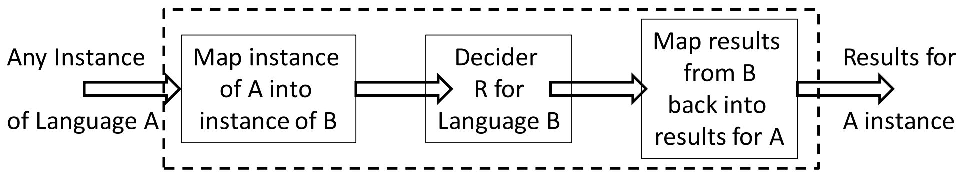
pp. 215-227. **Undecidable Language Problems** (Sec. 5.1)

- Remember A_{TM}={<M,w>| M accepts w} is undecidable
  - When M does not accept w cannot decide if it is because it will eventually reject or loop
- **Reduction**: converting one problem A into another problem B, where we can use solver for B to solve A

Decider S for Language A
Using assumed decider for Language B

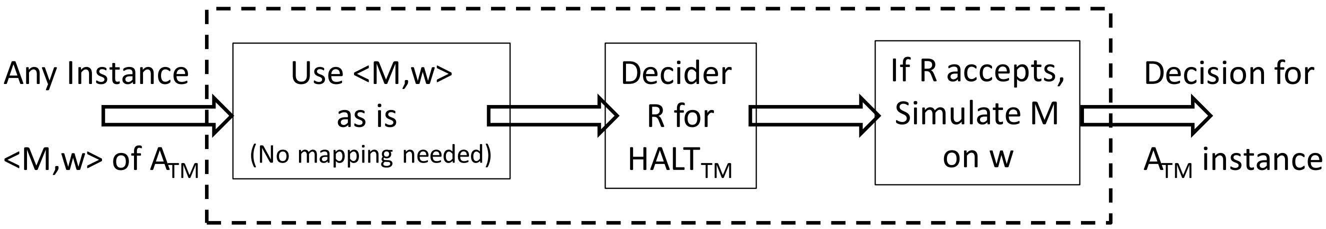


If we chose A as a known undecidable language,
then if R exists as decider for B, then we have built a decider for A
- which we know is impossible

  - Also A clearly cannot be "harder" than B, so if B is "decidable" then so is A.
  - Standard reduction:
    - ***Assume language L*** of interest ***decidable by TM R***
    - Show that solving L means we can solve A_{TM}
      - By mapping any instance of A_{TM} into L
    - Thus if R exists, then we can construct a TM S so that A_{TM} is decidable
    - But this is impossible, so no such R can exist

- **$HALT_{TM}$ = {<M,w>| M is a TM that halts on w}**
- (p. 216) **Theorem 5.1. $HALT_{TM}$ is undecidable**
  - Proof by contradiction. Assume $HALT_{TM}$ is decidable by R
  - Build a decider for A_{TM}
    - Given <M,w> instance from A_{TM}, pass unchanged to R
    - If R finds M halts on w, R halts and accepts
    - If R finds M doesn't halt on w, R halts and rejects

Decider S for A_{tm} ***if Decider R for Language $HALT_{TM}$ exists***

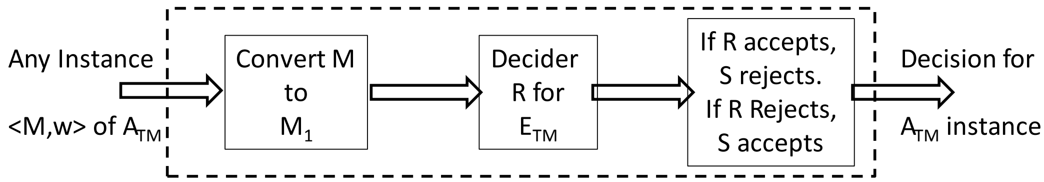


If R rejects, reject <M,w>
If R accepts, Use Sim Results

  - Construct TM S to decide A_{TM} from R as follows
      - Run R on <M,w>
      - If R rejects, reject (we know M loops on w)
      - If R accepts (we know M halts on w):
        - Simulate M on w until it halts
        - If M accepts w then S accepts
        - If M rejects w, then S rejects
    - If R exists, then S as constructed above decides A_{TM}
    - ***But A_{TM} is undecidable, so R cannot exist***

- **E_{TM} = {<M>| M is a TM and L(M)=Φ}**
- (p. 217) **Theorem 5.2 E_{TM} is undecidable**
  - Assume R decides E_{TM}, i.e. given <M> as input, R
    - accepts if L(M) is empty
    - rejects if L(M) is not

Decider S for A_{tm} ***if Decider R for Language E_{TM} exists***



If R accepts, L(M1) is empty
If R rejects, Use Sim Results

  - Use R to construct an S that decides A_{TM} as follows
    - Given any <M,w>, first convert M to <u>**M_1**</u> as follows
      - On any input x, If x != w, M_1 rejects
      - If x = w, run M on w and accept if M does
      - Only string M_1 can possibly accept is w
    - Now define S on an input <M,w> as follows
      - Construct M_1 from M
      - Run R on <M_1> (We are assuming R exists)
      - If R accepts (i.e. L(M) = Φ), S rejects (w not in L(M))
      - else if R rejects (L(M_1) not empty), S accepts
        - w accepted by M
  - If R were decider for E_{TM}, then S is a decider for A_{TM}

- (p. 218) **REGULAR$_{TM}$={<M>|M a TM & L(M) is regular}**
- **Theorem 5.3 REGULAR$_{TM}$ is undecidable**
  - Assume REGULAR$_{TM}$ is decidable by some TM R
    - Given some M, R accepts if L(M) is regular
    - R rejects if L(M) is NOT regular
  - Construct S from R as decider for A_{TM} ={<M,w>} as follows
    - Take M from its input <M,w> and modify M to M_2 that
      - recognizes non-regular language $\{0^n1^n|n\geq0\}$ if M does not accept w
      - recognizes regular language ∑* if M accepts w
      - M_2 constructed ONLY for purpose of feeding its description into assumed decider R for REGULAR$_{TM}$
    - Run R on <M_2>
      - If R accepts, then <M_2> recognizes a regular language
        - Which means M accepts w
      - If R rejects, then M_2 recognizes a non-reg language
        - Which means that M does not accept w
    - Which makes R a decider for A_{TM}

- (p. 219 & Prob. 5.28) **Rice's Theorem**:
  - Let P be any property of the language of a TM
  - L_P= {<M>| M a TM such that L(M) has property P}
    - L_P contains some but not all TMs
    - Whenever L(M1)=L(M2), <M1> ε L_P iff <M2> ε L_P
  - Thus L_P is undecidable
- Above proved undecidability from A_{TM}
  - but other undecidable languages such as E_{TM} usable

- **EQ_{TM} = {<M1, M2>| M1, M2 TMs, and L(M1)=L(M2)}**
- (p. 220) **Theorem 5.4 EQ_{TM} is undecidable**
  - Assume TM R decides EQ_{TM}
  - Construct S to decide **E_{TM}** (not A_{TM}) as follows:
    - On input <M> to E_{TM}
    - Run R on <M,M1> where M1 a TM that rejects all inputs
    - If R accepts (i.e. M matches machine with empty language), then S accepts (L(M) is emoty)
    - If R rejects (M!=M1) then S rejects (M accepts something)
  - If R exists we now have in S a decider for E_{TM}
  - Not possible, so R cannot exist

- (p. 220) Reductions via Computational Histories
- **Accepting Computational History** of M given w
  - Sequence of configurations C_1, … C_l where
    - C_1 is start, C_l is accepting, and C_i legally follows from C_{i-1}
  - Remember a configuration = ua q_i bv, b under tape head
  - Note this is finite in length
- **Rejection Computational History** is similar
- (p. 221) **Linear Bounded Automata (LBA)**
  - TM with finite tape
  - Cannot move off of original tape: Off left or into "blanks"
- (p. 222) **Lemma 5.8. Assume** M is an LBA with exactly q states & g symbols in Γ. There are exactly qng^n possible configurations of tape of length n.
- **A_{LBA} = {<M,w>| M an LBA that accepts w}**
- (p. 222) Theorem 5.9 A_{LBA} is decidable
  - Have decider L keep track of each configuration that M enters while processing w
  - If we ever enter same configuration a 2nd time, reject
    - This is after at most qng^n steps of simulating M
  - If M accepts, L accepts
  - If M rejects, L rejects

- (p. 223) **E_{LBA} = {<M>| M an LBA where L(M) is empty}**
- **Theorem 5.10 E_{LBA} is undecidable**
  - Assume TM R decides E_{LBA}
  - (p. 224) Construct an LBA B that recognizes all accepting computational histories for M on w
    - If M accepts w, L(B) = 1 string
    - If M does not accept w, then L(B) is empty
  - Given <M,w> B constructs all valid histories as strings separated by #s
  - Construct S to decide A_{TM} as follows
    - Construct LBA B from <M,w>
    - Run R on <B>
    - If R rejects, S accepts
    - If R accepts, S rejects
- (p. 5.13) Theorem 5.12 Likewise **ALL_{CFG} = {<G>| G is CFG where L(G)=Σ* is undecidable**

- **(p. 227) PCP: POST CORRESPONDENCE PROBLEM**
  - Consider a set of dominoes with 2 strings on each
  - A **match**: list of dominoes where concatenated string on top is same as concatenated string on bottom
    - Repetitions allowed
  - PCP: Given a set of dominoes, is there a match?
    - Can use duplicates
    - Try Exercise 5.3 p. 239
  - PCP is undecidable (see book for proof details)
    - Reduction from A_{TM} via accepting histories
    - Given any <M,w> build a matching PCP instance
    - IF PCP is decidable, so is A_{TM}