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Lattice Points Inside Rational Simplices and the Casson Invariant of Brieskorn Spheres

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Abstract. We express the number of lattice points inside certain simplices with vertices in $\mathbb{Q}^3$ or $\mathbb{Q}^4$ in terms of Dedekind–Rademacher sums. This leads to an elementary proof of a formula relating the Euler characteristic of the Seiberg–Witten-Floer homology of a Brieskorn $\mathbb{Z}$-homology sphere to the Casson invariant.

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0. Introduction

From the very beginning, it was apparent that the Seiberg–Witten analogue of the instanton Floer homology of a ($\mathbb{Z}$-)homology 3-sphere is no longer a topological invariant, since it can vary with the metric. W. Chen [2], Y. Lim [7] and Marcolli–Wang [8] have explained the metric dependence of the Euler characteristic of the SWF (= Seiberg–Witten–Floer) homology.

More precisely, if g_i $(i = 0, 1)$ are two generic Riemann metrics on a homology 3-sphere N and $\lambda_{\mathrm{SW}}(N, g_i)$ is the Euler characteristic of the SWF homology of (N, g_i), the results of [2, 7, 8] imply that

$$\chi_{\mathrm{SW}}(N, g_1) - \chi_{\mathrm{SW}}(N, g_0) = \tfrac{1}{8}\mathbf{F}(g_1) - \tfrac{1}{8}\mathbf{F}(g_0),$$

where

$$\mathbf{F}(g) = 4\eta_{\mathrm{dir}}(g) + \eta_{\mathrm{sign}}(g)$$

$\eta_{\mathrm{dir}}(g)$ denotes the eta invariant of the Dirac operator of (N, g) while $\eta_{\mathrm{sign}}(g)$ denotes the eta invariant of the odd signature operator on (N, g). In particular, the above equality shows that the quantity

$$\alpha(N) = -\chi_{\mathrm{SW}}(N, g) + \tfrac{1}{8}\mathbf{F}(g)$$

is independent of g and thus is a topological quantity.

In 1996, W. Chen, P. Kronheimer and T. Mrowka have conjectured that this quantity coincides (up to a sign) with the Casson invariant of N. This has been established recently by Y. Lim, [6].

In the present paper we consider in greater detail the special case of Brieskorn homology spheres $\Sigma(a_1, \ldots, a_n)$ with at most 4 singular fibers, $n \leqslant 4$ and explore the rich arithmetic hiding behind these objects. More precisely, the results [13] show that for a certain natural metric g_0 on $\Sigma(a_1, a_2, \ldots, a_n)$ (realizing the Thurston geometric of this Seifert manifold), and we have

$$-\chi_{\mathrm{SW}}(\Sigma(a_1, \ldots, a_n), g_0) = 2C_{a_1,\ldots,a_n}, \quad n = 3, 4$$

where $C_{a_1,\ldots,a_n}$ is the number of lattice points in the simplex

$$\Delta(a_1, \ldots, a_n) := \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_i \geqslant 0, \sum_{i=1}^{n} \frac{x_i}{a_i} < \frac{1}{2}\left(n - 2 - \sum_{i=1}^{n} \frac{1}{a_i}\right)\right\}.$$

The first goal of this paper is the direct, explicit and elementary determination of $C_{a_1,\ldots,a_n}$ when $n = 3, 4$. This arithmetic problem intrigued the author since the vertices of the simplex $\Delta(a_1, \ldots, a_n)$ are *not lattice points* and some of the counting techniques using Riemann–Roch theorem on toric varieties seem to apply.⋆ Still, this is not the most general rational simplex. It displays a miraculous symmetry (Lemma 2.2) which, when used in conjunction with a generalization of an idea of Mordell (see [10, 15]), leads to an *explicit formula* of this number in terms of Dedekind–Rademacher sums (see below). In fact, very little additional effort is needed to compute the entire Ehrhart polynomial of this polytope.

On the topological side of the story, R. Fintushel and R. Stern have shown in [3] that the Casson invariant $\lambda(\Sigma(a, b, c,))$ of the Brieskorn sphere $\Sigma(a, b, c)$ is $\frac{1}{8}\sigma(a, b, c)$ where $\sigma(a, b, c)$ denotes the signature of the Milnor fiber of $\Sigma(a, b, c)$. This result was extended to arbitrary $\Sigma(a_1, \ldots, a_n)$ by Neuwmann–Wahl in [11],

$$\lambda(\Sigma(a_1, a_2, \ldots, a_n)) = \tfrac{1}{8}\sigma(a_1, \ldots, a_n). \tag{0.1}$$

The Chen–Kronheimer–Mrowka formula in this case is equivalent to

$$-2C_{a_1,\ldots,a_n} - \tfrac{1}{8}\mathbf{F}(g_0) = \tfrac{1}{8}\sigma(a_1, \ldots, a_n). \tag{0.2}$$

According to Zagier (see [4, 11]) the signature of $\sigma(a_1, \ldots, a_n)$ can be expressed in terms of Dedekind sums. In [13] we have expressed $\mathbf{F}(g_0)$ in terms of Dedekind–Rademacher sums as well. Thus (0.2) becomes an identity between Dedekind sums. The second goal of this paper is to prove the identity between Dedekind–Rademacher sums by elementary means.

⋆M. Vergne has kindly pointed out to the author that the paper [1] contains a description of the Ehrhart polynomial of a general rational polytope. However, we believe that the amount of work required to translate the general formulae of [1] into the very explicit language of Dedekind–Rademacher sums needed for the topological applications in this paper would involve about the same amount of work as our ad-hoc proof based on Mordell's trick.

The *Dedekind–Rademacher* sums are defined for every coprime positive integers h, k, and any real numbers x, y by

$$s(h, k; x, y) = \sum_{\mu=0}^{k-1} \left(\left(\frac{\mu + y}{k}\right)\right)\left(\left(\frac{h(\mu + y)}{k} + x\right)\right)$$

where for any $r \in \mathbb{R}$ we set

$$((r)) = \begin{cases} 0, & r \in \mathbb{Z}, \\ \{r\} - \frac{1}{2}, & r \in \mathbb{R} \setminus \mathbb{Z}, \end{cases} \qquad (\{r\} := r - [r]).$$

The above description is theoretically very convenient but computationally cumbersome. Due to the reciprocity law (see [14] or the Appendix) the numerical determination of these sums in concrete cases is as computationally complex as the Euclid's algorithm for the pair (h, k). The sums $s(h, k; 0, 0)$ are precisely the Dedekind sums $s(h, k)$ discussed in great detail in [4,15].

The present paper consists of three sections and an appendix. In the first section we survey the results of [13] which express $\mathbf{F}$ in terms of Dedekind–Rademacher sums and reduce the computation of χ_{SW} to a lattice point count. In the next section, we describe a generalization of an idea of Mordell which reduces the lattice point count to a certain arithmetic expression. The third section describes this arithematic expression in terms of Dedekind–Rademacher sums and completes the proof of (0.2). For the reader's convenience we have included a brief appendix containing the basic properties of Dedekind–Rademacher sums used in this paper.

1. Geometric Preliminaries

For pairwise coprime integers $a_1, \ldots, a_n \geqslant 2$, $n \geqslant 3$ we denote by $\Sigma(\vec{a})$, $\vec{a} = (a_1, \ldots, a_n)$, the Brieskorn homology sphere $\Sigma(a_1, \ldots, a_n)$ with n singular fibers (see [5] for a precise definition). We orient $\Sigma(\vec{a})$ as the boundary of a complex manifold. $\Sigma(\vec{a})$ is a Seifert manifold. With respect to the above orientation it is a singular S^1 fibration over the orbi-sphere $S^2(\vec{a})$ which has n cone points of isotropies $\mathbb{Z}_{a_i}$, $1 \leqslant i \leqslant n$. This fibration has rational degree

$$\ell = -\frac{1}{A}, \quad A := a_1 a_2, \ldots, a_n.$$

Set $b_i := A/a_i$. The Seifert invariants $\vec{\beta} = (\beta_1, \ldots, \beta_n)$ (normalized as in [13]) are defined by

$$\beta_i b_i \equiv -1 \,(\mathrm{mod}\; a), \quad 0 \leqslant \beta_i < a_i.$$

Set $q_i := \beta_i^{-1} = -b_i (\mathrm{mod}\; a_i)$, $i = 1, \ldots, n$. The canonical line bundle of $S^2(\vec{a})$ has rational degree

$$\kappa = \kappa(\vec{a}) := (n-2) - \sum_{i=1}^{n} \frac{1}{a_i}.$$

The universal covering space of $\Sigma(\vec{a})$ is the Lie group $G = G(\vec{a})$

$$G(\vec{a}) = \begin{cases} SU(2), & \kappa(\vec{a}) < 0, \\ \widetilde{PSL}_2(\mathbb{R}), & \kappa(\vec{a}) > 0, \end{cases}$$

where $\widetilde{PSL}_2(\mathbb{R})$ denotes the universal cover of $PSL_2(\mathbb{R})$. Moreover, $\Sigma(\vec{a}) \cong \Gamma/G$ where Γ is a discrete subgroup of G. The natural left invariant metrics on G (see [17]) induce a natural metric g_0 on $\Sigma(\vec{a})$. All the geometric quantities discussed in the sequel are computed with respect to this metric and for simplicity we will omit g_0 from the various notations. Thus $\mathbf{F}(\vec{a})$ is $\mathbf{F}(g_0)$.

Set

$$\rho := \left\{\frac{\kappa}{2\ell}\right\} = \begin{cases} \frac{1}{2}, & A \text{ even}, \\ 0, & A \text{ odd}, \end{cases}$$

and define $\vec{\gamma} = (\gamma_1, \ldots, \gamma_n)$ by the equalities

$$\gamma_i = m\beta_i \,(\mathrm{mod}\ \alpha_i), \quad 1 \leqslant i \leqslant n,$$

where m is the integer

$$m := \frac{\kappa}{2\ell} - \rho = \frac{u - (n-2)A - 2\rho}{2}, \quad u := \sum_i b_i.$$

In [13] we have proved the following:

- *If A is even* then

$$\mathbf{F}(\vec{a}) = 1 - 4\sum_i s(\beta_i, a_i) - \\ - 4\sum_i \left(\left(\left(\frac{q_i\gamma_i + \rho}{a_i} \right) \right) + 2s\left(\beta_i, a_i; \frac{\gamma_i + \beta_i\rho}{a_i}, -\rho \right) \right). \tag{1.3}$$

The above expression can be further simplified using the identities

$$s(\beta_i, a_i) = -s(b_i, a_i), \tag{1.4}$$

$$s\left(\beta_i, a_i; \frac{\gamma_i + \beta_i/2}{a_i}, -\frac{1}{2}\right) \\ = -s\left(b_i, a_i, \frac{1}{2}, \frac{1}{2}\right) - \frac{1}{2}\left(\left(\frac{q_i\gamma_i + \frac{1}{2}}{a_i}\right)\right). \tag{1.5}$$

The identity (1.4) is elementary and can be safely left to the reader. The identity (1.5) is proved in the Appendix. Putting the above together we deduce that when A is even we have

$$\mathbf{F}(\vec{a}) = 1 + 4\sum_i s(b_i, a_i) + 8\sum_i s\left(b_i, a_i; \frac{1}{2}, \frac{1}{2}\right). \tag{1.6}$$

- *If A is odd* then

$$\mathbf{F}(\vec{a}) = 1 - \frac{1}{A} - 4\sum_i s(\beta_i, a_i) - \\ - 4\sum_{i=1}^{n}\left(2s\left(\beta_i, a_i; \frac{\gamma_i + \beta_i\rho}{a_i}, -\rho\right) + \left(\left(\frac{q_i\gamma_i + \rho}{a_i}\right)\right)\right). \tag{1.7}$$

Similarly, we have an identity

$$s\left(\beta_i, a_i; \frac{\gamma_i}{a_i}, 0\right) + \frac{1}{2}\left(\left(\frac{q_i\gamma_i}{a_i}\right)\right) = -s\left(b_i, a_i; \frac{1}{2}, \frac{1}{2}\right), \tag{1.8}$$

and we deduce

$$\mathbf{F}(\vec{a}) = 1 - \frac{1}{A} + 4\sum_i s(b_i, a_i) + 8\sum_i s\left(b_i, a_i; \frac{1}{2}, \frac{1}{2}\right). \tag{1.9}$$

The signature $\sigma(\vec{a})$ of the Milnor fiber of $\Sigma(\vec{a})$ can be expressed in terms of Dedekind sums as well (see [11, Section 1])

$$\sigma(\vec{a}) = -1 - \frac{(n-2)A}{3} + \frac{1}{3A} + \frac{1}{3}\sum_i \frac{b_i}{a_i} - 4\sum_i s(b_i, a_i). \tag{1.10}$$

We deduce that

$$\mathbf{F}(\vec{a}) + \sigma(\vec{a}) = -\frac{(n-2)A}{3} + \frac{\varepsilon}{3A} + \frac{1}{3}\sum_i \frac{b_i}{a_i} + 8\sum_i s\left(b_i, a_i; \frac{1}{2}, \frac{1}{2}\right), \tag{1.11}$$

where

$$\varepsilon = \begin{cases} 1 & A \text{ even}, \\ -2 & A \text{ odd}. \end{cases}$$

The description of χ_{SW} requires a bit more work. Introduce the simplex

$$\Delta(\vec{a}) = \left\{\vec{x} \in \mathbb{Z}^n \ ; \ x_i \geqslant 0, \ \sum_i \frac{x_i}{a_i} < \frac{\kappa}{2}\right\}.$$

For each $\vec{x} \in \Delta(\vec{a})$ set

$$d(\vec{x}) = \sum_i \left[\frac{x_i}{a_i}\right],$$

and denote by $\mathcal{S}_{\vec{x}}$ the symmetric product of $d(\vec{x})$ copies of S^2. Note that if $n = 3, 4$ then $d(\vec{x}) = 0$ for all $\vec{x} \in \Delta(\vec{a})$ so that $\mathcal{S}_{\vec{x}}$ consists of single point.

The irreducible part of the adiabatic Seibert–Witten equations on $\Sigma(\vec{a})$ was studied in [12, 13] and can be described as $\mathfrak{M}_{\vec{a}} = \bigcup_{\vec{x}} \mathfrak{M}_{\vec{x}}$ where $\mathfrak{M}_{\vec{x}} = \mathfrak{M}_{\vec{x}}^+ \cup \mathfrak{M}_{\vec{x}}^-$, $\mathfrak{M}_{\vec{x}}^\pm \cong \mathcal{S}_{\vec{x}}$. Moreover, the virtual dimensions of the spaces of finite energy gradient flows originating at the unique reducible solution and ending at one of the $\mathfrak{M}_{\vec{x}}$ are all odd. Using the adiabatic argument in §3.3 of [13] we deduce that if all $d(\vec{x})$ are zero the Seiberg–Witten–Floer homology obtained using the usual Seiberg–Witten equations is isomorphic with the Seiberg–Witten–Floer homology obtained using the adiabatic equations. Moreover, all the even dimensional Betti numbers of the Seiberg–Witten–Floer homology are zero and we deduce

$$\chi_{\mathrm{SW}}(\vec{a}) = -2C_{\vec{a}} := -2\#\Delta(\vec{a}). \tag{1.12}$$

The main result of this paper is the following:

THEOREM 1.1. *If* $\vec{a} = (a_1, \ldots, , a_n) \in \mathbb{Z}_+^n$, $n = 3, 4$ *has mutually coprime entries then*

$$\begin{aligned}-16C_{\vec{a}} &= \mathbf{F}(\vec{a}) + \sigma(\vec{a}) \\ &= -\frac{(n-2)A}{3} + \frac{\varepsilon}{3A} + \frac{1}{3}\sum_i \frac{b_i}{a_i} + 8\sum_i s\left(b_i, a_i; \frac{1}{2}, \frac{1}{2}\right).\end{aligned} \tag{1.13}$$

According to (0.1), (1.12) *and* (1.11) *this is equivalent to*

$$\tfrac{1}{8}\mathbf{F}(\vec{a}) + 2C_{\vec{a}} = -\alpha(\Sigma(\vec{a})) = \lambda(\Sigma(\vec{a})),$$

where λ denotes the Casson invariant.

Remark 1.2. As indicated in [13], Rohlin's theorem implies that the term $\mathbf{F}(\vec{a})$ is divisible by 8. The results of [11] show the signature $\sigma(\vec{a})$ is also divisible by 8. Thus the right-hand side of (1.13) is an integer divisible by 8. The above theorem shows that $\mathbf{F}(\vec{a}) + \sigma(\vec{a})$ is in effect divisible by 16!

2. The Mordell Trick

Let $\vec{a} \in \mathbb{Z}^n$ be as in the previous section. Denote by $\mathcal{P} = \mathcal{P}_{\vec{a}}$ the parallelipiped

$$\mathcal{P} := ([0, a_1 - 1] \times \cdots \times [0, a_n - 1]) \cap \mathbb{Z}^n.$$

When $n = 3$ we will use the notation $\vec{a} = (a, b, c)$. Define $q\colon \mathcal{P} \to \mathbb{R}$ by

$$q(\vec{x}) = \sum_i \frac{x_i + \frac{1}{2}}{a_i} = \sum_i \frac{x_i}{a_i} + \frac{u}{2A}.$$

Remark 2.1.

(a) Suppose $n = 3$, $\vec{a} = (a, b, c)$. Note that $q(\mathbf{p}) \in \frac{1}{2}\mathbb{Z}$ for some $\mathbf{p} \in \mathcal{P}$ if and only if abc is odd and

$$\mathbf{p} = \mathbf{p}_0 := \left(\frac{a-1}{2}, \frac{b-1}{2}, \frac{c-1}{2}\right).$$

In this case $q(\mathbf{p}_0) = \frac{3}{2}$.

(b) Suppose $n = 4$, $\vec{a} = (a_1, \ldots, a_4)$. Then $q(\mathbf{p}) \in \mathbb{Z}$ for some $\mathbf{p} \in \mathcal{P}$ if and only if A is odd and

$$\mathbf{p} = \mathbf{p}_0 = \left(\frac{a_1 - 1}{2}, \ldots, \frac{a_4 - 1}{2}\right)$$

in which case $q(\mathbf{p}_0) = 2$.

For every interval $I \subset \mathbb{R}$ we put $N_I := \# q^{-1}(I)$. Note that if $n = 3$

$$C_{a,b,c} = N_{(0,\frac{1}{2})}. \tag{2.1}$$

while if $n = 4$

$$C_{a_1,\ldots,a_4} = N_{(0,1)}. \tag{2.2}$$

For every $r \in \mathbf{R}$ define $\|r\| = [r + \frac{1}{2}]$ where $[\cdot]$ denotes the integer part function. Note that $\|r\|$ is the integer closest to r. We now discuss separately the two cases, $n = 3$ and $n = 4$

- *The case* $n = 3$, $\vec{a} = (a, b, c)$. Imitating Mordell (see [10, 15) we introduce the quantity $\alpha := \sum_{\mathcal{P}} (\|q\| - 1)(\|q\| - 2)$.

  Observe that

$$\alpha = 2N_{[0,\frac{1}{2})} + 2N_{[\frac{5}{2},3)} = 2N_{(0,\frac{1}{2})} + 2N_{(\frac{5}{2},3)}. \tag{2.3}$$

The importance of the last equality follows from the following elementary result.

LEMMA 2.2. $N_{(0,\frac{1}{2})} = N_{(\frac{5}{2},3)}$.

Proof. Consider the involution

$$\omega : \mathcal{P} \to \mathcal{P}, \quad (x, y, z) \mapsto (a - 1 - x, b - 1 - y, c - 1 - z).$$

It has the property $q(\omega(\mathbf{p})) = 3 - q(\mathbf{p})$ from which the lemma follows immediately. □

Using the lemma and the equalities (2.1), (2.3) we deduce

$$4C_{a,b,c} = \sum_{\mathcal{P}} (\|q\| - 1)(\|q\| - 2). \tag{2.4}$$

- *The case* $n = 4$, $\vec{a} = (a_1, \dots, a_4)$. Arguing exactly as above we deduce

$$4C_{\vec{a}} = 4N_{(0,1)} = \sum_{\mathbf{p}} ([q] - 1)([q] - 2). \tag{2.5}$$

The proof of Theorem 1.1 will be completed by providing an expression for the above sums in terms of Dedekind–Rademacher sums. This will be achieved in the next section following the strategy of [10] (see also [15]).

3. The Proof of Theorem 1.1

We will consider separately the two cases $n = 3$ and $n = 4$.

3.1. THE CASE $n = 3$

Set $\vec{a} = (a, b, c)$ so that $A = abc$, $u = ab + bc + ca$. We will distinguish two cases: A is even and A is odd.

- *A is even.* In this case $q(\mathbf{p}) + \frac{1}{2} \notin \mathbb{Z}$ so that

$$\|q\| = q + \tfrac{1}{2} - \left\{q + \tfrac{1}{2}\right\} = q - \left(\left(q + \tfrac{1}{2}\right)\right).$$

The sum can be rewritten as

$$\begin{aligned}
&\sum_{\mathcal{P}} \left(q - \left(\left(q + \tfrac{1}{2}\right)\right) - 1\right)\left(q - \left(\left(q + \tfrac{1}{2}\right)\right) - 2\right) \\
&\quad = \sum_{\mathcal{P}} q^2 - 3\sum_{\mathcal{P}} q + 2\sum_{\mathcal{P}} 1 - \\
&\qquad - 2\sum_{\mathcal{P}} q\left(\left(q + \tfrac{1}{2}\right)\right) + \sum_{\mathcal{P}} \left(\left(q + \tfrac{1}{2}\right)\right)^2 + 3\sum_{\mathcal{P}} \left(\left(q + \tfrac{1}{2}\right)\right).
\end{aligned}$$

We compute each of these 6 sums separately.

$$\sum_{\mathcal{P}} 1 = \#\mathcal{P} = abc,$$

$$\begin{aligned}
\sum_{\mathcal{P}} q &= \sum_{\mathcal{P}} \frac{x + \frac{1}{2}}{a} + \sum_{\mathcal{P}} \frac{y + \frac{1}{2}}{b} + \sum_{\mathcal{P}} \frac{z + \frac{1}{2}}{c} \\
&= \frac{bc}{2a} \sum_{x=0}^{a-1} (2x + 1) + \frac{ca}{2b} \sum_{y=0}^{b-1} (2y + 1) + \frac{ab}{2c} \sum_{z=0}^{c-1} (2z + 1) \\
&= \frac{3abc}{2},
\end{aligned}$$

$$\sum_{\mathcal{P}} q^2 = \sum_{\mathcal{P}} \left(\frac{x+\frac{1}{2}}{a}\right)^2 + \sum_{\mathcal{P}} \left(\frac{y+\frac{1}{2}}{b}\right)^2 + \sum_{\mathcal{P}} \left(\frac{z+\frac{1}{2}}{c}\right)^2 + \\ + 2c\left(\sum_{x=0}^{a-1} \frac{x+\frac{1}{2}}{a}\right)\left(\sum_{y=0}^{b-1} \frac{y+\frac{1}{2}}{b}\right) + 2b\left(\sum_{x=0}^{a-1} \frac{x+\frac{1}{2}}{a}\right)\left(\sum_{z=0}^{c-1} \frac{z+\frac{1}{2}}{c}\right) + \\ + 2a\left(\sum_{z=0}^{c-1} \frac{z+\frac{1}{2}}{c}\right)\left(\sum_{y=0}^{b-1} \frac{y+\frac{1}{2}}{b}\right).$$

Using basic properties of Bernoulli polynomials (see [16]) we deduce

$$\sum_{x=0}^{a-1} \left(\frac{x+\frac{1}{2}}{a}\right)^2 = \frac{1}{3a^2}\left(B_3(a+\tfrac{1}{2}) - B_3(\tfrac{1}{2})\right),$$

where

$$B_3(t) = \frac{t(2t-1)(t-1)}{2}$$

is the third Bernoulli polynomial. Note that $B_3(\frac{1}{2}) = 0$ and

$$B_3\left(t+\tfrac{1}{2}\right) = t\left(t^2 - \tfrac{1}{4}\right).$$

Using the identity

$$\sum_{k=0}^{n-1} \frac{k+\frac{1}{2}}{n} = \frac{n}{2},$$

we conclude

$$\sum_{\mathcal{P}} q^2 = \frac{5abc}{2} - \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right).$$

Next

$$\sum_{\mathcal{P}} ((q+\tfrac{1}{2})) = \sum_{\mathcal{P}} \left(\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u+abc}{2abc}\right)\right) \\ = \sum_{k=0}^{abc-1} \left(\left(\frac{k}{abc} + \frac{u+abc}{2abc}\right)\right)$$

According to the Kubert identity (A.4) in the Appendix the last sum is equal to $(((u+abc)/2))$ which is zero. Thus

$$\sum_{\mathcal{P}} \left((q+\tfrac{1}{2})\right) = 0.$$

The sum $\sum q((q+\frac{1}{2}))$ requires a bit more work. Note first that it decomposes as

$$\begin{aligned}
&\sum_{x=0}^{a-1}\frac{x+\frac{1}{2}}{a}\sum_{y,z}\Big(\Big(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u+abc}{2abc}\Big)\Big)+\\
&\quad+\sum_{y=0}^{b-1}\frac{y+\frac{1}{2}}{b}\sum_{z,x}\Big(\Big(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u+abc}{2abc}\Big)\Big)+\\
&\quad+\sum_{x=0}^{c-1}\frac{z+\frac{1}{2}}{c}\sum_{x,y}\Big(\Big(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u+abc}{2abc}\Big)\Big)+\\
&=S_1+S_2+S_3.
\end{aligned}$$

We describe in detail the computation of S_1. The other two sums are entirely similar. Note first that

$$\begin{aligned}
&\sum_{y,z}\Big(\Big(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u+abc}{2abc}\Big)\Big)\\
&\quad=\sum_{y,z}\Big(\Big(\frac{y}{b}+\frac{z}{c}+\frac{x}{a}+\frac{u+abc}{2abc}\Big)\Big)\\
&\quad=\sum_{k=0}^{bc-1}\Big(\Big(\frac{k}{bc}+\frac{x}{a}+\frac{u+abc}{2abc}\Big)\Big)
\end{aligned}$$

(use the Kubert identity (A.4))

$$\begin{aligned}
&\quad=\Big(\Big(\frac{bcx}{a}+\frac{u+abc}{2a}\Big)\Big).\\
&\quad=\Big(\Big(\frac{bc(x+\frac{1}{2})}{a}+\frac{bc+b+c}{2}\Big)\Big)=\Big(\Big(\frac{bc(x+\frac{1}{2})}{a}+\frac{1}{2}\Big)\Big).
\end{aligned}$$

We conclude

$$\begin{aligned}
S_1&=\sum_{x=0}^{a-1}\frac{x+\frac{1}{2}}{a}\Big(\Big(\frac{bc(x+\frac{1}{2})}{a}+\frac{1}{2}\Big)\Big)\\
&=\sum_{x=0}^{a-1}\Big(\Big(\frac{x+\frac{1}{2}}{a}\Big)\Big)\Big(\Big(\frac{bc(x+\frac{1}{2})}{a}+\frac{1}{2}\Big)\Big)+\frac{1}{2}\sum_{x=0}^{a-1}\Big(\Big(\frac{bc(x+\frac{1}{2})}{a}+\frac{1}{2}\Big)\Big)
\end{aligned}$$

(use the Kubert identity)

$$\begin{aligned}
&=\sum_{x=0}^{a-1}\Big(\Big(\frac{x+\frac{1}{2}}{a}\Big)\Big)\Big(\Big(\frac{bc(x+\frac{1}{2})}{a}+\frac{1}{2}\Big)\Big)\\
&=s\big(bc,a;\tfrac{1}{2},\tfrac{1}{2}\big).
\end{aligned}$$

Hence

$$\sum_{\mathcal{P}} q\big((q+\tfrac{1}{2})\big) = s\big(bc, a; \tfrac{1}{2}, \tfrac{1}{2}\big) + s\big(ca, b; \tfrac{1}{2}, \tfrac{1}{2}\big) + s\big(ab, c; \tfrac{1}{2}, \tfrac{1}{2}\big).$$

Finally

$$\sum_{\mathcal{P}} ((q+\tfrac{1}{2}))^2 = \sum_{\mathcal{P}} \left(\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u+abc}{2abc}\right)\right)^2$$
$$= \sum_{k=0}^{abc-1} \left(\left(\frac{k+(u+abc/2)}{abc}\right)\right)^2$$

(use the fact that $u + abc$ is odd in this case)

$$= \sum_{k=0}^{abc-1} \left(\left(\frac{k+\frac{1}{2}}{abc}\right)\right)^2 = s(1, abc; 0, \tfrac{1}{2})$$

(use (A.1)

$$= \frac{abc}{12} - \frac{1}{12abc}.$$

Putting together the above information we deduce that if abc is even then

$$4C_{a,b,c} = \frac{abc}{12} - \frac{1}{12abc} - \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right) - \\ - 2(s(bc, a; \tfrac{1}{2}, \tfrac{1}{2}) + s(ca, b; \tfrac{1}{2}, \tfrac{1}{2}) + s(ab, c; \tfrac{1}{2}, \tfrac{1}{2})). \tag{3.1}$$

The identity (1.13) is now obvious.

- *A is odd.* In this case, using Remark 2.1 we deduce

$$\|q(p)\| = \begin{cases} q - ((q+\frac{1}{2})), & \mathbf{p} \neq \mathbf{p}_0, \\ q - (q+\frac{1}{2})) + \frac{1}{2}, & \mathbf{p} = \mathbf{p}_0. \end{cases}$$

Thus

$$(\|q\|-1)(\|q\|-2) = \begin{cases} (q - ((q+\frac{1}{2})) - 1)(q - ((q+\frac{1}{2})) - 2), & \mathbf{p} \neq \mathbf{p}_0, \\ (q - ((q+\frac{1}{2})) - \frac{1}{2})(q - ((q+\frac{1}{2})) - \frac{3}{2}), & \mathbf{p} = \mathbf{p}_0. \end{cases}$$

Hence

$$
\begin{aligned}
4C_{a,b,c} &= \sum_{\mathcal{P}} (q - ((q+\tfrac{1}{2})) - 1)(q - ((q+\tfrac{1}{2})) - 2) + \\
&\quad + (q - ((q+\tfrac{1}{2})) - \tfrac{1}{2})(q - ((q+\tfrac{1}{2})) - \tfrac{3}{2})\,|_{\mathbf{p}_0} - \\
&\quad - (q - ((q+\tfrac{1}{2})) - 1)(q - ((q+\tfrac{1}{2})) - 2)\,|_{\mathbf{p}_0} \\
&= \sum_{\mathcal{P}} (q - ((q+\tfrac{1}{2})) - 1)(q - ((q+\tfrac{1}{2})) - 2) + \tfrac{1}{4}.
\end{aligned}
\tag{3.2}
$$

The above sum can be computed exactly as in the even case with one notable difference, namely

$$
\sum ((q+\tfrac{1}{2}))^2 = \sum_{k=0}^{abc-1} \left(\left(\frac{k + (u + abc/2)}{abc} \right) \right)^2
$$

($u + abc$ is even)

$$
= \sum_{k=0}^{abc-1} \left(\left(\frac{k}{abc} \right) \right)^2 = s(1, abc; 0, 0) = \frac{abc}{12} + \frac{1}{6abc} - \frac{1}{4}.
$$

Thus, when abc is odd we have

$$
\begin{aligned}
4C_{a,b,c} &= \frac{abc}{12} + \frac{1}{6abc} - \frac{1}{12}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right) - \\
&\quad - 2(s(bc, a; \tfrac{1}{2}, \tfrac{1}{2}) + s(ca, b; \tfrac{1}{2}, \tfrac{1}{2}) + s(ab, c; \tfrac{1}{2}, \tfrac{1}{2})).
\end{aligned}
$$

This completes the proof of Theorem 1.1 when $n = 3$.

3.2. THE CASE $n = 4$

We follow a similar strategy with some obvious modifications. Set $\vec{a} = (a_1, \ldots, a_4)$, $A = a_1 a_2 a_3 a_4$, $u = b_1 + \ldots + b_4$ and

$$
S_{\vec{a}} = \sum_{\mathcal{P}_{\vec{a}}} ([q] - 1)([q] - 2).
$$

As in the previous subsection will distinguish two situations.

- *A is even.* Note that for every $\mathbf{p} \in \mathcal{P}$ we have $q(\mathbf{p}) \notin \mathbb{Z}$ so that

$$
[q] = q - ((q)) - \tfrac{1}{2}.
$$

Thus

$$
\begin{aligned}
S_{\vec{a}} &= \sum_{\mathcal{P}} (q - ((q)) - 3/2)(q - (q)) - 5/2) \\
&= \sum_{\mathcal{P}} (q^2 - 4q + 15/4) - 2\sum_{\mathcal{P}} q((q)) + \sum_{\mathcal{P}} ((q))^2 + 4\sum_{\mathcal{P}} ((q)).
\end{aligned}
$$

The computation of the above terms follows the same pattern as in the previous subsection.

$$
\begin{aligned}
&\sum_{\mathcal{P}} ((q)) = 0, \\
&\sum_{\mathcal{P}} 15/4 = 15\#\mathcal{P}/4 = 15A/4, \\
&\sum_{\mathcal{P}} q = \sum_{i=1}^{4} b_i \sum_{x_i=0}^{a_i-1} \frac{x_i + \frac{1}{2}}{a_i} = \sum_{i=1}^{4} \frac{b_i a_i}{2} = 2A, \\
&\sum_{p} q^2 = \sum_{i=1}^{4} b_i \sum_{x_i=0}^{a_i-1} \left(\frac{x_i + \frac{1}{2}}{a_i}\right)^2 + 2\sum_{i<j} \frac{A}{a_i a_j} \left(\sum_{x_i=0}^{a_i-1} \frac{x_i + \frac{1}{2}}{a_i}\right)\left(\sum_{x_j=0}^{a_j-1} \frac{x_j + \frac{1}{2}}{a_j}\right) \\
&\quad = \sum_{i=1}^{4} \frac{b_i a_i (a_i^2 - \frac{1}{4})}{3a_i^2} + \sum_{1 \leqslant i<j \leqslant 4} \frac{A}{2} \\
&\quad = \sum_{i=1}^{4} \left(\frac{A}{3} - \frac{b_i}{12a_i}\right) + 3A = \frac{13A}{3} - \frac{1}{12} \sum_{i} \frac{b_i}{a_i}.
\end{aligned}
$$

When A is even u is odd and we have

$$
\sum_{\mathcal{P}} ((q))^2 = \sum_{k=0}^{A-1} \left(\left(\frac{k + u/2}{A}\right)\right)^2 = s(1, A; 0, \tfrac{1}{2}) = \frac{A}{12} - \frac{1}{2A}.
$$

Finally, $\sum_{\mathcal{P}} q((q)) = S_1 + \cdots + S_4$ where

$$
S_1 = \sum_{x_1=0}^{a_1-1} \frac{x_1 + \frac{1}{2}}{a_1} \cdot \sum_{x_2, x_3, x_4} \left(\left(\frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_1}{a_1} + \frac{u}{2A}\right)\right).
$$

S_2, S_3, S_4 are defined similarly. To compute S_1 note that

$$
\sum_{x_2, x_3, x_4} \left(\left(\frac{x_2}{a_2} + \frac{x_3}{a_3} + \frac{x_4}{a_4} + \frac{x_1}{a_1} + \frac{u}{2A}\right)\right) = \sum_{k=0}^{b_1-1} \left(\left(\frac{k}{b_1} + \frac{x_1}{a_1} + \frac{u}{2A}\right)\right)
$$

(use the Kubert identity)

$$
= \left(\left(\frac{b_1 x_1}{a_1} + \frac{u}{2a_1}\right)\right) = \left(\left(\frac{b_1(x_1 + \frac{1}{2})}{a_1} + \frac{u - b_1}{2a_1}\right)\right)
$$

($(u-b_1)/a_1$ is odd)

$$= \left(\left(\frac{b_1(x_1+\frac{1}{2})}{a_1} + \frac{1}{2}\right)\right).$$

Thus

$$S_1 = \sum_{x_1=0}^{a_1-1} \frac{x_1+1/2}{a_1}\left(\left(\frac{b_1(x_1+\frac{1}{2})}{a_1} + \frac{1}{2}\right)\right)$$

and we deduce as in the previous subsection that $S_1 = s(b_1, a_1; \frac{1}{2}, \frac{1}{2})$. By adding all the above together we deduce that if A is even then

$$4C_{\vec{a}} = S_{\vec{a}} = \frac{A}{6} - \frac{1}{12A} - \frac{1}{12}\sum_i \frac{b_i}{a_i} - 2\sum_i s(b_i, a_1; \tfrac{1}{2}, \tfrac{1}{2}).$$

The identity (1.13) is now obvious.

- *A is odd.* In this case u is even. Arguing as in the previous subsection, we deduce

$$S_{\vec{a}} = \sum_{\mathcal{P}} (q - ((q)) - 3/2)(q - ((q)) - 5/2) + \tfrac{1}{4}.$$

The only term in the previous computations which is influenced by the parity of A is

$$\sum_{\mathcal{P}} ((q))^2 = \sum_{k=0}^{A-1}\left(\left(\frac{k+u/2}{A}\right)\right)^2 = s(1, A)$$
$$= \frac{A}{12} + \frac{1}{6A} - \frac{1}{4}.$$

Putting together all the terms we obtain again the identity (1.13). The Theorem 1.1 is proved. □

Appendix: Basic Facts Concerning Dedekind–Rademacher Sums

In the paper [14], Hans Rademacher introduced for every pair of coprime integers h, k and any real numbers x, y the following generalization of the classical Dedekind sums

$$s(h, k; x, y) = \sum_{\mu=0}^{k-1}\left(\left(\frac{\mu+y}{k}\right)\right)\left(\left(\frac{h(\mu+y)}{k} + x\right)\right).$$

A simple computations shows that $s(h, k; x, y)$ depends only on x, y mod 1. When

$h = 1$ and $x = 0$ one can prove (see [14])

$$s(1, k; 0, y) = \begin{cases} \dfrac{k}{12} + \dfrac{1}{6k} - \dfrac{1}{4}, & y \in \mathbb{Z}, \\ \dfrac{k}{12} + \dfrac{1}{k} B_2(\{y\}), & y \in \mathbb{R} \setminus \mathbb{Z}, \end{cases} \tag{A.1}$$

where $B_2(t) = t^2 - t + \frac{1}{6}$ is the second Bernoulli polynomial.

Perhaps the most important property of these Dedekind–Rademacher sums is their reciprocity law which makes them computationally very friendly: their computational complexity is comparable with the complexity of the classical Euclid's algorithm. To formulate it we must distinguish two cases.

- Both x and y are integers. Then

$$s(\beta, \alpha; x, y) + s(\alpha, \beta; y, x) = -\tfrac{1}{4} + \frac{\alpha^2 + \beta^2 + 1}{12\alpha\beta}. \tag{A.2}$$

- x and/or y is not an integer. Then

$$s(\beta, \alpha; x, y) + s(\alpha, \beta; y, x) = ((x)) \cdot ((y)) + \frac{\beta^2 \psi_2(y) + \psi_2(\beta y + \alpha x) + \alpha^2 \psi_2(x)}{2\alpha\beta} \tag{A.3}$$

  where $\psi_2(x) := B_2(\{x\})$.

An important ingredient behind the reciprocity law is the following identity ([14, Lemma 1])

$$\sum_{\mu=0}^{k-1} \left(\left(\frac{\mu + w}{k} \right) \right) = ((w)) \quad \forall w \in \mathbb{R}. \tag{A.4}$$

Following the terminology in [9], we will call the above equality the *Kubert identity*.

We conclude with a proof of the identity (1.5). For simplicity we consider only the case $n = 3$ and $i = 1$. Set $\vec{a} = (a, b, c)$. Thus $A = abc$ is even, $u = bc + ca + ab$ and $b_1 = bc$. For arbitrary n the proof is only notationally more complicated.

The proof of (1.5) goes as follows:

$$s\left(\beta_1, a; \frac{\gamma_1 + \beta_1/2}{a}, -\frac{1}{2}\right) = \sum_{x=0}^{a-1} \left(\left(\frac{x - \frac{1}{2}}{a} \right) \right) \left(\left(\frac{\beta_1 x + \gamma_1}{a} \right) \right)$$

$(\gamma_1 = \beta_1(u - abc - 1)/2 \bmod a)$

$$= \sum_{x=0}^{a-1} \left(\left(\frac{x - \frac{1}{2}}{a}\right)\right)\left(\left(\frac{\beta_1(x - (abc - u + 1/2))}{a}\right)\right)$$

$(y := x - (abc - u + 1/2) \bmod a)$

$$= \sum_{y=0}^{a-1} \left(\left(\frac{y + (abc - u + 1/2) - \frac{1}{2}}{a}\right)\right)\left(\left(\frac{\beta_1 y}{a}\right)\right)$$

(use $y = -bcz \bmod a$ and $\beta_1 bc \equiv -1 \bmod a_1$)

$$= -\sum_{z=0}^{a-1} \left(\left(\frac{bcz - (abc - u/2)}{a}\right)\right)\left(\left(\frac{z}{a}\right)\right)$$

$$= -\sum_{z=0}^{a-1} \left(\left(\frac{bc(z + \frac{1}{2})}{a} + \frac{b + c - bc}{2}\right)\right)\left(\left(\frac{z}{a}\right)\right)$$

$$= -\sum_{z=0}^{a-1} \left(\left(\frac{bc(z + \frac{1}{2})}{a} + \frac{1}{2}\right)\right)\left(\left(\frac{z}{a}\right)\right).$$

At this point we use the elementary identity

$$\left(\left(\frac{z}{a}\right)\right) = \left(\left(\frac{z + \frac{1}{2}}{a}\right)\right) - \frac{1}{2a} + \frac{1}{2}\delta(z),$$

where

$$\delta(z) = \begin{cases} 1 & z \equiv 0 \pmod{a}, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce

$$s\left(\beta_1, \alpha_1; \frac{\gamma_1 + \beta_1/2}{\alpha_1}, -\frac{1}{2}\right)$$
$$= -\sum_{z=0}^{a-1} \left(\left(\frac{bc(z + \frac{1}{2})}{a} + \frac{1}{2}\right)\right)\left(\left(\frac{z + \frac{1}{2}}{a}\right)\right) +$$
$$+ \frac{1}{2a} \sum_{z=0}^{a-1} \left(\left(\frac{bc(z + \frac{1}{2})}{a} + \frac{1}{2}\right)\right) - \frac{1}{2}\left(\left(\frac{bc}{2a} + \frac{1}{2}\right)\right).$$

The Kubert identity shows that the second sum above vanishes. Also

$$\left(\left(\frac{q_1\gamma_1 + \frac{1}{2}}{a}\right)\right) = \left(\left(\frac{u - abc/2}{a}\right)\right) = \left(\left(\frac{b + c - bc}{2} + \frac{bc}{2a}\right)\right)$$
$$= \left(\left(\frac{bc}{2a} + \frac{1}{2}\right)\right).$$

The identity (1.5) is proved. The proof of (1.8) is similar and is left to the reader

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