

# BEAD SLIDING AND CONVEX INEQUALITIES

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ABSTRACT. We analyze a simple game of beads on a rod and relate it to some classical convex inequalities..

We consider distributions (or configurations) of  $n$  beads on the real semiaxis  $[\mu, \infty)$ . Any bead in such a distribution is capable of sliding to the right (in the positive direction) but not allowed to slide to the left. We indicate such a distribution of beads by a vector

$$\vec{A} = (A_1, \dots, A_n), \quad \mu \leq A_1 < A_2 < \dots < A_n,$$

where the coordinates  $A_i$  indicate the positions of the beads. The  $i$ -th bead is the bead located at  $A_i$ . A distribution is called *monotone* if

$$A_1 - \mu \leq A_2 - A_1 \leq \dots \leq A_n - A_{n-1}.$$

We denote by  $\mathcal{B}_n = \mathcal{B}_n(\mu)$  the collection of monotone distributions of  $n$  beads on the semiaxis  $[\mu, \infty)$ . Clearly, we can view  $\mathcal{B}_n(\mu)$  as a closed convex set in  $\mathbb{R}^n$ .

We will indicate the elements of  $\mathcal{B}_n(\mu)$  using capital letters  $\vec{A}, \vec{B}$  etc. To a configuration  $\vec{A} \in \mathcal{B}_n(\mu)$  we associate the vector of differences  $\vec{a} = \Delta\vec{A}$ ,

$$\vec{a} = (a_1, \dots, a_n), \quad a_1 = A_1 - \mu, \dots, a_k = A_k - A_{k-1}, \forall k = 2, \dots, n.$$

We have a natural partial order on  $\mathcal{B}_n(\mu)$

$$\vec{A} \leq \vec{B} \iff A_k \leq B_k, \quad \forall k = 1, \dots, n.$$

Let  $e_1, \dots, e_n$  denote the canonical basis of  $\mathbb{R}^n$ . Given a bead distribution  $\vec{A} \in \mathcal{B}_n(\mu)$  we define an *admissible bead slide* to be a transformation

$$\vec{A} \mapsto \vec{A}' = \vec{A} + \delta e_k,$$

where  $\delta \geq 0$ ,  $1 \leq k \leq n$  and the distribution  $\vec{A}'$  is monotone. Intuitively, this means that we slide to the right by a distance  $\delta$  the  $k$ -th bead of the distribution  $\vec{A}$ . The admissibility of the move means that the resulting distribution of beads continues to be monotone.

We define a new partial relation  $\preceq$  on  $\mathcal{B}_n(\mu)$  by declaring  $\vec{A} \preceq \vec{B}$  if the distribution  $\vec{B}$  can be obtained from  $\vec{A}$  via a finite sequence of admissible bead slides. When  $\vec{A} \preceq \vec{B}$  we say that *we can slide the distribution  $\vec{A}$  to the distribution  $\vec{B}$*

If we think of  $\mathcal{B}_n(\mu)$  as a closed convex set in  $\mathbb{R}^n$  and  $\vec{A}, \vec{B} \in \mathcal{B}_n(\mu)$ , then  $\vec{A} \preceq \vec{B}$  if and only if we can travel from  $\vec{A}$  to  $\vec{B}$  *inside*  $\mathcal{B}_n(\mu)$  along a positive zig-zag, i.e., a continuous path consisting of finitely many segments parallel to the coordinate axes and oriented in the positive directions of the axes.

The goal of this note is to investigate when can we slide one mononote distribution of beads to another monotone distribution. Clearly if we can slide  $\vec{A}$  to  $\vec{B}$  then  $\vec{A} \leq \vec{B}$ .

**Remark 1.** The converse implication is true if  $n = 1, 2$ , but false if  $n \geq 3$ . Indeed if  $n \geq 3$ , and  $\vec{B} \in \mathcal{B}_n(\mu)$  is an equidistant distribution, i.e.,

$$B_1 - \mu = B_2 - B_1 = \cdots = B_n - B_{n-1}$$

then there is no distribution  $\vec{A} \prec \vec{B}$ . To see this observe that there is no distribution  $\vec{A}$  such that  $\vec{B}$  is obtained from  $\vec{A}$  by a single admissible bead slide.  $\square$

Define  $\lambda_n : \mathcal{B}_n(\mu) \rightarrow [0, \infty)$  by setting

$$\lambda_n(\vec{A}) := a_n - a_1 = (a_k - a_{n-1}) + \cdots (a_2 - a_1) + a_1,$$

where we recall that

$$a_1 = A_1 - \mu, \quad a_k = A_k - A_{k-1}, \quad k > 1.$$

Clearly  $\lambda_n(\vec{A}) = 0$  if and only if the beads described by the distribution  $\vec{A}$  are equidistant, i.e.,

$$A_n - A_{n-1} = \cdots = A_2 - A_1 = A_1 - \mu.$$

The following is the main result of this note.

**Theorem 2.** *Let  $\mu \in \mathbb{R}$  and  $\vec{B} \in \mathcal{B}_n(\mu)$ . Then*

$$b_k > b_{k-2}, \quad \forall k \geq 3 \iff \vec{A} \preceq \vec{B}, \quad \forall \vec{A} < \vec{B} \in \mathcal{B}_n(\mu), \quad (1)$$

where

$$b_1 = B_1 - \mu, \quad b_k = B_k - B_{k-1}, \quad \forall k \geq 2.$$

**Remark 3.** The condition  $b_k > b_{k-2}, \forall k \geq 3$  signifies that no string of four consecutive beads of the distribution  $\vec{B}$  is equidistant.  $\square$

*Proof.* We first prove the implication  $\Rightarrow$ ,

$$b_k > b_{k-2}, \quad \forall k \geq 3 \implies \vec{A} \preceq \vec{B}, \quad \forall \vec{A} \leq \vec{B}. \quad (S_n)$$

We argue by induction on  $n$ . The cases  $n = 1$  and  $n = 2$  are trivial.

To complete the inductive step note first that the assumption  $b_k > b_{k-2}$  for all  $k \geq 2$  implies  $\lambda(\vec{B}) > 0$ . We have the following key estimate.

**Lemma 4.** *If  $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$  are such that  $\vec{A} \leq \vec{B}$  and  $A_{n+1} = B_{n+1}$  then*

$$\lambda_{n+1}(\vec{A}) \geq \frac{1}{n} \lambda_{n+1}(\vec{B}). \quad (2)$$

*Proof.* For  $k = 2, \dots, n+1$  we set

$$\alpha_k := a_k - a_{k-1}, \quad \beta_k := b_k - b_{k-1}.$$

Note that  $\alpha_k, \beta_k \geq 0$ ,

$$\begin{aligned} \lambda_{n+1}(\vec{A}) &= \sum_{k=2}^{n+1} \alpha_k, & \lambda_{n+1}(\vec{B}) &= \sum_{k=2}^{n+1} \beta_k, \\ a_k &= a_1 + \sum_{i=2}^k \alpha_i, & b_k &= a_1 + \sum_{i=2}^k \beta_i, \end{aligned}$$

and

$$(n+1)a_1 + \sum_{k=2}^{n+1} (n-k+1)\alpha_k = A_{n+1} - \mu = B_{n+1} - \mu = (n+1)b_1 + \sum_{k=2}^{n+1} (n-k+1)\beta_k.$$

Hence

$$\sum_{k=2}^{n+1} (n-k+1)\alpha_k = (n+1)(b_1 - a_1) + \sum_{k=2}^{n+1} (n-k+1)\beta_k \geq \sum_{k=2}^{n+1} (n-k+1)\beta_k.$$

We deduce

$$n\lambda_{n+1}(\vec{A}) = n \sum_{k=2}^{n+1} \alpha_k \geq \sum_{k=2}^{n+1} (n-k+1)\alpha_k \geq \sum_{k=2}^{n+1} (n-k+1)\beta_k \geq \sum_{k=2}^{n+2} \beta_k = \lambda_{n+1}(\vec{B}).$$

□

Consider two distributions  $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$ . Then we can slide the last bead of  $\vec{A}$  until it reaches the position of the last bead of  $\vec{B}$ .

$$\vec{A} \mapsto \vec{A}' := \vec{A} + (B_{n+1} - A_{n+1})\mathbf{e}_{n+1}$$

Clearly this slide is admissible. This shows that it suffices to prove  $(S_{n+1})$  only in the special case  $A_{n+1} = B_{n+1}$ . To prove the implication  $(S_{n+1})$  we will rely on the following simple observation.

**Lemma 5.** *Assume that the implication  $(S_k)$  holds for every  $k \leq n$ . If  $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$  are two distributions such that  $\vec{A} \leq \vec{B}$ , and  $A_k = B_k$  for some  $k \leq n$  then  $\vec{A} \preceq \vec{B}$ .*

*Proof.* Note that

$$(A_1, \dots, A_k) \leq (B_1, \dots, B_k) \quad \text{and} \quad (A_{k+1}, \dots, A_{n+1}) \leq (B_{k+1}, \dots, B_{n+1}).$$

According to  $S_k$ , we can slide the first  $k$ -beads of the distribution  $\vec{A}$  to the first  $k$  beads of the distribution  $\vec{B}$ . Using  $S_{n-k+1}$  we can then slide the last  $(n-k+1)$  beads of the distribution  $\vec{A}$  to the last  $(n-k+1)$  beads of the distribution  $\vec{B}$ .

□

Using the above observations we deduce that the implication  $S_{n+1}$  is a consequence of the following result.

**Lemma 6.** *Assume that the implication  $(S_k)$  holds for every  $k \leq n$ . If  $\vec{A}, \vec{B} \in \mathcal{B}_{n+1}(\mu)$  are two distributions such that  $\vec{A} \leq \vec{B}$  and  $A_{n+1} = B_{n+1}$  then we can slide  $\vec{A}$  to a configuration  $\vec{C} \in \mathcal{B}_{n+1}(\mu)$  that crosses  $\vec{B}$ , i.e.,*

- (a)  $\vec{C} \leq \vec{B}$ ,
- (b)  $C_{n+1} = B_{n+1}$ ,
- (c)  $C_k = B_k$  for some  $k \leq n$ .

*Proof.* Define

$$\mathcal{B}_{n+1}(\vec{B}) := \{ \vec{T} \in \mathcal{B}_{n+1}(\mu); \vec{T} \leq \vec{B}, T_{n+1} = B_{n+1} \}.$$

Note that  $\vec{A} \in \mathcal{B}_{n+1}(\vec{B})$ . We define a  $\vec{B}$ -move, to be a bead slide on a configuration  $\vec{T} \in \mathcal{B}_{n+1}(\vec{B})$  that produces another configuration in  $\mathcal{B}_{n+1}(\vec{B})$ . We need to prove that by a sequence of  $\vec{B}$ -moves starting with  $\vec{A}$  we can produce a configuration  $\vec{C} \in \mathcal{B}_{n+1}(\vec{B})$  that crosses  $\vec{B}$ .

We argue by contradiction. Hence we will work under the following assumption.

*We cannot produce crossing configurations via any sequence of  $\vec{B}$ -moves starting with  $\vec{A}$ .* †

We show that this implies that there exists a sequence of configurations  $\vec{A}_\nu \in \mathcal{B}_{n+1}(\vec{B})$ ,  $\nu \geq 1$ , such that

$$\lim_{\nu \rightarrow \infty} \lambda_{n+1}(\vec{A}_\nu) = 0.$$

In view of the assumption  $\lambda(\vec{B}) > 0$  this sequence contradicts the inequality (2).

Denote by  $\mathcal{B}_{n+1}(\vec{A}, \vec{B})$  the set of configurations in  $\mathcal{B}_{n+1}(\vec{B})$  that can be obtained from  $\vec{A}$  by a sequence of  $\vec{B}$ -moves. We will produce a real number  $\kappa \in (0, 1)$  and a map

$$\mathcal{T} : \mathcal{B}_{n+1}(\vec{A}, \vec{B}) \rightarrow \mathcal{B}_{n+1}(\vec{A}, \vec{B})$$

such that

$$\lambda(\mathcal{T}(\vec{X})) \leq \kappa \lambda(\vec{X}), \quad \forall \vec{X} \in \mathcal{B}_{n+1}(\vec{A}, \vec{B}).$$

The sequence

$$\vec{A}_\nu := \mathcal{T}^\nu(\vec{A})$$

will then produce the sought for contradiction.

We begin by constructing maps

$$\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_n : \mathcal{B}_{n+1}(\mu) \rightarrow \mathcal{B}_{n+1}(\mu)$$

so that for any  $k = 1, \dots, n$  and any  $\vec{X} \in \mathcal{B}_{n+1}(\mu)$  we have

$$\mathcal{M}_k(\vec{X}) = \left( X_1, \dots, x_{k-1}, \frac{1}{2}(X_{k-1} + X_{k+1}), X_{k+1}, \dots, X_{n+1} \right),$$

where for uniformity we set  $X_0 = \mu$ . In other words,  $\mathcal{M}_k(\vec{X})$  is obtained from  $\vec{X}$  by sliding the  $k$ -th bead of  $\vec{X}$  to the midpoint of the interval  $(X_{k-1}, X_{k+1})$ . In the new configuration the beads  $(k-1)$ ,  $k$  and  $(k+1)$  are equidistant.

Now define

$$\mathcal{T} : \mathcal{B}_{n+1}(\mu) \rightarrow \mathcal{B}_{n+1}(\mu), \quad \mathcal{T} = \mathcal{M}_1 \circ \mathcal{M}_2 \circ \dots \circ \mathcal{M}_n.$$

Note that

$$\mathcal{M}_n(X_1, \dots, X_{n+1}) = \left( X_1, \dots, X_{n-1}, \frac{1}{2}(X_{n-1} + X_{n+1}), X_{n+1} \right).$$

The configuration  $\mathcal{M}_{n-1} \circ \mathcal{M}_n(\vec{X})$  differs from  $\mathcal{M}_n(\vec{X})$  only at the  $(n-1)$ -th component which is

$$\frac{1}{2}X_{n-2} + \frac{1}{4}X_{n-1} + \frac{1}{4}X_{n+1}.$$

The  $(n-k)$ -th component of  $\mathcal{M}_{n-k} \circ \dots \circ \mathcal{M}_n(\vec{X})$  is

$$\frac{1}{2}X_{n-k-1} + \frac{1}{4}X_{n-k} + \dots + \frac{1}{2^{k+1}}X_{n-1} + \frac{1}{2^{k+1}}X_{n+1}.$$

The first component of  $\vec{Y} := \mathcal{T}(\vec{X})$  is

$$Y_1 = \frac{1}{2}X_0 + \frac{1}{4}X_1 + \dots + \frac{1}{2^n}X_{n-1} + \frac{1}{2^n}X_{n+1}.$$

If we set

$$x_1 = X_1 - X_0 = X_1 - \mu, \quad x_2 = X_2 - X_1, \dots, x_{n+1} = X_{n+1} - X_n$$

we deduce

$$Y_1 = \frac{1}{2^n}X_{n+1} + \sum_{k=0}^{n-1} \frac{1}{2^{k+1}}X_k = \frac{1}{2^n}X_{n+1} + \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \left( \mu + \sum_{i=1}^k x_i \right)$$

$$\begin{aligned}
 &= \frac{1}{2^n} \left( \mu + \sum_{i=1}^{n+1} x_i \right) + \left( 1 - \frac{1}{2^n} \right) \mu + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} \sum_{i=1}^k x_i \\
 &= \mu + \frac{1}{2^n} \sum_{i=1}^{n+1} x_i + \left( \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} \right) x_1 + \left( \sum_{k=2}^{n-1} \frac{1}{2^{k+1}} \right) x_2 + \cdots + \frac{1}{2^n} x_{n-1} \\
 &= \mu + \frac{1}{2} x_1 + \frac{1}{4} x_2 + \cdots + \frac{1}{2^{n-1}} x_{n-1} + \frac{1}{2^n} x_n + \frac{1}{2^n} x_{n+1}.
 \end{aligned}$$

Observe that

$$\lambda_{n+1}(X) = x_{n+1} - x_1, \quad \lambda_{n+1}(Y) = y_{n+1} - y_1 = Y_{n+1} - Y_n - Y_1 + Y_0.$$

We have

$$\begin{aligned}
 \lambda_{n+1}(Y) &= X_{n+1} - \frac{1}{2}(X_{n+1} + X_{n-1}) - Y_1 + \mu \\
 &= \sum_{i=1}^{n+1} x_i - \frac{1}{2} \left( \sum_{i=1}^{n+1} x_i + \sum_{i=1}^{n-1} x_i \right) - \left( \frac{1}{2^n} x_{n+1} + \sum_{k=1}^n \frac{1}{2^k} x_k \right) \\
 &= \frac{1}{2} (x_{n+1} + x_n) - \left( \frac{1}{2^n} x_{n+1} + \sum_{k=1}^n \frac{1}{2^k} x_k \right) \\
 &\leq \left( 1 - \frac{1}{2^n} \right) x_{n+1} - \sum_{k=1}^n \frac{1}{2^k} x_k = \sum_{k=1}^n \frac{1}{2^k} (x_{n+1} - x_k) \\
 &\leq \left( \sum_{k=1}^n \frac{1}{2^k} \right) (x_{n+1} - x_1) = \left( 1 - \frac{1}{2^n} \right) \lambda_{n+1}(X).
 \end{aligned}$$

Hence

$$\lambda_{n+1}(\mathcal{T}(\vec{X})) \leq (1 - 2^{-n}) \lambda_{n+1}(\vec{X}), \quad \forall \vec{X} \in \mathcal{B}_{n+1}(\mu). \quad (3)$$

To conclude the proof it suffices to show that

$$\mathcal{M}_k(\vec{X}) \in \mathcal{B}(\vec{B}), \quad \forall \vec{X} \in \mathcal{B}(\vec{A}, \vec{B}), \quad k = 1, \dots, n. \quad (4)$$

Let  $\vec{X} = (X_1, \dots, X_{n+1}) \in \mathcal{B}(\vec{A}, \vec{B})$  and set  $\vec{Y} = \mathcal{M}_k(\vec{X})$ . Then

$$Y_i = \begin{cases} X_i, & i \neq k \\ \frac{1}{2}(X_{k-1} + X_{k+1}), & i = k. \end{cases}$$

To prove that  $\vec{Y} \in \mathcal{B}(\vec{A}, \vec{B})$  we have to prove that  $Y_k \leq B_k$ . If this were not the case, then  $Y_k > B_k$ . Since  $X_k < B_k$ , we deduce  $(B_k - X_k) < (Y_k - X_k)$ . This implies that sliding the the  $k$ -th bead of  $\vec{X}$  by distance  $(B_k - X_k)$  is an admissible slide, and it is obviously a  $\vec{B}$ -move since the resulting configuration  $\vec{X}'$  is in  $\mathcal{B}_{n+1}(\vec{B})$ . Clearly, the configuration  $\vec{X}'$  crosses  $\vec{B}$  since  $X'_k = B_k$ . This contradicts the assumption  $(\dagger)$  and finishes the proof of Lemma 6 and of the implication  $\Rightarrow$  in (1).  $\square$

To prove the converse implication  $\Leftarrow$  we argue by induction. The cases  $n = 1, 2$  are trivial, while the case  $n = 3$  follows from Remark 1.

For the inductive step suppose  $\vec{A} \prec \vec{B}$  in  $\mathcal{B}_{n+1}(\mu)$ ,  $\forall \vec{A} < \vec{B}$ . Then

$$(B_1, \dots, A_n) \prec (B_1, \dots, B_n) \in \mathcal{B}_n(\mu), \quad \forall (A_1, \dots, A_n) < (B_1, \dots, B_n),$$

and the inductive assumption implies

$$b_k > b_{k-2}, \quad \forall 2 \leq k \leq n.$$

To prove that  $b_{n+1} > b_{n-1}$  we argue by contradiction. Suppose  $b_{n+1} = b_{n-1}$  so that

$$b_{n+1} = b_n = b_{n-1}.$$

The condition  $b_n > b_{n-2}$  implies that  $b_{n-2} < b_{n-1}$ . Consider the bead distribution  $\vec{C} \in \mathcal{B}_{n+1}(\mu)$  described by

$$\begin{aligned} C_k &= B_k, \quad \forall k \leq n-2, \\ C_{n-1} &= C_{n-2} + b_{n-2} = B_{n-2} + b_{n-2} < B_{n-1}, \\ C_n &= C_{n-1} + b_{n-1} < B_n, \quad C_{n+1} = C_n + b_n < B_n. \end{aligned}$$

Then  $\vec{C} < \vec{B}$ , yet arguing as in Remark 1 we see that  $\vec{C} \not\prec \vec{B}$ . This contradiction completes the proof of Theorem 2.  $\square$

The partial order  $\preceq$  on  $\mathcal{B}_n(\mu)$  is a binary relation and thus can be identified with a subset of  $\mathcal{B}_n(\mu) \times \mathcal{B}_n(\mu)$ . We denote by  $\preceq_t$  its (topological) closure in  $\mathcal{B}_n(\mu) \times \mathcal{B}_n(\mu)$ .

**Corollary 7.** *The binary relation  $\preceq_t$  is a partial order relation. More precisely*

$$\vec{A} \preceq_t \vec{B} \iff \vec{A} \leq \vec{B}.$$

*Proof.* Clearly  $\vec{A} \preceq_t \vec{B} \implies \vec{A} \leq \vec{B}$ . Conversely, suppose  $\vec{A} \leq \vec{B}$ . For every  $\varepsilon > 0$  we define

$$\vec{B}(\varepsilon) = (B_1(\varepsilon), \dots, B_n(\varepsilon)),$$

where  $B_k(\varepsilon) = 2^k \varepsilon$ . Then

$$B_{k+1}(\varepsilon) - B_k(\varepsilon) = b_{k+1} + 2^k \varepsilon > b_k + 2^{k-1} \varepsilon = B_k(\varepsilon) - B_{k-1}(\varepsilon).$$

Theorem 2 implies that  $\vec{A} \prec \vec{B}(\varepsilon)$ . Letting  $\varepsilon \rightarrow 0$  we deduce  $\vec{A} \preceq_t \vec{B}$ .  $\square$

The above corollary can be used to produce various interesting inequalities.

For simplicity we set  $\mathcal{B}_n := \mathcal{B}_n(0)$ . The bead distributions in  $\mathcal{B}_n$  are described by nondecreasing strings of nonnegative numbers

$$\vec{a} = (a_1, \dots, a_n), \quad 0 \leq a_1 \leq \dots \leq a_n$$

To such a vector we associate the monotone bead distribution

$$\vec{A} = (A_1, \dots, A_n), \quad A_k = a_1 + \dots + a_k.$$

The condition  $\vec{A} \leq \vec{B}$  in  $\mathcal{B}_n$  can then be rewritten as

$$a_1 + \dots + a_k \leq b_1 + \dots + b_k, \quad \forall k = 1, \dots, n.$$

In this notation, and admissible bead slide is a transformation of the form

$$(a_1, \dots, a_k, a_{k+1}, \dots, a_n) \mapsto (a_1, \dots, a_k + \delta, a_{k+1} - \delta, \dots, a_n), \quad 2\delta \leq a_{k+1} - a_k. \quad (5)$$

Suppose  $f : [0, \infty) \rightarrow [0, \infty)$  is a nondecreasing  $C^1$  function. We then get a map  $\mathcal{T}_f : \mathcal{B}_n \rightarrow \mathcal{B}_n$ ,

$$(a_1, a_1 + a_2, \dots, a_1 + \dots + a_n) \mapsto (f(a_1), f(a_1) + f(a_2), \dots, f(a_1) + \dots + f(a_n)).$$

**Theorem 8.** *Suppose  $f : [0, \infty) \rightarrow [0, \infty)$  is  $C^1$  and nondecreasing. Then the induced map  $\mathcal{T}_f : \mathcal{B}_n \rightarrow \mathcal{B}_n$  preserves the order relation  $\leq$  if and only if  $f$  is concave, i.e., the derivative  $f'$  is nonincreasing.*

*Proof.* In view of Corollary 7 and the continuity of  $f$  we deduce that  $\mathcal{T}_f$  preserves the order  $\leq$  if and only if  $\mathcal{T}_f(\vec{A}) \leq \mathcal{T}_f(\vec{B})$  whenever  $\vec{B}$  is obtained from  $\vec{A}$  via a single admissible bead slide. Using (5) we see that this means that for any  $0 \leq x \leq y$ ,  $0 \leq \delta \leq \frac{1}{2}(y-x)$  we have

$$f(x+\delta) \geq f(x), \quad f(x+\delta) + f(y-\delta) \geq f(x) + f(y).$$

The first inequality follows from the fact that  $f$  is nondecreasing. The second inequality can be rephrased as

$$\int_x^{x+\delta} f'(t)dt = f(x+\delta) - f(x) \geq f(y) - f(y-\delta) = \int_{y-\delta}^y f'(s)ds,$$

for any  $x, y, \delta \geq 0$  such that  $x \leq x+\delta \leq y-\delta \leq y$ . This clearly happens if and only if  $f'$  is nonincreasing.  $\square$

**Remark 9.** In the above result we can drop the  $C^1$  assumption on  $f$ , but the last step in the proof requires a slightly longer and less transparent argument.  $\square$

**Corollary 10.** Suppose  $f : [\mu, \infty) \rightarrow \mathbb{R}$  is  $C^1$ , nondecreasing and concave, and  $(y_i)_{1 \leq i}$  is a nondecreasing sequence of real numbers

$$\mu \leq y_1 \leq \dots \leq y_n.$$

Then for any numbers  $x_1, \dots, x_n \in [\mu, \infty)$  such that

$$x_1 + \dots + x_k \leq y_1 + \dots + y_k, \quad \forall k = 1, \dots, n$$

we have

$$f(x_1) + \dots + f(x_n) \leq f(y_1) + \dots + f(y_n). \quad (6)$$

*Proof.* Denote by  $(x'_k)$  the increasing rearrangement of the numbers  $x_1, \dots, x_n$ . Then

$$x'_1 + \dots + x'_k \leq x_1 + \dots + x_k \leq y_1 + \dots + y_k, \quad \forall k = 1, \dots, n,$$

$$f(x'_1) + \dots + f(x'_n) = f(x_1) + \dots + f(x_n),$$

so it suffices to prove (2) in the special case when the sequence  $(x_k)$  is nondecreasing. Define

$$a_k := x_k - \mu, \quad b_k := y_k - \mu, \quad 1 \leq k \leq n,$$

$$A_k = a_1 + \dots + a_k, \quad B_k = b_1 + \dots + b_k, \quad 1 \leq k \leq n,$$

$$g : [0, \infty) \rightarrow [0, \infty), \quad g(t) = f(t + \mu) - f(\mu).$$

Then  $(A_1, \dots, A_n) \leq (B_1, \dots, B_n) \in \mathcal{B}_n$ , and the function  $g$  is  $C^1$ , nondecreasing and concave. It follows that the induced map  $\mathcal{T}_g : \mathcal{B}_n \rightarrow \mathcal{B}_n$  is order preserving. In particular, we conclude that

$$g(a_1) + \dots + g(a_n) \leq g(b_1) + \dots + g(b_n).$$

This clearly implies (6).  $\square$

**Corollary 11.** Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$ , concave function and  $y_1 \leq \dots \leq y_n$ . Then for any sequence  $x_1, \dots, x_n$  such that

$$x_1 + \dots + x_k \leq y_1 + \dots + y_k, \quad \forall k = 1, \dots, n-1,$$

and

$$x_1 + \dots + x_n = y_1 + \dots + y_n \quad (7)$$

we have

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n). \quad (8)$$

*Proof.* Choose  $L > \max\{x_i, y_j; 1 \leq i, j \leq n\}$  and define

$$g : \mathbb{R} \rightarrow \mathbb{R}, \quad g(t) = \begin{cases} f(t) - f'(L)t, & t \leq L \\ f(L) - f'(L)L, & t > L. \end{cases}$$

Then  $g$  is  $C^1$ , nondecreasing and concave and Corollary 10 implies that

$$f(x_1) + \cdots + f(x_n) - f'(L) \sum_{k=1}^n x_k \leq f(y_1) + \cdots + f(y_n) - f'(L) \sum_{k=1}^n y_k.$$

The inequality (8) now follows by invoking the equality (7). □

Corollary 11 implies the Schur majorization inequalities [1, 2.19-20], [2, Chap. 13]. More precisely, we have the following result.

**Corollary 12** (Schur majorization). *Suppose  $b_1 \geq \cdots \geq b_n$  is a nonincreasing sequence of real numbers and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$ , convex function, i.e.,  $g'$  is nondecreasing. Then for any sequence  $a_1, \dots, a_n$  satisfying*

$$a_1 + \cdots + a_k \geq b_1 + \cdots + b_k, \quad k = 1, \dots, n-1,$$

and

$$a_1 + \cdots + a_n = b_1 + \cdots + b_n$$

we have

$$g(a_1) + \cdots + g(a_n) \geq g(b_1) + \cdots + g(b_n).$$

*Proof.* Use Corollary 11 with the sequences  $x_k = -a_k$ ,  $y_j = -b_j$  and  $f(t) = -g(-t)$ . □

## REFERENCES

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