Boolean Satisfiability: The Central Problem of Computation

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(p. 299) SAT: Boolean Satisfiability

- **wff**: well-formed-formula constructed from
  - A set V of Boolean variables
  - Boolean operations AND, OR, NOT
- **Satisfiability**: is there a substitution of 0s and 1s to variables that makes the wff true
  - i.e. makes all clauses simultaneously true
- **Unsatisfiability**: no substitution makes all clauses true at same time
- See references in “Links” class page

CNF: Clausal Normal Form

- wff restructured as AND of a set of clauses
  - Each clause an OR of a set of literals
  - Each literal a variable or its negation
- For a wff in clausal form to be true
  - All clauses must be true
  - For any clause to be true at least one literal must be true
- Example: (~x v y) & (x v y) & (x v ~y)
  - x=1, y=1 makes expression true
- (~x v y) & (x v y) & (x v ~y) & (~x v ~y)
  - No assignment of values make this true

Why Does SAT Matter

- Huge range of direct applications
- Will show that ***ALL*** computable functions can be converted into a SAT problem
- If we can solve SAT quickly, we can solve ***any*** computable problem quickly
- But ***no one*** has been able to find such a solution!

Applications

Following list taken from http://logos.ucd.ie/~jpms/talks/talksite/jpms-wodes08.pdf

- **Circuit construction and simulation**
- **Model checking: H/W, S/W, test patterns**
- **AI: Planning; Knowledge representation; Games**
- **Bioinformatics: Haplotype inference; Pedigree checking; Maximum quartet consistency; etc.**
- **Design automation:**
- **Equivalence checking; Delay computation; Fault diagnosis; Noise analysis; etc.**
- **Security: Cryptanalysis; Inversion attacks on hash functions; etc.**
- **Computationally hard problems: Graph coloring; Traveling salesperson; etc.**
- **Mathematical problems: van der Waerden numbers; etc**
- **Core engine for many other problem domains**

SAT Problem Sizes

- Hundreds of thousands to millions of variables
- Huge numbers of clauses
- Often very large numbers of literals per clause
- Sample problem sources:
  - http://www.cs.ubc.ca/~hoos/SATLIB/benchm.html
- There is even a yearly competition that has been going on for decades
  - Current 2017: https://baldur.iti.kit.edu/sat-competition-2017/index.php?cat=certificates
  - 2016: https://baldur.iti.kit.edu/sat-competition-2016/index.php?cat=certificates

Example: Sudoku to SAT

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

Fill in all blanks so 1…9 appear on every row, column, and 3x3 grid

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

- Define 729 variables $x_{i,j,d}$ ($1 \le i,j,d \le 9$) such that
  - $x_{i,j,d}$ = 1 if cell (i,j) has digit d, 0 otherwise
- 81 clauses: 1 for each cell (i,j) to ensure it has a digit:
  - $(x_{i,j,1} \vee x_{i,j,2} \vee \dots x_{i,j,9})$
- 81 sets of 36 clauses to ensure no cell has 2 digits:
  - For each of $1 \le d < d' \le 9$: $(\sim x_{i,j,d} \vee \sim x_{i,j,d'})$
- To state that row i, for example, has all 9 digits:
  - AND of 9 clauses (1 for each value of d) where d'th clause is $(x_{i,1,d} \vee \dots \vee x_{i,9,d})$
  - And 9 sets of 36 = 324 clauses to ensure uniqueness $(\sim x_{i,j,d} \vee \sim x_{i,j',d})$
- Repeat construction for all rows, columns, grids
- Total of 11,745 clauses (most with 2 literals/clause, rest have 9)
- Initialize cells by setting certain variables, e.g. $x_{1,1,5}$=1 and $x_{1,1,d}$ = 0 for $d \neq 5$

A 2x2 Sudoku

1	

1	2
2	1

- 8 variables: $x_{1,1,1}$, $x_{1,1,2}$, $x_{1,2,1}$, $x_{1,2,2}$, $x_{2,1,1}$, $x_{2,1,2}$, $x_{2,2,1}$, $x_{2,2,2}$
- 4 clauses to ensure a digit/cell:
  – $(x_{1,1,1} \vee x_{1,1,2})\&(x_{1,2,1} \vee x_{1,2,2})\&(x_{2,1,1} \vee x_{2,1,2})\&(x_{2,2,1} \vee x_{2,2,2})$
- 4 sets of 1 clause to ensure no duplicates:
  – $(\sim x_{1,1,1} \vee \sim x_{1,1,2})\&(\sim x_{1,2,1} \vee \sim x_{1,2,2})\&(\sim x_{2,1,1} \vee \sim x_{2,1,2})\&(\sim x_{2,2,1} \vee \sim x_{2,2,2})$
- 4 clauses for row 1:
  – $(x_{1,1,1} \vee x_{1,2,1})\&(x_{1,1,2} \vee x_{1,2,2})\&(\sim x_{1,1,1} \vee \sim x_{1,2,1})\&(\sim x_{1,1,2} \vee \sim x_{1,2,2})$
- 4 clauses for row 2:
  – $(x_{2,1,1} \vee x_{2,2,1})\&(x_{2,1,2} \vee x_{2,2,2})\&(\sim x_{2,1,1} \vee \sim x_{2,2,1})\&(\sim x_{2,1,2} \vee \sim x_{2,2,2})$
- 4 clauses for column 1:
  – $(x_{1,1,1} \vee x_{2,1,1})\&(x_{1,1,2} \vee x_{2,1,2})\&(\sim x_{1,1,1} \vee \sim x_{2,1,1})\&(\sim x_{1,1,2} \vee \sim x_{2,1,2})$
- 4 clauses for column 2:
  – $(x_{1,2,1} \vee x_{2,2,1})\&(x_{1,2,2} \vee x_{2,2,2})\&(\sim x_{1,2,1} \vee \sim x_{2,2,1})\&(\sim x_{1,2,2} \vee \sim x_{2,2,2})$
- 2 Initialization clauses: $x_{1,1,1}$ & $\sim x_{1,1,2}$

Variants of SAT in CNF

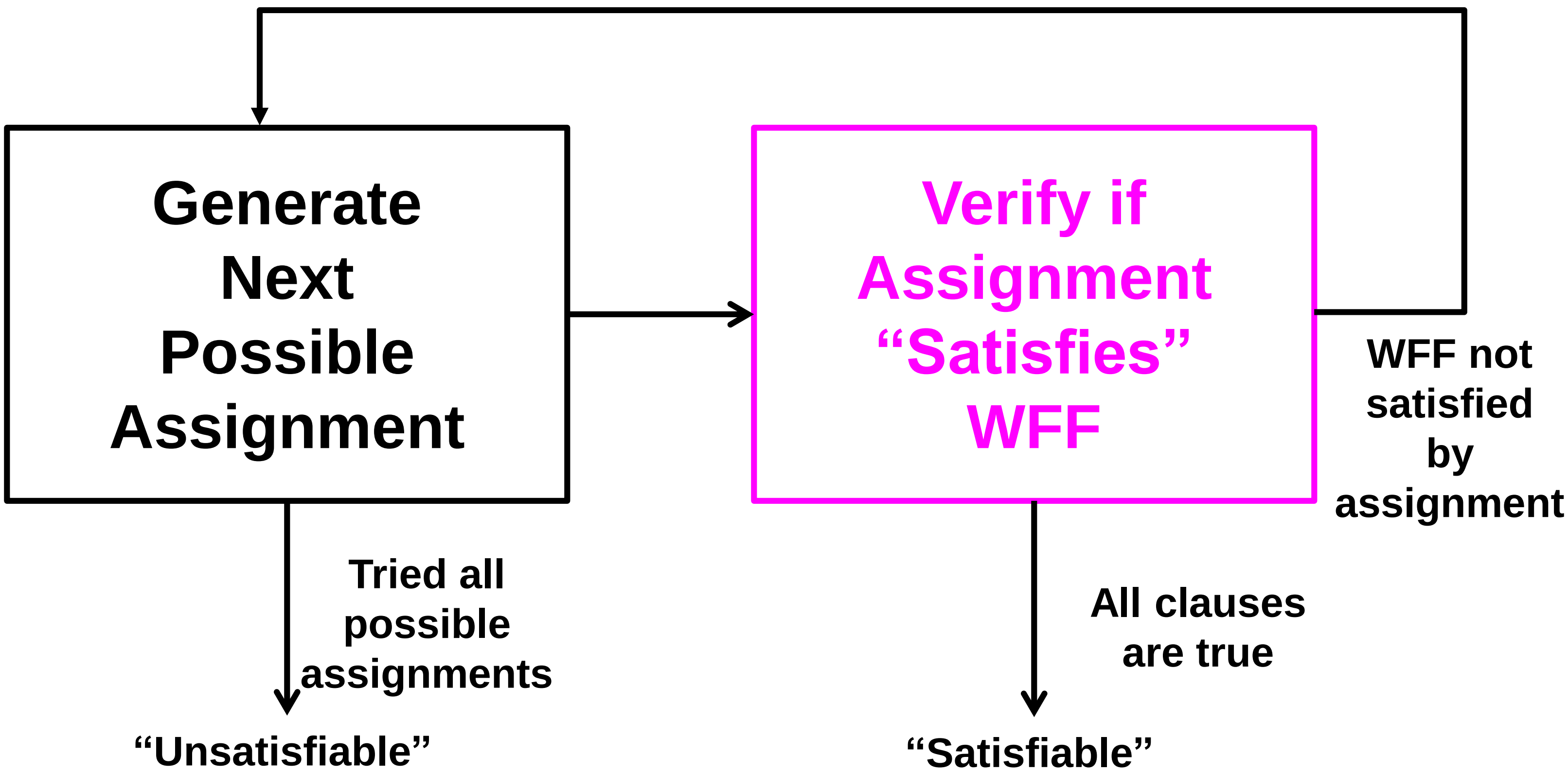
- **1-SAT**: all clauses have <u>exactly</u> 1 literal
  - Each clause is one literal
  - If any 2 clauses are a variable & its complement, then reject
  - E.g. x_1 & x_2 & $\sim x_3$ satisfied by $x_1 = 1, x_2 = 1, x_3 = 0$
  - But add on clause $\sim x_1$ and unsatisfiable
- **2-SAT**: all clauses have *at most* 2 literals
  - Clause: $(L_{i1} \vee L_{i2})$
- **3-SAT**: all clauses have *at most* 3 literals
  - Clause: $(L_{i1} \vee L_{i2} \vee L_{i3})$
  - At least one literal in each clause must be true

The Simplest SAT Solver

- Generate all 2^V assignments to V variables
- For each assignment, check each clause
- **Satisfiable**: Some assignment makes all clauses true
- **Unsatisfiable**: no assignment works

x	y	z	x V ~y	y V z	~xV~z	~xV~yVz	xVyV~z	All Clauses	All but last
0	0	0	1	0	1	1	1	0	0
0	0	1	1	1	1	1	0	0	1
0	1	0	0	1	1	1	1	0	0
0	1	1	0	1	1	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	0	1	1	1	0	1	1	0	0
1	1	0	1	1	1	0	1	0	0
1	1	1	1	1	0	1	1	0	0

Brute Force Approach



Brute Force Algorithm

for each of 2^V combinations of variable values

Verifier

for each clause in wff

C

for each literal in clause

K

look up variable in assignment

if literal is true: break to next clause

if all literals are false:

break to next combination

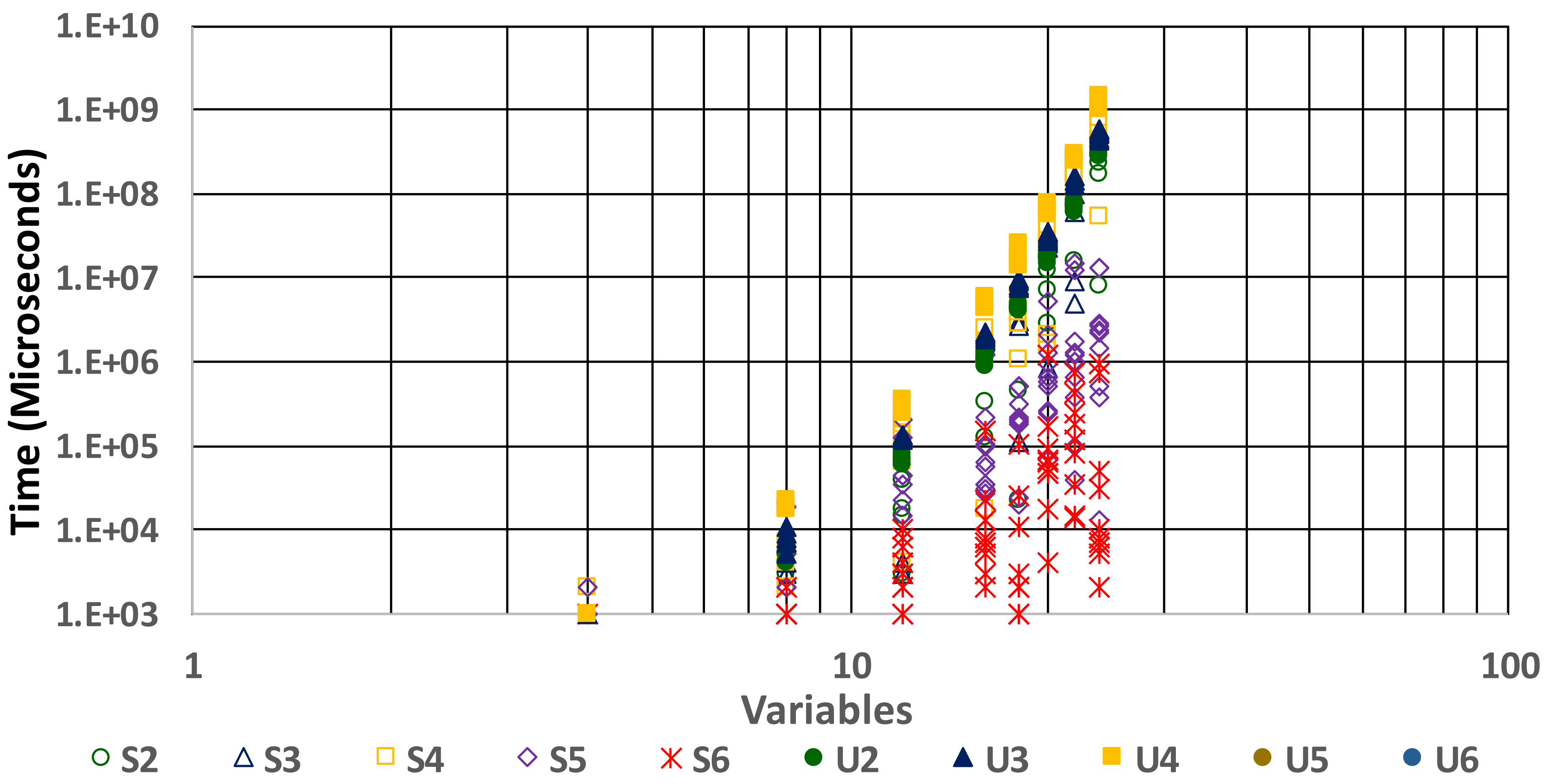
if all clauses are true: break "Satisfiable"

if no combination satisfied: "Unsatisfiable"

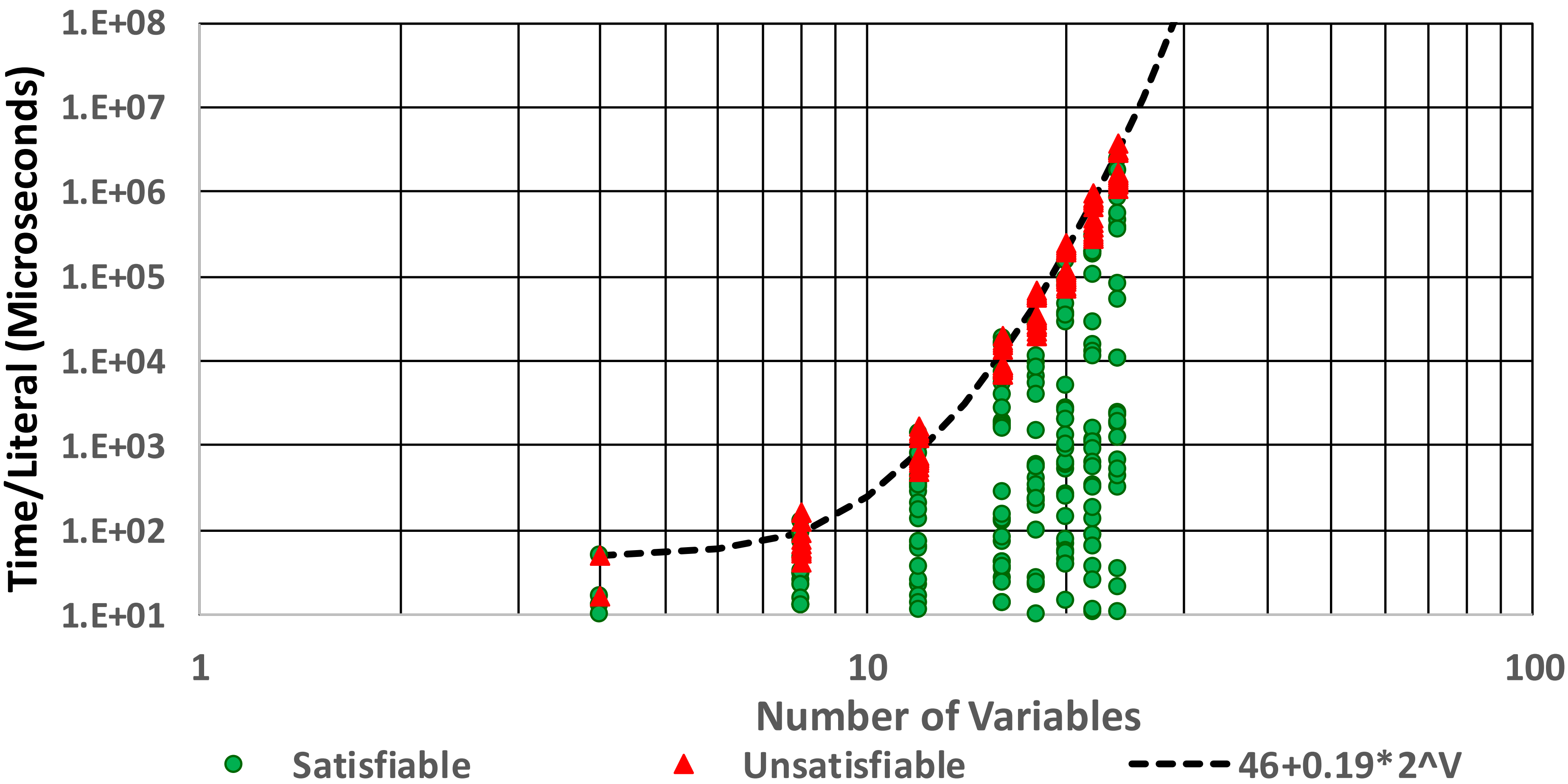
Time Complexity: $O(2^V*C*K)$

- V = # variables
- C = # Clauses
- K = # Literals per Clause

A Python Implementation



Dividing by CK=# Literals



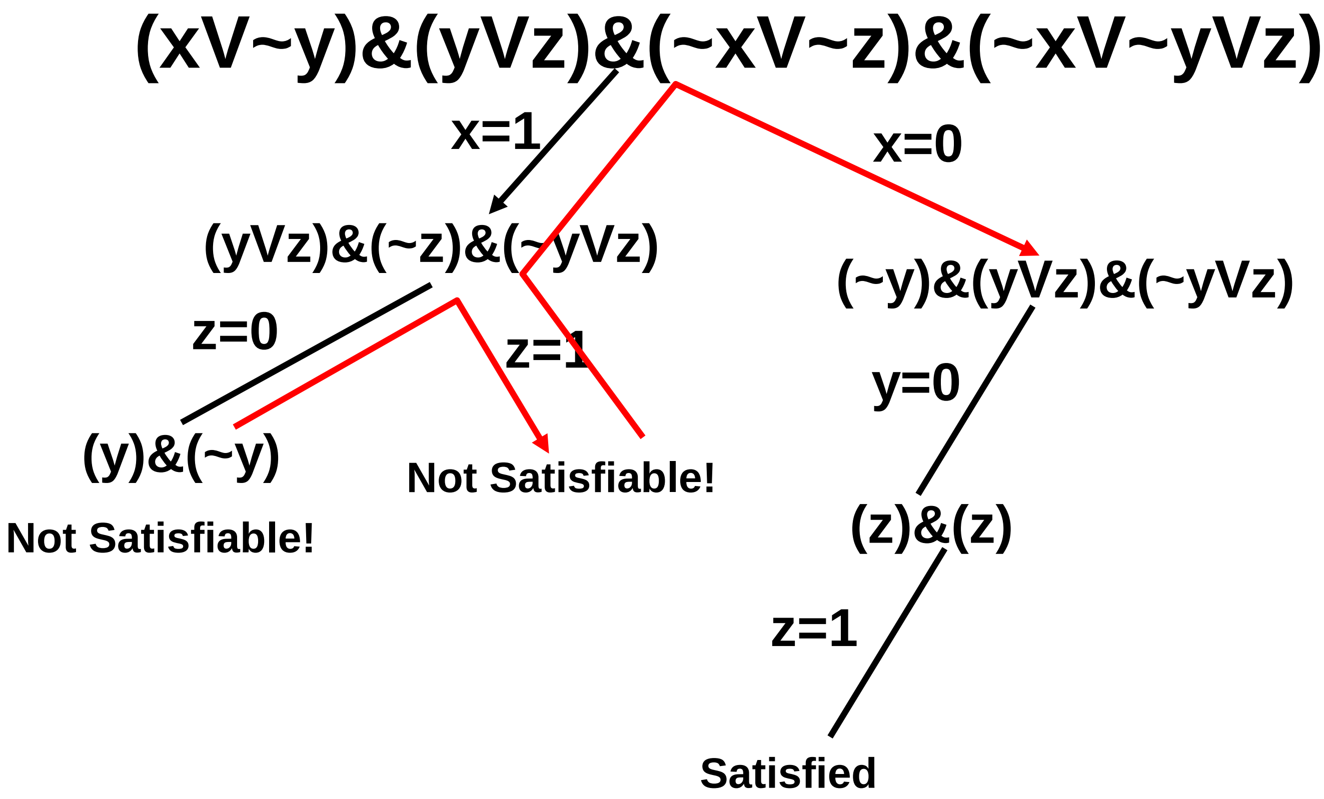
Backtracking: Core to Real Solvers

- Consider "incremental" approach that generates assignment dynamically
- Keep track of state of clauses under current partial assignment; clauses may be
  - **True**: some literal in clause has a variable value that makes it true
  - **False**: all literals in clause have variable values that make literals false
  - **Undetermined**: one or more literals have variables without any current assigned value
- Keep "stack" of order of assignments to allow backtrack if current assignment doesn't work

Basic Backtracking

- Select some variable
- Select value to give to that variable (to make some clause true)
  - Save (on stack) variable and value as a "CHOICE POINT"
- Ignore all clauses now true
- If no clause remains, declare "Satisfied"
  - Values on stack are satisfying assignment
- If some clause is now "false":
  - Go to top choice point, reverse value and try again
  - If top variable has tried both values, pop choice point, and repeat on choice point below below
  - If stack is now empty, declare "Unsatisfiable"
- If no clauses false and some still undetermined, repeat above on a different variable that has no value

Equivalent to a “Tree Traversal”



Red: Backtrack to last Choice Point and try another

Another Example

(xV~y)&(yVz)&(~xV~z)&(~xV~yVz)&(xVyV~z)

The Unit Clause Rule

- Additional trick: When a clause has *only one* undetermined literal
  - Add a choice point entry with that variable
  - Assign value to variable to make literal true
  - With flag that reversing value need not be tried
- Many other heuristics have been developed
- Average complexity *greatly* reduced
- But for kSAT, k>2, worst case still $O(2^V)$

Special Case: 2SAT

- **Speedup observation:**
  - **Assume we guess x_i = 1 (build a choice point)**
  - **All clauses with x_i as a literal are now true**
- **Now look at all clauses of form ($\sim x_i$ V L_j)**
  - **$\sim x_i$ is false from assignment**
  - **so L_i *must be* true => *new assignment***
  - **Can repeat as long as we generate new assignments**
- **Backtrack when we get conflicting assignments to same variable**
- **Variations are *polynomial* even in worst case**
  - **Possible to get linear time**

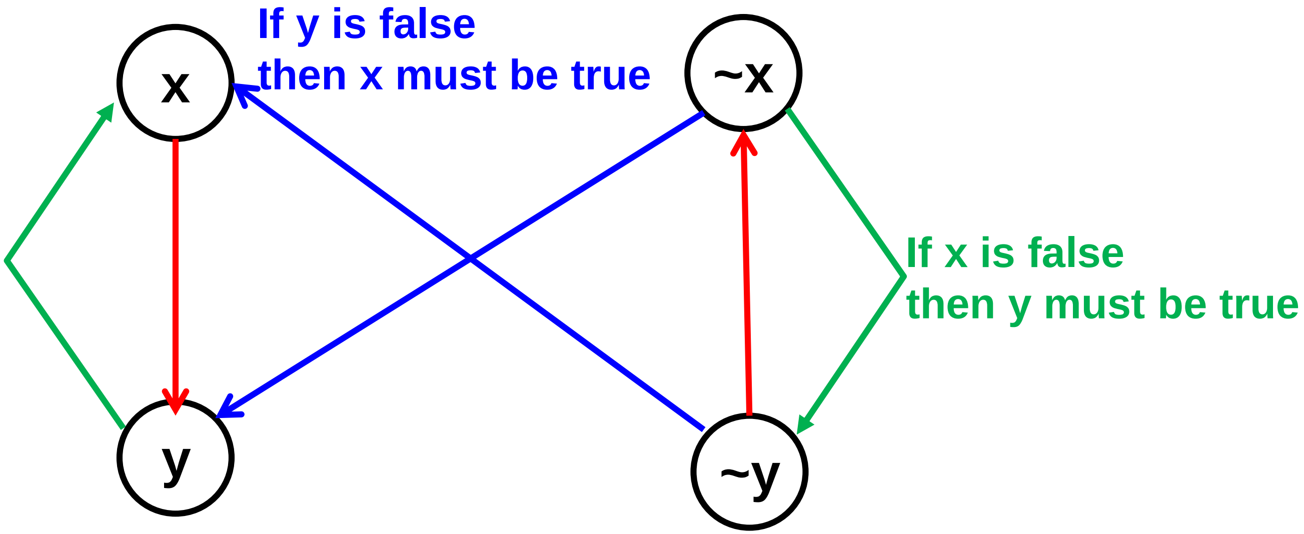
Alternative 2SAT Graph Algorithm

- **If V variables, generate 2V vertices**
  - **pairs labelled x_i and $\sim x_i$**
- **For each clause (L_i V L_k) using variables x_i and x_k, generate 2 edges in the graph**
  - **$\sim L_i$ to L_k**
  - **$\sim L_k$ to L_i**
- **Unsatisfiable if for any x_i there is a path**
  - **from x_i to $\sim x_i$**
  - **and $\sim x_i$ to x_i**
- **Satisfiable if no such path**

2SAT as Domino Chains

Example:

(~x V y) & (x V y) & (x V ~y)



What happens when we add clause (~x V ~y)?

Your Turn: Bipartite Matching

- What are variables?
- How to guarantee at least one match per vertex?
- How to guarantee only 1 match per vertex?