymbol{p})\Psi_k(\boldsymbol{q}) = \frac{2n+1}{4\pi}P_n(\boldsymbol{p}\bullet\boldsymbol{q}),\;\; \forall \boldsymbol{p},\boldsymbol{q}\in S^2,$$

where $\bullet$ denotes the inner product in $\mathbb{R}^3$, and P_n denotes the n-th Legendre polynomial,

$$P_n(t) = (-1)^n\frac{1}{2^n n!}\frac{d^n}{dt^n}(1-t^2)^n.$$

In this case the stochastic metric $\boldsymbol{\sigma}=\boldsymbol{\sigma}_n$ is obviously $SO(3)$-invariant and it is a (constant) multiple of the round metric. In view of Remark 2.7 this implies that for any $\boldsymbol{p}\in S^2$ the random variables

$$\boldsymbol{U}_n \ni \boldsymbol{u} \mapsto \operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u},g)\;\text{ and }\; \boldsymbol{U}_n\ni\boldsymbol{u}\mapsto d\boldsymbol{u}(\boldsymbol{p})$$

are independent and we deduce that

$$\mathcal{N}(\boldsymbol{U}_n) = \frac{1}{2\pi}\int_{S^2}\frac{1}{J_g(p)}\left(\int_{\boldsymbol{U}_n}|\det\operatorname{Hess}_{\boldsymbol{p}}(\boldsymbol{u},g))|\underbrace{\frac{e^{-\frac{1}{2}|\boldsymbol{u}|^2}}{(2\pi)^{\frac{\dim\boldsymbol{U}_n}{2}}}|d\boldsymbol{u}|}_{=:d\boldsymbol{\gamma}_n(\boldsymbol{u})}\right)dV_g(\boldsymbol{p}).$$

Clearly, the integrand in the above formula is invariant with respect to the $SO(3)$-action on S^2 and we thus have

$$\mathcal{N}(\boldsymbol{U}_n) = \frac{2}{J_g(\boldsymbol{p}_0)}\int_{\boldsymbol{U}_n}\bigl|\det\operatorname{Hess}_{\boldsymbol{p}_0}(\boldsymbol{u},g)\bigr|\,d\gamma_n(\boldsymbol{u}), \tag{2.29}$$

where $\boldsymbol{p}_0$ a fixed (but arbitrary) point on S^2. To compute the term in the right-hand side of the above equality we use the equalities (2.26) and (2.27).

Fix normal coordinates (x^1,x^2) in a neighborhood $\mathcal{O}$ of $\boldsymbol{p}_0$ so we can view $\mathcal{E}_n$ as a function $\mathcal{E}_n(x,y)$. The location of a point $\boldsymbol{p}\in\mathcal{O}$ is described by a smooth function

$$\mathcal{O}\ni(x^1,x^2)\mapsto\boldsymbol{p}(x^1,x^2)\in\mathbb{R}^3.$$

The tangent vector ∂_{x^i}, viewed as a vector in $\mathbb{R}^3$, corresponds with the derivative $\boldsymbol{p}_{x^i} := \partial_{x^i}\boldsymbol{p}$ of the above function. At $\boldsymbol{p}_0$ we have

$$\boldsymbol{p}_{x^i}\bullet\boldsymbol{p}_{x^j} = \delta_{ij}\;\text{ and }\;\boldsymbol{p}_{x^i}\bullet\boldsymbol{p}_0 = 0,\;\;\forall i,j. \tag{2.30}$$

The arcs $C_1 = \{x^2=0\}$ and $C_2=\{x^1=0\}$ are portions of great circles intersecting orthogonally at $\boldsymbol{p}_0$. Note that x^1 is the arclength parameter along C_i, $i=1,2$. The vectors $\boldsymbol{p}_{x^i}$ are unit tangent vectors along these arcs. This shows that at $\boldsymbol{p}_0$ we have

$$\boldsymbol{p}_{x^ix^i} = -\boldsymbol{p}_0.$$

Since the arcs C_1 and C_2 are planar their torsion is trivial and the Frenet formulæ imply that at $\boldsymbol{p}_0$ we have

$$\boldsymbol{p}_{x^i x^j} = 0,\ \ \forall i \neq j.$$

The last two equalities can be rewritten in compact form as

$$\boldsymbol{p}_{x^i x^j} = -\delta_{ij}\boldsymbol{p}_0,\ \ \forall i,j \tag{2.31}$$

We set

$$\begin{aligned} s_n &:= \frac{2n+1}{\pi}P_n'(1) = \frac{2n+1}{4\pi} \times \frac{n(n+1)}{2} \sim \frac{1}{4\pi}n^3 \\ t_n &:= \frac{2n+1}{\pi}P_n''(1) = \frac{(2n+1)}{4\pi} \times \frac{(n+2)(n+1)n(n-1)}{16} \sim \frac{1}{32\pi}n^5. \end{aligned} \tag{2.32}$$

We deduce

$$\begin{aligned} \boldsymbol{\sigma}(\partial_{x^j}, \partial_{x^k}) &= \partial_{x_j}\partial_{y^k}\mathcal{E}(\boldsymbol{p},\boldsymbol{q})|_{\boldsymbol{p}=\boldsymbol{q}=\boldsymbol{p}_0} \\ &= \frac{2n+1}{4\pi}\Big(P_n'(\boldsymbol{p}\bullet\boldsymbol{q})\boldsymbol{p}_{x^j}\bullet\boldsymbol{q}_{y_k} + P_n^{(2)}(\boldsymbol{p}\bullet\boldsymbol{q})(\boldsymbol{p}_{x^j}\bullet\boldsymbol{q})(\boldsymbol{p}\bullet\boldsymbol{q}_{y^k})\Big)_{\boldsymbol{p}=\boldsymbol{q}=\boldsymbol{p}_0} \\ &= s_n\delta_{jk}, \end{aligned} \tag{2.33}$$

and

$$J_g(\boldsymbol{p}_0) = s_n. \tag{2.34}$$

To compute $\partial^4_{x^i x^j y^k y^\ell}\mathcal{E}_n(\boldsymbol{p},\boldsymbol{q})$ at $\boldsymbol{p}=\boldsymbol{q}=\boldsymbol{p}_0$ we will use (2.30) and (2.31) to cut down the complexity of the final formula. We deduce that at $\boldsymbol{p}=\boldsymbol{q}=\boldsymbol{p}_0$ we have

$$\begin{aligned} \partial^4_{x^i x^j y^k y^\ell}\mathcal{E}_n(\boldsymbol{p},\boldsymbol{q}) &= \frac{2n+1}{4\pi}\Big(P_n'(\boldsymbol{p}\bullet\boldsymbol{q})\boldsymbol{p}_{x^i x^j}\bullet\boldsymbol{q}_{y^\ell y^k} + P_n^{(2)}(\boldsymbol{p}\bullet\boldsymbol{q})(\boldsymbol{p}_{x^i x^j}\bullet\boldsymbol{q})(\boldsymbol{p}\bullet\boldsymbol{q}_{y^k y^\ell})\Big)_{\boldsymbol{p}=\boldsymbol{q}} \\ &+\frac{2n+1}{4\pi}\Big(P_n^{(2)}(\boldsymbol{p}\bullet\boldsymbol{q})(\boldsymbol{p}_{x^i}\bullet\boldsymbol{q}_{y^\ell})(\boldsymbol{p}_{x^j}\bullet\boldsymbol{q}_{y^k}) + P_n^{(2)}(\boldsymbol{p}\bullet\boldsymbol{q})(\boldsymbol{p}_{x^j}\bullet\boldsymbol{q}_{y^\ell})(\boldsymbol{p}_{x^i}\bullet\boldsymbol{q}_{y^k})\Big)_{\boldsymbol{p}=\boldsymbol{q}}, \end{aligned}$$

and thus

$$\partial^4_{x^i x^j y^k y^\ell}\mathcal{E}_n(\boldsymbol{p},\boldsymbol{q})_{\boldsymbol{p}=\boldsymbol{q}} = \big(s_n+t_n\big)\delta_{ij}\delta_{k\ell} + t_n\big(\,\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}\,\big). \tag{2.35}$$

Denote by $d\boldsymbol{\Gamma}_n$ the pushforward of the Gaussian measure $d\gamma_n$ via the Hessian map

$$\boldsymbol{U}_n \ni \boldsymbol{u} \mapsto \operatorname{Hess}_{\boldsymbol{p}_0}(\boldsymbol{u},g) \in \mathcal{S}(T_{\boldsymbol{p}_0}S^2) = \mathcal{S}_2.$$

We deduce from (2.35) that the covariance form $\boldsymbol{\Sigma}_n$ of $d\boldsymbol{\Gamma}_n$ satisfies the equality

$$\boldsymbol{\Sigma}_n = \boldsymbol{\Sigma}_{a_n,b_n,c_n},\ \ a_n = s_n + 3t_n,\ \ b_n = s_n+t_n,\ \ c_n = t_n,$$

where $\boldsymbol{\Sigma}_{a,b,c}$ is defined by the conditions (B.2a) and (B.2b). Observe that a_n, b_n, c_n satisfy (B.4),

$$a_n = b_n + 2c_n.$$

As explained in Appendix B, this implies that $d\boldsymbol{\Gamma}_n$ is $O(2)$-invariant. Set

$$a_n^* = \frac{a_n}{t_n},\ \ b_n^* = \frac{b_n}{t_n},\ \ c_n^* = \frac{c_n}{t_n},$$

and denote by $d\boldsymbol{\Gamma}_n^*$ the Gaussian measure on $\mathcal{S}_2$ with covariance matrix $\boldsymbol{\Sigma}_{a_n^*,b_n^*,c_n^*}$. Using (A.7) we deduce that

$$\int_{\mathcal{S}_2} |\det X|\, d\boldsymbol{\Gamma}_n(X) = t_n\int_{\mathcal{S}_2}|\det X|\, d\boldsymbol{\Gamma}_n^*(X).$$

From (2.29) and (2.34) we now deduce

$$\mathcal{N}(\boldsymbol{U}_n) = \frac{2t_n}{s_n}\int_{\mathcal{S}_2}|\det X|\, d\boldsymbol{\Gamma}_n^*(X). \tag{2.36}$$

Observe that as $n \to \infty$ we have

$$\frac{2t_n}{s_n} \sim \frac{n^2}{4}, \;\; a_n^* \sim 3, \;\; b_n^* \sim 1 \;\; c_n^* \sim 1,$$

so that

$$\mathcal{N}(\boldsymbol{U}_n) \sim \frac{n^2}{4} \int_{\mathcal{S}_2} |\det X|\, d\boldsymbol{\Gamma}_{3,1,1}(X), \tag{2.37}$$

where $d\boldsymbol{\Gamma}_{3,1,1}(X)$ is the Gaussian measure on $\mathcal{S}_2$ with covariance form $\boldsymbol{\Sigma}_{3,1,1}$. More precisely (see (B.11))

$$d\boldsymbol{\Gamma}_{3,1,1}(X) = \frac{1}{4(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{4}\left(\operatorname{tr} X^2 - \frac{1}{4}(\operatorname{tr} X)^2\right)} \cdot \sqrt{2} \prod_{1 \le i \le j \le 2} dx_{ij}.$$

In Appendix C we show that

$$\int_{\mathcal{S}_2} |\det X|\, d\boldsymbol{\Gamma}_{3,1,1}(X) = \frac{4}{\sqrt{3}}, \tag{2.38}$$

and we deduce from (2.29) that

$$\mathcal{N}(\boldsymbol{U}_n) \sim \frac{n^2}{\sqrt{3}} \;\text{ as }\; n \to \infty. \tag{2.39}$$

Let us observe that for n very large, a typical spherical harmonic $\boldsymbol{u} \in \boldsymbol{U}_n$ is a Morse function on S^2 and 0 is a regular value. The nodal set $\{\boldsymbol{u} = 0\}$ is disjoint union of smoothly embedded circles. We denote by $\mathcal{D}_{\boldsymbol{u}}$ the set of connected components of the complement of the nodal set are called the *nodal domains* of $\boldsymbol{u}$ and we denote $\delta(\boldsymbol{u})$ the cardinality of $\mathcal{D}_{\boldsymbol{u}}$. A result of Pleijel and Peetre, [6, 35, 38], shows that

$$\delta(\boldsymbol{u}) \le \frac{4}{j_0^2} n^2 \approx 0.692 n^2, \tag{2.40}$$

where j_0 denotes the first positive zero of the Bessel function J_0.

We think of $\delta(\boldsymbol{u})$ as a random variable and we denote by δ_n its expectation,

$$\delta_n = \frac{1}{(2\pi)^{\dim \boldsymbol{U}_n} 2} \int_{\boldsymbol{U}_n} \delta(\boldsymbol{u}) e^{-\frac{1}{2}|\boldsymbol{u}|^2} |d\boldsymbol{u}|.$$

Recently, Nazarov and Sodin [30], have proved that there exists a positive constant $a > 0$ such that

$$\delta_n \sim a n^2 \;\text{ as }\; n \to \infty.$$

Additionally, for large n, with high probability, $\delta(\boldsymbol{u})$ is close to an^2 (see [30] for a precise statement). The equality (2.40) implies that

$$a \le \frac{4}{j_0} \approx 0.692.$$

We can improve this a little bit.

Denote by $p(\boldsymbol{u})$ the number of local minima and maxima of $\boldsymbol{u}$, and by $s(\boldsymbol{u})$ the number of saddle points. Then

$$\mathcal{N}(\boldsymbol{y}) = p(\boldsymbol{u}) + s(\boldsymbol{u}), \;\; p(\boldsymbol{u}) - s(\boldsymbol{u}) = \chi(S^2) = 2.$$

This proves that

$$p(\boldsymbol{y}) = \frac{1}{2}\big(\,\mathcal{N}(\boldsymbol{u}) + 2\,\big).$$

For every nodal region D, we denote by $p(\boldsymbol{u}, D)$ the number of local minima and maxima[4] of $\boldsymbol{u}$ on D. Note that $p(\boldsymbol{u}, D) > 0$ for any D and thus the number $p(\boldsymbol{u}) = \sum_{D\in\mathcal{D}_{\boldsymbol{u}}} p(\boldsymbol{u}, D)$ can be viewed as a weighted count of nodal domains. We set

$$p(\boldsymbol{U}_n) := \frac{1}{(2\pi)^{\frac{\dim \boldsymbol{U}_n}{2}}} \int_{\boldsymbol{U}_n} e^{-\frac{1}{2}|\boldsymbol{u}|^2} p(\boldsymbol{u})\, |d\boldsymbol{u}|.$$

The equality (2.39) implies that

$$p(\boldsymbol{U}_n) \sim \frac{1}{2\sqrt{3}} n^2 \;\; \text{as} \;\; n \to \infty.$$

This shows that

$$a \leq \frac{1}{2\sqrt{3}} \approx 0.288. \tag{2.41}$$

Remark 2.8. Using (2.13) we deduce in a similar fashion a stochastic Gauss-Bonnet formula

$$2 = \chi(S^2) = \frac{2}{s_n} \int_{\boldsymbol{U}_n} \det \mathrm{Hess}_{\boldsymbol{p}_0}(\boldsymbol{u}, g)\big|\, d\gamma_n(\boldsymbol{u}).$$

In Appendix C we give a direct proof of this equality to test the correctness of the various normalization constants in the above computations. □

3. The proof of Theorem 1.1

3.1. **Asymptotic estimates of the spectral function.** We fix an orthonormal basis of $L^2(M, g)$ consisting of eigenfunctions Ψ_n of Δ_g,

$$\Delta_g \Psi_n = \lambda_n \Psi_n, \;\; n = 0, 1, \dots, \;\; \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq \cdots .$$

The collection $(\Psi_n)_{\lambda_n \leq L}$ is therefore an orthonormal basis of $\boldsymbol{U}_L$ so that the covariance kernel of the Gaussian field determined by $\boldsymbol{U}_L$ is

$$\mathcal{E}_L(\boldsymbol{p}, \boldsymbol{q}) = \sum_{\lambda_n \leq L} \Psi_n(\boldsymbol{p}) \Psi_n(\boldsymbol{q}).$$

This function is also known as the *spectral function* associated to the Laplacian. Equivalently, $\mathcal{E}_L$ can be identified with the Schwartz kernel of the orthogonal projection onto $\boldsymbol{U}_L$. Observe that

$$\int_M \mathcal{E}_L(\boldsymbol{p}, \boldsymbol{p})\, |dV_g(\boldsymbol{p})| = \dim \boldsymbol{U}_L.$$

In the groundbreaking work [22], L. Hörmander used the kernel of the wave group $e^{\boldsymbol{i}t\sqrt{\Delta}}$ to produce refined asymptotic estimates for the spectral function. More precisely he showed (see [22] or [24, §17.5])

$$\mathcal{E}_L(p, p) = \frac{\boldsymbol{\omega}_m}{(2\pi)^m} L^{\frac{m}{2}} + O\big(L^{\frac{m-1}{2}}\big) \;\; \text{as} \;\; L \to \infty, \tag{3.1}$$

uniformly with respect to $\boldsymbol{p} \in M$. Above, $\boldsymbol{\omega}_m$ denotes the volume of the unit ball in $\mathbb{R}^m$. This implies immediately the classical Weyl estimates

$$\dim \boldsymbol{U}_L \sim \frac{\boldsymbol{\omega}_m}{(2\pi)^m} \mathrm{vol}_g(M) L^{\frac{m}{2}}. \tag{3.2}$$

[4]A simple application of the maximum principle shows that on each nodal domain, all the local extrema of $\boldsymbol{y}$ are of the same type: either all local minima or all local maxima. Thus $p(\boldsymbol{u}, D)$ can be visualized as the number of $\boldsymbol{p}$eaks of $|\boldsymbol{u}|$ on D.

Hörmander's approach can be refined to produce asymptotic estimates for the behavior of the derivatives the spectral function in a neighborhood of the diagonal. We describe below these estimates following closely the presentation in [7]. For more general results we refer to [40, Thm. 1.8.5, 1.8.7].

We set $\lambda := L^{\frac{1}{2}}$. Fix a point $\boldsymbol{p}$ and normal coordinates $x = (x^1, \dots, x^m)$ at $\boldsymbol{p}$. Note that $x(\boldsymbol{p}) = 0$. For any multi-indices $\alpha, \beta \in \mathbb{Z}^m_{\geq 0}$ we have (see [7, Thm. 1.1, Prop. 2.3])

$$\frac{\partial^{\alpha+\beta}\mathcal{E}_L(x,y)}{\partial x^\alpha \partial y^\beta}|_{x=y=0} = C_m(\alpha,\beta)\lambda^{m+|\alpha|+|\beta|} + O\big(\lambda^{m+|\alpha|+|\beta|-1}\big), \tag{3.3}$$

where

$$C_m(\alpha,\beta) = \begin{cases} 0, & \alpha-\beta \notin (2\mathbb{Z})^m \\ & \\ \frac{(-1)^{\frac{|\alpha|-|\beta|}{2}}}{(2\pi)^m}\int_{\boldsymbol{B}^m} \boldsymbol{x}^{\alpha+\beta}|d\boldsymbol{x}|, & \alpha-\beta \in (2\mathbb{Z})^m, \end{cases} \tag{3.4}$$

and $\boldsymbol{B}^m$ denotes the unit ball

$$\boldsymbol{B}^m = \big\{ \boldsymbol{x} \in \mathbb{R}^m;\ \ |\boldsymbol{x}| = 1 \,\big\}.$$

The estimates (3.3) are uniform in $\boldsymbol{p} \in M$. Using (A.6) we deduce (compare with (B.13))

$$\frac{1}{(2\pi)^m}\int_{\boldsymbol{B}^m} \boldsymbol{x}^{\alpha+\beta}|d\boldsymbol{x}| = \frac{1}{(4\pi)^{\frac{m}{2}}\Gamma\big(1+\frac{|\alpha|+|\beta|+m}{2}\big)}\int_{\mathbb{R}^m}\boldsymbol{x}^{\alpha+\beta}\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}|.$$

We set

$$K_m = C_m(\alpha,\alpha),\ \ |\alpha| = 1,$$

so that

$$K_m = \frac{1}{(4\pi)^{\frac{m}{2}}\Gamma\big(2+\frac{m}{2}\big)}\int_{\mathbb{R}^m} x_1^2\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| = \frac{1}{2(4\pi)^{\frac{m}{2}}\Gamma\big(2+\frac{m}{2}\big)}. \tag{3.5}$$

For any $i \leq j$ define $\alpha_{ij} \in \mathbb{Z}^m$ so that

$$\boldsymbol{x}^{\alpha_{ij}} = x_i x_j.$$

For $i \leq j$ and $k \leq \ell$ we set

$$C_m(i,j;k,\ell) = C_m(\alpha_{ij},\alpha_{k\ell}) = \frac{1}{(4\pi)^{\frac{m}{2}}\Gamma\big(3+\frac{m}{2}\big)}\int_{\mathbb{R}^m} x_i x_j x_k x_\ell \frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}|. \tag{3.6}$$

For $i < j$ we have

$$C_m(i,i;j,j) = \frac{1}{(4\pi)^{\frac{m}{2}}\Gamma\big(3+\frac{m}{2}\big)}\int_{\mathbb{R}^m} x_i^2 x_j^2\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| = \frac{1}{4(4\pi)^{\frac{m}{2}}\Gamma\big(3+\frac{m}{2}\big)} =: c_m. \tag{3.7}$$

$$C_m(i,j;i,j) = C_m(i,i;j,j),$$

Finally

$$C_m(i,i;i,i) = \frac{1}{(4\pi)^{\frac{m}{2}}\Gamma\big(3+\frac{m}{2}\big)}\int_{\mathbb{R}^m} x_i^4\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| = \frac{3}{4(4\pi)^{\frac{m}{2}}\Gamma\big(3+\frac{m}{2}\big)} = 3c_m, \tag{3.8}$$

and

$$C_m(i,j;k,\ell) = 0,\ \ \forall k \leq \ell,\ \ (i,j) \neq (k,\ell).$$

3.2. **Probabilistic consequences of the previous estimates.** We denote by $\boldsymbol{\sigma}^L$ the stochastic metric on M determiner by the sample space $\boldsymbol{U}_L$, $L \gg 0$. As explained in Subsection 2.3 the covariance form of the random vector $\boldsymbol{U}_L \ni \boldsymbol{u} \mapsto d\boldsymbol{u}(\boldsymbol{p}) \in T^*_{\boldsymbol{p}}M$ is $\boldsymbol{\sigma}^L_{\boldsymbol{p}}$, and from (3.3) we deduce

$$\begin{aligned}\boldsymbol{\sigma}^L_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}) &= \frac{\partial^2 \mathcal{E}_L(x,y)}{\partial x^i \partial y^j}|_{x=y=0} = K_m \lambda^{m+2}\delta_{ij} + O(\lambda^{m+1}) \\ &= K_m\lambda^{m+2} g_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}) + O(\lambda^{m+1}) \;\text{ as }\; L\to\infty, \;\text{ uniformly in } \boldsymbol{p}.\end{aligned} \tag{3.9}$$

In particular, if $\boldsymbol{S}^L_{d\boldsymbol{u}(\boldsymbol{p})}$ denotes the covariance operator of the random vector $d\boldsymbol{u}(\boldsymbol{p})$, then we deduce from the above equality that

$$\boldsymbol{S}^L_{d\boldsymbol{u}(\boldsymbol{p})} = K_m \lambda^{m+2}\mathbb{1}_m + O(\lambda^{m+1}), \;\text{ uniformly in } \boldsymbol{p}, \tag{3.10}$$

and invoking (2.20) we deduce

$$J^L_g(\boldsymbol{p}) = (\det \boldsymbol{S}^L_{d\boldsymbol{u}(\boldsymbol{p})})^{\frac{1}{2}} = K_m^{\frac{m}{2}}\lambda^{\frac{m(m+2)}{2}} + O\big(\lambda^{\frac{m(m+2)}{2}-1}\big), \;\text{ uniformly in } \boldsymbol{p}. \tag{3.11}$$

Denote by $\boldsymbol{\Sigma}^L_{H\boldsymbol{p}}$ the covariance form of the random matrix

$$\boldsymbol{U}_L \ni \boldsymbol{u} \mapsto \nabla^2_{\boldsymbol{p}}\boldsymbol{u} \in \mathcal{S}(T_{\boldsymbol{p}}M) = \mathcal{S}_m.$$

Using (2.27) and (3.3) we deduce

$$\boldsymbol{\Sigma}^L_{H\boldsymbol{p}} = c_m\lambda^{m+4}\Sigma_{3,1,1} + O(\lambda^{m+3}), \;\text{ uniformly in } \boldsymbol{p}, \tag{3.12}$$

where the positive definite, symmetric bilinear form $\boldsymbol{\Sigma}_{3,1,1} : \mathcal{S}^\vee_m \times \mathcal{S}^\vee_m \to \mathbb{R}$ is described by the equalities (B.2a) and (B.2b). We denote by $\Gamma_{3,1,1}$ the centered Gaussian measure on $\mathcal{S}_m$ with covariance form $\Sigma_{3,1,1}$.

The equality (2.28) coupled with (3.3) imply that the covariance operator $\boldsymbol{\Omega}^L_{\boldsymbol{p}}$ satisfies

$$\boldsymbol{\Omega}^L_{\boldsymbol{p}} = O(\lambda^{m+2}), \;\text{ uniformly in } \boldsymbol{p}. \tag{3.13}$$

Using (3.10), (3.12) and (3.13) we deduce that the covariance operator $\boldsymbol{\Xi}^L_{\boldsymbol{p}}$ defined as in (2.23) satisfies the estimate

$$\boldsymbol{\Xi}^L_p = c_m\lambda^{m+4}\widehat{Q}_{3,1,1} + O(\lambda^{m+2}), \;\text{ as }\; L\to\infty, \;\text{ uniformly in } \boldsymbol{p}, \tag{3.14}$$

where $\widehat{Q}_{3,1,1}$ is the covariance operator associated to the covariance form $\Sigma_{3,1,1}$ and it is described explicitly in (B.3). If we denote by $d\Gamma_L$ the Gaussian measure on $\mathcal{S}_m$ with covariance operator $\boldsymbol{\Xi}^L_{\boldsymbol{p}}$, we deduce that

$$d\Gamma_L(Y) = \frac{1}{(2\pi)^{\frac{N_m}{2}}(\det \boldsymbol{\Xi}^L_{\boldsymbol{p}})^{\frac{1}{2}}} e^{-\frac{(\boldsymbol{\Xi}^L_{\boldsymbol{p}}Y,Y)}{2}} \cdot \underbrace{2^{\frac{1}{2}\binom{m}{2}}\prod_{i\leq j} dy_{ij}}_{|dY|},$$

where

$$N_m = \dim \mathcal{S}_m = \frac{m(m+1)}{2}.$$

Let us observe that $|dY|$ is the Euclidean volume element on $\mathcal{S}_m$ defined by the natural inner product on $\mathcal{S}_m$, $(X,Y) = \operatorname{tr}(XY)$. We set

$$c_L := c_m\lambda^{m+4}, \;\; Q^L_{\boldsymbol{p}} = \frac{1}{c_L}\boldsymbol{\Xi}^L_{\boldsymbol{p}}.$$

Using (A.7) we deduce that

$$\frac{1}{(2\pi)^{\frac{N_m}{2}}(\det \boldsymbol{\Xi}^L_{\boldsymbol{p}})^{\frac{1}{2}}}\int_{\mathcal{S}_m} |\det Y| e^{-\frac{(\boldsymbol{\Xi}^L_{\boldsymbol{p}}Y,Y)}{2}}|dY| = \frac{(c_L)^{\frac{m}{2}}}{(2\pi)^{\frac{N_m}{2}}(\det Q^L_{\boldsymbol{p}})^{\frac{1}{2}}}\int_{\mathcal{S}_m} |\det Y| e^{-\frac{(Q^L_{\boldsymbol{p}}Y,Y)}{2}}|dY|.$$

From the estimate (3.14) we deduce that

$$Q^L_{\boldsymbol{p}} \to \widehat{Q}_{3,1,1} \text{ as } L \to \infty, \text{ uniformly in } \boldsymbol{p}.$$

We conclude that

$$\boldsymbol{E}(|\det Y_{\boldsymbol{p}}|) = \int_{\mathcal{S}_m} |\det Y| d\Gamma_L(Y) \sim c_m^{\frac{m}{2}} \lambda^{\frac{m(m+4)}{2}} \int_{\mathcal{S}_m} |\det Y| d\Gamma_{3,1,1}(Y). \tag{3.15}$$

The measure $d\Gamma_{3,1,1}$ is described explicitly in (B.11), more precisely

$$d\Gamma_{3,1,1}(Y) = \frac{1}{(2\pi)^{\frac{N_m}{2}}\sqrt{\mu_m}} \cdot e^{-\frac{1}{4}\left(\operatorname{tr} Y^2 - \frac{1}{m+2}(\operatorname{tr} Y)^2\right)} |dY|,$$

where μ_m is given by (B.12). Using (2.25), (3.11) and (3.15) we deduce that

$$\begin{aligned}\boldsymbol{E}(\mathcal{N}_L) &\sim \left(\frac{c_m}{K_m}\right)^{\frac{m}{2}} \lambda^{\frac{m(m+4)}{2} - \frac{m(m+2)}{2}} \operatorname{vol}_g(M) \int_{\mathcal{S}_m} |\det Y| d\Gamma_{3,1,1}(Y) \\ &\overset{(3.2)}{\sim} \left(\frac{c_m}{K_m}\right)^{\frac{m}{2}} \frac{(2\pi)^m}{\boldsymbol{\omega}_m} \dim \boldsymbol{U}_L.\end{aligned}$$

Observe that

$$\frac{c_m}{K_m} = \frac{\Gamma(2+\frac{m}{2})}{2\Gamma(3+\frac{m}{2})} = \frac{1}{m+4}, \quad \boldsymbol{\omega}_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(1+\frac{m}{2})} \quad \frac{(2\pi)^m}{\boldsymbol{\omega}_m} = (4\pi)^{\frac{m}{2}} \Gamma\Big(1+\frac{m}{2}\Big).$$

This completes the proof of (1.1) and (1.7). □

3.3. **On the asymptotic behavior of the stochastic metric.** We denote by $g(L)$ the metric

$$g(L) := \lambda^{-(m+2)} \sigma^L = L^{-\frac{(m+2)}{2}} \boldsymbol{\sigma}^L,$$

where K_m is described by (3.5).The estimate (3.9) shows that

$$g(L) \xrightarrow{C^0} g \text{ as } L \to \infty,$$

where K_m is described by (3.5). The metrics $g(L)$ are closely related to the metrics constructed in [5, Thm. 5]. We want to discuss here possible ways to improve the topology of the convergence.

Observe that if $g(L)$ were to converge in the C^2-topology to K_m then the sectional curvatures of $g(L)$ would have to be uniformly bounded. Conversely, the results of S. Peters [36] show that the C^0 convergence coupled with an uniform bound on the sectional curvatures would yield a $C^{1,\alpha}$ convergence.

The results in [1, §12.2.1] describe a simple way of expressing the sectional curvatures of σ^L in terms of the spectral function $\mathcal{E}_L$. Here are the details.

Denote by ∇^L the Levi-Civita connection of the metric σ^L. Fix a point $\boldsymbol{p} \in M$ and g-normal coordinates $(x^1, \dots, x^m)$ at $\boldsymbol{p}$. We set

$$\mathcal{E}^L_{i_1,\dots,i_a;j_1,\dots,j_b} := \frac{\partial^{a+b}\mathcal{E}_L(x,y)}{\partial x^{i_1}\cdots\partial x^{i_a}\partial y^{j_1}\cdots\partial y^{j_b}}|_{x=y=0},$$

$$\boldsymbol{\sigma}(L)_{ij} := \boldsymbol{\sigma}^L_{\boldsymbol{p}}(\partial_{x^i}, \partial_{x^j}), \ \ 1 \le i,j \le m,$$

and we denote by $(\boldsymbol{\sigma}(L)^{ij})_{1\le i,j\le m}$ the inverse matrix of $(\boldsymbol{\sigma}(L)_{ij})_{1\le i,j\le m}$. From [1, Eq. (12.2.6)] we deduce

$$\Gamma(L)_{ijk} := \boldsymbol{\sigma}^L_{\boldsymbol{p}}(\nabla^L_{\partial_{x^i}}\partial_{x^j}, \partial_{x^k}) = \mathcal{E}^L_{ij;k}.$$

We set

$$\Gamma(L)^k_{ij} := \sum_\ell \boldsymbol{\sigma}(L)^{k\ell}\Gamma(L)_{ij\ell} = \sum_\ell \boldsymbol{\sigma}(L)^{k\ell}\mathcal{E}^L_{ij;\ell},$$

so that

$$\left(\nabla^L_{\partial_{x^i}}\partial_{x^j}\right)_{\boldsymbol{p}} = \sum_k \Gamma(L)^k_{ij}\partial_{x^k}.$$

For $\boldsymbol{u}\in \boldsymbol{U}_L$ we set

$$H^L_{ij}(\boldsymbol{u}) := \left(\partial_{\partial^i_x}\partial_{x^j}\boldsymbol{u} - (\nabla^L_{\partial_{x^i}}\partial_{x^j})\boldsymbol{u}\right)_{\boldsymbol{p}} = \partial_{\partial^i_x}\partial_{x^j}\boldsymbol{u}(\boldsymbol{p}) - \sum_k \Gamma(L)^k_{ij}\partial_{x_k}\boldsymbol{u}(\boldsymbol{p}). \tag{3.16}$$

We think of the matrix $H^L_{ij}(\boldsymbol{u})$ as an element $H^L(\boldsymbol{u})\in T^*_{\boldsymbol{p}}M\otimes T^*_{\boldsymbol{p}}M$,

$$H^L(\boldsymbol{u}) = \sum_{i,j} H^L_{ij}dx^i\otimes dx^j$$

and we set

$$H^L(\boldsymbol{u})\wedge H^L(\boldsymbol{u}) := \sum_{i,j,k\ell} H^L_{ij}(\boldsymbol{u})H^L_{k\ell}(\boldsymbol{u})dx^i\wedge dx^k\otimes dx^j\wedge dx^\ell$$
$$=: \sum_{i<k,j<\ell} Q^L_{ikj\ell}(\boldsymbol{u})dx^i\wedge dx^k\otimes dx^j\wedge dx^\ell.$$

Note that

$$Q^L_{ikj\ell}(\boldsymbol{u}) = 2\left(H^L_{ij}(\boldsymbol{u})H^L_{k\ell}(\boldsymbol{u}) - H^L_{kj}(\boldsymbol{u})H^L_{i\ell}(\boldsymbol{u})\right).$$

We denote by R^L the Riemann tensor of $\boldsymbol{\sigma}^L$ and we set

$$R^L_{ijk\ell} := \boldsymbol{\sigma}^L\left(R^L(\partial_{x^i},\partial_{x^j})\partial_{x^k},\partial_{x^\ell}\right)_{\boldsymbol{p}}.$$

The map $\boldsymbol{U}_L\ni\boldsymbol{u}\mapsto Q^L_{ikj\ell}(\boldsymbol{u})\in\mathbb{R}$ is a random variable and according to [1, Lemma 12.2.1] we have[5]

$$2R^L_{ikj\ell} = -\boldsymbol{E}\left(Q^L_{ikj\ell}\right). \tag{3.17}$$

In particular we deduce that

$$-R^L_{ijij} = \boldsymbol{E}\left(H^L_{ii}H^L_{jj} - (H^L_{ij})^2\right).$$

From (3.9) we deduce that

$$\boldsymbol{\sigma}(L)_{ij} = \mathcal{E}^L_{i;j} \sim K_m\lambda^{m+2}\delta_{ij} + O(\lambda^{m+1})\ \text{ as } L\to\infty.$$

Hence

$$\boldsymbol{\sigma}(L)^{ij} \sim \frac{1}{K_m\lambda^{m+2}}\Big(\delta^{ij} + O(\lambda^{-1})\Big).$$

From (3.3) we deduce that as $\lambda\to\infty$ we have

$$\Gamma(L)^k_{ij} \sim \sum_\ell \frac{1}{K_m\lambda^{m+2}}\Big(\delta^{k\ell} + O(\lambda^{-1})\Big)\mathcal{E}^L_{ij;\ell} \sim \frac{1}{K_m\lambda^{m+2}}\mathcal{E}^L_{ij;k} + O(\lambda^{-1}) = O(1), \tag{3.18a}$$

$$\boldsymbol{E}(\partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{p}),\partial_{x^k}\boldsymbol{u}(\boldsymbol{p})) = \mathcal{E}^L_{ij;k} = O(\lambda^{m+2}) \tag{3.18b}$$

Using the estimates (2.27), (3.16), (3.18a) and (3.18b) in (3.17) we deduce

$$\boldsymbol{E}\left(H^L_{ii}H^L_{jj} - (H^L_{ij})^2\right) = \left(\mathcal{E}^L_{ii;jj} - \mathcal{E}^L_{ij;ij}\right) + O(\lambda^{m+2}).$$

[5] Alternatively, in our case, the equalities (3.17) are simple consequences of Theorema Egregium, [31, §4.2.4, Eq. (4.2.12)].

We deduce that the sectional curvature of $\boldsymbol{\sigma}^L$ along the plane spanned by $\partial_{x^i}, \partial_{x^k}$ is

$$K_{ij}^L = -\frac{R_{ijij}}{\boldsymbol{\sigma}(L)_{ii}\boldsymbol{\sigma}(L)_{jj} - \boldsymbol{\sigma}(L)_{ij}^2} = \frac{1}{K_m^2\lambda^{2m+4}}\left(\mathcal{E}_{ii;jj}^L - \mathcal{E}_{ij;ij}^L\right) + O\left(\frac{\mathcal{E}_{ii;jj}^L - \mathcal{E}_{ij;ij}^L}{\lambda^{2m+5}}\right).$$

On the other hand

$$\boldsymbol{E}(\partial^2_{x^ix^j}\boldsymbol{u}(\boldsymbol{p}), \partial^2_{x^kx^\ell}\boldsymbol{u}(\boldsymbol{p})\,) = \mathcal{E}^L_{ij;k\ell} \sim C_m(i,j;k,\ell)\lambda^{m+4} + O(\lambda^{m+3}),\quad i\leq j,\quad k\leq \ell,$$

where $C_m(i,j;k,\ell)$ is defined by (3.6), and we deduce

$$\mathcal{E}_{ii;jj}^L - \mathcal{E}_{ij;ij}^L = \big(C_m(i,i;j,j) - C_m(i,j;i,j)\big)\lambda^{m+4} + O(\lambda^{m+3}) = O(\lambda^{m+3}). \tag{3.19}$$

Hence

$$K_{ij}^L = \frac{1}{K_m^2\lambda^{2m+4}}\left(\mathcal{E}_{ii;jj}^L - \mathcal{E}_{ij;ij}^L\right) + O(\lambda^{-m-2}).$$

The sectional curvature of $g(L) = \lambda^{-m-2}\boldsymbol{\sigma}^L$ along the plane spanned by $\partial_{x^i}, \partial_{x^j}$ is

$$\overline{K}_{ij}^L = \lambda^{m+2}K_{ij}^L = \frac{1}{K_m^2\lambda^{m+2}}\left(\mathcal{E}_{ii;jj}^L - \mathcal{E}_{ij;ij}^L\right) + O(1).$$

We deduce that the sectional curvatures of $g(L)$ are uniformly bounded if and only if

$$\mathcal{E}_{ii;jj}^L - \mathcal{E}_{ij;ij}^L = O(\lambda^{m+2}) \;\text{ uniformly over } M. \tag{3.20}$$

Note that the estimates (3.20) are stronger than the estimates (3.19) which are direct consequences of the Bin-Hörmander estimates (3.3).

Let us point our that (3.20) hold when (M,g) is a homogeneous space equipped with an invariant metric. Indeed, in this case the metric $g(L)$ has the same symmetries as g and thus there exists a constant $c_L > 0$ such that $g(L) = c_L g$. Then $c_L \to K_m$ as $L\to\infty$ so that $g(L)\to K_m g$ in the C^∞-topology and therefore $\overline{K}_{ij}^L = O(1)$.

4. THE PROOF OF THEOREM 1.2

4.1. **Reduction to the classical Gaussian orthogonal ensemble.** We begin by describing the large m behavior of the integral

$$I_m := \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}\sqrt{\mu_m}}\int_{\mathcal{S}_m} |\det X| e^{-\frac{1}{4}\left(\operatorname{tr} X^2 - \frac{1}{m+2}(\operatorname{tr} X)^2\right)}|dX|,$$

where we recall that

$$\mu_m = 2^{\binom{m}{2}+m-1}(m+2).$$

We will use a trick of Fyodorov [18]; see also [17, §1.5]. Recall first the classical equality

$$\int_{\mathbb{R}} e^{-(at^2+bt+c)}|dt| = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} e^{\frac{\Delta}{4a}},\quad \Delta = b^2-4ac,\quad a>0.$$

For any real numbers u,v,w, we have

$$\begin{aligned} ut^2 + v\operatorname{tr}(X+wt\mathbb{1}_m)^2 &= (u+mw^2)t^2 + 2vw(\operatorname{tr} X)t + v\operatorname{tr} X^2\\ &=: a(u,v,w)t^2 + b(u,v,w)t + c(u,v,w).\end{aligned}$$

We seek u,v,w such that

$$\frac{v^2w^2}{u+mw^2}(\operatorname{tr} X)^2 - \frac{v}{u+mw^2}\operatorname{tr} X^2 = \frac{b^2-4ac}{4a} = -\frac{1}{4}\left(\operatorname{tr} X^2 - \frac{1}{m+2}(\operatorname{tr} X)^2\right).$$

We have

$$\frac{v}{u+mw^2}=\frac{1}{4},\quad \frac{v^2w^2}{u+mw^2}=\frac{1}{4(m+2)},$$

and we deduce

$$vw^2=\frac{1}{(m+2)},\quad v=\frac{1}{4}(u+mw^2)\Longleftrightarrow u=4v-mw^2.$$

Hence

$$w^2=\frac{1}{v(m+2)},\quad u=4v-\frac{m}{v(m+2)}.$$

We choose $v=\frac{1}{2}$ so that

$$w^2=\frac{2}{(m+2)},\quad u=2-\frac{2m}{(m+2)}=\frac{4}{m+2},\quad a(u,v,w)=4v=2,$$

$$e^{-\frac{1}{4}\left(\operatorname{tr}X^2-\frac{1}{m+2}(\operatorname{tr}X)^2\right)}=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}}e^{-\frac{4t^2}{m+2}}e^{-\frac{1}{2}\operatorname{tr}(X+t\sqrt{\frac{2}{m+2}}\mathbb{1}_m)^2}dt$$

$(s=\sqrt{\frac{2}{m+2}}t)$

$$=\left(\frac{2(m+2)}{\pi}\right)^{\frac{1}{2}}\int_{\mathbb{R}}e^{-\frac{1}{2}\operatorname{tr}(X+\frac{s}{\sqrt{2}}\mathbb{1}_m)^2}e^{-s^2}ds=\left(\frac{m+2}{2}\right)^{\frac{1}{2}}\int_{\mathbb{R}}e^{-\frac{1}{2}\operatorname{tr}(X-s\mathbb{1}_m)^2}\cdot\underbrace{\frac{e^{-2s^2}}{\sqrt{\frac{\pi}{2}}}ds}_{d\boldsymbol{\gamma}(s)}.$$

Hence

$$\begin{aligned}I_m&=\underbrace{\frac{(m+2)^{\frac{1}{2}}}{2^{\frac{1}{2}}(2\pi)^{\frac{m(m+1)}{4}}\sqrt{\mu_m}}}_{=:A_m}\int_{\mathbb{R}}\left(\int_{\mathcal{S}_m}|\det X|e^{-\frac{1}{2}\operatorname{tr}(X-s\mathbb{1}_m)^2}|dX|\right)d\boldsymbol{\gamma}(s)\\&=A_m\int_{\mathbb{R}}\underbrace{\left(\int_{\mathcal{S}_m}|\det(x\mathbb{1}_m-Y)|e^{-\frac{1}{2}\operatorname{tr}Y^2}|dY|\right)}_{=:f_m(x)}d\boldsymbol{\gamma}(x).\end{aligned}\tag{4.1}$$

For any $O(n)$-invariant function : $\mathcal{S}_n\to\mathbb{R}$ we have a Weyl integration formula (see [3, 17, 27]),

$$\frac{1}{(2\pi)^{\frac{\dim\mathcal{S}_m}{2}}}\int_{\mathcal{S}_n}f(X)|dX|=\frac{1}{\boldsymbol{Z}_n}\int_{\mathbb{R}^n}f(\lambda)|\Delta_m(\lambda)|\,|d\lambda|,$$

where

$$\Delta_n(\lambda):=\prod_{1\le i<j\le n}(\lambda_j-\lambda_i),$$

and the constant $\boldsymbol{Z}_n$ is defined by the equality [3, Eq. (2.5.11)] ,

$$\boldsymbol{Z}_m:=\int_{\mathbb{R}^n}e^{-\frac{1}{2}|\lambda|^2}|\Delta_m(\lambda)|\,|d\lambda|=2^{\frac{n}{2}}n!\prod_{j=1}^n\Gamma\left(\frac{j}{2}\right).\tag{4.2}$$

Now observe that for any $\lambda_0\in\mathbb{R}$ we have (with f_m defined in (4.1))

$$f_m(\lambda_0)=\frac{(2\pi)^{\frac{\dim\mathcal{S}_m}{2}}}{\boldsymbol{Z}_m}\int_{\mathbb{R}^m}e^{-\frac{|\lambda|^2}{2}}\left(\prod_{j=1}^n|\lambda_j-\lambda_0|\right)\big|\,\Delta_m(\lambda)\,\big|\,|d\lambda|$$

$$= \frac{e^{\frac{1}{2}\lambda_0^2}(2\pi)^{\frac{\dim \mathcal{S}_m}{2}}}{\boldsymbol{Z}_m}\int_{\mathbb{R}^m} e^{-\frac{1}{2}\sum_{i=1}^m \lambda_i^2}\big|\Delta_{m+1}(\lambda_0,\lambda_1,\dots,\lambda_m)\big|\,|d\lambda_1\cdots d\lambda_m|$$

$$= \frac{e^{\frac{1}{2}\lambda_0^2}(2\pi)^{\frac{\dim \mathcal{S}_m}{2}}\boldsymbol{Z}_{m+1}}{\boldsymbol{Z}_m}\underbrace{\frac{1}{\boldsymbol{Z}_{m+1}}\int_{\mathbb{R}^m} e^{-\frac{1}{2}\sum_{i=1}^m \lambda_i^2}\big|\,\Delta_{m+1}(\lambda_0,\lambda_1,\dots,\lambda_m)\big|\,|d\lambda_1\cdots d\lambda_m|}_{=:\rho_{m+1}(\lambda_0)}.$$

The function $R_n(x) = n\rho_n(x)$ is known in random matrix theory as the 1-*point correlation function* of the Gaussian orthogonal ensemble of symmetric $n\times n$ matrices, [13, §4.4.1], [19, §3], [27, §4.2]. We conclude that

$$I_m = \frac{(2\pi)^{\frac{\dim \mathcal{S}_m}{2}}A_m\boldsymbol{Z}_{m+1}}{\boldsymbol{Z}_m}\int_{\mathbb{R}}\rho_{m+1}(x)e^{\frac{x^2}{2}}d\boldsymbol{\gamma}(x) = \frac{A_m\boldsymbol{Z}_{m+1}}{\boldsymbol{Z}_m}\int_{\mathbb{R}}\rho_{m+1}(x)\sqrt{\frac{2}{\pi}}e^{-\frac{3x^2}{2}}dx.$$

We have

$$\frac{\boldsymbol{Z}_{m+1}}{\boldsymbol{Z}_m} = 2^{\frac{1}{2}}(m+1)\Gamma\left(\frac{m+1}{2}\right),$$

$$\sqrt{\frac{2}{\pi}}\frac{(2\pi)^{\frac{\dim \mathcal{S}_m}{2}}A_m\boldsymbol{Z}_{m+1}}{\boldsymbol{Z}_m} = (2\pi)^{\frac{\dim \mathcal{S}_m}{2}}\sqrt{\frac{2}{\pi}}2^{\frac{1}{2}}(m+1)\Gamma\left(\frac{m+1}{2}\right)\frac{(m+2)^{\frac{1}{2}}}{2^{\frac{1}{2}}(2\pi)^{\frac{m(m+1)}{4}}\sqrt{\mu_m}}$$

$$= \sqrt{\frac{2}{\pi}}(m+1)\Gamma\left(\frac{m+1}{2}\right)\frac{(m+2)^{\frac{1}{2}}}{\sqrt{\mu_m}}$$

$$= \sqrt{\frac{2}{\pi}}(m+1)\Gamma\left(\frac{m+1}{2}\right)\frac{(m+2)^{\frac{1}{2}}}{2^{\frac{1}{2}\binom{m-1}{2}+\frac{m-1}{2}}(m+2)^{\frac{1}{2}}}$$

$$= \sqrt{\frac{2}{\pi}}(m+1)\frac{\Gamma\left(\frac{m+1}{2}\right)}{2^{\frac{1}{2}\binom{m-1}{2}+\frac{m-1}{2}}} = \sqrt{\frac{2}{\pi}}\frac{2\Gamma\left(\frac{m+3}{2}\right)}{2^{\frac{1}{2}\binom{m-1}{2}+\frac{m-1}{2}}}.$$

We deduce

$$I_m = \sqrt{\frac{2}{\pi}}\frac{2\Gamma\left(\frac{m+3}{2}\right)}{2^{\frac{1}{2}\binom{m-1}{2}+\frac{m-1}{2}}}\int_{\mathbb{R}}\rho_{m+1}(x)e^{-\frac{3x^2}{2}}dx. \tag{4.3}$$

We set

$$\bar{\rho}_n(s) := \sqrt{n}\rho_n(\sqrt{n}s),$$

and we deduce

$$\int_{\mathbb{R}}\rho_n(x)e^{-\frac{3x^2}{2}}dx = \int_{\mathbb{R}}\rho_n(\sqrt{n}s)e^{-\frac{3ns^2}{2}}ds = n\int_{\mathbb{R}}e^{-\frac{3ns^2}{2}}\bar{\rho}_n(s)ds$$
$$= \left(\frac{2\pi}{3n}\right)^{\frac{1}{2}}\int_{\mathbb{R}}\underbrace{\frac{(3n)^{\frac{1}{2}}e^{-\frac{3ns^2}{2}}}{(2\pi)^{\frac{1}{2}}}}_{=:w_n(s)}\cdot\bar{\rho}_n(s)ds. \tag{4.4}$$

To proceed further we use as guide Wigner's theorem, [3, 13, 17, 27] stating that the sequences of probability measures

$$\bar{\rho}_n(x)dx = \sqrt{n}\rho_n(\sqrt{n}x)dx = \frac{1}{\sqrt{n}}R_n(\sqrt{n}x)dx$$

converges weakly to the semi-circle probability measure[6] $\rho(x)dx$,

$$\rho(x) = \frac{1}{\pi}\begin{cases}\sqrt{2-x^2}, & |x|\leq\sqrt{2}\\ 0, & |x|>\sqrt{2}.\end{cases} \tag{4.5}$$

We observe that the Gaussian measures $w_n(s)ds$ converge to the Dirac delta measure concentrated at the origin. This suggests that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\bar{\rho}_n(s)w_n(s)ds = \rho(0) = \frac{\sqrt{2}}{\pi}. \tag{4.6}$$

We will show that this is indeed the case by slightly refining the arguments in one particular proof of Wigner's theorem; see [17, §7.1.6],[19, §6.1] or [27, A.9]. For the moment we will take (4.6) for granted and show that it immediately implies (1.4).

Using (4.6) in (4.3) and (4.4) we deduce that

$$I_m \sim \sqrt{\frac{2}{\pi}}\frac{2\Gamma\left(\frac{m+3}{2}\right)}{2^{\frac{1}{2}\left(\frac{m-1}{2}\right)+\frac{m-1}{2}}}\times\left(\frac{2\pi}{3(m+1)}\right)^{\frac{1}{2}}\times\frac{\sqrt{2}}{\pi}\ \text{ as }\ m\to\infty.$$

We now invoke Stirling's formula to conclude that

$$\log I_m \sim\sim \log\Gamma\left(\frac{m+3}{2}\right)\sim\frac{m}{2}\log m,\ \text{ as }\ m\to\infty. \tag{4.7}$$

Form (1.3) we deduce that

$$\log C(m) = \log I_m + \frac{m}{2}\log 4\pi + \log\Gamma\Big(1+\frac{m}{2}\Big) - \frac{m}{2}\log(m+4).$$

Stirling's formula and (4.7) imply that

$$\log C(m)\sim\log I_m\sim\frac{m}{2}\log m\ \text{ as }\ m\to\infty.$$

This proves (1.4). □

4.2. **Wigner's semicircle law revisited.** We can now present the postponed proof of (4.6). The 1-point correlation function $R_n(x)$ can be expressed explicitly in terms of Hermite polynomials, [27, Eq. (7.2.32) and §A.9],

$$R_n(x) = \underbrace{\sum_{k=0}^{n-1}\psi_k(x)^2}_{=:\boldsymbol{k}_n(x)} + \underbrace{\left(\frac{n}{2}\right)^{\frac{1}{2}}\psi_{n-1}(x)\int_{\mathbb{R}}\varepsilon(x-t)\psi_n(t)dt + \alpha_n(x)}_{=:\boldsymbol{\ell}_n(x)}, \tag{4.8}$$

where

$$\psi_n(x) = \frac{1}{(2^n n!\sqrt{\pi})^{\frac{1}{2}}}e^{-\frac{x^2}{2}}H_n(x),\quad H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2}),$$

$$\alpha_n(x) = \begin{cases}0, & n\in 2\mathbb{Z},\\ \frac{\psi_{n-1}(x)}{\int_{\mathbb{R}}\psi_{n-1}(x)dx}, & n\in 2\mathbb{Z}+1,\end{cases}$$

and

$$\varepsilon(x) = \begin{cases}\frac{1}{2}, & x>0\\ 0, & x=0,\\ -\frac{1}{2}, & x<0.\end{cases}$$

[6]There are different rescalings of the semicircle measures in the literature. Our conventions agree with those in [27].

From the Christoffel-Darboux formula [42, Eq. (5.5.9)] we deduce

$$\pi^{\frac{1}{2}}e^{x^2}\sum_{k=0}^{n-1}\psi_k(x)^2=\sum_{k=1}^{n-1}\frac{1}{2^k k!}H_k(x)^2=\frac{1}{2^n(n-1)!}\big(\,H_n'(x)H_{n-1}(x)-H_n(x)H_n'(x)\,\big)$$

Using the recurrence formula $H_n'=2xH_n-H_{n+1}$ we deduce

$$H_n'(x)H_{n-1}(x)-H_n(x)H_n'(x)=H_n^2(x)-H_{n-1}(x)H_{n+1}(x)$$

and

$$\boldsymbol{k}_n(x)=\frac{e^{-x^2}}{2^n(n-1)!\pi^{\frac{1}{2}}}\Big(H_n^2(x)-H_{n-1}(x)H_{n+1}(x)\Big).$$

We set

$$\bar{\boldsymbol{k}}_n(x):=\frac{\boldsymbol{k}_n\big(\sqrt{n}x\big)}{\sqrt{n}},\;\;\bar{\boldsymbol{\ell}}_n(x):=\frac{\boldsymbol{\ell}_n\big(\sqrt{n}x\big)}{\sqrt{n}},\;\;\bar{R}_n(x)=\frac{1}{\sqrt{n}}R_n(\sqrt{n}x)=\bar{\rho}_n(x)$$

so that

$$\bar{R}_n(x)=\bar{\boldsymbol{k}}_n(x)+\bar{\boldsymbol{\ell}}_n(x).$$

Lemma 4.1.

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|\bar{\boldsymbol{\ell}}_n(x)|=0. \tag{4.9}$$

Proof. Using the generating series [42, Eq. (5.5.7)]

$$\sum_{n=0}^{\infty}H_n(x)\frac{T^n}{n!}=e^{2Tx-T^2}$$

we deduce that

$$\sum_{n=0}^{\infty}\left(\int_{\mathbb{R}}e^{-\frac{x^2}{2}}H_n(x)dx\right)\frac{T^n}{n!}=e^{T^2}\int_{\mathbb{R}}e^{-\frac{(x-2T)^2}{2}}dx=\sqrt{2\pi}e^{T^2},$$

so that

$$\frac{1}{(2n)!}\int_{\mathbb{R}}e^{-\frac{x^2}{2}}H_{2n}(x)dx=\frac{\sqrt{2\pi}}{n!}\;\;\text{and}\;\;\int_{\mathbb{R}}\psi_{2n}(x)dx=\frac{\sqrt{2(2n)!}}{2^n n!\pi^{\frac{1}{4}}}\sim\text{const}\cdot n^{\frac{1}{4}}\;\;\text{as } n\to\infty.$$

Using [13, Thm. 6.55] or [42, Thm. 8.91.3] we deduce that

$$\sup_{x\in\mathbb{R}}|\psi_n(x)|=O(n^{-\frac{1}{12}})$$

and thus

$$\sup_{x\in\mathbb{R}}|\alpha_n(x)|=O(n^{-\frac{1}{12}-\frac{1}{4}})=O(n^{-\frac{1}{3}})\;\;\text{as } n\to\infty.$$

We set

$$F_n(x)=\int_{\mathbb{R}}\varepsilon(x-t)\psi_n(t)dt.$$

Using [13, Thm. 6.55 + Eq. (6.26)] we deduce $\sup_{x\in\mathbb{R}}|F_n(x)|=O\big(n^{-\frac{1}{12}}\big)$. This proves (4.9). □

From the above lemma we deduce that

$$\int_{\mathbb{R}}\big(\,\bar{\rho}_n(s)-\rho(s)\,\big)w_n(s)ds=\int_{\mathbb{R}}\big(\,\bar{\boldsymbol{k}}_n(s)-\rho(s)\,\big)w_n(s)ds+O\big(\,n^{-\frac{1}{12}}\,\big)\;\;\text{as } n\to\infty.$$

Lemma 4.2.

$$\lim_{n\to\infty}\int_{\mathbb{R}}\big(\,\bar{\boldsymbol{k}}_n(s)-\rho(s)\,\big)w_n(s)ds=0.$$

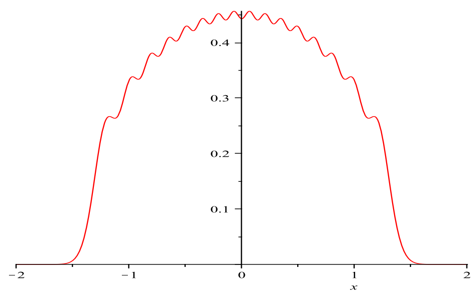

FIGURE 1. *The graph of* $\bar{\boldsymbol{k}}_{16}(x)$, $|x|\leq 2$.

Proof. Fix $c\in(0,\sqrt{2})$ so that the interval $(-c,c)$ lies inside the oscillatory regime of $H_n(\sqrt{n}t)$. We have

$$\int_{\mathbb{R}}\big(\,\bar{\boldsymbol{k}}_n(s)-\rho(s)\,\big)w_n(s)ds$$

$$=\int_{|s|\leq c}\big(\,\bar{\boldsymbol{k}}_n(s)-\rho(s)\,\big)w_n(s)ds+\int_{|s|>c}\big(\,\bar{\boldsymbol{k}}_n(s)-\rho(s)\,\big)w_n(s)ds$$

$$\leq\sup_{|s|\leq c}|\bar{\boldsymbol{k}}_n(s)-\rho(s)|+\sup_{|s|>c}\big|\big(\,\bar{\boldsymbol{k}}_n(s)-\rho(s)\,\big)\big|\int_{|s|>c}w_n(s)ds.$$

Using the Plancherel-Rotach formulæ ([13, Eq. (6.126)], [37], [42, Thm. 8.22.9]) and arguing as in [17, §7.1.6] or [19, §6.1] we deduce that

$$\lim_{n\to\infty}\sup_{|s|\leq c}|\bar{\boldsymbol{k}}_n(s)-\rho(s)|=0.$$

On the other hand

$$\lim_{n\to\infty}\int_{|s|>c}w_n(s)ds=0,$$

and [42, Thm.8.91.3] implies that

$$\sup_{|s|>c}\big|\big(\,\bar{\boldsymbol{k}}_n(s)-\rho(s)\,\big)\big|=O(1)\ \text{ as } n\to\infty.$$

□

Since $w_n(s)ds$ converges to the δ-measure concentrated at the origin we deduce

$$\lim_{n\to\infty}\int_{\mathbb{R}}\rho(s)w_n(s)ds=\rho(0)=\frac{\sqrt{2}}{\pi}.$$

This proves (4.6).

APPENDIX A. GAUSSIAN MEASURES AND GAUSSIAN RANDOM FIELDS

For the reader's convenience we survey here a few basic facts about Gaussian measures. For more details we refer to [9]. A *Gaussian measure* on $\mathbb{R}$ is a Borel measure $\gamma_{m,\sigma}$ of the form

$$\gamma_{m,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx.$$

The scalar m is called the *mean* while σ is called the *standard deviation*. We allow σ to be zero in which case

$$\gamma_{m,0} = \delta_m = \text{the Dirac measure on } \mathbb{R} \text{ concentrated at } m.$$

Suppose that $\boldsymbol{V}$ is a finite dimensional vector space. A *Gaussian measure* on $\boldsymbol{V}$ is a Borel measure γ on $\boldsymbol{V}$ such that, for any $\xi \in \boldsymbol{V}^\vee$, the pushforward $\xi_*(\gamma)$ is a Gaussian measure on $\mathbb{R}$, $\xi_*(\gamma) = \gamma_{m(\xi),\sigma(\xi)}$.

One can show that the map $\boldsymbol{V}^\vee \ni \xi \mapsto m(\xi) \in \mathbb{R}$ is linear, and thus can be identified with a vector $\boldsymbol{m}_\gamma \in \boldsymbol{V}$ called the *barycenter* or *expectation* of γ that can be alternatively defined by the equality $\boldsymbol{m}_\gamma = \int_{\boldsymbol{V}} \boldsymbol{v} d\gamma(\boldsymbol{v})$. Moreover, there exists a nonnegative definite, symmetric bilinear map

$$\boldsymbol{\Sigma} : \boldsymbol{V}^\vee \times \boldsymbol{V}^\vee \to \mathbb{R} \;\text{ such that }\; \sigma(\xi)^2 = \boldsymbol{\Sigma}(\xi,\xi), \;\; \forall \xi \in \boldsymbol{V}^\vee.$$

The form $\boldsymbol{\Sigma}$ is called the *covariance form* and can be identified with a linear operator $\boldsymbol{S} : \boldsymbol{V}^\vee \to \boldsymbol{V}$ such that

$$\boldsymbol{\Sigma}(\xi,\eta) = \langle \xi, \boldsymbol{S}\eta \rangle, \;\; \forall \xi,\eta \in \boldsymbol{V}^\vee,$$

where $\langle -,- \rangle : \boldsymbol{V}^\vee \times \boldsymbol{V} \to \mathbb{R}$ denotes the natural bilinear pairing between a vector space and its dual. The operator $\boldsymbol{S}$ is called the *covariance operator* and it is explicitly described by the integral formula

$$\langle \xi, \boldsymbol{S}\eta \rangle = \Lambda(\xi,\eta) = \int_{\boldsymbol{V}} \langle \xi, \boldsymbol{v} - \boldsymbol{m}_\gamma \rangle \langle \eta, \boldsymbol{v} - \boldsymbol{m}_\gamma \rangle d\gamma(\boldsymbol{v}).$$

The Gaussian measure is said to be *nondegenerate* if $\boldsymbol{\Sigma}$ is nondegenerate, and it is called *centered* if $\boldsymbol{m} = 0$. A nondegenerate Gaussian measure on $\boldsymbol{V}$ is uniquely determined by its covariance form and its barycenter.

Example A.1. Suppose that $\boldsymbol{U}$ is an n-dimensional Euclidean space with inner product $(-,-)$. We use the inner product to identify $\boldsymbol{U}$ with its dual $\boldsymbol{U}^\vee$. If $A : \boldsymbol{U} \to \boldsymbol{U}$ is a symmetric, positive definite operator, then

$$d\boldsymbol{\gamma}_A(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det A}} e^{-\frac{1}{2}(A^{-1}\boldsymbol{u},\boldsymbol{u})} \, |d\boldsymbol{u}| \tag{A.1}$$

is a centered Gaussian measure on $\boldsymbol{U}$ with covariance form described by the operator A. □

If $\boldsymbol{V}$ is a finite dimensional vector space equipped with a Gaussian measure γ and $\boldsymbol{L} : \boldsymbol{V} \to \boldsymbol{U}$ is a linear map then the pushforward $\boldsymbol{L}_*\gamma$ is a Gaussian measure on $\boldsymbol{U}$ with barycenter

$$\boldsymbol{m}_{\boldsymbol{L}_*\gamma} = \boldsymbol{L}(\boldsymbol{m}_\gamma)$$

and covariance form

$$\boldsymbol{\Sigma}_{\boldsymbol{L}_*\gamma} : \boldsymbol{U}^\vee \times \boldsymbol{U}^\vee \to \mathbb{R}, \;\; \boldsymbol{\Sigma}_{\boldsymbol{L}_*\gamma}(\eta,\eta) = \boldsymbol{\Sigma}_\gamma(\boldsymbol{L}^\vee\eta, \boldsymbol{L}^\vee\eta), \;\; \forall \eta \in \boldsymbol{U}^\vee,$$

where $\boldsymbol{L}^\vee : \boldsymbol{U}^\vee \to \boldsymbol{V}^\vee$ is the dual (transpose) of the linear map $\boldsymbol{L}$. Observe that if γ is nondegenerate and $\boldsymbol{L}$ is surjective, then $\boldsymbol{L}_*\gamma$ is also nondegenerate.

Suppose $(\mathcal{S},\mu)$ is a probability space. A *Gaussian* random vector on $(\mathcal{S},\mu)$ is a (Borel) measurable map

$$X : \mathcal{S} \to \boldsymbol{V}, \;\; \boldsymbol{V} \text{ finite dimensional vector space}$$

such that $X_*\mu$ is a Gaussian measure on $\boldsymbol{V}$. We will refer to this measure as the *associated Gaussian measure*, we denote it by γ_X and we denote by $\boldsymbol{\Sigma}_X$ (respectively $\boldsymbol{S}_X$) its covariance form (respectively operator),

$$\boldsymbol{\Sigma}_X(\xi_1,\xi_2)=\boldsymbol{E}\big(\,\langle \xi_1, X-\boldsymbol{E}(X)\,\rangle\,\langle \xi_2, X-\boldsymbol{E}(X)\,\rangle\,\big).$$

Note that the expectation of γ_X is precisely the expectation of X. The random vector is called *nondegenerate*, respectively *centered*, if the Gaussian measure γ_X is such.

Suppose that $X_j:\mathcal{S}\to\boldsymbol{V}_1$, $j=1,2$, are two *centered* Gaussian random vectors such that the direct sum $X_1\oplus X_2:\mathcal{S}\to\boldsymbol{V}_1\oplus\boldsymbol{V}_2$ is also a centered Gaussian random vector with associated Gaussian measure

$$\gamma_{X_1\oplus X_2}=p_{X_1\oplus X_2}(\boldsymbol{x}_1,\boldsymbol{x}_2)|d\boldsymbol{x}_1 d\boldsymbol{x}_2|.$$

We obtain a bilinear form

$$\boldsymbol{cov}(X_1,X_2):\boldsymbol{V}_1^\vee\times\boldsymbol{V}_2^\vee\to\mathbb{R},\;\; \boldsymbol{cov}(X_1,X_2)(\xi_1,\xi_2)=\boldsymbol{\Sigma}(\xi_1,\xi_2),$$

called the *covariance form*. The random vectors X_1 and X_2 are independent if and only if they are uncorrelated, i.e.,

$$\boldsymbol{cov}(X_1,X_2)=0.$$

We can form the random vector $\boldsymbol{E}(X_1|X_2)$, the conditional expectation of X_1 given X_2. If X_1 and X_2 are independent then $\boldsymbol{E}(X_1|X_2)=\boldsymbol{E}(X_1)$, while at the other extreme we have $\boldsymbol{E}(X_1|X_1)=X_1$.

To find a formula for $\boldsymbol{E}(X_1|X_2)$ in general we fix Euclidean metrics $(-,-)_{\boldsymbol{V}_j}$ on $\boldsymbol{V}_j$. We can then identify $\boldsymbol{cov}(X_1,X_2)$ with a linear operator $\boldsymbol{Cov}(X_1,X_2):\boldsymbol{V}_2\to\boldsymbol{V}_1$, via the equality

$$\begin{aligned}\boldsymbol{E}\big(\,\langle\xi_1,X_1\rangle\langle\xi_2,X_2\rangle\,\big)&=\boldsymbol{cov}(X_1,X_2)(\xi_1,\xi_2)\\&=\big\langle\,\xi_1,\boldsymbol{Cov}(X_1,X_2)\xi_2^\dagger\,\big\rangle,\;\;\forall\xi_1\in\boldsymbol{V}_1^\vee,\;\;\xi_2\in\boldsymbol{V}_2^\vee,\end{aligned}$$

where $\xi_2^\dagger\in\boldsymbol{V}_2$ denotes the vector metric dual to ξ_2. The operator $\boldsymbol{Cov}(X_1,X_2)$ is called the *covariance operator* of X_1,X_2.

Lemma A.2 (Regression formula)**.** *If X_1 and X_2 are as above and, additionally, X_2 is nondegenerate, then*

$$\boldsymbol{E}(X_1|X_2)=\boldsymbol{Cov}(X_1,X_2)\boldsymbol{S}_{X_2}^{-1}\big(\,X_2-\boldsymbol{E}(X_2)\,\big)+\boldsymbol{E}(X_1).\tag{A.2}$$

Proof. We follow the elegant argument in [4, Prop. 1.2]. We seek a linear operator $C:\boldsymbol{V}_2\to\boldsymbol{V}_1$ such that the random vector $Y=X_1-CX_2$ is independent of X_2. If such an operator exists then

$$\boldsymbol{E}(X_1|X_2)=\boldsymbol{E}(Y|X_2)+\boldsymbol{E}(CX_2|X_2)=\boldsymbol{E}(Y)+CX_2=CX_2.$$

Since the random vector X_1-CX_2 is Gaussian the operator C must satisfy the constraint

$$\boldsymbol{cov}(X_1-CX_2,X_2)=0\Longleftrightarrow 0=\boldsymbol{Cov}(X_1-CX_2,X_2)=\boldsymbol{Cov}(X_1,X_2)-\boldsymbol{Cov}(CX_2,X_2)$$

To find C we note that

$$\big\langle\,\xi_1,\boldsymbol{Cov}(CX_2,X_2)\xi_2^\dagger\,\big\rangle=\boldsymbol{E}\big(\,\langle\xi_1,CX_2\rangle\langle\xi_2,X_2\rangle\,\big)$$

$$=\boldsymbol{E}(\,\langle C^\vee\xi_1,X_2\rangle\langle\xi_2,X_2\rangle)=\boldsymbol{\Sigma}_{X_2}(C^\vee\xi_1,\xi_2)=\langle\xi_1,C\boldsymbol{S}_{X_2}\xi_2\rangle.$$

Identifying $\boldsymbol{V}_2$ with $\boldsymbol{V}_2^\vee$ via the Euclidean metric $(-,-)_{\boldsymbol{V}_2}$, we can regard $\boldsymbol{S}_{X_2}$ as a linear, symmetric nonnegative operator $\boldsymbol{V}_2\to\boldsymbol{V}_2$, and we deduce that $\boldsymbol{Cov}(CX_2,X_2)=C\boldsymbol{S}_{X_2}=\boldsymbol{Cov}(X_1,X_2)$ which shows that

$$C=\boldsymbol{Cov}(X_1,X_2)\boldsymbol{S}_{X_2}^{-1}.\tag{A.3}$$

□

The conditional probability density of X_1 given that $X_2 = \boldsymbol{x}_2$ is the function

$$p_{(X_1|X_2=\boldsymbol{x}_2)}(\boldsymbol{x}_1) = \frac{p_{X_1\oplus X_2}(\boldsymbol{x}_1,\boldsymbol{x}_2)}{\int_{\boldsymbol{V}_1} p_{X_1\oplus X_2}(\boldsymbol{x}_1,\boldsymbol{x}_2)|d\boldsymbol{x}_1|}.$$

For a measurable function $f : \boldsymbol{V}_1 \to \mathbb{R}$ the conditional expectation $\boldsymbol{E}(f(X_1)|X_2 = \boldsymbol{x}_2)$ is the (deterministic) scalar

$$\boldsymbol{E}(f(X_1)|X_2 = \boldsymbol{x}_2) = \int_{\boldsymbol{V}_1} f(\boldsymbol{x}_1)p_{(X_1|X_2=\boldsymbol{x}_2)}(\boldsymbol{x}_1)|d\boldsymbol{x}_1|.$$

Again, if X_2 is nondegenerate, then we have the *regression formula*

$$\boldsymbol{E}(f(X_1)|X_2 = \boldsymbol{x}_2) = \boldsymbol{E}\big(\, f(Y + C\boldsymbol{x}_2)\,\big) \tag{A.4}$$

where $Y : \mathcal{S} \to \boldsymbol{V}_1$ is a Gaussian vector with

$$\boldsymbol{E}(Y) = \boldsymbol{E}(X_1) - C\boldsymbol{E}(X_2),\;\; \boldsymbol{S}_Y = \boldsymbol{S}_{X_1} - \boldsymbol{Cov}(X_1,X_2)\boldsymbol{S}_{X_2}^{-1}\boldsymbol{Cov}(X_2,X_1), \tag{A.5}$$

and C is given by (A.3).

Let us point out that if $X : \mathcal{S} \to \boldsymbol{U}$ is a Gaussian random vector and $\boldsymbol{L} : \boldsymbol{U} \to \boldsymbol{V}$ is a linear map, then the random vector $\boldsymbol{L}X : \mathcal{S} \to \boldsymbol{V}$ is also Gaussian. Moreover

$$\boldsymbol{E}(\boldsymbol{L}X) = \boldsymbol{L}\boldsymbol{E}(X),\;\; \boldsymbol{\Sigma}_{\boldsymbol{L}X}(\xi,\xi) = \boldsymbol{\Sigma}_X(\boldsymbol{L}^\vee\xi,\boldsymbol{L}^\vee\xi),\;\; \forall \xi \in \boldsymbol{V}^\vee,$$

where $\boldsymbol{L}^\vee : \boldsymbol{V}^\vee \to \boldsymbol{U}^\vee$ is the linear map dual to $\boldsymbol{L}$. Equivalently, $\boldsymbol{S}_{\boldsymbol{L}X} = \boldsymbol{L}\boldsymbol{S}_X\boldsymbol{L}^\vee$.

A *random field* (or *function*) on a set $\boldsymbol{T}$ is a map $\xi : \boldsymbol{T} \times (\mathcal{S},\mu) \to \mathbb{R},\;\; (t,s) \mapsto \xi_t(s)$ such that

- $(\mathcal{S},\mu)$ is a probability space, and
- for any $t \in \boldsymbol{T}$ the function $\xi_t : \mathcal{S} \to \mathbb{R}$ is measurable, i.e., it is a random variable.

Thus, a random field on $\boldsymbol{T}$ is a family of random variables ξ_t parameterized by the set $\boldsymbol{T}$. For simplicity we will assume that all these random variables have finite second moments. For any $t \in \boldsymbol{T}$ we denote by μ_{t_1} the expectation of ξ_t. The *covariance function* or *kernel* of the field is the function $C_\xi : \boldsymbol{T} \times \boldsymbol{T} \to \mathbb{R}$ defined by

$$C_\xi(t_1,t_2) = \boldsymbol{E}\big(\,(\xi_{t_1} - \mu_{t_1})\,(\,\xi_{t_2} - \mu_{t_2})\,\big) = \int_{\mathcal{S}} \big(\xi_{t_1}(s) - \mu_{t_1}\big)\big(\xi_{t_2}(s) - \mu_{t_2}\,\big)\,d\mu(s).$$

The field is called *Gaussian* if for any finite subset $F \subset \boldsymbol{T}$ the random vector

$$\mathcal{S} \in s \mapsto \big(\,\xi_t(s)\,\big)_{t\in F} \in \mathbb{R}^F$$

is a Gaussian random vector. Almost all the important information concerning a Gaussian random field can be extracted from its covariance kernel.

Here is a simple method of producing Gaussian random fields on a set $\boldsymbol{T}$. Choose a finite dimensional space $\boldsymbol{U}$ of real valued functions on $\boldsymbol{T}$. Once we fix a Gaussian measure $d\gamma$ on $\boldsymbol{U}$ we obtain tautologically a random field

$$\xi : \boldsymbol{T} \times \boldsymbol{U} \to \mathbb{R},\;\; (\boldsymbol{t},\boldsymbol{u}) \mapsto \xi_{\boldsymbol{t}}(\boldsymbol{u}) = \boldsymbol{u}(\boldsymbol{t}).$$

This is a Gaussian field since for any finite subset $F \subset \boldsymbol{T}$ the random vector

$$\Xi : \boldsymbol{U} \to \mathbb{R}^F,\;\; \boldsymbol{u} \mapsto (\boldsymbol{u}(\boldsymbol{t}))_{\boldsymbol{t}\in F}$$

is Gaussian because the map Ξ is linear and thus the pushforward $\Xi_* d\gamma$ is a Gaussian measure on $\mathbb{R}^F$. For more information about random fields we refer to [1, 4, 12, 21].

In the conclusion of this section we want to describe a few simple integral formulas.

Proposition A.3. *Suppose $\boldsymbol{V}$ is an Euclidean space of dimension N, $f:\boldsymbol{U}\to\mathbb{R}$ is a locally integrable, positively homogeneous function of degree $k\geq 0$, and $A:\boldsymbol{U}\to\boldsymbol{U}$ is a positive definite symmetric operator. Denote by $B(\boldsymbol{U})$ the unit ball of $\boldsymbol{V}$ centered at the origin, and by $S(\boldsymbol{U})$ its boundary. Then the following hold*

$$\begin{aligned}\frac{1}{\pi^{\frac{N}{2}}(k+N)}\int_{S(\boldsymbol{U})}f(\boldsymbol{u})|dA(\boldsymbol{u})| &= \frac{1}{\pi^{\frac{N}{2}}}\int_{B(\boldsymbol{U})}f(\boldsymbol{u})|d\boldsymbol{u}| \\ &= \frac{1}{\Gamma\big(1+\frac{k+N}{2}\big)}\int_{\boldsymbol{U}}f(\boldsymbol{u})\frac{e^{-|\boldsymbol{u}|^2}}{\pi^{\frac{N}{2}}}|d\boldsymbol{u}|.\end{aligned} \tag{A.6}$$

$$\int_{\boldsymbol{U}}f(\boldsymbol{u})d\boldsymbol{\gamma}_{tA}(\boldsymbol{u}) = t^{\frac{k}{2}}\int_{\boldsymbol{U}}f(\boldsymbol{u})d\boldsymbol{\gamma}_A(\boldsymbol{u})\ \ \forall t>0, \tag{A.7}$$

where $d\boldsymbol{\gamma}_A$ is the Gaussian measure defined by (A.1).

Proof. We have

$$\int_{B(\boldsymbol{U})}f(\boldsymbol{u})|d\boldsymbol{u}| = \int_0^1 t^{k+N-1}\left(\int_{S(\boldsymbol{U})}f(\boldsymbol{u})|dA(\boldsymbol{u})|\right) = \frac{1}{k+N}\int_{S(\boldsymbol{U})}f(\boldsymbol{u})|dA(\boldsymbol{u})|.$$

On the other hand

$$\frac{1}{\pi^{\frac{N}{2}}}\int_{\boldsymbol{U}}f(u)e^{-|\boldsymbol{u}|^2}|d\boldsymbol{u}| = \frac{1}{\pi^{\frac{N}{2}}}\left(\int_0^\infty t^{k+N-1}e^{-t^2}dt\right)\int_{S(\boldsymbol{U})}f(\boldsymbol{u})|dA(\boldsymbol{u})|$$

$$= \frac{1}{2\pi^{\frac{N}{2}}}\Gamma\left(\frac{k+N}{2}\right)\int_{S(\boldsymbol{U})}f(\boldsymbol{u})|dA(\boldsymbol{u})| = \frac{k+N}{2\pi^{\frac{N}{2}}}\Gamma\left(\frac{k+N}{2}\right)\int_{B(\boldsymbol{U})}f(\boldsymbol{u})|d\boldsymbol{u}|.$$

$$= \frac{1}{\pi^{\frac{N}{2}}}\Gamma\left(1+\frac{k+N}{2}\right)\int_{B(\boldsymbol{U})}f(\boldsymbol{u})|d\boldsymbol{u}|.$$

This proves (A.6). The equality (A.7) follows by using the change in variables $\boldsymbol{u}=t^{\frac{1}{2}}\boldsymbol{v}$. □

APPENDIX B. GAUSSIAN RANDOM SYMMETRIC MATRICES

We want to describe in some detail a 3-parameter family of centered Gaussian measures on $\mathcal{S}_m$, the vector space of real symmetric $m\times m$ matrices, $m>1$.

For any $1\leq i\leq j$ define $\xi_{ij}\in\mathcal{S}_m^\vee$ so that for any $A\in\mathcal{S}_m$

$$\xi_{ij}(A)=a_{ij}=\text{the }(i,j)\text{-th entry of the matrix }A.$$

The collection $(\xi_{ij})_{1\leq i\leq j\leq m}$ is a basis of the dual space $\mathcal{S}_m^\vee$. We denote by $(E_{ij})_{1\leq i\leq j}$ the dual basis of $\mathcal{S}_m$. More precisely, E_{ij} is the symmetric matrix whose (i,j) and (j,i) entries are 1 while all the other entries are equal to zero. For any $A\in\mathcal{S}_m$ we have

$$A=\sum_{i\leq j}\xi_{ij}(A)E_{ij}.$$

The space $\mathcal{S}_m$ is equipped with an inner product

$$(-,-):\mathcal{S}_m\times\mathcal{S}_m\to\mathbb{R},\ \ (A,B)=\operatorname{tr}(AB),\ \ \forall A,B\in\mathcal{S}_m.$$

This inner product is invariant with respect to the action of $\mathrm{SO}(m)$ on $\mathcal{S}_m$. We set

$$\widehat{E}_{ij}:=\begin{cases}E_{ij}, & i=j\\ \frac{1}{\sqrt{2}}E_{ij}, & i<j.\end{cases}.$$

The collection $(\widehat{E}_{ij})_{i\leq j}$ is a basis of $\mathcal{S}_m$ orthonormal with respect to the above inner product. We set

$$\hat{\xi}_{ij} := \begin{cases} \xi_{ij}, & i=j \\ \sqrt{2}\xi_{ij}, & i<j. \end{cases}$$

The collection $(\hat{\xi}_{ij})_{i\leq j}$ the orthonormal basis of $\mathcal{S}_m^\vee$ dual to $(\widehat{E}_{ij})$. The volume density induced by this metric is

$$|dX| := \prod_{i\leq j} d\widehat{\xi}_{ij} = 2^{\frac{1}{2}\binom{m}{2}} \prod_{i\leq j} dx_{ij}.$$

To any numbers a,b,c satisfying the inequalities

$$a-b,\ \ c,\ \ a+(m-1)b>0. \tag{B.1}$$

we will associate a centered Gaussian measure $\Gamma_{a,b,c}$ on $\mathcal{S}_m$ uniquely determined by its covariance form

$$\mathbf{\Sigma} = \mathbf{\Sigma}_{a,b,c} : \mathcal{S}_m^\vee \times \mathcal{S}_m^\vee \to \mathbb{R}$$

defined as follows:

$$\mathbf{\Sigma}(\xi_{ii},\xi_{ii}) = a,\ \ \mathbf{\Sigma}(\xi_{ii},\xi_{jj}) = b,\ \ \forall i\neq j, \tag{B.2a}$$

$$\mathbf{\Sigma}(\xi_{ij},\xi_{ij}) = c,\ \ \mathbf{\Sigma}(\xi_{ij},\xi_{k\ell}) = 0,\ \ \forall i<j,\ \ k\leq \ell,\ \ (i,j)\neq(k,\ell). \tag{B.2b}$$

To see that $\mathbf{\Sigma}_{a,b,c}$ is positive definite if a,b,c satisfy (B.1) we decompose $\mathcal{S}_m^\vee$ as a direct sum of subspaces

$$\mathcal{S}_m^\vee = \mathcal{D}_m \oplus \mathcal{O}_m,$$

$$\mathcal{D}_m = \operatorname{span}\{\xi_{ii};\ \ 1\leq i\leq m\},\ \ \mathcal{O}_m = \operatorname{span}\{\xi_{ij};\ \ 1\leq i<j\leq m\},\ \ \dim \mathcal{O}_m = \binom{m}{2}$$

With respect to this decomposition, and the corresponding bases of these subspaces the matrix $Q_{a,b,c}$ describing $\mathbf{\Sigma}_{a,b,c}$ with respect to the basis (ξ_{ij}) has a direct sum decomposition

$$Q_{a,b,c} = G_m(a,b) \oplus c\mathbb{1}_{\binom{m}{2}},$$

where $G_m(a,b)$ is the $m\times m$ symmetric matrix whose diagonal entries are equal to a while all the off diagonal entries are all equal to b.

The the spectrum of $G_m(a,b)$ consists of two eigenvalues: $(a-b)$ with multiplicity $(m-1)$ and the simple eigenvalue $a-b+mb$. Indeed, if C_m denotes the $m\times m$ matrix with all entries equal to 1, then $G_m(a,b) = (a-b)\mathbb{1}_m + bC_m$. The matrix C_m has rank 1 and a single nonzero eigenvalue equal to m with multiplicity 1. This proves that $Q_{a,b,c}$ is positive definite since its spectrum is positive. We denote by $d\Gamma_{a,b,c}$ the centered Gaussian measure on $\mathcal{S}_m$ with covariance form $\mathbf{\Sigma}_{a,b,c}$.

Since $\mathcal{S}_m$ is equipped with an inner product we can identify $\mathbf{\Sigma}_{a,b,c}$ with a symmetric, positive definite bilinear form on $\mathcal{S}_m$. We would like to compute the matrix $\widehat{Q} = \widehat{Q}_{a,b,c}$ that describes $\mathbf{\Sigma}_{a,b,c}$ with respect to the orthonormal basis $(\widehat{E}_{ij})_{1\leq i\leq j}$. We have

$$\widehat{Q}(\widehat{E}_{ii},\widehat{E}_{ii}) = Q(\hat{\xi}_{ii},\hat{\xi}_{ii}) = a,\ \ \widehat{Q}(\widehat{E}_{ii},\widehat{E}_{jj}) = b,\ \ \forall i\neq j,$$

$$\widehat{Q}(\widehat{E}_{ij},\widehat{E}_{ij}) = Q(\hat{\xi}_{ij},\hat{\xi}_{ij}) = 2Q(\xi_{ij},\xi_{ij}) = 2c,\ \ \forall i<j,$$

Thus

$$\widehat{Q}_{a,b,c} = G_m(a,b) \oplus 2c\mathbb{1}_{\binom{m}{2}}. \tag{B.3}$$

If $|-|_{a,b,c}$ denotes the Euclidean norm on $\mathcal{S}_m$ determined by $\mathbf{\Sigma}_{a,b,c}$ then for

$$A = \sum_{i\leq j} a_{ij}E_{ij} = \sum_i a_{ii}\widehat{E}_{ii} + \sqrt{2}\sum_{i<j} a_{ij}\widehat{E}_{ij}.$$

we have

$$|A|^2_{a,b,c} = a\sum_i a_{ii}^2 + 2b\sum_{i<j} a_{ii}a_{jj} + 4c\sum_{i<j} a_{ij}^2$$

$$= (a-b-2c)\sum_i a_{ii}^2 + b\left(\sum_i a_{ii}\right)^2 + 2c\left(\sum_i a_{ii}^2 + 2\sum_{i<j} a_{ij}^2\right)$$

$$= (a-b-2c)\sum_i a_{ii}^2 + b(\operatorname{tr} A)^2 + 2c\operatorname{tr} A^2.$$

Observe that when

$$a - b = 2c \tag{B.4}$$

we have

$$|A|^2_{a,b,c} = b(\operatorname{tr} A)^2 + 2c\operatorname{tr} A^2 \tag{B.5}$$

so that the norm $|-|_{a,b,c}$ and the Gaussian measure $d\Gamma_{a,b,c}$ are $O(m)$-invariant. Let us point out that the space $\mathcal{S}_m$ equipped with the Gaussian measure $d\Gamma_{2,0,1}$ is the well known GOE, the *Gaussian orthogonal ensemble*.

To obtain a more concrete description of $\Gamma_{a,b,c}$ we first identify $\boldsymbol{\Sigma}_{a,b,c}$ with a symmetric operator $\widehat{Q}_{a,b,c} : \mathcal{S}_m \to \mathcal{S}_m$. Using (B.3) we deduce that

$$\widehat{Q}_{a,b,c} = G(a,b) \oplus 2c\mathbb{1}_{\binom{m}{2}}.$$

Observe that

$$\det \widehat{Q}_{a,b,c} = (a-b)\big(a+(m-1)b\big)^{m-1}(2c)^{\binom{m}{2}}, \tag{B.6}$$

and

$$\widehat{Q}^{-1}_{a,b,c} = \widehat{Q}_{a',b',c'} = G_m(a',b') \oplus 2c'\mathbb{1}_{\binom{m}{2}}, \tag{B.7}$$

where $2c' = \frac{1}{2c}$ and the real numbers a', b' are determined from the linear system

$$\begin{cases} a' - b' &= \frac{1}{a-b} \\ a' + (m-1)b' &= \frac{1}{a+(m-1)b}. \end{cases} \tag{B.8}$$

We then have

$$d\Gamma_{a,b,c}(X) = \frac{1}{(2\pi)^{\frac{m(m+1)}{4}}(\det \widehat{Q}_{a,b,c})^{\frac{1}{2}}} e^{-\frac{1}{2}(\widehat{Q}^{-1}_{a,b,c}X,X)} 2^{\frac{1}{2}\binom{m}{2}} \prod_{i\leq j} dx_{ij}, \tag{B.9}$$

where

$$(\widehat{Q}^{-1}_{a,b,c}X, X) = \Big(a' - b' - \frac{1}{2c}\Big)\sum_i x_{ii}^2 + b'(\operatorname{tr} X)^2 + \frac{1}{2c}\operatorname{tr} X^2. \tag{B.10}$$

The special case $b = c > 0$, $a = 3c$ is particularly important for our considerations. We denote by $(-,-)c$ and respectively $d\Gamma_c$ the inner product and respectively the Gaussian measure on $\mathcal{S}_m$ corresponding to the covariance form $\Sigma_{3c,c,c}$.

If we set $\widehat{Q}_c := \widehat{Q}_{3c,c,c}$ then we deduce from (B.7) that

$$\widehat{Q}_c^{-1} = \widehat{Q}_{a',b',c'} = G_m(a',b') \oplus \frac{1}{2c}\mathbb{1}_{\binom{m}{2}},$$

where

$$\begin{cases} a' - b' &= \frac{1}{2c} = 2c' \\ a' + (m-1)b' &= \frac{1}{(m+2)c}. \end{cases}$$

We deduce

$$mb' = \frac{1}{(m+2)c} - \frac{1}{2c} = -\frac{m}{2c(m+2)} \Rightarrow b' = -\frac{1}{2c(m+2)}.$$

Note that the invariance condition (B.4) $a' - b' = 2c'$ is automatically satisfied so that

$$\big(\widehat{Q}_c^{-1}X, X\big) = \frac{1}{2c}\operatorname{tr} X^2 - \frac{1}{2c(m+2)}(\operatorname{tr} X)^2.$$

Using (B.6) and (B.9) we deduce

$$d\Gamma_c(X) = \frac{1}{(2\pi c)^{\frac{m(m+1)}{4}}\sqrt{\mu_m}} \cdot e^{-\frac{1}{4c}\left(\operatorname{tr} X^2 - \frac{1}{m+2}(\operatorname{tr} X)^2\right)} \underbrace{2^{\frac{1}{2}\binom{m}{2}}\prod_{i\leq j}|dx_{ij}|}_{|dX|}, \tag{B.11}$$

where

$$\mu_m := 2^{\binom{m}{2}+(m-1)}(m+2). \tag{B.12}$$

The inner product $(-,-)_c$ has the alternate description

$$\begin{aligned}(A,B)_c = I_c(A,B) &:= 4c\int_{\mathbb{R}^m}(A\boldsymbol{x},\boldsymbol{x})(B\boldsymbol{x},\boldsymbol{x})\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| \\ &= c\int_{\mathbb{R}^m}(A\boldsymbol{x},\boldsymbol{x})(B\boldsymbol{x},\boldsymbol{x})\frac{e^{-\frac{|\boldsymbol{x}|^2}{2}}}{(2\pi)^{\frac{m}{2}}}|d\boldsymbol{x}|, \quad \forall A,B\in \mathcal{S}_m.\end{aligned} \tag{B.13}$$

To verify (B.13) it suffices to show that

$$I_c(E_{ii},E_{ii}) = 3c,\;\; I_c(E_{ii},E_{jj}) = c,\;\; I_c(E_{ij},E_{ij}) = 4c,\;\; \forall 1\leq i<j\leq m,$$
$$I_c(E_{ij},E_{k\ell}) = 0,\;\; \forall 1\leq i<j\leq m,\;\; k\leq \ell,\;\; (i,j)\neq(k,\ell).$$

To achieve this we need to use the classical identity

$$\int_{\mathbb{R}^m} x_1^{2p_1}\cdots x_m^{2p_m}e^{-|\boldsymbol{x}|^2}|d\boldsymbol{x}| = \prod_{k=1}^m\int_{\mathbb{R}} x_k^{2p_k}e^{-t^2}dx_k = \prod_{k=1}^m\Gamma\left(p_k+\frac{1}{2}\right).$$

Observe that

$$\int_{\mathbb{R}^m}(E_{ii}\boldsymbol{x},\boldsymbol{x})(E_{jj}\boldsymbol{x},\boldsymbol{x})\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| = \int_{\mathbb{R}^m}x_i^2x_j^2\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}|$$
$$= \pi^{-\frac{m}{2}}\Gamma\Big(\frac{1}{2}\Big)^{m-2}\begin{cases}\Gamma\big(\frac{5}{2}\big)\Gamma\big(\frac{1}{2}\big), & i=j,\\ \Gamma\big(\frac{3}{2}\big)^2, & i\neq j\end{cases} = \frac{1}{4}\times\begin{cases}3, & i=j\\ 1, & i\neq j.\end{cases}$$

Next, if $i<j$ we have

$$\int_{\mathbb{R}^m}(E_{ij}\boldsymbol{x},\boldsymbol{x})(E_{ij}\boldsymbol{x},\boldsymbol{x})\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| = 4\int_{\mathbb{R}^m}x_i^2x_j^2\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| = 1.$$

Finally, if $i<j$, $k\leq\ell$ and $(i,j)\neq(k,\ell)$, then the quartic polynomial

$$(E_{ij}\boldsymbol{x},\boldsymbol{x})(E_{k\ell}\boldsymbol{x},\boldsymbol{x})$$

is odd with respect to a reflection $x_p\mapsto -x_p$ for some $p=\{i,j,k,\ell\}$ and thus

$$\int_{\mathbb{R}^m}(E_{ij}\boldsymbol{x},\boldsymbol{x})(E_{k\ell}\boldsymbol{x},\boldsymbol{x})\frac{e^{-|\boldsymbol{x}|^2}}{\pi^{\frac{m}{2}}}|d\boldsymbol{x}| = 0.$$

APPENDIX C. SOME GAUSSIAN INTEGRALS

The proof of (2.38). We want to find the value of the integral

$$\boldsymbol{I} = \frac{1}{4(2\pi)^{\frac{3}{2}}} \int_{\mathcal{S}_2} |\det X| e^{-\frac{1}{4}\left(\operatorname{tr} X^2 - \frac{1}{4}(\operatorname{tr} X)^2\right)} \cdot \sqrt{2} \prod_{1\leq i\leq j\leq 2} dx_{ij}.$$

We first make the change in coordinates

$$x_{11} = x+y,\ \ x_{22} = x-y,\ \ x_{12} = z.$$

Then

$$\det X = x^2 - y^2 - z^2,\ \ \operatorname{tr} X = 2x,\ \ \operatorname{tr} X^2 = 2(x^2+y^2+z^2).$$

Hence

$$\boldsymbol{I} = \frac{1}{2(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |x^2-y^2-z^2| e^{-\frac{1}{4}(x^2+2y^2+2z^2)} \sqrt{2} |dxdydz|$$

$(x = \sqrt{2}u)$

$$= \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} |2u^2-y^2-z^2| e^{-\frac{1}{2}(u^2+y^2+z^2)} |dudydz|$$

$$= \frac{2}{\pi^{\frac{3}{2}}} \int_{\mathbb{R}^3} |2u^2-y^2-z^2| e^{-(u^2+y^2+z^2)} |dudydz|.$$

We now make the change in variables $y = r\cos\theta$, $y = r\sin\theta$, $r>0$ $\theta\in[0,2\pi)$ and we deduce

$$\boldsymbol{I} = \frac{2}{\pi^{\frac{3}{2}}} \int_{\mathbb{R}} \int_0^\infty \left(\int_0^{2\pi} |2u^2-r^2| e^{-u^2+r^2} d\theta \right) rdrdu$$
$$= \frac{8\pi}{\pi^{\frac{3}{2}}} \int_0^\infty \int_0^\infty |2u^2-r^2| e^{-(u^2+r^2)} rdrdu.$$

We now make the change in variables

$$u = t\sin\varphi,\ \ r = t\cos\varphi,\ \ t>0,\ \ 0\leq\varphi\leq\frac{\pi}{2},$$

and we conclude

$$\boldsymbol{I} = \frac{8}{\pi^{\frac{1}{2}}} \left(\int_0^\infty e^{-t^2} t^4 dt \right) \left(\int_0^{\frac{\pi}{2}} |3\sin^2\varphi - 1| \cos\varphi d\varphi \right)$$

$(t = \sqrt{s},\ x = \sin\varphi)$

$$= \frac{4}{\pi^{\frac{1}{2}}} \left(\int_0^\infty e^{-s} s^{\frac{3}{2}} ds \right) \left(\int_0^1 |3x^2-1| dx \right) = \frac{4}{\pi^{\frac{1}{2}}} \times \Gamma\Big(\frac{5}{2}\Big) \times \frac{4}{3\sqrt{3}} = \frac{4}{\sqrt{3}}. \qquad \square$$

Proposition C.1. *Consider the Gaussian measure* $\boldsymbol{\Gamma}_n$ *on* $\mathcal{S}_2$ *with covariance form* $\boldsymbol{\Sigma}_{a_n,b_n,c_n}$*, where*

$$a_n = s_n + 3t_n,\ \ b_n = s_n + t_n,\ \ c_n = t_n,\ \ s_n, t_n > 0.$$

Then

$$\boldsymbol{I}_n := \frac{2}{s_n} \int_{\mathcal{S}_2} \det X d\boldsymbol{\Gamma}_n(X) = 2. \tag{C.1}$$

Proof. Using (B.9) we deduce

$$d\boldsymbol{\Gamma}_n(X)=\frac{1}{(2\pi)^{\frac{3}{2}}(\det\widehat{Q}_{a_n,b_n,c_n})^{\frac{1}{2}}}e^{-\frac{1}{2}\left(\widehat{Q}_{a_n,b_n,c_n}X,X\right)}\sqrt{2}\prod_{1\leq i\leq j\leq 2}dx_{ij},$$

where

$$\det\widehat{Q}_{a_n,b_n,c_n}\stackrel{(B.6)}{=}2(a_n-b_n)(a_n+b_n)c_n=8t_n^2(s_n+2t_n)=:\delta_n,$$

$$\left(\widehat{Q}_{a_n,b_n,c_n}X,X\right)\stackrel{(B.10)}{=}\left(a_n'-b_n'-\frac{1}{2c_n}\right)(x_{11}^2+x_{22}^2)+b_n'(\operatorname{tr}X)^2+\frac{1}{2c_n}\operatorname{tr}X^2,$$

and, according to (B.8), the parameters a_n', b_n' are determined by the linear system

$$\begin{cases} a_n'-b_n' &= \frac{1}{a_n-b_n}\\ a_n'+b_n' &= \frac{1}{a_n+b_n}.\end{cases}$$

We deduce that

$$b_n'=-\frac{b_n}{a_n^2-b_n^2}$$

and since $a_n-b_n=2c_n$ we have

$$\left(a_n'-b_n'-\frac{1}{2c_n}\right)=0.$$

Hence

$$\left(\widehat{Q}_{a_n,b_n,c_n}X,X\right)=\frac{1}{2c_n}\operatorname{tr}X^2-\frac{b_n}{a_n^2-b_n^2}\left(\operatorname{tr}X\right)^2=\frac{1}{2t_n}\left(\operatorname{tr}X^2-\frac{s_n+t_n}{2(s_n+2t_n)}\left(\operatorname{tr}X\right)^2\right).$$

Using the change in variables $X:=\sqrt{t_n}X$ we deduce

$$\boldsymbol{I}_n:=\frac{2t_n^{\frac{3}{2}}}{s_n}\times\underbrace{\frac{1}{8\pi^{\frac{3}{2}}(s_n+2t_n)^{\frac{1}{2}}}\int_{\mathcal{S}^2}\det Xe^{-\frac{1}{4}\left(\operatorname{tr}X^2-\frac{s_n+t_n}{2(s_n+2t_n)}(\operatorname{tr}X)^2\right)}\sqrt{2}\prod_{1\leq i\leq j\leq 1}dx_{ij}}_{=:\boldsymbol{J}_n}. \tag{C.2}$$

We make the change in coordinates

$$x_{11}=x+y,\;\; x_{22}=x-y,\;\; x_{12}=z.$$

Then

$$\det X=x^2-y^2-z^2,\;\;\operatorname{tr}X=2x,\;\;\operatorname{tr}X^2=2(x^2+y^2+z^2).$$

$$\begin{aligned}\operatorname{tr}X^2-\frac{s_n+t_n}{2(s_n+2t_n)}(\operatorname{tr}X)^2&=\frac{4(s_n+2t_n)(x^2+y^2+z^2)-4(s_n+t_n)x^2}{2(s_n+2t_n)}\\&=\frac{4(s_n+2t_n)(y^2+z^2)+4t_nx^2}{2(s_n+2t_n)},\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{J}_n&=2\sqrt{2}\int_{\mathbb{R}^3}\left(x^2-(y^2+z^2)\right)e^{-\frac{(s_n+2t_n)(y^2+z^2)+t_nx^2}{2(s_n+2t_n)}}dxdydz\\&=2\sqrt{2}(s_n+2t_n)^{\frac{5}{2}}\int_{\mathbb{R}^3}\left(x^2-(y^2+z^2)\right)e^{-\frac{(s_n+2t_n)(y^2+z^2)+t_nx^2}{2}}dxdydz\end{aligned}$$

$(x:=x\sqrt{2},\, y:=y\sqrt{2},\, z:=z\sqrt{2})$

$$=16(s_n+2t_n)^{\frac{5}{2}}\int_{\mathbb{R}^3}\left(x^2-(y^2+z^2)\right)e^{-(s_n+2t_n)(y^2+z^2)-t_nx^2}dxdydz$$

$(x := t_n^{-\frac{1}{2}}x,\ y = (s_n+2t_n)^{-\frac{1}{2}}y,\ z = (s_n+2t_n)^{-\frac{1}{2}}z)$

$$= \frac{16(s_n+2t_n)^{\frac{3}{2}}}{t_n^{\frac{1}{2}}}\int_{\mathbb{R}^3}\left(\frac{x^2}{t_n} - \frac{y^2+z^2}{s_n+2t_n}\right)e^{-(x^2+y^2+z^2)}dxdydz.$$

We now use the identity

$$\int_{\mathbb{R}^3} u^2 e^{-(u^2+v^2+w^2)}dudvdw = \left(\int_{\mathbb{R}} u^2 e^{-u^2}du\right)\times\underbrace{\left(\int_{\mathbb{R}} e^{-t^2}dt\right)^2}_{=\pi}$$

$$= 2\pi\int_0^\infty u^2e^{-u^2}du = \pi\int_0^\infty s^{\frac{1}{2}}e^{-s}ds = \frac{\pi^{\frac{3}{2}}}{2}.$$

Hence

$$\boldsymbol{J}_n = \frac{16(s_n+2t_n)^{\frac{3}{2}}}{t_n^{\frac{1}{2}}}\times\frac{\pi^{\frac{3}{2}}}{2}\left(\frac{1}{t_n} - \frac{2}{s_n+2t_n}\right) = \frac{8\pi^{\frac{3}{2}}(s_n+2t_n)^{\frac{3}{2}}}{t_n^{\frac{1}{2}}}\times\frac{s_n}{t_n(s_n+2t_n)}.$$

Using the last equality in (C.2) we obtain (C.1) . □

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