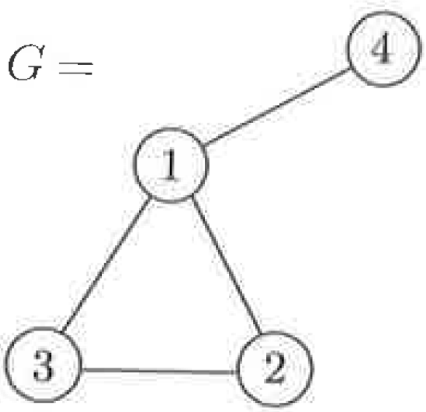
Graphs

- (p 186) **Graphs G = (V,E)**
  - set V of **vertices**, each with a unique name
    - Note: book calls vertices as **nodes**
  - set E of **edges** between vertices, each encoded as tuple of 2 vertices as in (u,v)
- Edges may be **directed** (from u to v) or **undirected**
  - Undirected edge eqvt to pair of directed edges
- Example of undirected graph

$G =$



$\langle G \rangle =$

(1,2,3,4)((1,2),(2,3),(3,1),(1,4))

- **Labeled graph**: each edge has a “name”
- **Weighted graph**: each edge has numerical value
  - E.g. in graph of V=cities, weight on (u,v) is distance from u to v

Terminology about Graphs (see P. 12)

- **Outdegree** of a vertex u: # of edges leaving it
  - i.e. |{(u, v)}| for some v
- **Indegree** of a vertex v: # of edges entering it
  - i.e. |{(u, v)}| for some u
  - For undirected graph outdegree(u) = indegree(u) = **degree**(u)
- **K-regular**: every vertex has degree k (p. 13)
- **Subgraph:** subsets V' and E' of V and E where all edges in E' are between vertices in V'
- **Path of length k:** from u to v if a set of k edges in G (u_i,v_i), $1 \le i \le k$, where $u_1 = u$, $v_i = u_{i+1}$, and $v_k = v$
- **Simple path:** no vertices are repeated
- **Shortest path:** between u & v is simple path of shortest lengthg
- **Diameter:** longest shortest path between any 2 vertices
- **Hamiltonian Path:** goes thru every vertex once
- **Cycle**: a path exists from u back to u

- **Connected** iff every vertex can be reached from every other vertex by some path
- **Strongly connected** iff a directed path from each vertex to every other
- Tree: graph with no simple paths: Has **root** & **leaves**
- **Vertex Cover of size k**: subset of k vertices where every edge touches at least one of them (p. 312)
- **K-Clique:** subset of k vertices where there is an edge between every pair in it.
- Two graphs G & H are **isomorphic** if vertices of one can be reordered so that graphs are identical
- **Spanning Tree**: subgraph that forms a tree that includes all vertices, but with minimum # of edges
- **Flow Network**: directed weighted graph where:
  - Each edge can carry a “flow” (a number)
  - Each edge has a “capacity” (max possible flow value)
  - For each vertex, ∑ incoming flow = ∑outgoing flow
    - Except for some **source** that generates out flow
    - And some **sink** which has no outgoing flow

Graph Data Structures

- Assume |V| = N, |E| = M
- **Adjacency Matrix**: NxN Boolean matrix A where
  - A[u,v] = 1 if (u,v) in E
  - A[u,v] = 0 otherwise
  - If graph is weighted, A[u,v] = weight on (u,v)
- **CSR (Compressed Sparse Row)**: 3 vectors
  - A: M vector of weights (one per edge)
  - JA: M vector of vertex indecies
  - IA: N+1 vector of indices into A, JA
    - IA[u] = index into A, JA for 1^{st} edge from u
    - JA[IA[u]] thru JA[IA[u+1]-1] are the v's for edges (u,v)
    - Matching elements in A are weights
- Lots of variations

Spanning Trees & BFS

- Common problem: starting at some vertex u, find tree that reaches as many vertices as possible
  - If all vertices reachable, then a "spanning tree"
- Common algorithm: **BFS** (**Breadth First Search**)
  - **Frontier**: set of all vertices that have been reached "for 1st time"
    - At start, just u
  - Also each vertex can be marked as "touched" or not
    - At start only u so marked
  - For each vertex in current frontier:
    - Follow each outgoing edge
    - If other vertex is not touched:
      - Mark as touched
      - Add to new frontier
  - When frontier empty, swap with new frontier and repeat
- Option: when first touched, mark vertex with "level"
  - Level = # of edges from root u
  - At end, final level = diameter of reached subgraph
- Clearly polynomial time

- Basis for the **GRAPH500** benchmark
  - www.graph500.org
  - Search graphs with up to trillions of vertices
  - Literally thousands of different implementations on different computers, esp. parallel
  - Established by an **ND quad-domer**
- **Beamer's Algorithm**:
  - When frontier too large, instead explore all non-touched
  - If any of them have an edge to current touched, mark them as touched
  - Can reduce time by 10X

- **Max flow**: given a flow network, find largest flow from source to sink where no edge exceeds its capacity
- Ford-Fulkerson Algorithm
  - https://en.wikipedia.org/wiki/Ford%E2%80%93Fulkerson_algorithm
  - Define:
    - c(u,v) = capacity of edge (u, v)
    - f(u,v) = flow on edge (u,v)
    - **Residual Network** $G_f(V,E_f)$ = network with capacity $c_f(u,v) = c(u,v) - f(u,v)$ ("residual flow")

Output Compute a flow f from s to t of maximum value

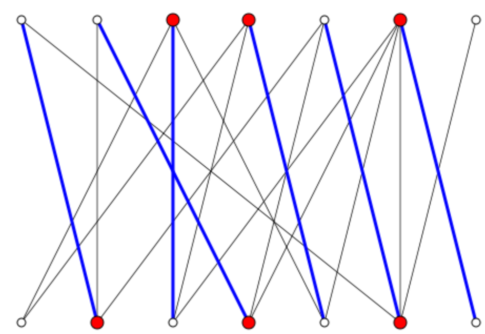
1. $f(u,v) \leftarrow 0$ for all edges (u,v)
2. While there is a path p from s to t in G_f, such that $c_f(u,v) > 0$ for all edges $(u,v) \in p$:
   1. Find $c_f(p) = \min\{c_f(u,v) : (u,v) \in p\}$
   2. For each edge $(u,v) \in p$
      1. $f(u,v) \leftarrow f(u,v) + c_f(p)$ (*Send flow along the path*)
      2. $f(v,u) \leftarrow f(v,u) - c_f(p)$ (*The flow might be "returned" later*)

- "←" denotes assignment. For instance, "*largest* ← *item*" means that the value of *largest* changes to the value of *item*.
- "**return**" terminates the algorithm and outputs the following value.

  - "Finding paths" can use BFS
  - Each new path chosen is called "**augmenting path**"

Bipartite Graphs

- References:
  - https://en.wikipedia.org/wiki/Bipartite_graph
- **Bipartite Graph**:
  - 2 disjoint sets of vertices U and V, called "parts"
  - Every edge connects vertex from U with one from V
- **Matching**: subset of edges where no two edges share an endpoint
  - **Maximal Matching**: edge set is largest possible matching
  - **Perfect Matching**: |U|=|V|=|matching set|
- Examples of Bipartite Graphs:
  - Athletes and Teams they played with
  - Actors and Movies they acted in
  - Trains and Stations
  - Social networks
  - All graphs that are trees
  - Graphs that form single cycles with even # vertices
  - Graphs that can be written "on a 2D plane" without edges crossing



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- Properties
  - No odd cycles
  - 2-colorable
- **Kőnig's theorem**: # of edges in maximum matching = # of vertices in a minimum vertex cover
- O(VE) algorithm: **Hopcroft-Karp**
  - https://en.wikipedia.org/wiki/Hopcroft%E2%80%93Karp_algorithm
  - Similar to Ford-Fulkerson

Input: Bipartite graph $G(U \cup V, E)$
Output: Matching $M \subseteq E$
$M \leftarrow \emptyset$
repeat
  $\mathcal{P} \leftarrow \{P_1, P_2, \ldots, P_k\}$ *maximal set of vertex-disjoint shortest augmenting paths*
  $M \leftarrow M \oplus (P_1 \cup P_2 \cup \cdots \cup P_k)$
until $\mathcal{P} = \emptyset$

  - P_k = augmenting path of length k
  - "Free vertex" – does not appear in current matching M
  - Iteratively find "augmenting path"
    - Use BFS to partition vertices into layers
      - Current free vertices from U are the 1st layer
    - Pick a free vertex from 1st level
    - Use BFS to create a tree
      - Alternate edges between not in M and in M
      - Stop tree if reach a free vertex u from U

- Ends at level k when ≥1 free vertices in V are reached
- Define free vertices v from V into a set F
- Choose some shortest tree and then shortest path
  - Called **augmenting path** (of length k)
- Remove all edges from M that are in the path
- Add in all edges from path that are not in M
- Repeat until no free vertex from U has an augmenting path to a free vertex in V

- Why does this work?
  - At each iteration, always adding 1 more edge to M than deleting
  - Guaranteed to stop at min(|U|, |V|)
- Complexity: for each vertex in V may look at each edge in E
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