THE CO-AREA FORMULA

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CONTENTS

1. STATEMENT OF THE FORMULA

We start with the simplest version.

Fubini theorem. Suppose φ is a integrable function on $\mathbb{R}^{n+k}$. Then

$$\int_{\mathbb{R}^{n+k}} \varphi(x^1,\dots,x^{n+k})|dx^1\cdots\wedge dx^{n+k}|$$
$$=\int_{\mathbb{R}^k}\left(\int_{\mathbb{R}^n}\varphi(x^1,\dots,x^k,x^{k+1},\dots x^{n+k}|dx^{k+1}\cdots dx^{n+k}|\right)|dx^1\cdots dx^k|.$$

We can reformulate this as follows. Set

$$\boldsymbol{y}=(x^1,\dots,x^k),\;\;\boldsymbol{x}=(x^{k+1},\dots,x^{n+k})$$

and define $A:\mathbb{R}^{n+k}\to\mathbb{R}^k,\;\;(\boldsymbol{y},\boldsymbol{x})\mapsto\boldsymbol{y}$. Then

$$\int_{\mathbb{R}^{n+k}}\varphi(\boldsymbol{x},\boldsymbol{y})dV_{n+k}(\boldsymbol{x},\boldsymbol{y})=\int_{\mathbb{R}^k}\left(\int_{A^{-1}(\boldsymbol{y})}\varphi(\boldsymbol{x},\boldsymbol{y})dV_n(\boldsymbol{x})\right)dV_k(\boldsymbol{y}). \tag{1.1}$$

where dV_m denotes the m-dimensional Lebesgue measure.

Consider now a slightly more general case of a linear map

$$A:\mathbb{R}^{n+k}\to\mathbb{R}^k,\;\;(x^1,\dots,x^k,x^{k+1},\dots,x^{n+k})\mapsto(y^1,\dots,y^k)=(\mu_1x^1,\dots,\mu_kx^k), \tag{1.2}$$

where $\mu_1,\dots,\mu_k$ are positive numbers. Applying the Fubini theorem we deduce

$$\begin{aligned}\int_{\mathbb{R}^{n+k}}\mu_1\cdots\mu_k\varphi(x^1,\dots,x^{n+k})dV_{n+k}(x^1,\dots,x^{n+k})\\
=\int_{\mathbb{R}^{n+k}}\varphi\Big(\frac{y^1}{\mu^1},\dots\frac{x^k}{\mu^k},x^{k+1},\dots,x^{n+k}\Big)|dy^1\cdots dy^kdx^{k+1}\cdots dx^{n+k}|\\
=\int_{\mathbb{R}^k}\left(\int_{A^{-1}(\boldsymbol{y})}\varphi(\boldsymbol{x},\boldsymbol{y})dV_n(\boldsymbol{x})\right)dV_k(\boldsymbol{y}).\end{aligned} \tag{1.3}$$

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Notes for the "*Blue collar seminar on geometric integration theory*".

But for the factor $\mu_1 \cdots \mu_k$, the formulæ (1.1) and (1.3) look similar. To give an invariant meaning to this quantity we need to use the following elementary fact of linear algebra.

Lemma 1.1. *Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are* Euclidean *space of dimensions $n+k$ and respectively k, $n \geq 0$, and $A : \boldsymbol{U} \to \boldsymbol{V}$ is a* surjective *linear map. Then there exist Euclidean coordinates $x^1, \dots, x^{n+k}$ on $\boldsymbol{U}$, Euclidean coordinates $y^1, \dots, y^k$ on $\boldsymbol{V}$ and positive numbers $\mu_1, \dots, \mu^k$ such that, in these coordinates the operator A is described by*

$$y^j = \mu_j x^j, \ \ 1 \leq j \leq k.$$

The numbers $\mu_1^2, \dots, \mu_k^2$ are the eigenvalues of the positive symmetric operator $AA^* : \boldsymbol{V} \to \boldsymbol{V}$ so that

$$\mu_1 \cdots \mu_k = \sqrt{\det AA^*}.$$

Proof. Let $\boldsymbol{W}$ denote the orthogonal complement of $\ker A$ in $\boldsymbol{U}$. Denote by A_0 the restriction of A to W so that $A_0 : \boldsymbol{W} \to \boldsymbol{V}$ is a linear isomorphism. Note that $\boldsymbol{W}$ coincides with the range of the adjoint operator $A^* : \boldsymbol{V} \to \boldsymbol{U}$ so that

$$A_0 A_0^* = AA^*.$$

We want to find a linear isometry $R : \boldsymbol{V} \to \boldsymbol{W}$ such that the operator

$$B = A_0 R : \boldsymbol{V} \to \boldsymbol{V}$$

is symmetric. Note that since R is an isometry we have $R^{-1} = R^*$. Moreover we have a commutative diagram

$$\begin{tikzcd} \boldsymbol{W} \arrow[r, "A_0"] & \boldsymbol{V} \\ \boldsymbol{V} \arrow[u, "R"] \arrow[r, "B"'] & \boldsymbol{V} \arrow[u, "\mathbb{1}_{\boldsymbol{V}}"'] \end{tikzcd}$$

Note that $A_0 A^* : \boldsymbol{V} \to \boldsymbol{V}$ is positive and symmetric. We define

$$R := A_0^*(A_0 A_0^*)^{-1/2} : \boldsymbol{V} \to \boldsymbol{W}.$$

Let us show that R is indeed an isometry. Indeed, for any $\boldsymbol{v} \in \boldsymbol{V}$ we have

$$(R\boldsymbol{v}, R\boldsymbol{v}) = \bigl(\, A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v},\ A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v}\,\bigr) = \bigl(\,(A_0A_0^*)^{-1/2}\boldsymbol{v},\ A_0A_0^*(A_0A_0^*)^{-1/2}\boldsymbol{v}\,\bigr)$$
$$= \bigl(\,(A_0A_0^*)^{-1/2}\boldsymbol{v},\ (A_0A_0^*)^{1/2}\boldsymbol{v}\,\bigr) = (\boldsymbol{v}, \boldsymbol{v}).$$

Clearly $A_0 R = A_0 A_0^*(A_0 A_0^*)^{-1/2} = (A_0 A_0^*)^{1/2}$ is symmetric. Now choose an orthonormal basis that diagonalizes B. Transport it via R to an orthonormal basis of $\boldsymbol{W}$. With respect to these bases of $\boldsymbol{W}$ and $\boldsymbol{V}$ the operator A is described by a diagonal matrix with entries consisting of the eigenvalues of $A_0 R = (A_0 A_0^*)^{1/2}$.

□

Returning to (1.3) we see that

$$\mu_1 \dots \mu_k = J_A := \sqrt{\det AA^*}.$$

The quantity J_A is called the Jacobian of A. Thus, we can rewrite (1.3) as

$$\int_{\mathbb{R}^{n+k}} J_A \varphi(x^1, \dots, x^{n+k}) dV_{n+k}(x^1, \dots, x^{n+k}) = \int_{\mathbb{R}^k} \left(\int_{A^{-1}(\boldsymbol{y})} \varphi(\boldsymbol{x}, \boldsymbol{y}) dV_n(\boldsymbol{x}) \right) dV_k(\boldsymbol{y}). \quad (1.4)$$

Lemma 1.1 shows that (1.4) holds for *any surjective* linear map $\mathbb{R}^{n+k} \to \mathbb{R}^k$.

It is convenient to give a more explicit description of J_A. This relies on the concept of Gramm determinant. More precisely, given a collection of vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ in an Euclidean space $\boldsymbol{U}$ we define their *Gramm determinant* (or *Grammian*) to be the quantity

$$\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) := \det\Big((\boldsymbol{u}_i, \boldsymbol{u}_j)_{\boldsymbol{U}} \Big)_{1 \le i,j \le k},$$

where $(-,-)_{\boldsymbol{U}}$ denotes the inner product in $\boldsymbol{U}$. Geometrically, $\sqrt{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)}$ is the k-dimensional volume of the paralelipiped spanned by the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$,

$$P(\boldsymbol{w}_1, \dots, \boldsymbol{w}_k) = \Big\{ \sum_{j=1}^k t_j \boldsymbol{w}_j; \;\; t_j \in [0,1] \Big\}.$$

Note that $\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = 0$ iff the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ are linearly dependent and $\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = 1$ if the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ form an orthonormal system.

Lemma 1.2. *Let $A : \boldsymbol{U} \to \boldsymbol{V}$ be as in Lemma 1.1. Fix a basis $\boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k}$ of $\boldsymbol{U}_0 := \ker A$ and vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ such that $A\boldsymbol{u}_1, \dots, A\boldsymbol{u}_k$ span $\boldsymbol{V}$. Then*

$$J_A^2 = \frac{\mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_k)\mathbb{G}(\boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}. \tag{1.5}$$

Proof. We first prove the result when $\dim \boldsymbol{U} = \dim \boldsymbol{V}$. In this case the collection $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ is a basis of $\boldsymbol{U}$. Fix an orthonormal basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ of $\boldsymbol{U}$ denote by $T : \boldsymbol{U} \to \boldsymbol{U}$ the linear operator

$$\boldsymbol{e}_j \mapsto \boldsymbol{u}_j.$$

Then

$$\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k) = \det T^*T,$$

$$\mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_k) = \det((AT)^*(AT)) = |\det T^*| \det AA^* |\det T| = J_A^2 \det TT^*.$$

To deal with the general case, we denote by P_0 the orthogonal projection onto $\boldsymbol{U}_0$. Now define

$$\widehat{A} : \boldsymbol{U} \to \widehat{\boldsymbol{V}} := \boldsymbol{V} \oplus \boldsymbol{U}_0, \;\; \boldsymbol{u} \mapsto A\boldsymbol{u} \oplus P_0\boldsymbol{u}.$$

we equip $\widehat{\boldsymbol{V}}$ with the product Euclidean structure.

Let us observe that

$$J_A = J_{\widehat{A}}.$$

Indeed, with respect to the direct sum decomposition $\widehat{\boldsymbol{V}} = \boldsymbol{V} \oplus \boldsymbol{U}_0$ the operator $\widehat{A}\widehat{A}^*$ has the block decomposition

$$\widehat{A}\widehat{A}^* = \begin{bmatrix} AA^* & 0 \\ 0 & \mathbb{1}_{\boldsymbol{U}_0} \end{bmatrix}$$

so that

$$\det \widehat{A}\widehat{A}^* = \det AA^*.$$

Now apply the first part of the proof to deduce that

$$J_A^2 = J_{\widehat{A}}^2 = \frac{\mathbb{G}(\widehat{A}\boldsymbol{u}_1, \dots, \widehat{A}\boldsymbol{u}_k, \widehat{A}\boldsymbol{f}_{k+1}, \dots, \widehat{A}\boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})} = \frac{\mathbb{G}(A\boldsymbol{u}_1, \dots A\boldsymbol{u}_k)\mathbb{G}(\boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}{\mathbb{G}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k, \boldsymbol{f}_{k+1}, \dots, \boldsymbol{f}_{n+k})}.$$

□

Suppose now that X and Y are C^1 manifolds of dimensions $n+k$ and respectively k, $n \geq 0$ equipped with Riemann metrics g_X and g_Y. We denote by $|dV_X|$ and $|dV_Y|$ the volume densities induced by g_M and respectively g_N.

Suppose that $F : X \to Y$ is a C^1-map such that for any $\boldsymbol{p} \in M$ the differential $D_{\boldsymbol{p}}F : T_{\boldsymbol{p}}X \to T_{F(\boldsymbol{p})}Y$ is surjective. We denote by $J_F(\boldsymbol{p})$ the Jacobian of this map.

Theorem 1.3 (The co-area formula: version 1)**.** *For any nonnegative function $\varphi : X \to \mathbb{R}$ which is measurable with respect to the measure defined by $|dV_X|$ we have*

$$\int_X J_F(\boldsymbol{p})\varphi(\boldsymbol{p})|dV_X(\boldsymbol{p})| = \int_Y \left(\int_{F^{-1}(\boldsymbol{q})} \varphi(\boldsymbol{p})|dV_{F^{-1}(\boldsymbol{q})}(\boldsymbol{p})| \right) |dV_Y(\boldsymbol{q}), \tag{1.6}$$

where $|dV_{F^{-1}(\boldsymbol{q})}|$ denotes the volume density on the fiber $F^{-1}(\boldsymbol{q})$ induced by the restriction of g_X to $F^{-1}(\boldsymbol{q})$.

Proof. We consider first the case when X is an open subset of $\mathbb{R}^{n+k}$ with coordinates $(x^1, \dots, x^{n+k})$ equipped with a C^1-metric g_X, Y is an open subset of $\mathbb{R}^k$ with coordinates $(y^1, \dots, y^k)$ equipped with a metric g_Y and the map F is given by

$$y^j = x^j, \;\; j = 1, \dots, k.$$

We have

$$|dV_X| = \sqrt{\mathbb{G}(\partial_{x^1}, \dots, \partial_{x^{n+k}})}|dx^1 \cdots dx^{n+k}|$$
$$= \underbrace{\sqrt{\mathbb{G}(\partial_{x^1}, \dots, \partial_{x^{n+k}})}}_{=:\rho_X}|dy^1 \cdots dy^k dx^{k+1} \cdots dx^{n+k}|,$$
$$|dV_{F^{-1}(q)}| = \underbrace{\sqrt{\mathbb{G}(\partial_{x^{k+1}}, \partial_{x^{n+k}})}}_{=:\rho_F}|dx^{k+1} \cdots dx^{n+k}|,$$

where the subscript X indicates that the inner product in the definition of the above Gramm determinants is the one determined by the Riemann metric on X. Similarly

$$|dV_Y| = \underbrace{\sqrt{G_Y(\partial_{y^1}, \dots, \partial_{y^k}))}}_{=:\rho_Y}|dy^1 \cdots dy^k| = \sqrt{G_Y(DF\partial_{x^1}, \dots, DF\partial_{x^k}))}|dy^1 \cdots dy^k|.$$

Using the Fubini theorem we deduce that for any nonnegative, measurable function $\phi : X \to \mathbb{R}$ we have

$$\int_X \rho_Y \phi \rho_X |dy^1 \cdots dy^k dx^{k+1} \cdots dx^{n+k}| = \int_Y \left(\int_{F^{-1}(\boldsymbol{y})} \rho_X \phi |dx^{k+1} \cdots dx^{n+k}| \right) \rho_Y |dy^1 \cdots dy^k|$$

$$= \int_Y \left(\int_{F^{-1}(\boldsymbol{y})} \frac{\rho_X}{\rho_F} \phi \rho_F |dx^{k+1} \cdots dx^{n+k}| \right) |dV_Y(\boldsymbol{y})| = \int_Y \left(\int_{F^{-1}(\boldsymbol{y})} \frac{\rho_X}{\rho_F} \phi |dV_{F^{-1}(\boldsymbol{y})}| \right) |dV_Y(\boldsymbol{y})|.$$

Suppose that above $\rho_Y \phi = J_F \varphi$, i.e., $\phi = \frac{J_F}{\rho_Y}\varphi$. Then the above equality can be rewritten

$$\int_X J_F(\boldsymbol{x})\varphi(\boldsymbol{x})\,|dV_X(\boldsymbol{x})| = \int_Y \left(\int_{F^{-1}(\boldsymbol{y})} \frac{\rho_X J_F}{\rho_F \rho_Y} \varphi |dV_{F^{-1}(\boldsymbol{y})}| \right) |dV_Y(\boldsymbol{y})|.$$

The co-area formula is proved once we show that

$$\frac{\rho_X J_F}{\rho_F \rho_Y} = 1, \;\; \text{i.e.,} \;\; J_F = \frac{\rho_Y \rho_F}{\rho_X}.$$

The last equality follows from (1.5).

The general case of the co-area formula can be reduced to the special case via partition of unity and the implicit function theorem. $\square$

Corollary 1.4. *Let X, Y and $F : X \to Y$ be as in Theorem 1.3. Then for any measurable function $\phi : X \to \mathbb{R}$ we have*

$$\int_X \phi(\boldsymbol{p})|dV_X(\boldsymbol{p})| = \int_Y \left(\int_{F^{-1}(\boldsymbol{q})} \frac{\phi(\boldsymbol{p})}{J_F(\boldsymbol{p})} |dV_{F^{-1}(\boldsymbol{q})}(\boldsymbol{p})| \right) |dV_Y(\boldsymbol{q}), \tag{1.7}$$

Proof. Apply (1.6) to $\varphi = \frac{\phi}{J_F}$. $\square$

Corollary 1.5. *Suppose X is a C^1 manifold equipped with a C^1-metric g_X, and $f : X \to \mathbb{R}$ is a C^1 function with no critical points . Then for any measurable function $\phi : X \to \mathbb{R}$ we have*

$$\int_X \phi(\boldsymbol{p})|dV_X(\boldsymbol{p})| = \int_{\mathbb{R}} \left(\int_{\{f=t\}} \frac{\phi(\boldsymbol{p})}{|\nabla f(\boldsymbol{p})|} |dV_{f^{-1}(t)}(\boldsymbol{p})| \right) dt. \tag{1.8}$$

In particular, by setting $\phi = 1$ we deduce

$$\operatorname{vol}(X) = \int_{\mathbb{R}} \left(\int_{\{f=t\}} \frac{1}{|\nabla f(\boldsymbol{p})|} |dV_{f^{-1}(t)}(\boldsymbol{p})| \right) dt. \tag{1.9}$$

$\square$

Example 1.6. We want to show how to use (1.9) to compute $\boldsymbol{\sigma}_n$, the "area" of the unit sphere

$$S^n = \Big\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^n; \ \ \sum_{j=0}^n x_j^2 = 1 \Big\}.$$

Let S^n_* denote the unit sphere with the poles $x_0 = \pm 1$. removed. Then $\boldsymbol{\sigma}_n = \operatorname{vol}(S^n_*)$.

Consider $f : S^n_* \to \mathbb{R}$, $f(x_0, \dots, x_n) = x_0$. This function has no critical points on S^n_*. Let $\boldsymbol{p} \in S^n_*$ such that $f(\boldsymbol{p}) = x_0(\boldsymbol{p}) = t$. Denote by θ the angle between the radius $O\boldsymbol{p}$ and the x_0-axis. Note that

$$\cos\varphi = x_0 = t.$$

The gradient of f is the projection of ∂_{x_0} on the tangent plane $T_{\boldsymbol{p}}S^n$. We deduce that

$$|\nabla f(\boldsymbol{p})| = |\partial_{x_0}| \sin\varphi = (1-t^2)^{1/2}.$$

The level set $\{f = t\}$ is an $(n-1)$-dimensional sphere of radius $(1-t^2)^{1/2}$ and we deduce

$$\int_{\{f=t\}} \frac{1}{|\nabla f(\boldsymbol{p})|} |dV_{f^{-1}(t)}(\boldsymbol{p})| = (1-t^2)^{-1/2} \operatorname{vol}(f = t) = \boldsymbol{\sigma}_{n-1}(1-t^2)^{\frac{n-2}{2}}.$$

Hence

$$\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_{n-1} \int_{-1}^1 (1-t^2)^{\frac{n-2}{2}} dt = 2\boldsymbol{\sigma}_{n-1} \int_0^1 (1-t^2)^{\frac{n-2}{2}} dt$$

$(t = \sqrt{s})$

$$= \boldsymbol{\sigma}_{n-1} \int_0^1 (1-s)^{\frac{n}{2}-1} s^{\frac{1}{2}-1} ds =: B\Big(\frac{n}{2} - 1, \frac{1}{2}\Big).$$

The integral $B(p,q)$ was computed by Euler who showed that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Hence

$$\frac{\boldsymbol{\sigma}_n}{\boldsymbol{\sigma}_{n-1}} = \frac{\Gamma(\frac{n}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{n+1}{2})}.$$

Using the equalities $\boldsymbol{\sigma}_0 = 2$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ we deduce

$$\boldsymbol{\sigma}_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}.$$

We can obtain easily $\boldsymbol{\omega}_n$, the volume of the unit n-dimensional ball,

$$\boldsymbol{\omega}_n = \frac{1}{n}\boldsymbol{\sigma}_{n-1} = \frac{\pi^{\frac{n}{2}}}{\frac{n}{2}\Gamma(\frac{n}{2})} = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{1.10}$$

$\square$

2. THE HAUSDORFF MEASURE

Suppose (X, d) is a separable metric space. Fix a nonnegative real number r. For any positive number δ and any set $S \subset X$ we set

$$H^r_\delta(S) := \frac{\boldsymbol{\omega}_r}{2^r} \inf \left\{ \sum_{j\geq 1} (\operatorname{diam} B_j)^r;\;\; S \subset \bigcup_{j\geq 1} B_j,\;\; \operatorname{diam} B_j < \delta \right\}.$$

Note that if

$$0 < \delta_0 < \delta_1 \Rightarrow H^r_{\delta_0}(S) \geq H^r_{\delta_1}(S).$$

Thus the limit

$$\lim_{\delta \searrow 0} H^r_\delta(S)$$

exists and we denote it by H^r. The correspondence $S \mapsto H^r(S)$ is an outer measure satisfying the Caratheodory condition, [6, Chap.12]

$$\operatorname{dist}(S_1, S_2) > 0 \Rightarrow H^r(S_1 \cup S_2) = \mathcal{H}^r(S_1) + H^r(S_2).$$

This implies, [6, Chap. 5], that any Borel set B is measurable with respect to $\mathcal{H}^r_+$, i.e.,

$$H^r_+(Y) = H^r_+(Y \cap B) + H^r_+(Y \setminus B),\;\; \forall Y \subset X.$$

We denote by $\sigma_r(X)$ the set of $\mathcal{H}^r_+$-measurable subset of X and by $\mathcal{H}^r$, or $\mathcal{H}^r_X$ the restriction of $\mathcal{H}^r_+$ to $\sigma_r(X)$. The measure $\mathcal{H}^r$ is called the r-the Hausdorff measure.

Example 2.1. (a) If M is a C^1-manifold of dimension m equipped with a C^0- Riemann metric g that induces a metric space structure on M, then for any Borel set $B \subset M$ we have

$$\mathcal{H}^m_M(B) = \operatorname{vol}_g(B).$$

In particular, $\mathcal{H}^m_M$ coincides with the measure induced by the volume density determined by g.
(b) If M is a C^1-submanifold of dimension k of C^1 Riemann manifold X of dimension n, then

$$\mathcal{H}^k_X(M) = \operatorname{vol}(M),$$

where $\operatorname{vol}(M)$ denotes the volume of M with respect to the Riemann metric induced by the Riemann metric on X.
(c) If X, Y are locally compact metric spaces, $F : X \to Y$ is a Lipschitz map with Lipschitz constant $\leq L$, and $B \subset X$ is a Borel set, then $F(B)$ is $\mathcal{H}^r_Y$-measurable and

$$\mathcal{H}^r_Y\big(F(B)\big) \leq L^r \mathcal{H}^r_X(B).$$

For proofs of the above statements (a), (b), (c) we refer to [6, Chap 12]. □

We have the following density result concerning Hausdorff measurable functions.

Theorem 2.2. *Suppose that X a separable metric space $S \subset X$ is a $\mathcal{H}^m$-measurable set such that $\mathcal{H}^m(S) < \infty$. Then for for $\mathcal{H}^m$-almost any $x \in X \setminus S$ we have*

$$\underbrace{\limsup_{r\searrow 0} \frac{\mathcal{H}^m\big(\, B_r(x) \cap S\,\big)}{\boldsymbol{\omega}_m r^m}}_{=:\Theta^{*m}(S,x)} = 0. \qquad \square$$

For a proof we refer to [3, §4.3] or [5, §3.].

Corollary 2.3. *Suppose that S is a C^1 manifold of dimension m equipped with a C^0 Riemann metric.There exists a subset $S^* \subset S$ such that the following hold.*

- $\mathcal{H}^m(S \setminus S^*) = 0$.
- *For any $x \in S^*$ we have*

$$\lim_{r\searrow 0} \frac{\mathcal{H}^m(S \cap B_r(x))}{\mathcal{H}^m(X \cap B_r(x))} = 1.$$

Proof. We have

$$\mathcal{H}^m(S \cap B_r(x)) = \mathcal{H}^m(X \cap B_r(x)) - \mathcal{H}^m\big(\, S^c \cap B_r(x)\,\big).$$

From Theorem 2.2 we deduce that there exits a subset $S^* \subset S$ such that $\mathcal{H}^m(S \setminus S^*) = 0$ and for any $x \in S^*$ we have

$$\Theta^{*m}(S^c, x) = 0, \;\; \text{i.e.} \;\; \limsup_{r\searrow 0} \frac{\mathcal{H}^m(S^c \cap B_r(x))}{\boldsymbol{\omega}_m r^m} = 0.$$

We deduce

$$\begin{aligned}\liminf_{r\searrow 0} \frac{\mathcal{H}^m(S \cap B_r(x))}{\boldsymbol{\omega}_m r^m} &= \liminf_{r\searrow 0} \frac{\mathcal{H}^m(X \cap B_r(x))}{\boldsymbol{\omega}_m r^m} - \limsup_{r\searrow 0} \frac{\mathcal{H}^m(S^c \cap B_r(x))}{\boldsymbol{\omega}_m r^m} \\ &= \liminf_{r\searrow 0} \frac{\mathcal{H}^m(X \cap B_r(x))}{\boldsymbol{\omega}_m r^m}.\end{aligned}$$

The desired conclusion follows by observing that

$$\lim_{r\searrow 0} \frac{\boldsymbol{\omega}_m r^m}{\mathcal{H}^m(X \cap B_r(x))} = 1 \geq \liminf_{r\searrow 0} \frac{\mathcal{H}^m(S \cap B_r(x))}{\mathcal{H}^m(X \cap B_r(x))}.$$

The result is now obvious. □

Theorem 2.4 (Eilenberg inequality)**.** *Suppose (X, d_X) is a separable metric space and Y is a C^1 manifold of dimension k equipped with a C^0-Riemann metric g. Denote by $d_Y : Y \times Y \to \mathbb{R}$ the metric on Y induced by g. Let $F : X \to Y$ be a map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\,\big) \leq L d_X(x_1, x_2), \;\; \forall x_1, x_2 \in X.$$

Then for any $m \geq k$ there exists a constant[1] *$C(m,k) > 0$ such that for any Borel set $A \subset X$ we have*

$$\int_Y^* \mathcal{H}_X^{m-k}\big(A \cap F^{-1}(y)\,\big) d\mathcal{H}^k(y) \leq C(m,k) L^k \mathcal{H}^m(A),$$

[1] We can $C(m,k) = \frac{\boldsymbol{\omega}_{m-k}\boldsymbol{\omega}_k}{\boldsymbol{\omega}_m}$

where $\int^*$ denotes the upper Lebesgue integral. □

For a proof of this inequality we refer to [1, §13.3] or [3, §5.2.1]. The strategy behind the proof is identical to the strategy behind the proof of Lemma 2.7 described a bit later. As explained in [3, §5.2.1], this inequality implies that the following technical result.

Corollary 2.5. *Let $F : X \to Y$ be as in Theorem 2.4. Then for any $m \geq k$ and any Borel subset $A \subset X$ the map*

$$Y \ni y \mapsto \mathcal{H}_X^{m-k}\big(A \cap F^{-1}(y)\,\big) \in [0,\infty]$$

is $\mathcal{H}_Y^k$-measurable. □

Theorem 2.6 (The co-area formula: version 2)**.** *Suppose X and Y are C^1 Riemann manifolds of dimensions $n+k$ and respectively k, $n \geq 0$. If $F : X \to Y$ is a C^1-map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\big) \leq L d_X(x_1, x_2), \;\; \forall x_1, x_2 \in X,$$

then, for any $\mathcal{H}_X^{n+k}$-measurable subset $A \subset X$ we have

$$\underbrace{\int_A J_F(x) d\mathcal{H}_X^{n+k}(x)}_{=:I(A)} = \underbrace{\int_Y \mathcal{H}_M^n\big(\,A \cap F^{-1}(y)\,\big) d\mathcal{H}_Y^k(y)}_{=:J(A)}. \tag{2.1}$$

Proof. We follow closely the strategy in [1, §13.4]. We prove (2.1) in several several gradually more general cases.

Step 1. *We prove that $I(A) = J(A)$ if A is compact and F has no critical points on A.* Choose a small open neighborhood $\mathcal{O}$ of A in M such that F has no critical points in $\mathcal{O}$. The equality $I(A) = J(A)$ follows from Theorem 1.3 applied to the map $F : \mathcal{O} \to Y$ and the function $\varphi = 1_A$, the indicator function of A.

Step 2. *We prove that $I(A) = J(A)$ if F has no critical points on A.* We choose a family of compact sets $C_\varepsilon \subset A$, $\varepsilon > 0$ such that

$$\mathcal{H}_X^{n+k}(A \setminus C_\varepsilon) \leq \varepsilon, \;\; C_\varepsilon \subset C_{\varepsilon'}, \;\; \forall \varepsilon \geq \varepsilon' > 0.$$

From the monotone convergence theorem we deduce that

$$\lim_{\varepsilon \to 0} I(C_\varepsilon) = I(A). \tag{2.2}$$

From Step 1 we deduce

$$J(A) = J(C_\varepsilon) + J(A \setminus C_\varepsilon) = I(C_\varepsilon) + J(A \setminus C_\varepsilon). \tag{2.3}$$

From the Eilenberg inequality we deduce

$$J(A \setminus C_\varepsilon) \leq C(m,k) L^k \varepsilon,$$

so that

$$\lim_{\varepsilon \to 0} J(A \setminus C_\varepsilon) = 0. \tag{2.4}$$

We obtain (2.1) by letting $\varepsilon \to 0$ in (2.3) and then invoking (2.2) and (2.4).

Step 3. *We prove that $I(A) = J(A)$ for any A.* Choose a compact set $C_\varepsilon \subset A$ such that

$$\mathcal{H}_X^{n+k}(A \setminus C_\varepsilon) < \varepsilon.$$

Define

$$C_\varepsilon^0 := \{ x \in C_\varepsilon;\;\; J_F(x) = 0 \}.$$

Then

$$\begin{aligned} J(A) - I(A) &= J(A \setminus C_\varepsilon) - I(A \setminus C_\varepsilon) + J(C_\varepsilon \setminus C_\varepsilon^0) - I(C_\varepsilon \setminus C_\varepsilon^0) + J(C_\varepsilon^0) - I(C_\varepsilon^0) \\ &= J(A \setminus C_\varepsilon) - I(A \setminus C_\varepsilon) + J(C_\varepsilon \setminus C_\varepsilon^0) - I(C_\varepsilon \setminus C_\varepsilon^0) + J(C_\varepsilon^0). \end{aligned}$$

From Step 2 we know that $J(C_\varepsilon \setminus C_\varepsilon^0) - I(C_\varepsilon \setminus C_\varepsilon^0) = 0$ and the proof of Step 2 shows that

$$\lim_{\varepsilon \to 0} \Big(J(A \setminus C_\varepsilon) - I(A \setminus C_\varepsilon) \Big) = 0.$$

Hence

$$J(A) - I(A) = \lim_{\varepsilon \to 0} J(C_\varepsilon^0).$$

The equality (2.1) now follows from the following Sard-like result.

Lemma 2.7. *If C is a compact subset of X such that $J_F(x) = 0$, for any $x \in C$ then*

$$\int_Y \mathcal{H}_X^n(C \cap F^{-1}(y))\, d\mathcal{H}_Y^k(y) = 0.$$

□

Proof of Lemma 2.7. Let us first observe that for any $\boldsymbol{p} \in C$ and any $\varepsilon > 0$ there exists $r_\varepsilon = r_\varepsilon(\boldsymbol{p})$ such that for any $0 < r < r_\varepsilon(x)$ we have

$$\mathcal{H}^k\big(F(B(\boldsymbol{p}, r)) \big) \leq \varepsilon L^{k-1} r^k. \tag{2.5}$$

Indeed, we have $\operatorname{rank} D_{\boldsymbol{p}} F \leq k-1$. The definition of the differential of F at x implies that, given a choice of coordinates x near $\boldsymbol{p}$ such that $x(\boldsymbol{p}) = 0$ we have

$$F(x) = F(0) + A_{\boldsymbol{p}} x + o(|x|), \;\; A_{\boldsymbol{p}} := D_{\boldsymbol{p}} F.$$

Hence, for any $\varepsilon > 0$, the set $F(B(\boldsymbol{p}, r))$ is contained in a k-dimensional polydisk of the form $\mathbb{D}^{k-1}(F(\boldsymbol{p}), Lr) \times [-\varepsilon r, \varepsilon r]$ if r is sufficiently small, $r < r_\varepsilon(\boldsymbol{p})$. Above, $\mathbb{D}^{k-1}(y, R)$ indicates a $(k-1)$-disk of center y and radius R. Since C is compact we can assume that

$$r_\varepsilon := \inf_{\boldsymbol{p} \in C} r_\varepsilon(\boldsymbol{p}) > 0.$$

We deduce that

$$\mathcal{H}^k\big(F(S \cap C) \big) \leq \varepsilon L^{k-1} \operatorname{diam}(S)^k, \;\; \forall S \subset X, \;\; \operatorname{diam} S < \frac{1}{2} r_\varepsilon. \tag{2.6}$$

For any $s > 0$ we can find a countable cover of C in X by measurable sets $(X_i^s)_{i \geq 1}$ such that

$$\operatorname{diam}(X_i^s) < \frac{1}{s} \;\text{ and }\; \mathcal{H}^{n+k}(C) \geq \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \sum_{i \geq 1} (\operatorname{diam} X_i^s)^{n+k} - \frac{1}{s}. \tag{2.7}$$

By definition

$$\mathcal{H}^n\big(C \cap f^{-1}(y)\big) \leq \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \liminf_{s \to \infty} \sum_{i \geq 1} \big(\operatorname{diam} X_i^s \cap f^{-1}(y)\big)^n.$$

For any set $E \subset X$ we denote by φ_E the characteristic function of the closure of $F(E)$. We can then rewrite the above equality as

$$\mathcal{H}^n\big(C \cap f^{-1}(y)\big) \leq \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}} \liminf_{s \to \infty} \sum_{i \geq 1} \big(\operatorname{diam} X_i^s\big)^n \varphi_{X_i^s}(y).$$

The Fatou lemma then implies

$$\int_Y^* \mathcal{H}^n\big(C\cap f^{-1}(y)\big)\,d\mathcal{H}^k_Y \le \frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}}\liminf_{s\to\infty}\sum_{i\ge 1}\big(\operatorname{diam} X_i^s\big)^n\int_Y \varphi_{X_i^s}(y)d\mathcal{H}^k_Y.$$

Fix $\varepsilon>0$. We deduce from (2.6) that for s sufficiently large, $s>s_\varepsilon$ we have

$$\int_Y \varphi_{X_i^s}(y)d\mathcal{H}^k_Y \le \varepsilon L^{k-1}(\operatorname{diam} X_i^s)^k.$$

Hence

$$\int_Y^* \mathcal{H}^n\big(C\cap f^{-1}(y)\big)\,d\mathcal{H}^k_Y \le \varepsilon L^{k-1}\frac{\boldsymbol{\omega}_{n+k}}{2^{n+k}}\liminf_{s\to\infty}\sum_{i\ge 1}\big(\operatorname{diam} X_i^s\big)^{n+k}$$

$$\stackrel{(2.7)}{\le} \varepsilon L^{k-1}\left(\mathcal{H}^{n_k}(C)+\frac{1}{s_\varepsilon}\right).$$

Now let $\varepsilon\to 0$. □

Remark 2.8. The proof of the Eilenberg inequality follows an identical strategy with the inequality (2.5) replaced by the inequality

$$\mathcal{H}^k\big(F(S)\big)\le C(m,k)(\dim S)^k$$

for any Borel set $S\subset X$ with sufficiently small diameter. □

Corollary 2.9. *Let $F:X\to Y$ be as in Theorem 2.6. Then for any nonnegative measurable function $\varphi:X\to\mathbb{R}$ we have*

$$\int_X \varphi(x)J_F(x)d\mathcal{H}^{n+k}_X(x)=\int_Y\left(\int_{F^{-1}(y)} g(x)d\mathcal{H}^n_X(x)\right)d\mathcal{H}^k_Y(y). \tag{2.8}$$

Proof. By Theorem 2.6 the equality (2.8) is true when φ is the characteristic function of a measurable subset of X. By linearity, (2.8) is trie for linear combinations of such functions. We now observe that for any measurable nonnegative function φ we can find a sequence of simple functions $(\varphi_\nu)_{\nu\ge 1}$ that converges increasingly and almost everywhere to φ. □

3. Lipschitz maps

To formulate our last and most general version of the co-area formula we need to recall a few facts about Lipschitz maps between locally Euclidean sets.

Theorem 3.1 (Rademacher)**.** *Suppose $U_k\subset\mathbb{R}^{n_k}$, $k=0,1$ are open sets and $F:U_0\to U_1$ is a Lipschitz map. Then the map F is almost everywhere differentiable and the differential is a measurable map $U_0\to\operatorname{Hom}(\mathbb{R}^{n_0},\mathbb{R}^{n_1})$. Moreover, for any $\varepsilon>0$ there exists a C^1 map $F_\varepsilon:U_0\to\mathbb{R}^{n_1}$ such that*

$$\operatorname{vol}\Big(\big\{x\in U_0;\ \ F(x)\ne F_\varepsilon(x)\big\}\Big)+\operatorname{vol}\Big(\big\{x\in U_0;\ \ DF(x)\ne DF_\varepsilon(x)\big\}\Big)<\varepsilon$$ □

For a proof we refer to [3, §5.1].

Theorem 3.2 (Extension theorem)**.** *Suppose that $S\subset\mathbb{R}^n$ is a closed subset and $F:S\to\mathbb{R}$ is a Lipschitz function. Then f admits an extension to a Lipschitz function $\widetilde{f}:\mathbb{R}^n\to\mathbb{R}$ that has the same Lipschitz constant as f.* □

For a proof we refer to [3, Thm. 5.1.12].

Theorem 3.3 (The co-area formula: version 3)**.** *Suppose X and Y are C^1 Riemann manifolds of dimensions $n+k$ and respectively k, $n \geq 0$. If $F : M \to N$ is a map satisfying the Lipschitz condition*

$$d_Y\big(F(x_1), F(x_2)\big) \leq L d_X(x_1, x_2), \;\; \forall x_1, x_2 \in X,$$

then, for any $\mathcal{H}_X^{n+k}$-measurable subset $A \subset X$ we have

$$\underbrace{\int_A J_F(x) d\mathcal{H}_X^{n+k}(x)}_{=:I(A)} = \underbrace{\int_Y \mathcal{H}_M^n\big(\, A \cap F^{-1}(y)\,\big) d\mathcal{H}_Y^k(y)}_{=:J(A)}. \tag{3.1}$$

Proof. Clearly, it suffices to prove the theorem for sets A with the following property: A is contained in a coordinate neighborhood $U_0 \subset X$ and $F(U_0)$ is contained in a coordinate neighborhood $U_1 \subset Y$ such that U_0 is bi-Lipschitz homeomorphic to a bounded open subset in $\mathbb{R}^{n+k}$ and U_1 is bi-Lipschitz homeomorphic to a bounded open set in $\mathbb{R}^k$. For any $\varepsilon > 0$ we can find a compact subset $C_\varepsilon \subset U_0$ and a C^1-map $F_\varepsilon : U_0 \to \mathbb{R}^k$ such that

$$\mathcal{H}_X^{n+k}(U_0 \setminus C_\varepsilon) < \varepsilon, \;\; F|_{C_\varepsilon} = F_\varepsilon|_{C_\varepsilon}, \;\; J_F|_{C_\varepsilon} = J_{F_\varepsilon}|_{C_\varepsilon}.$$

Then

$$I(A) - J(A) = I(A \cap C_\varepsilon) - J(A \cap C_\varepsilon) + I(A \setminus C_\varepsilon) - J(A \setminus C_\varepsilon).$$

The monotone convergence theorem implies that

$$\lim_{\varepsilon \searrow 0} I(A \setminus C_\varepsilon) = 0$$

while the Eilenberg inequality implies that

$$\lim_{\varepsilon \searrow 0} J(A \setminus C_\varepsilon) = 0.$$

On the other hand, there exists an open neighborhood V_0 of C_ε in U_0 such that $F_\varepsilon(V_0) \subset U_1$. Applying Theorem 2.6 to the C^1-map $F_\varepsilon : V_0 \to U_1$ we deduce that

$$I(C_\varepsilon) = \int_{C_\varepsilon} J_F(x) d\mathcal{H}_X^{n+k}(x) = \int_{C_\varepsilon} J_{F_\varepsilon}(x) d\mathcal{H}_X^{n+k}(x)$$

$$\stackrel{(2.1)}{=} \int_Y \mathcal{H}_M^n\big(\, C_\varepsilon \cap F_\varepsilon^{-1}(y)\,\big) d\mathcal{H}_Y^k(y) = \int_Y \mathcal{H}_M^n\big(\, C_\varepsilon \cap F^{-1}(y)\,\big) d\mathcal{H}_Y^k(y) = J(C_\varepsilon).$$

□

Corollary 3.4 (Area formula)**.** *Let X, Y be two n-dimensional C^1-manifolds equipped with C^0-Riemann metrics and $F : X \to Y$ a Lipschitz map. Then*

$$\int_Y \# F^{-1}(y) d\mathcal{H}^n(y) = \int_X J_F(x) d\mathcal{H}^n(x). \tag{3.2}$$

□

4. RECTIFIABLE SETS

A set $S \subset \mathbb{R}^n$ is said to be *countably m-rectifiable* if it is $\mathcal{H}^m$-measurable and

$$S \subset S_0 \cup \left(\bigcup_{j \geq 1} F_j(\mathbb{R}^m) \right),$$

where

- $\mathcal{H}^m(S_0) = 0$;
- the functions $F_j : \mathbb{R}^m \to \mathbb{R}^n$ are Lipshitz, $\forall j \geq 1$.

We have the following result, [3, §5.4].

Proposition 4.1. *Suppose that $S \subset \mathbb{R}^n$ is $\mathcal{H}^m$-measurable and countably m-rectifiable. Then*

$$S = \bigcup_{j=0}^{\infty} S_j,$$

where

- $\mathcal{H}^m(X_0) = 0$*;*
- $S_i \cap S_j = \emptyset$ *if* $i \neq j$*;*
- *for* $j \geq 1$ *there exists an* m*-dimensional* C^1*-submanifold* $X_j \subset \mathbb{R}^n$ *such that* $S_j \subset X_j$*.*

Definition 4.2. If S is a $\mathcal{H}^m$-measurable subset of $\mathbb{R}^n$, then we say that an m-dimensional vector subspace $W \subset \mathbb{R}^n$ is the approximate tangent space for S at $x \in \mathbb{R}^n$ if

$$\lim_{r \searrow 0} \int_{r^{-1}(S-x)} f(y) d\mathcal{H}^m(y) = \int_W f(y) d\mathcal{H}^m(y), \;\; \forall f \in C^0_{cpt}(\mathbb{R}^n).$$

Proposition 4.3. *Suppose that $S \subset \mathbb{R}^n$ is a countably m-rectifiable set such that $\mathcal{H}^m(S \cap K) < \infty$ for any compact subset $K \subset \mathbb{R}^m$. Then there exists a subset $S_{\mathrm{sing}} \subset S$ such that*

- $\mathcal{H}^m(S_{\mathrm{sing}}) = 0$ *and*
- *for any* $x \in S \setminus S_{\mathrm{sing}}$ *there exist an approximate tangent space to* S *at* x*.*

Proof. We write S as in Proposition 4.1

$$S = \bigcup_{j=0}^{\infty} S_j$$

where S_j is contained in a C^1, m-dimensional submanifold $X_j \subset \mathbb{R}^n$, $S_i \cap S_j = \emptyset$, $\forall i \neq j$, $\mathcal{H}^m(S_0) = 0$. For $j > 0$ we denote by S_j^* the set of points $x \in S_j$ such that

$$\lim_{r \searrow 0} \frac{\mathcal{H}^m((S - S_j) \cap B_r(x))}{r^m} = \lim_{r \searrow 0} \frac{\mathcal{H}^m((X_j - S_j) \cap B_r(x))}{r^m} = 0.$$

By Theorem 2.2 we have $\mathcal{H}^m(S_j \setminus S_j^*) = 0$. We will show that S admits an approximate tangent space at any point $x \in S_j^*$. Indeed, suppose $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^n)$. For simplicity assume that $\operatorname{supp} f \subset B_1(0)$, and $f \geq 0$. Then using the change in variables $y = \frac{1}{r}(z - x)$

$$\int_{\frac{1}{r}(S-x)} f(y) d\mathcal{H}^m(y) = \frac{1}{r^m} \int_S f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z)$$

$$= \frac{1}{r^m} \int_{B_r(x) \cap S} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z)$$

Now observe that

$$\frac{1}{r^m}\left|\int_{B_r(x)\cap S} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) - \int_{B_r(x)\cap S_j} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z)\right|$$
$$\leq \sup f \frac{\mathcal{H}^m(B_r(x)\cap(S_j\setminus S)}{r^m} \to 0,$$

and

$$\frac{1}{r^m}\left|\int_{B_r(x)\cap S_j} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z) - \int_{B_r(x)\cap X_j} f\left(\frac{1}{r}(z-x)\right) d\mathcal{H}^m(z)\right|$$
$$\leq \sup f \frac{\mathcal{H}^m(B_r(x)\cap(X_j\setminus S_j)}{r^m} \to 0$$

Hence

$$\lim_{r\searrow 0}\left(\int_{\frac{1}{r}(S-x)} f(y)d\mathcal{H}^m(y) - \int_{\frac{1}{r}(X_j-x)} f(y)d\mathcal{H}^m(y)\right) = 0.$$

□

Suppose $X \subset \mathbb{R}^n$ is a C^1 m-dimensional manifold of finite (induced) volume and $S \subset X$ is a $\mathcal{H}^m$-measurable subset. We set $S^c := X \setminus S$.

Suppose that $S \subset \mathbb{R}^N$ is a countably $(n+k)$-rectifiable subset of $\mathbb{R}^N$. We can then express S as a disjoint union

$$S = \bigsqcup_{j=0}^{\infty} S_j$$

where $\mathcal{H}^{n+k}(S_0) = 0$, and S_j is contained in a C^1-submanifold $X_j \subset \mathbb{R}^N$, $\dim X_j = n+k$, $\forall j \geq 1$.

A Lipschitz map on $f : S \to \mathbb{R}^M$ admits a Lipschitz extension to the closure of S and thus a Lipschitz extension to a map $F : \mathbb{R}^N \to \mathbb{R}^M$. The restriction of F to each X_j is $\mathcal{H}^{n+k}$ a.e. differentiable and for $\mathcal{H}^{n+k}$ a.e. point $\boldsymbol{p} \in S_j$ we have a differential

$$D_{\boldsymbol{p}}F : T_{\boldsymbol{p}}S \to \mathbb{R}^m.$$

One can show that if $F(S)$ is contained in a countably k-rectifiable subset $Z \subset \mathbb{R}^n$, then fo $\mathcal{H}^{n+k}$ a.e. point $\boldsymbol{p} \in S$ the set Z admits a tangent space at $\boldsymbol{q} = F(\boldsymbol{p})$ and moreover

$$D_{\boldsymbol{p}}F(T_{\boldsymbol{p}}S) \subset T_{\boldsymbol{q}}Z.$$

We denote by $J_F(\boldsymbol{p})$ the Jacobian of this map. We can now state the fibnal version of the coarea formula.

Theorem 4.4 (The co-area formula: final version)**.** *Suppose that $S \subset \mathbb{R}^N$ is $(n+k)$-rectifiable, $Z \subset \mathbb{R}^M$ is k- rectifiable and $F : S \to Z$ is a Lipschitz map. Then, for any $\mathcal{H}^{n+k}$-measurable subset $A \subset S$ we have*[2]

$$\int_A J_F(\boldsymbol{p})d\mathcal{H}^{n+k}(\boldsymbol{p}) = \int_Z \mathcal{H}^n\big(\,F^{-1}(z)\,\big)\cap A)d\mathcal{H}^k(z). \tag{4.1}$$

□

The proof is obtained by putting together all the facts we have gathered so far. For details we refer to [2, Thm. 3.2.22] or [5, §12]

[2] Implicit in the statement of (4.1) is the fact that the various integrands are measurable with respect to the appropriate measures.

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