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ON THE CAPPELL–LEE–MILLER GLUING THEOREM

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We prove a generalization of the Cappell-Lee-Miller theorem which we formulate using a new language, of asymptotic maps and asymptotic exact sequences. We also present applications to eigenvalue estimates, approximation of obstruction bundles and gluing of determinant line bundles arising frequently in gauge theory.

Introduction.

Many surgery problems arising in gauge theory require the understanding of the kernels of selfadjoint, possibly $\mathbb{Z}_2$-graded, Dirac type operators on manifolds with very long necks. As is well-known, the dimension of the kernel is a very unstable quantity: Small perturbations can destroy it completely. Stated in this fashion, this problem has no chance of being solved. A more reasonable approach is to replace the kernel with a more stable object, such as the space spanned by eigenfunctions corresponding to "small" eigenvalues.

The range of the "small" attribute is vague but a simple heuristic argument offers an idea about its size. More precisely, on a manifold of large diameter r, a self-adjoint elliptic operator of order k is to be expected to have eigenvalues of size $\sim r^{-k}$. If $k = 1$, the smallness attribute ought to refer to sizes $c(r)$ such that

$$(*) \qquad c(r) = o(r^{-1}) \ \text{ as } \ r \to \infty.$$

Thus, if M_r is a compact, oriented Riemann manifold with a long neck of length $\sim r$ (see Figure 1), $c(r)$ is a function satisfying $(*)$ and $\mathfrak{D}_r$ is a selfadjoint Dirac operator, we would like to have a good approximation of the space $\mathcal{K}_r$ spanned by eigenfunctions of $\mathfrak{D}_r$ corresponding to eigenvalues $|\lambda| \leq c(r)$. We should think of $\mathfrak{D}_r$ as a family of operators, varying slowly with $r \to \infty$. The physicists would call such a situation, an adiabatic deformation.

T. Yoshida solved this problem in [**13**] for a special class of operators $\mathfrak{D}_r$ arising in Floer theory, namely the so-called odd signature operators. A bit later, Cappell-Lee-Miller have shown in [**3**] that all of Yoshida's ideas extend with no essential modifications to arbitrary Dirac type operators $\mathfrak{D}_r$, assuming as Yoshida did that along the cylindrical neck of M_r the coefficients of $\mathfrak{D}_r$ are *independent* of the longitudinal coordinate. (Such an operator is called cylindrical.)

If an operator *slightly* deviates from being cylindrical, it is natural to believe that the results in [**3**] continue to hold. However, the ad-hoc arguments of [**3**, **13**] do not extend to this more general situation.

In this paper we propose a new, more transparent and considerably shorter proof of the Cappell-Lee-Miller gluing theorem in this more general context. More concretely, we show that the results of [**3**, **13**] continue to hold if $\mathfrak{D}_r$ differs from a cylindrical type operator by an exponentially small quantity. What we believe warrants publicity is the new extremely versatile point of view we adopt.

As in [**3**] and [**13**], we want to approximate $\mathcal{K}_r$ by a space over which we have some control. We propose a formulation which closely resembles the well-known Mayer-Vietoris theorem. In that classical situation, the cohomology of a union of two spaces fits in a long exact sequence of vector spaces and linear maps. In gauge theoretic applications, $\mathcal{K}_r$ has a cohomological interpretation, in terms of deformation complexes. We sought to include $\mathcal{K}_r$ in a long exact sequence of *finite dimensional* vector spaces and linear maps and we were very pleasantly surprised to discover that such a formulation is possible, provided we *slightly adjust* our requirements.

More precisely, we prove the existence of a sequence

$$(**) \qquad 0 \to \mathcal{K}_r \stackrel{f_r}{\longrightarrow}{}^a X_r \stackrel{g_r}{\longrightarrow}{}^a Y_r \to 0$$

which is *"approximately"* exact.

To explain the meaning of the above statement, let us first mention that the symbol $\mathcal{K}_r \longrightarrow^a X_r$ does not denote a linear map from $\mathcal{K}_r$ to X_r. It denotes *an asymptotic map*, that is a linear map from $\mathcal{K}_r$ to the ambient space of X_r whose range is "very close" to a subspace of X_r as $r \to \infty$. The sequence $(**)$ is not quite exact, it is *asymptotically exact*, meaning that the range of f_r is "very close" to the kernel of g_r. This may not sound satisfactory, but a simple argument shows that an asymptotically exact sequence of asymptotic maps can be perturbed to a *genuinely* exact sequence of *genuine* maps. The advantage of working with asymptotic maps comes from the fact that f_r and g_r are the maps the intuition tells us they ought to be.

More "precisely", the first arrow in $(**)$ is called the (*adiabatic*) *splitting map*. It says that, in the adiabatic limit $r \to \infty$, the sections in $\mathcal{K}_r$ split into a pair of harmonic spinors on the manifolds with cylindrical ends $M_i(\infty)$, $i = 1, 2$, depicted in Figure 1. X_r is precisely the space of pairs of *asymptotically cylindrical* harmonic spinors, i.e., harmonic spinors which have an asymptotic limit as $t \to \infty$ along the cylindrical end of $M_i(\infty)$. According to the Fredholm results of Lockhart and McOwen, X_r is a finite dimensional space.

The second arrow in (∗∗) is telling us that not any pair of asymptotically cylindrical spinors arises in such a splitting. Y_r is the *obstruction space* which indicates that a pair of asymptotically cylindrical harmonic spinors arises in an adiabatic splitting if and only if they have matching asymptotics. The second arrow is the analogue of the difference map in the usual Mayer-Vietoris sequence.

From a technical point of view, the asymptotic maps and sequences are as easy to use as their traditional counterparts. On the other hand, the asymptotic language makes many of the gluing arguments in gauge theory much more transparent.

The present paper is divided as follows. In Section 1 we describe the geometric context of our gluing theorem. It consists of cylindrical objects: Manifolds, bundles, sections etc. In Section 2 we introduce the asymptotic language and formulate the main result. In Section 3 we list some basic analytical facts about elliptic equations on manifolds with cylindrical ends. We mention in particular the Key Estimate which adds a bit of compactness to the situation. Its completely elementary proof is deferred to the Appendix. Section 4 contains the proof of the Main Theorem itself.

In Section 5 we present several applications. The first one is concerned with small eigenvalues of selfadjoint elliptic operators on manifolds containing long necks of the type considered by W. Chen in [4]. His result is equivalent with the statement that if $\ker g_r = 0$ then the operators $\mathfrak{D}_r$ are invertible for $r \gg 0$ and the norms of their inverses are $O(r)$. The sequence (∗∗) makes this result nearly obvious.

We consider next super-symmetric operators, so that $\mathfrak{D}_r$ can be represented as

$$\mathfrak{D}_r = \begin{bmatrix} 0 & \not{\mathfrak{D}}_r^* \\ \not{\mathfrak{D}}_r & 0 \end{bmatrix}$$

where $\not{\mathfrak{D}}_r$ is a first order elliptic operator

$$C^\infty(\mathcal{E}^+) \to C^\infty(\mathcal{E}^-)$$

and $\mathcal{E}_\pm \to M$ are Hermitian vector bundles over M. We study what happens if the odd part of the kernel of g_r is trivial, $(\ker g_r)_- = 0$. We show that $\not{\mathfrak{D}}_r$ admits a L^2-bounded right inverse of norm $O(r)$ as $r \to \infty$. This result is often needed in gauge theoretic gluing problems over *even* dimensional manifolds; see [**11**].

We conclude this section with some asymptotic lower estimates for the first eigenvalue of the Hodge-DeRham Laplacian acting on forms of a given degree. Most of the known lower estimates for eigenvalues have a geometric origin. Our estimates have a *topological* origin. For degree one forms these estimates play an important role in gauge theory. These estimates on 0-forms also follow from classical results of Li and Yau.

We believe the asymptotic language will find applications in other problems involving adiabatic deformations. It is not difficult to introduce the notion of asymptotic (co)chain complexes and asymptotic cohomology. Many of the basic results in homological algebra have an asymptotic counterpart.

1. Cylindrical objects.

A *cylindrical* $(n+1)$-manifold is an oriented Riemannian $(n+1)$-manifold $(\hat{N}, \hat{g})$ with a cylindrical end modeled by $\mathbb{R}_+ \times N$, where (N, g) is an oriented compact Riemannian n-manifold. In more precise terms, this means that the complement of an open precompact subset of $\hat{N}$ is *isometric* in an orientation preserving fashion to the cylinder $\mathbb{R}_+ \times N$. We will denote the canonical projection $\mathbb{R}_+ \times N \to N$ by π while t will denote the *outgoing* longitudinal coordinate along the neck. We will regularly denote the "slice" N by $\partial_\infty \hat{N}$ and the metric g by $\partial_\infty \hat{g}$. For each $t \geq 0$ we set $\hat{N}_t := \hat{N} \setminus (t, \infty) \times N$.

A *cylindrical structure* on a vector bundle $\hat{E} \to \hat{N}$ consists of a vector bundle $E \to N$ and a bundle isomorphism

$$\hat{\vartheta} : \hat{E}\,|_{\mathbb{R}_+ \times N} \to \pi^* E.$$

We will use the notation $E := \partial_\infty \hat{E}$.

A cylindrical vector bundle will be a vector bundle together with a cylindrical structure $(\hat{\vartheta}, E)$. Observe that the cotangent bundle $T^*\hat{N}$ has a natural cylindrical structure such that

$$\partial_\infty T^* \hat{N} \cong \mathbb{R}\langle dt \rangle \oplus T^* N.$$

A section $\hat{u}$ of a cylindrical vector bundle $(\hat{E}, \hat{\vartheta}, E)$ is said to be *cylindrical* if there exists a section u of $\partial_\infty \hat{E}$ such that along the neck

$$\hat{\vartheta} \hat{u} = \pi^* u.$$

When there is no danger of confusion, we will write the above equality simply as $\hat{u} = \pi^* u$. We will use the notation $u := \partial_\infty \hat{u}$.

Given any cylindrical vector bundle $(\hat{E}, \hat{\vartheta}, E)$ there exists a canonical first order partial differential operator P, defined over the cylindrical end, uniquely determined by the conditions

$$P(\hat{f}\hat{u}) = \frac{d\hat{f}}{dt}\hat{u} + \hat{f} P \hat{u}, \;\; \forall \hat{f} \in C^\infty(\mathbb{R}_+ \times N), \;\; \hat{u} \in \hat{E}\,|_{\mathbb{R}_+ \times N},$$

and $P\hat{v} = 0$ for any cylindrical section $\hat{v}$ of $\hat{E}\,|_{\mathbb{R}_+ \times N}$. We will denote this operator by ∂_t.

It is now clear that we can organize the set of cylindrical bundles over a given cylindrical manifold as a category. Moreover, we can perform all the standard tensorial operations in this category such as direct sums, tensor products, duals, etc.

A *cylindrical partial differential operator* (*p.d.o.*) will be a *first order* p.d.o. $\hat{L}$ between two cylindrical bundles $\hat{E}$, $\hat{F}$ such that along the neck $[T,\infty)\times N$ ($T \gg 0$) it can be written as

$$\hat{L} = G\partial_t + L$$

where $L : C^\infty(E) \to C^\infty(E)$ is a first order p.d.o., $E = \hat{E}|_N$, $F = \hat{F}|_N$ and $G : E \to F$ is a bundle morphism. We will use the notation

$$L := \partial_\infty \hat{L}.$$

If $\hat{\sigma}$ denotes the symbol of $\hat{L}$ then we see that $G = \hat{\sigma}(dt)$ and[1]

$$\partial_\infty \hat{L} = \hat{L} - G\partial_t.$$

A connection on a cylindrical vector bundle $\hat{E} \to \hat{N}$ is a special example of first order p.d.o.

$$\hat{\nabla} : C^\infty(\hat{E}) \to C^\infty(T^*\hat{N} \otimes \hat{E})$$

between cylindrical bundles. The connection is called *strongly cylindrical* if it is cylindrical as a p.d.o. and *temporal* i.e.,

$$\hat{\nabla}_t \hat{u} = \partial_t \hat{u}, \;\; \forall \hat{u} \in C^\infty(\hat{E}).$$

Two cylindrical manifolds $(\hat{N}_i, \hat{g}_i)$, $i = 1,2$ are called *compatible* if there exists an orientation reversing diffeomorphism

$$\varphi : N_1 \to N_2$$

such that

$$g_1 = \varphi^* g_2.$$

Two cylindrical vector bundles $(\hat{E}_i, \hat{\vartheta}_i, E_i = \partial_\infty \hat{E}_i) \to \hat{N}_i$ are said to be compatible if there exists a vector bundle isomorphism

$$\gamma : E_1 \to E_2$$

covering φ.

For simplicity, we set $N := N_1$, we will fix some (ghost) reference, orientation *reversing* diffeomorphism $\Phi_0 : N \to N_2$ so that we can identify φ with an orientation *preserving* self-diffeomorphism of N. Also, in this case it is very convenient to think of the end of $\hat{N}_2$ as the cylinder $(-\infty, 0) \times N$ so that the outgoing coordinate on $\hat{N}_2$ is $-t$. Note that the compatibility conditions provide a way of identifying $\partial_\infty \hat{E}_1$ with $\partial_\infty \hat{E}_2$ so that we can decide when a section of $\partial_\infty \hat{E}_1$ is equal to a section of $\partial_\infty \hat{E}_2$.

[1]The operator G is orientation sensitive because it depends on the choice of *outgoing coordinate* t.

The sections $\hat{u}_i$ of the compatible cylindrical bundle $\hat{E}_i$ are called compatible if $\partial_\infty \hat{u}_1 = \partial_\infty \hat{u}_2$. The cylindrical partial differential operators $\hat{L}_i$ on $\hat{N}_i$, $i = 1, 2$, are compatible if along their necks they have the form

$$\hat{L}_1 = G_1\partial_t - L_1, \;\; \hat{L}_2 = G_2\partial_t - L_2, \;\; G_1 + G_2 = L_1 - L_2 = 0.$$

Consider two compatible cylindrical manifolds $\hat{N}_i$, $i = 1, 2$. For every orientation preserving diffeomorphism $\varphi : N \to N$ and every $r \gg 0$ we denote by $\hat{N}(r) = \hat{N}(r, \varphi)$ the manifold obtained by attaching $\hat{N}_1(r) := \hat{N}_1 \setminus (r+1, \infty) \times N$ to $\hat{N}_2(r) := \hat{N}_2 \setminus (-\infty, -r-1) \times N$ (see Figure 1) using the obvious orientation preserving identification

$$(\dagger) \qquad \imath_r \times \Phi_0 \circ \varphi : [r+1, r+2] \times N \to [-r-2, -r-1] \times N,$$

$$(t, x) \mapsto \Big(t - 2r - 3, \Phi_0 \circ \varphi(x)\Big).$$

Two compatible cylindrical bundles $\hat{E}_i$ can be glued in an obvious way to form a bundle $\hat{E}(r) = \hat{E}_1 \#_r \hat{E}_2$ for all $r \gg 0$. We want to emphasize that the topological types of the resulting manifold $\hat{N}(r)$ and bundle $\hat{E}(r)$ depend on the gluing isomorphisms φ and γ. In the sequel, to simplify the presentation, we will not include φ and γ in our notations.

2. The main result.

Consider the following set-up:

- Two compatible cylindrical manifolds $M_i(\infty)$, $i = 1, 2$

$$\partial_\infty M_1(\infty) = N, \;\; \partial_\infty M_2(\infty) = -N.$$

We assume the cylindrical end of $M_1(\infty)$ is $\mathbb{R}_+ \times N$ and the cylindrical end of $M_2(\infty)$ is $\mathbb{R}_- \times N$.

- $\hat{E}_i \to M_i(\infty)$ are compatible cylindrical bundles equipped with compatible cylindrical Hermitian metrics and compatible strongly cylindrical connections $\hat{\nabla}^i$. All the Sobolev norms will be defined in terms of these connections and the Levi-Civita connection.

- $\hat{D}_i : C^\infty(\hat{E}_i) \to C^\infty(\hat{E}_i)$ are compatible self-adjoint cylindrical, Dirac-type operators, in the sense of [**9**]. Recall that the Dirac-type condition means that the squares $\hat{D}_i^2$ have the same principal symbols as a Laplacian. The principal symbols of D_i define Clifford multiplications on $\hat{E}_i$ which we denote by $\hat{\mathbf{c}}_i$. Set

$$\hat{\mathbf{c}} := \hat{\mathbf{c}}_1, \;\; J := \hat{\mathbf{c}}(dt)$$

where t denotes the outgoing longitudinal coordinate on the end of $M_1(\infty)$. Observe that $J = -J^* = J^{-1}$. Along the neck $\hat{D}_1$ has the form

$$\hat{D}_1 = J(\partial_t - D)$$

where $D := J\partial_\infty \hat{D}_1$. We assume that D is a selfadjoint Dirac-type operator[2] on the bundle $E := \hat{E}_1\,|_N \cong \hat{E}_2\,|_N$. The compatibility condition implies

$$\partial_\infty \hat{D}_1 = \partial_\infty \hat{D}_2.$$

Thus, along the neck $\hat{D}_2$ has the form

$$\hat{D}_2 = J(\partial_t - D)$$

where we recall that $-t$ is the outgoing coordinate on the cylindrical end of $M_2(\infty)$. We want to emphasize that D is *independent of the longitudinal coordinate* t along the necks.

- Two smooth self-adjoint endomorphisms $\hat{B}_i$. Set

$$B_i(t) := \hat{B}_i\,|_{t\times N}, \quad A_i(t) := JB_i(t).$$

We assume the following additional facts about $A_i(t)$.

  - $A_i(t)$ anti-commutes with J

$$\{J, A_i(t)\} = JA_i(t) + A_i(t)J = 0.$$

  - There exist $C, \lambda > 0$ such that[3]

$$\sup\Big\{ |\hat{A}_i(x)|; \;\; x \in [t, t+1] \times N \Big\} \le C \exp(-\lambda |t|). \tag{2.1}$$

Consider a smooth, decreasing, cut-off function $\eta : \mathbb{R} \to [0, 1]$ such that

$$\eta(t) \equiv 1, \;\; t \le 1/4$$

$$\eta(t) \equiv 0, \;\; t \ge 3/4$$

and

$$\left| \frac{d\eta}{dt} \right| \le 4, \;\; \forall t \ge 0.$$

For each $r > 0$ and $t > 0$ set $\eta_r(t) := \eta(t-r)$. Now extend η_r by symmetry to a function on $\mathbb{R}$ still denoted by η_r. We can regard $\eta_r(t)$, $t \ge 0$, as a smooth function on $M_1(\infty)$ and $\eta_r(t)$, $t \le 0$ as a smooth function on $M_2(\infty)$ so we can form the perturbations

$$\hat{D}_{i,r} := \hat{D}_i + \eta_r \hat{B}_i$$

[2]This is a very mild restriction. As explained in [**8**], this happens for **all** Dirac type operators which appear in geometric problems.

[3]This *pointwise* condition on the zeroth order perturbation $\hat{A}_i$ can be relaxed to a condition involving a weighted Sobolev norm. However the *exponential type* control on the size is **essential**. It is needed to invoke the Fredholm results of [**6**].

and

$$\hat{D}_{i,\infty} := \hat{D}_i + \hat{B}_i.$$

Note that via the diffeomorphism (†) we have the identification

$$D_{1,r}\,|_{C(r+1)} = D_{2,r}\,|_{C(-r-2)}, \quad C(t) := [t, t+1] \times N.$$

Denote by $M_1(r)$ the manifold obtained from $M_1(\infty)$ by chopping off the cylinder $(r+2, \infty) \times N$, by $M_2(r)$ the manifold $M_2(\infty) \setminus (-\infty, -r-2) \times N$ and by $M(r)$ the manifold $M_1(\infty)\#_r M_2(\infty)$ described in the previous subsection; see Figure 1.

We can regard $M_1(r)$ and $M_2(r)$ in a natural way as submanifolds of $M(r)$ which intersect over a cylinder $C(r+1)$. Hence the operators $\hat{D}_{i,r}$ can be glued together to produce a Dirac type operator $\mathfrak{D}_r$ on the glued bundle $\hat{E}(r) := \hat{E}_1 \#_r \hat{E}_2 \to M(r)$.

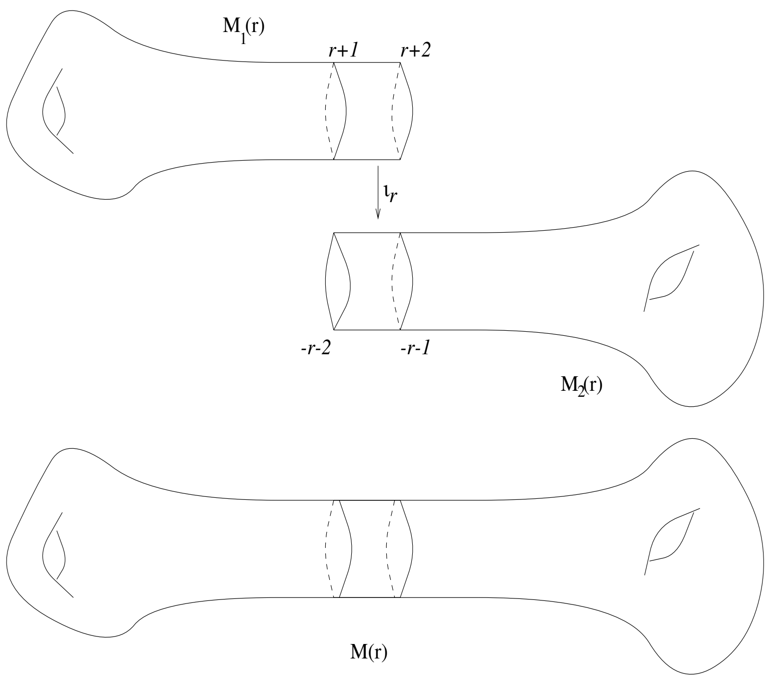

Figure 1. Gluing two manifolds with cylindrical ends.

The operators $\hat{D}_i$ may have additional symmetries. We will be particularly interested in *super-symmetric* operators. This means the bundles $\hat{E}_i$ are equipped with orthogonal (unitary) decompositions

$$\hat{E}_i = \hat{E}_i^+ \oplus \hat{E}_i^- \tag{2.2}$$

which determine the chiral operators $\hat{C}_i := \hat{P}_i^+ - \hat{P}_i^-$ where $\hat{P}_i^\pm$ denotes the orthogonal projection $\hat{E}_i \to \hat{E}_i^\pm$. The Dirac operator $\hat{D}_i$ is said to be *super-symmetric* if

$$\left\{\hat{C}_i, \hat{D}_i\right\} = 0. \tag{2.3}$$

Equivalently, in terms of the splitting (2.2) it has the block decomposition

$$\hat{D}_i = \left[\begin{array}{cc} 0 & \hat{\mathbf{D}}_i^* \\ \hat{\mathbf{D}}_i & 0 \end{array}\right]$$

where $\hat{\mathbf{D}}_i$ is a first order elliptic operator $C^\infty(\hat{E}_i^+) \to C^\infty(\hat{E}_i^-)$. Condition (2.3) implies that for any 1-form α on $\hat{M}_i(\infty)$ the Clifford multiplication by α anti-commutes with $\hat{C}_i$

$$\{\hat{\mathbf{c}}(\hat{\alpha}), \hat{C}_i\} = 0. \tag{2.4}$$

Note that along the neck the operator $\hat{\mathbf{D}}_i$ has the form

$$\hat{\mathbf{D}}_i = G(\nabla_t - \mathbf{D})$$

where $G : E^+ \to E^-$ is the bundle isomorphism given by the Clifford multiplication by dt and $\mathbf{D} : C^\infty(E_i^+) \to C^\infty(E_i^+)$ is a self-adjoint, Dirac-type operator.

We will further assume that the two super-symmetries are compatible along the "boundary" N i.e.,

$$\hat{C}_1 |_N = \hat{C}_2 |_N =: C.$$

Thus the bundle E is super-symmetric with chiral operator C. Conditions (2.3) and (2.4) imply that

$$[C, D] = CD - DC = 0. \tag{2.5}$$

In this case we assume the perturbations $\hat{B}_i$ are compatible with the chiral operators in an obvious sense. Clearly the super-symmetry is transmitted to the glued bundle $\mathcal{E}_r$ and the glued operator $\mathfrak{D}_r$. The space $\ker \mathfrak{D}_r$ is naturally a finite dimensional $\mathbb{Z}_2$-graded space.

In this paper we will address the following question.

Main Problem. *Understand the behavior of* $\ker \mathfrak{D}_r$ *as* $r \to \infty$.

The kernel of an operator is a notoriously unstable object so it is unrealistic to be able to solve the Main Problem as stated. We need to "stabilize" $\ker \mathfrak{D}_r$ if we expect to say something of significance.

To formulate the main result we need to introduce some additional notions. We begin with the notions of asymptotic map and asymptotic exactness. An *asymptotic map* is a sequence $(U_r, V_r, f_r)_{r>0}$ with the following properties:

(a) There exist Hilbert spaces H_0 and H_1 such that U_r is a closed subspace of H_0 and V_r is a closed subspace of H_1, $\forall r > 0$.
(b) f_r is a densely defined linear map $f_r : U_r \to H_1$ with closed graph and range $R(f_r)$, $\forall r > 0$.
(c) $\lim_{r\to\infty} \hat{\delta}(R(f_r), V_r) = 0$ where, following [**5**], we set

$$\hat{\delta}(U, V) = \sup\Big\{ \mathrm{dist}\,(u, V);\ \ u \in U,\ |u| = 1 \Big\}.$$

We will denote asymptotic maps by $U_r \xrightarrow{f_r}{}^{a} V_r$. There is a super-version of this notion when U_r and V_r are $\mathbb{Z}_2$-graded and are closed subspaces in $\mathbb{Z}_2$-graded Hilbert spaces such that the natural inclusions are even.

The next result, proved in [**5**, IV.§2], explains the motivation behind the above definition.

Lemma 2.1. *If*

$$\hat{\delta}(U, V) < 1$$

then the orthogonal projection P_V onto V induces a one-to one map $U \to V$. If additionally

$$\hat{\delta}(V, U) < 1$$

then $P_V : U \to V$ is a linear isomorphism.

Define the gap between two closed subspaces U, V in a Hilbert space H by

$$\delta(U, V) = \max\Big\{ \hat{\delta}(U, V), \hat{\delta}(V, U) \Big\}.$$

The sequence of asymptotic maps

$$U_r \xrightarrow{f_r}{}^{a} V_r \xrightarrow{g_r}{}^{a} W_r,\ \ r \to \infty$$

is said to be *asymptotically exact* if

$$\lim_{r\to\infty} \delta(R(f_r), \ker g_r) = 0.$$

We have the following consequence of Lemma 2.1.

Lemma 2.2. *If the sequence*

$$U_r \xrightarrow{f_r}{}^{a} V_r \xrightarrow{g_r}{}^{a} W_r,\ \ r \to \infty$$

is asymptotically exact, P_r denotes the orthogonal projection onto $\ker g_r$ and Q_r the orthogonal projection onto W_r then there exists $r_0 > 0$ such that the sequence

$$U_r \xrightarrow{P_r \circ f_r} V_r \xrightarrow{Q_r \circ g_r} W_r$$

is exact for all $r > r_0$.

If the spaces H_j are $\mathbb{Z}_2$-graded $H_j = H_j^+ \oplus H_j^-$ we say the sequence is *super-symmetric* if the maps f_r and g_r are even, i.e., are compatible with the splitting. In this case we get two asymptotically exact sequences

$$U_r^\pm \to V_r^\pm \to W_r^\pm.$$

Next we need to introduce suitable functional spaces. For brevity we discuss only distributions on $M_1(\infty)$. Define the *extended* L^2-space $L^2_{\mathrm{ex}}(\hat{E}_1)$ as the space of sections $\hat{u} \in L^2_{\mathrm{loc}}(\hat{E}_1)$ such that there exists $u_\infty \in L^2(E)$ such that

$$\hat{u} - \hat{u}_\infty \in L^2(\hat{E}_1).$$

Above, $\hat{u}_\infty$ denotes the section in $L^2_{\mathrm{loc}}(\hat{E}_1)$ which is identically zero on $M_1(0)$ and coincides with the translation invariant section u_∞ on the infinite cylinder $\mathbb{R}_+ \times N$. u_∞ is uniquely determined by $\hat{u}$ and thus we get well-defined map

$$\partial_\infty : L^2_{\mathrm{ex}}(\hat{E}_1) \ni \hat{u} \mapsto u_\infty \in L^2(E)$$

called *asymptotic limit* (*trace*) map. For simplicity, denote by $\| \bullet \|$ the L^2-norm of distributions on N. $L^2_{\mathrm{ex}}(\hat{E}_1)$ is naturally equipped with a norm

$$\|\hat{u}\|^2_{\mathrm{ex}} := \|\hat{u} - \hat{u}_\infty\|^2_{L^2(\hat{E}_1)} + \|u_\infty\|^2.$$

Clearly $L^2_{\mathrm{ex}}(\hat{E}_1)$ with the above norm is a Hilbert space and we have a short exact sequence

$$0 \hookrightarrow L^2(\hat{E}_1) \hookrightarrow L^2_{\mathrm{ex}}(\hat{E}_1) \xrightarrow{\partial_\infty} L^2(E) \to 0.$$

The map

$$L^2(E) \ni u_\infty \mapsto \hat{u}_\infty \in L^2_{\mathrm{ex}}(\hat{E}_1)$$

defines a splitting of this sequence.

Define the space of extended L^2-solutions of $\hat{D}_{i,\infty}$ as

$$K_i := \ker \hat{D}_{i,\infty} \cap L^2_{\mathrm{ex}}(\hat{E}_i).$$

The results of [1] and [6] show that these are finite dimensional spaces and the spaces of asymptotic traces $L_i := \partial_\infty(K_i)$ are subspaces in $\mathcal{H} := \ker D \subset L^2(E)$. We have a difference map

$$\Delta : K_1 \oplus K_2 \to \mathcal{H}, \quad \hat{u}_1 \oplus \hat{u}_2 \mapsto \partial_\infty \hat{u}_1 - \partial_\infty \hat{u}_2.$$

We denote its kernel by $\mathcal{K}_\infty$. It is a finite dimensional subspace of $L^2_{\mathrm{ex}}(\hat{E}_1) \oplus L^2_{\mathrm{ex}}(\hat{E}_2)$.

Finally we define the *splitting map*

$$S_r : C^\infty(\mathcal{E}_r) \to L^2_{\mathrm{ex}}(\hat{E}_1) \oplus L^2_{\mathrm{ex}}(\hat{E}_2)$$

by $\psi \mapsto S_r^1\psi \oplus S_r^2\psi$ where

$$S_r^1\psi = \psi \ \text{ on } \ M_1(r) \subset M_1(\infty)$$

and

$$(S_r^1\psi)(t,x) = \psi(r,x), \ \ \forall t \geq r, \ x \in N.$$

S_r^2 is defined similarly. We can now formulate the main result of this paper. When $\hat{B}_i \equiv 0$, i.e., the operators $\hat{D}_{i,\infty}$ are translation invariant along the neck, this result was proved by Cappell-Lee-Miller in [3] and is implicitly contained in [13]. The super-symmetric situation was discussed in [7] in the special case of the anti-selfduality operators, and in a different analytic setup.

Main Theorem. *Fix a positive real number δ such that*

$$0 < \delta < \min(\gamma, \lambda)$$

where we recall that λ controls the sizes[4] *of the perturbations $\hat{A}_i$ and we denoted by γ the spectral gap of D, i.e.,*

$$\gamma := \mathrm{dist}\Big(0, \ \mathrm{spec}\,(D) \setminus \{0\}\Big).$$

For every function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$c(r) = o(1/r), \ \ c(r) \geq Ce^{-\delta r} \ \text{ as } r \to \infty$$

denote by $\mathcal{K}_r = \mathcal{K}_r(c)$ the subspace $L^2(\mathcal{E}_r)$ of spanned by the eigenvectors of $\mathfrak{D}_r$ corresponding to eigenvalues $|\lambda| \leq c(r)$. Then the following hold:

(a) *The splitting map S_r induces an asymptotic map*

$$\mathcal{K}_r(c) {\longrightarrow}^a K_1 \oplus K_2.$$

(b) *The sequence*

$$0 \to \mathcal{K}_r(c) \stackrel{S_r}{\longrightarrow}^a K_1 \oplus K_2 \stackrel{\Delta}{\to} \mathcal{H} \to \mathcal{H}/(L_1 + L_2) \to 0$$

is asymptotically exact. Furthermore, if all the operators involved are super-symmetric, the above sequence is super-symmetric as well.

In the course of the proof we will construct an asymptotic inverse Ψ_r to the splitting map which we call the *gluing map*. This is an asymptotic map $\mathcal{K}_\infty \to \mathcal{K}_r$ such that if P_r denotes the orthogonal projection onto $\mathcal{K}_r$ and P_∞ the orthogonal projection onto $\mathcal{K}_\infty$ then

$$\|(P_\infty S_r) \circ (P_r \Psi_r) - \mathbf{id}_{\mathcal{K}_\infty}\|_{L^2(M(r))} + \|(P_r \Psi_r) \circ (P_\infty S_r) - \mathbf{id}_{\widetilde{\mathcal{K}}_r}\|_{L^2(M(r))} = o(1)$$

as $r \to \infty$.

[4] Large $\lambda \Longleftrightarrow$ small perturbation.

3. First order elliptic equations on manifolds with cylindrical ends.

In this section we will survey a few analytical facts which are needed in the proof of the Main Theorem. The adequate functional background will be that of the Sobolev spaces $L^{k,p}$ consisting of distributions k-times differentiable with derivatives in L^p.

For any L^2_{loc} distribution $\hat{u} : t \mapsto u(t)$ on a cylinder $[0, L) \times N$ (where L can be ∞) we denote by $\rho_t(\hat{u})$ the function $[0, L) \to \mathbb{R}_+$ defined by

$$t \mapsto \rho_t(\hat{u}) := \left(\int_{C(t)} |u|^2 d\,\text{vol} \right)^{1/2}, \quad C(t) = [t, t+1] \times N.$$

Additionally, define

$$q : [0, L) \to [0, \infty], \quad t \mapsto q_{t,L}(\hat{u}) = \sup_{t<s<L} \rho_s(\hat{u}).$$

Note that if finite, $q_{t,L}$ is a decreasing function and thus belongs to $L^\infty_{\text{loc}}(0, L)$. When $L = \infty$ we set

$$q_t := q_{t,\infty}.$$

Now let us observe that the operator J induces a symplectic structure on $L^2(E)$ defined by

$$\omega(u, v) := \int_N (Ju, v) d\,\text{vol}.$$

The spectrum of D is real and consists only of discrete eigenvalues with finite multiplicities. Set

$$\mathcal{H}_\mu := \ker(\mu - D),$$

and denote by P_μ the orthogonal projection onto $\mathcal{H}_\mu$. Since $\{J, D\} = 0$ we deduce $J\mathcal{H}_\mu = \mathcal{H}_{-\mu}$. The spectral gap of D is the positive real number $\gamma = \gamma(D)$ defined as the smallest positive eigenvalue of D. Note that due to the spectral symmetry, $-\gamma(D)$ is also an eigenvalue of D. In particular, $\mathcal{H} = \mathcal{H}_0$ is J invariant and thus has an induced symplectic structure. We have the following result (see [**1**, **3**, **4**, **8**]).

Lemma 3.1. *The spaces $L_i = \partial_\infty(K_i)$ of asymptotic traces of extended L^2 solutions are Lagrangian subspaces of $\mathcal{H}$ i.e., $L_i^\perp = JL_i$.*

In the super-symmetric case we have $E = E^+ \oplus E^-$ and $G^*G = \mathbf{1}_{E^+}$, $GG^* = \mathbf{1}_{E^-}$

$$J = \begin{bmatrix} 0 & -G^* \\ G & 0 \end{bmatrix}.$$

Then $J(E^{\pm}) = JE^{\mp}$ and

$$D = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & J\mathbf{D}J^{-1} \end{bmatrix}.$$

The space $\mathcal{H}$ is $\mathbb{Z}_2$-graded

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-,$$

and $G\mathcal{H}^+ = \mathcal{H}^-$. The asymptotic limit spaces L_i now have decompositions

$$L_i = L_i^+ \oplus L_i^-, \;\; L_i^{\pm} \subset \mathcal{H}^{\pm},$$

and the Lagrangian condition translates into

$$(L_i^+)^{\perp} = G^* L_i^-, \;\; (L_i^-)^{\perp} = GL_i^+, \tag{3.1}$$

where $\perp$ denotes the orthogonal complement in $\mathcal{H}^{\pm}$.

Consider a cylinder $[0, L) \times N$. Denote by $\hat{E}$ the pullback of $E \to N$ to this cylinder and by $\hat{D}$ the partial differential operator on $C^{\infty}(\hat{E})$

$$\hat{D} = \partial_t - D.$$

For any eigenvalue μ of D and any smooth section $\hat{u}$ of $\hat{E}$ define a new section $\hat{u}_{\mu}$ by the condition

$$\hat{u}_{\mu}\,|_{t\times N} = P_{\mu} u(t) \quad u(t) := \hat{u}\,|_{t \times N}.$$

Clearly $\hat{u}_{\mu}$ is a smooth section which we will regard as a smooth map

$$\hat{u}_{\mu} = u_{\mu}(t) : [0, L) \to \mathcal{H}_{\mu}.$$

Set

$$u^{\perp}(t) = u(t) - u_0(t).$$

The following result is an elementary consequence of the method of separation of variables. To keep the flow of the arguments uninterrupted we defer the proof of this result to the Appendix.

Proposition 3.2 (Key Estimate)**.** *Fix $\lambda > 0$. There exists a constant $C > 0$ depending (continuously) only on the geometry of N,λ and the coefficients of D with the following property: For any smooth sections $\hat{u}$, $\hat{f}$ of $\hat{E}$ such that*

$$\hat{D}\hat{u} = \hat{f}$$

and

$$\|f(t)\| = O(e^{-\lambda t}), \;\; \text{as } t \to \infty, \tag{3.2}$$

the following inequalities hold:

$$\|u_0(t) - u_0(t+n)\| \le Ce^{-\lambda t} \quad \forall n \in \mathbb{Z} \cap [0, L-t), \tag{3.3}$$

$$\rho_{t+n}(\hat{u}^{\perp}) \le C\Big(e^{-\gamma n}\rho_t(\hat{u}^{\perp}) + e^{-\gamma n}\rho_{t+2n}(\hat{u}^{\perp}) + e^{-\lambda t}\Big), \tag{3.4}$$

$\forall n \in \mathbb{Z} \cap [0, (L-t)/2)$. *Above, γ denotes the spectral gap of D, i.e.,*

$$\gamma := \operatorname{dist}\Big(0,\ \operatorname{spec}(D) \setminus \{0\}\Big).$$

We have the following immediate consequence whose proof is left to the reader.

Corollary 3.3. *Let $L = \infty$ in the* Key Estimate *and fix $\lambda > 0$. There exists a constant $C > 0$ which depends only on λ, the geometry of N and D with the following property. If*

$$\hat{D}\hat{u} = \hat{f},$$

where both $\hat{u}$ and $\hat{f}$ are smooth and satisfy

$$\rho_t(\hat{u}) \in L^\infty(\mathbb{R}_+), \quad \|f(t)\| = O(e^{-\lambda t}), \tag{3.5}$$

then

$$\hat{u} \in L^2_{\rm ex}(\hat{E})$$

and

$$\|\partial_\infty \hat{u} - u_0(t)\| \leq C\Big(e^{-\lambda t} + e^{-\gamma t} q_t(\hat{u}^\perp)\Big).$$

Suppose $\hat{A}$ is a smooth selfadjoint endomorphism of $\hat{E} \to \mathbb{R}_+ \times N$ such that for some $\lambda > 0$ we have

$$\sup\{|A(t,x)|\ ;\ x \in N\} = O(e^{-\lambda t}). \tag{3.6}$$

Set $\hat{A}_r := \eta_r(t)\hat{A}$. The next result explains the role of the condition

$$c(r) = o(1/r)$$

in the statement of the Main Theorem.

Proposition 3.4. *Suppose that we have a sequence of smooth sections $\hat{u}_r$ satisfying the following conditions:*

(a) *There exists $C > 0$ such that $\rho_t(\hat{u}_r) < C$ for all $t, r > 0$.*
(b) *The sections $\hat{u}_r$ and their derivatives are uniformly bounded on $C(0)$.*
(c) *There exists a sequence of smooth endomorphisms B_r of E such that*

$$m(r) := \sup\{|B_r(x)|\ ;\ x \in N\} = o(1/r) \quad \textit{as} \quad r \to \infty$$

and $\hat{D} - \hat{A}_r\hat{u}_r - B_r\hat{u}_r = 0$ on the cylinder $[0, r] \times N$.
(d) *$u_r(t) = u_r(r)$, $\forall t \geq r \geq 0$.*

Then a subsequence of $\hat{u}_r$ converges in the norm of $L^2_{\rm ex}$ *to a section $\hat{u}$ satisfying $\hat{D} - \hat{A}\hat{u} = 0$ on $\mathbb{R}_+ \times N$. Moreover, on a subsequence*

$$u_r(r) \to \partial_\infty \hat{u} \quad \textit{in the norm of } L^2(E). \tag{3.7}$$

Proof. In the sequel we will use the same symbol C to denote positive constants independent of $t, r > 0$. Set $\hat{f}_r = \hat{A}_r \hat{u}_r + B_r \hat{u}_r$. Then

$$\hat{D}\hat{u}_r = \hat{f}_r \text{ on } [0, r] \times N. \tag{3.8}$$

Conditions (3.6), (a) and (b) coupled with a standard bootstrap argument imply that there exists a constant $C > 0$ such that

$$\sup\{|\hat{u}_r(t, x)| \,;\, (t, x) \in [0, r-1] \times N\} \leq C, \quad \forall r > 0. \tag{3.9}$$

This implies that a subsequence of $\hat{u}_r\,|_{[0,r]\times N}$ converges weakly in L^2_{loc} to a section $\hat{u}$ defined over the entire cylinder. Clearly $B_r \hat{u}_r \to 0$ in L^2_{loc} so that $\hat{u}$ is a weak solution of

$$\hat{D}\hat{u} - \hat{A}\hat{u} = 0 \text{ on } \mathbb{R}_+ \times N.$$

We can now conclude via elliptic estimates that we can extract a subsequence which converges is strongly in $L^{k,2}_{\mathrm{loc}}$. Moreover, according to (3.9) we deduce $\hat{u} \in L^\infty$. If we now set $\hat{f} = \hat{A}\hat{u}$ we deduce

$$\rho_t(f) \leq \|\hat{u}\|_\infty \rho_t(\hat{A}) = O(e^{-\lambda t}).$$

Corollary 3.3 implies $\hat{u} \in L^2_{\mathrm{ex}}$ and

$$\|u(t) - \partial_\infty \hat{u}\| \leq C(e^{-\gamma t} + e^{-\lambda t}). \tag{3.10}$$

The Key Estimate for (3.8), where

$$q_{t,r}(\hat{f}_r) \leq C(rm(r) + q_t(A_r)) \leq C(rm(r) + e^{-\lambda t}), \quad r \geq t \geq 0$$

implies that for all $0 \leq t \leq r$ we have

$$\|u_r(t) - u_r(r)\| \leq C(rm(r) + e^{-\lambda t}). \tag{3.11}$$

This proves (3.7) since $rm(r) = o(1)$. To show that the convergence $\hat{u}_r \to \hat{u}$ also takes place in the norm of L^2_{ex} we only need to establish that on a subsequence

$$\lim_{r\to\infty} \int_0^\infty dt \int_N |(u_r(t) - u(t)) - (u_r(r) - \partial_\infty \hat{u})|^2 d\,\mathrm{vol} \to 0.$$

We extract the subsequence using the following argument. For every $n > 0$ pick $r = r_n > n$ such that the following inequalities hold:

$$\int_0^n dt \int_N |u_{r_n}(t) - u(t)|^2 d\,\mathrm{vol} \leq \frac{1}{n^2} \tag{3.12}$$

$$\int_N |u_{r_n}(n) - u_{r_n}(r_n)|^2 d\,\mathrm{vol} < \frac{1}{n^2} \tag{3.13}$$

$$\int_n^\infty dt \int_N |u(t) - \partial_\infty \hat{u}|^2 d\,\mathrm{vol} \leq \frac{1}{n^2}. \tag{3.14}$$

The choice (3.12) is possible because the sequence $\hat{u}_r$ converges to $\hat{u}$ in the norm $L^2([0,n]\times N)$. The choice (3.13) is possible because $rm(r)=o(1)$ and (3.11). Finally, the choice (3.14) is possible because of (3.10). The subsequence $\hat{u}_{r_n}$ chosen as above converges to $\hat{u}$ in the norm of L^2_{ex}. □

4. Proof of the Main Theorem.

To show that $\lim_{r\to\infty}\delta(S_r(\mathcal{K}_r),\mathcal{K}_\infty)=0$ we will use the following elementary result which follows immediately from Lemma 2.1.

Lemma 4.1. *Suppose U is a finite dimensional subspace in a Hilbert space and U_r is a sequence of finite dimensional subspaces such that*

$$\lim_{r\to\infty}\hat{\delta}(U_r,U)=0 \tag{4.1}$$

and

$$\operatorname{liminf}\,\dim U_r\geq\dim U. \tag{4.2}$$

Then

$$\lim_{r\to\infty}\delta(U_r,U)=0.$$

We will show that the two assumptions in the lemma are satisfied if $U_r=S_r\mathcal{K}_r$ and $U=\mathcal{K}_\infty$. The proof of the Main Theorem is thus divided in two steps.

Step 1.

$$\lim_{r\to\infty}\hat{\delta}(S_r(\mathcal{K}_r),\mathcal{K}_\infty)=0.$$

We argue by contradiction. Thus we assume there exists a sequence $\psi_r\in\mathcal{K}_r$ such that

$$\|S_r\psi_r\|_{\text{ex}}=O(1)\ \text{ as } r\to\infty \tag{4.3}$$

and there exists $d_0>0$ such that

$$\operatorname{dist}(S_r\psi_r,\mathcal{K}_\infty)>d_0,\ \ \forall r>0. \tag{4.4}$$

Set $\psi_r^i:=S_r^i\psi_r$, $i=1,2$. We study only the behavior of ψ_r^1. The sequence ψ_r^2 behaves similarly. Condition (4.4) shows there exists a constant $c>0$ such that

$$\|\psi_r^1\|_{\text{ex}}\geq c\ \ \forall r>0.$$

Thus we can normalize ψ_r^1 so that $\|\psi_r^1\|_{\text{ex}}=1$ and (4.4) continues to hold (with an eventually smaller $d_0>0$).

Note first that using standard elliptic estimates and (4.3) we deduce that ψ_r^1 and its derivatives are uniformly bounded on $M_1(0)$. Thus a subsequence of ψ_r^1 converges to a solution of $\hat{D}_{1,\infty}\hat{u}=0$ on $M_1(0)$. Using Proposition 3.4

we deduce that a further subsequence of the restriction of ψ_r^1 to $\mathbb{R}_+ \times N$ converges in the norm of $L^2_{\rm ex}$ to a solution of $\hat{D}_{1,\infty}\hat{\psi} = 0$ on this semi-infinite cylinder. Clearly we have produced a section $\psi_\infty^1 \in \ker \hat{D}_{1,\infty} \cap L^2_{\rm ex}$ of norm 1. We proceed similarly with ψ_r^2. We now have a pair

$$\Psi := \psi_\infty^1 \oplus \psi_\infty^2 \in K_1 \oplus K_2$$

of norm 2 which according to (3.7) in Proposition 3.4 have the same asymptotic limit. Thus $\Psi \in \mathcal{K}_\infty$. However, this contradicts (4.4). Step 1 is completed.

Step 2. We will prove that

$$\dim \mathcal{K}_\infty \leq \dim S_r(\mathcal{K}_r) \ \ \forall r \gg 0.$$

We will rely on the following auxiliary result.

Lemma 4.2. *Suppose $\hat{u} \in L^{1,2}(\mathcal{E}_r)$ is such that*

$$\|\mathfrak{D}_r \hat{u}\|_{L^2(M(r))} < (1-\varepsilon)c(r)\|u\|_{L^2(M(r))}.$$

Then $\operatorname{dist}(u, \mathcal{K}_r(c)) < (1-\varepsilon)\|u\|_{L^2(M(r))}$.

Proof. Using the orthogonal decomposition

$$L^2(\mathcal{E}_r) = \mathcal{K}_r(c) \oplus \mathcal{K}_r(c)^\perp$$

we can write $\hat{u} = v + v^\perp$. Then $\operatorname{dist}(u, \mathcal{K}_r) = \|v^\perp\|_{L^2(M(r))}$. On the other hand

$$\begin{aligned}(1-\varepsilon)^2 c(r)^2 \|\hat{u}\|^2_{L^2(M(r))} &> \|\mathfrak{D}_r \hat{u}\|^2_{L^2(M(r))} \\ &\geq \|\mathfrak{D}_r v^\perp\|^2_{L^2(M(r))} \\ &\geq \Lambda^2 \|v^\perp\|^2_{L^2(M(r))}\end{aligned}$$

where $\Lambda^2 > c(r)^2$. The lemma is proved. □

To conclude the proof of Step 2 we will construct for $r \gg 0$ a space $V_r \subset L^2(\mathcal{E}_r)$ isomorphic to $\mathcal{K}_\infty$ such that

$$\hat{\delta}(V_r, \mathcal{K}_r) < 1. \tag{4.5}$$

According to Lemma 2.3, Chap. IV, §2 in [**5**] this means that the orthogonal projection onto $\mathcal{K}_r$ induces an injection $V_r \to \mathcal{K}_r$ so that

$$\dim \mathcal{K}_\infty = \dim V_r \leq \dim \mathcal{K}_r, \ \ \forall r \gg 0.$$

Condition (4.5) is satisfied provided $\operatorname{dist}(v, \mathcal{K}_r) < v$, for all $v \in V_r \setminus \{0\}$. According to Lemma 4.2 is suffices to construct a subspace $V_r \subset L^{1,2}(\mathcal{E}_r)$ isomorphic to $\mathcal{K}_\infty$ such that

$$\sup_{v \in V_r \setminus \{0\}} \frac{\|\mathfrak{D}_r v\|^2_{L^2(M(r))}}{\|v\|^2_{L^2(M(r))}} < c(r). \tag{4.6}$$

Such a subspace is obtained via a simple gluing construction.

We construct a gluing map

$$\Psi_r : \mathcal{K}_\infty \to L^{1,2}(\mathcal{E}_r), \ \ \hat{u}_1 \oplus \hat{u}_2 \mapsto \Psi_r$$

uniquely determined by the following conditions. Let u_∞ denote the common asymptotic limit of $\hat{u}_i$. Now set

$$\hat{v}_1 = \eta_r(t)\hat{u}_1 + \Big(1 - \eta_r(t)\Big)u_\infty.$$

Define $\hat{v}_2$ similarly. Clearly on the overlap

$$\imath_r : [r+1, r+2] \times N \to [-r-2, -r-1] \times N$$

we have

$$\hat{v}_1 = \hat{v}_2 = u_\infty$$

so we can glue these two sections on the overlap to produce a smooth section $\Psi_r \in C^\infty(\mathcal{E}_r)$. Clearly the map Ψ_r is linear. Set $V_r := \Psi_r(\mathcal{K}_r)$.

Note that Ψ_r is injective because if $\Psi_r(\hat{u}_1, \hat{u}_2) \equiv 0$ then both $\hat{u}_i$ must vanish on $M_i(0)$ and by unique continuation they must vanish everywhere. We claim V_r satisfies (4.6).

Clearly $\mathfrak{D}_r\Psi_r \equiv 0$ on $M_1(r-1), M_2(-r+1) \subset M(r)$ so we only need an estimate of $\hat{D}_{1,r}\hat{v}_1$ on the cylinder $[r-1, r+2] \times N$ and a similar one for $\hat{D}_{2,r}\hat{v}_2$. On this cylinder we have $\hat{D}_{1,r} = J(\partial_t - D - \eta_r\hat{A}_1)$ so that we have

$$\begin{aligned} -J\hat{D}_{1,r}\hat{v}_1 &= \Big(\hat{D} - \eta_r\hat{A}_1\Big)\Big(\eta_r\hat{u}_1 + (1-\eta_r)u_\infty\Big) \\ &= \Big(\hat{D} - \hat{A}_1\Big)\Big(\eta_r\hat{u}_1 + (1-\eta_r)u_\infty\Big) \\ &\quad + (1-\eta_r)\hat{A}_1\Big(\eta_r\hat{u}_1 + (1-\eta_r)u_\infty\Big). \end{aligned}$$

The first term can be rewritten as

$$\begin{aligned} &\Big(\hat{D} - \hat{A}_1\Big)\Big(\eta_r\hat{u}_1 + (1-\eta_r)u_\infty\Big) \\ &= \big[\hat{D} - \hat{A}_1, \eta_r\big]\hat{u}_1 + \Big[\hat{D} - \hat{A}_1, (1-\eta_r)\Big]u_\infty \\ &\quad + \eta_r(\hat{D} - \hat{A}_1)\hat{u}_1 + (1-\eta_r)\Big(\hat{D} - \hat{A}_1\Big)u_\infty \end{aligned}$$

($\hat{\mathbf{c}}$=Clifford multiplication on $\mathbb{R}_+ \times N$)

$$= \hat{\mathbf{c}}(d\eta_r)\Big(\hat{u}_1(t) - u_\infty\Big) - (1-\eta_r)\hat{A}_1u_\infty.$$

Thus

$$-J\hat{D}_{1,r}\hat{v}_1 = \hat{\mathbf{c}}(d\eta_r)\Big(\hat{u}_1(t) - u_\infty\Big) + \eta_r(1-\eta_r)\hat{A}_1\Big(\hat{u}_1(t) - u_\infty\Big).$$

We can now use Corollary 3.3 for the equation

$$\hat{D}\hat{u}_1 = \hat{f} := \hat{\mathbf{c}}(d\eta_r)\Big(\hat{u}_1(t) - u_\infty\Big) + \eta_r(1-\eta_r)\hat{A}_1\Big(\hat{u}_1(t) - u_\infty\Big) \ \text{ on } \ \mathbb{R}_+ \times N.$$

The decay rate of $A_1(t)$ shows that $\rho_t(\hat{f}) = O\Big(e^{-\lambda t}\rho_t(\hat{u}_1)\Big)$. Hence

$$\|\mathfrak{D}_r\hat{v}_1\|_{2,[r-1,r+2]\times N} \leq C(e^{-\lambda r} + e^{-\gamma r})q_t(\hat{v}_1).$$

We obtain a similar estimate involving $\hat{v}_2$. Since obviously $q_t(v_i) \leq C\|\Psi_r\|_{L^2(M(r))}$ (for $r \gg 0$) we deduce

$$\|\mathfrak{D}_r\Psi_r\|_{2,[r-1,r+2]\times N} \leq C(e^{-\lambda r} + e^{-\gamma r})\|\Psi_r\|_{L^2(M(r))}. \tag{4.7}$$

The last estimate implies (4.6) since

$$c(r) > C(e^{-\lambda r} + e^{-\gamma r}), \;\; \forall r \gg 0.$$

The Main Theorem is proved. □

Remark 4.3. We can rewrite the conclusion of the Main Theorem as a short asymptotically exact sequence

$$0 \to \mathcal{K}_r \overset{S_r}{\longrightarrow}{}^{a} K_1 \oplus K_2 \overset{\Delta}{\to} L_1 \oplus L_2 \to 0.$$

The gluing map Ψ_r is an asymptotic splitting of this sequence, in the sense described in the introduction.

Remark 4.4. The Main Theorem extends easily to families of operators. Suppose X is a compact CW-complex and all the constructions in the introduction depend continuously on the parameter $x \in X$ such that the spectral gaps of the boundary operators D_x are bounded from below

$$\gamma_0 := \inf_{x\in X}\gamma(D_x) > 0.$$

Then $h(x) := \dim\mathcal{H}_x$ is independent of x. We denote this common dimension by h. Assume also that the functions

$$\kappa_i : X \to \mathbb{Z}, \;\; \kappa_i(x) := \dim K_i(x) \;\; (i = 1, 2)$$

$$\ell : X \to \mathbb{Z}, \;\; \ell(x) = \dim L_1(x) \cap L_2(x)$$

are constant, $k_i(x) \equiv \kappa_i$, $\ell(x) \equiv \ell$. One then can show that $K_i(x)$ and $L_1(x) \cap L_2(x)$ depend continuously upon x in the gap topology. Thus they can be viewed as continuous maps in grassmannians of finite dimensional subspaces in Hilbert spaces and as such they define vector bundles over X. The Main Theorem for families states that for $r \gg 0$ the spaces $\widetilde{\mathcal{K}}_{r,x}(c)$ form a vector bundle over X and we have and exact sequence of vector bundles

$$0 \to \mathcal{K}_r \overset{\Gamma_r}{\to} K_1 \oplus K_2 \overset{\Delta}{\to} \mathcal{H} \to \mathcal{H}/(L_1 + L_2) \to 0.$$

Since L_1, L_2 are Lagrangian then

$$\mathcal{H}/(L_1 + L_2) \cong (L_1 + L_2)^{\perp} = L_1^{\perp} \cap L_2^{\perp} = J(L_1 \cap L_2).$$

A similar statement is true in the super-symmetric case. In [**10**] we describe a general gluing formula for index of families when the functions $h(x)$, $\ell(x)$ and $\kappa_i(x)$ are not necessarily constant.

5. Applications.

As promised, we will include some simple applications of the Main Theorem. For more sophisticated applications in gauge theory we refer to [**11**].

A. Suppose first that $\mathcal{K}_\infty = 0$. This is possible if and only if

$$L_1 \cap L_2 = 0$$

and $\ker(\partial_\infty : K_i \to L_i) = 0$, $i = 1, 2$. These kernels consist of the L^2-solutions of $\hat{D}_{i,\infty}$. This shows that the operators $\mathfrak{D}_r$ cannot have eigenvalues λ_r such that $|\lambda_r| = o(1/r)$ as $r \to \infty$. We have thus established the following result (proved for the first time in [**4**]).

Corollary 5.1. *Suppose that*

$$L_1 \cap L_2 = \{0\} \text{ and } \ker \hat{D}_{i,\infty} \cap L^2(\hat{E}_i) = \{0\}.$$

Then for $r \gg 0$ the operator $\mathfrak{D}_r$ has a bounded inverse

$$\mathfrak{D}_r^{-1} : L^2(\mathcal{E}_r) \to L^2(\mathcal{E}_r)$$

and

$$\|\mathfrak{D}_r^{-1}\|_{L^2,L^2} = O(r) \quad \textit{as} \quad r \to \infty.$$

B. Suppose now the entire situation is super-symmetric. Thus, we have decompositions

$$K_i = K_i^+ \oplus K_i^-, \;\; \mathcal{K}_\infty = \mathcal{K}_\infty^+ \oplus \mathcal{K}_\infty^-.$$

We assume

(5.1) $$K_i^- = \{0\}, \;\; i = 1, 2.$$

This implies $L_i^- = \{0\}$ and $\mathcal{K}_\infty^- = \{0\}$. The equality (3.1) shows that $L_1^+ = L_2^+ = \mathcal{H}^+$. We deduce[5]

(5.2) $$\mathcal{K}_r^- = \{0\}, \;\; \forall r \gg 0,$$

while the even part $\mathcal{K}_r$ fits in an exact sequence

$$0 \to \widetilde{\mathcal{K}}_t \stackrel{\Gamma_r^+}{\to} K_1^+ \oplus K_2^+ \stackrel{\Delta^+}{\to} \mathcal{H}^+ \to 0.$$

The bundle $\mathcal{E}_r$ has a decomposition

$$\mathcal{E}_r = \mathcal{E}_r^+ \oplus \mathcal{E}_r^-$$

with respect to which $\mathfrak{D}_r$ has the super-symmetric block decomposition

$$\mathfrak{D}_r = \begin{bmatrix} 0 & \mathscr{D}_r^* \\ \mathscr{D}_r & 0 \end{bmatrix}$$

[5] In [**11**] we have explained the relationship between $\mathcal{K}_r^-$ and the obstruction space of [**7**].

where $\mathfrak{D}_r : C^\infty(\mathcal{E}_r^+) \to C^\infty(\mathcal{E}_r^-)$. The equality (5.2) implies that $\mathfrak{D}_r$ is onto since

$$\mathcal{K}_r^- = \ker \mathfrak{D}_r^* \cong \operatorname{coker} \mathfrak{D}_r.$$

Thus $\mathfrak{D}_r\mathfrak{D}_r^*$ is one-to-one and onto and admits a bounded inverse $L^2(\mathcal{E}_r^-) \to L^2(\mathcal{E}_r^-)$. We claim that

$$\|(\mathfrak{D}_r\mathfrak{D}_r^*)^{-1}\|_{L^2,L^2} = O(r^2). \tag{5.3}$$

To prove this claim we argue by contradiction.

Because $\mathfrak{D}_r\mathfrak{D}_r^*$ is self-adjoint, positive and has compact resolvent, the norm of its inverse is $m(r)^{-1}$ where

$$m(r) = \inf\Big\{ \langle \mathfrak{D}_r\mathfrak{D}_r^* u\, ,\, u\, \rangle;\ \ \|u\|_{L^2(M(r))} = 1 \Big\}.$$

Suppose that for every $r \gg 0$ we can find $\phi_r \in L^2(\mathcal{E}_r^-)$ such that

$$\|\phi_r\|_{L^2(M(r))} = 1 \ \text{ and } \ m(r) = \|\mathfrak{D}_r^*\phi_r\|^2_{L^2(M(r))}$$

$$= \Big\langle \mathfrak{D}_r\mathfrak{D}_r^*\phi_r, \phi_r \Big\rangle_{L^2(M(r))} = o(1/r^2) \ \text{ as } \to \infty.$$

Now pick $c(r) > \exp(-\delta(r))$ such that

$$c(r) = o(1/r), \ \ m(r) = o(c(r)^2) \ \text{ as } \ r \to \infty.$$

The above δ is the same exponent as in the Main Theorem.

Now apply $\mathfrak{D}_r$ to the vector $u_r := 0 \oplus \phi_r \in L^2(\mathcal{E}_r^+ \oplus \mathcal{E}_r^-)$. We deduce

$$\|u_r\|_{L^2(M(r))} = 1 \ \text{ and } \ \frac{\|\mathfrak{D}_r u_r\|_{L^2(M(r))}}{\|u_r\|_{L^2(M(r))}} = \sqrt{m(r)}.$$

Thus, according to Lemma 4.2 we can conclude

$$\operatorname{dist}(u_r, \mathcal{K}_r(c)) \le \frac{\sqrt{m(r)}}{c(r)} = o(1).$$

On the other hand, u_r is purely odd which implies $\mathcal{K}_r^-(c) \neq 0$ for all $r \gg 0$. This contradicts (5.2) and thus proves (5.3). This estimate also shows that $\mathfrak{D}_r$ has a right inverse $R_r : L^2(\mathcal{E}_r^-) \to L^2(\mathcal{E}_r^+)$ of norm $O(r)$. We can now state our next result.

Proposition 5.2. *Suppose Condition (5.1) is satisfied. Then for $r \gg 0$ the operator $\mathfrak{D}_r$ is onto and admits a bounded right inverse of norm $O(r)$. Moreover*

$$\ker \mathfrak{D}_r = \mathcal{K}_r^+ \ \ \textit{for} \ \ r \gg 0. \tag{5.4}$$

Proof. The only thing left to prove is the equality (5.4) which follows immediately from the fact that the index of $\mathfrak{D}_r$ is independent of r and

$$\dim \mathcal{K}_r^+ - \dim \mathcal{K}_r^- = \operatorname{ind} \mathfrak{D} = \dim \ker \mathfrak{D}_r - \dim \ker \mathfrak{D}_r^*.$$

□

Suppose now that in Proposition 5.2 we have a family of operators, each satisfying (5.1) and subject to the restrictions listed in Remark 4.4. We deduce immediately the following consequence.

Corollary 5.3. *Under the above assumptions, there exists an exact sequence of vector bundles*

$$0 \to \ker \mathfrak{D}_r \xrightarrow{\Gamma_r} K_1 \oplus K_2 \xrightarrow{\Delta^+} \mathcal{H}^+ \to 0.$$

In particular, by passing to determinant line bundles we deduce an isomorphism of line bundles over X

$$\det(\operatorname{ind}(\mathfrak{D}_r)) \cong \det K_1 \otimes \det K_2 \otimes (\det \mathcal{H}^+)^*$$

where the left hand side term $\operatorname{ind}(\mathfrak{D}_r)$ *is viewed as an element in an appropriate K-theory of the parameter space X.*

Remark 5.4. (a) The terms $\det K_i$ are also determinant line bundles associated to the indices of the families of Atiyah-Patodi-Singer problems determined by $\hat{\mathbf{D}}_i$, $i = 1, 2$.

(b) Corollary 5.3 is also useful in orientability issues involving various moduli spaces arising in gauge theory.

The final application of the gluing theory we want to discuss has to do with lower estimates of the eigenvalues of the Hodge-DeRham Laplacian.

Denote by $\lambda_{1,k} = \lambda_{1,k}(r)$ the first nonzero eigenvalue of the Laplacian acting on the k-forms on $M(r)$. It is known (see [**12**, Chap. III, §4, Thm. 4]) that lower bounds on the Ricci curvature of $M(r)$ produce lower bounds $\mu_0(r)$ on $\lambda_{1,0}(r)$. In our geometric context this lower bound has the asymptotic behavior

$$\mu_0(r) \sim C_0 e^{-\nu r} r^{-2} \text{ as } r \to \infty,$$

where C_0 and ν are positive constants. The gluing theorem allows us to prove similar asymptotic lower estimates for all $\lambda_{1,k}(r)$ and surprisingly, the reason for these estimates is *topological* rather than geometric. Such estimates for $\lambda_{1,1}(r)$ are particularly useful in gauge theory.

Proposition 5.5. *Denote by δ the smallest nonnegative eigenvalue of the Hodge-DeRham Laplacian on the hypersurface N. Fix a continuous function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$c(r) = o(r^{-1}), \;\; c(r) > e^{-\delta r/2} \quad \textit{as } r \to \infty.$$

Then there exists $R > 0$ such that for all $r > R$ we have

$$\lambda_{1,k}(r) \geq c(r)^2, \quad \forall k = 0, 1, \ldots, \dim M(r).$$

Proof. Denote by $\mathfrak{D}_r$ the Hodge-DeRham operator

$$d + d^* : \Omega^*(M(r)) \to \Omega^*(M(r)).$$

The Hodge-DeRham Laplacian is the operator $\mathfrak{D}_r^2$. The eigenvalues of $\mathfrak{D}_r^2$ smaller than $c(r)^2$ span the vector space $\mathcal{K}_r(c)$. Thus, to prove the above estimate it suffices to show that

$$\mathcal{K}_r = \ker \mathfrak{D}_r \ \ \forall r \gg 0.$$

In this case the spaces K_i have a *topological* description (see [**1**]) and Mayer-Vietoris theorem allows us to identify

$$\ker\big(\Delta : K_1 \oplus K_2 \to L_1 + L_2\big)$$

with the DeRham cohomology of $M(r)$ (see [**11**, §4.1.6]). This means

$$\dim \mathcal{K}_r(c) = \dim H^*(M(r)) = \dim \ker \mathfrak{D}_r$$

so that $\mathcal{K}_r(c) = \ker \mathfrak{D}_r$. □

Appendix A. Some technical proofs.

The Key Estimate is a consequence of the following elementary result.

Lemma A.1. *Fix $\mu \in \mathbb{R}$. Suppose U is a finite dimensional Hilbert space and $u(t), f(t) : [0, L) \to U$ are two smooth functions satisfying the ordinary differential equation*

$$\dot{u} = \mu u + f. \tag{A.1}$$

Then there exists a constant $C > 0$ independent of μ, u and f such that the following hold:

(a) *If $\mu = 0$ then*

$$|u(t) - u(t+n)| \leq \int_t^{t+n} q_{s,L}(f) ds, \quad \forall t \in [0, L-n). \tag{A.2}$$

(b) *If $\mu > 0$ then*

$$|u(t)| \leq e^{-n|\mu|}|u(t+n)| + \frac{C}{\mu(1-e^{-2\mu})^{1/2}} \left(\int_t^{t+n} |f|^2 ds \right)^{1/2}, \quad \forall t \in [0, L-n). \tag{A.3}$$

(c) *If $\mu < 0$ then*

$$|u(t+n)| \leq e^{-n|\mu|}|u(t)| + \frac{C}{\mu(1-e^{-2\mu})^{1/2}} \left(\int_t^{t+n} |f|^2 ds \right)^{1/2}, \quad \forall t \in [0, L-n). \tag{A.4}$$

Proof. We prove only (a) and (b). (c) follows from (b) by time reversal.

Proof of (a). We have

$$|u(t+1) - u(t)| \leq \int_t^{t+1} |f(s)| ds \leq \rho_t(f).$$

Thus

$$\begin{aligned} |u(t+n) - u(t)| &\leq \sum_{k=1}^{n} |u(t+k) - u(t+k-1)| \\ &\leq \sum_{k=1}^{n} \rho_{t+k-1}(f) \leq \int_t^{t+n} q_{t,L}(f). \end{aligned}$$

Proof of (b). Denote by $\mathbf{e}_\mu$ the exponential function $e^{\mu t}$. We have

$$u(t+1) = e^\mu u(t) + \int_0^1 \mathbf{e}_\mu(1-s) f(t+s) ds$$

so that by Cauchy's inequality

$$|u(t+1) - e^\mu u(t)| \leq \rho_0(\mathbf{e}_\mu)\rho_t(f).$$

Hence

$$|u(t)| \leq e^{-\mu}|u(t+1)| + e^{-\mu}\rho_0(\mathbf{e}_\mu)\rho_t(f).$$

Now observe that

$$e^{-\mu}\rho_0(\mathbf{e}_\mu) = e^{-\mu}\frac{e^\mu - 1}{\mu} \leq \frac{1}{\mu}.$$

Set $x_k := |u(t+k)|$. The sequence x_k satisfies the difference inequality

$$x_k \leq e^{-\mu} x_{k+1} + \frac{1}{\mu}\rho_{t+k}(f).$$

Thus

$$x_0 \leq e^{-n\mu} x_n + \frac{1}{\mu} \sum_{k=0}^{n-1} \rho_{t+k}(f) e^{-(n-1-k)\mu}$$

(use the Cauchy-Schwartz inequality)

$$\leq e^{-n\mu} + \frac{1}{\mu}\left(\sum_{k=0}^{n-1} \rho_t(f)^2\right)^{1/2} \left(\sum_{k=0}^{n-1} e^{-2k\mu}\right)^{1/2}$$

$$\leq e^{-n\mu} + \frac{1}{\mu(1-e^{-2\mu})^{1/2}} \left(\int_t^{t+n} |f|^2 ds\right)^{1/2}.$$

This proves (A.3) and the lemma. □

Proof of the Key Estimate. Set

$$\nu(\gamma) := \sup_{\mu \geq \gamma} \frac{1}{\mu(1-e^{-2\mu})^{1/2}}.$$

Clearly $\nu(\gamma) \leq \infty$. Let $\hat{u}$ and $\hat{f}$ as in the statement of Proposition 3.2. Form the spectral decompositions

$$\hat{u} = \sum_{\mu} \hat{u}_\mu, \quad \hat{f} := \sum_{\mu} \hat{f}_\mu,$$

where $\hat{u}_\mu(t) := P_\mu \hat{u}(t)$ and $\hat{f}_\mu(t) := P_\mu \hat{f}(t)$. Because the sections $\hat{u}$ and $\hat{f}$ are smooth the above series converges in any Sobolev norm. The components $\hat{u}_\mu$ and $\hat{f}_\mu$ satisfy the ordinary differential equation (A.1). The inequality (3.3) is an immediate consequence of (A.2).

Using Lemma A.1 and the inequality $|\mu| \geq \gamma$ for every nonzero eigenvalue μ of D we deduce

$$\|\hat{u}_\mu(s+n)\|^2 \leq 2\left(e^{-2n\gamma}\|\hat{u}_\mu(s+n+\epsilon_\mu n)\|^2 + C\nu(\gamma)^2 \int_s^{s+n} \|\hat{f}_\mu(\tau)\|^2 d\tau\right),$$

where $\epsilon_\mu := \operatorname{sign}(\mu)$. Integrating with respect to $s \in (t, t+1)$, and using the Pythagorean theorem

$$\|\hat{u}^\perp(t)\|^2 = \sum_{\mu \neq 0} \|\hat{u}_\mu(t)\|^2, \quad \|\hat{f}^\perp(t)\|^2 = \sum_{\mu \neq 0} \|\hat{f}_\mu(t)\|^2$$

we deduce

$$\rho_{t+n}(\hat{u}^\perp)^2 \leq 2\Big(e^{-2\gamma n}\rho_t(\hat{u}^\perp)^2 + e^{-2\gamma n}\rho_{t+2n}(\hat{u}^\perp)^2\Big)$$

$$+C\nu(\gamma)^2 \int_t^{t+n+1} \|\hat{f}^\perp(s)\|^2 ds.$$

The estimate (3.4) now follows from the elementary inequality

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}.$$

□

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