TMs & Languages (end of Sec. 3.1 & Sec. 4.1)

- (p. 170) TMs and Languages
  - L(M) = set of strings accepted by TM M
  - L is **Turing-recognizable** if some TM M accepts it
  - When M started, 3 outcomes: Accept, Reject, Loops
    - M can fail to accept if it enters q_{reject} or loops
  - (p. 170) M is a **decider** is it never loops
    - I.E. always stops, regardless of input string
    - I.e. always ends up in either q_{accept} or q_{reject}
  - (p. 170) L is **Turing-decidable** (or simply **decidable**) if some Turing Machine decides it.
- Examples
  - (p. 171 Ex. 3.7) A = (0^k | $k=2^n$, $n \geq 0$}
    - Multiple iterations, each cuts # 0s in half
  - (p.173 Ex. 3.9) B = {w#w | w in {0,1}*}
  - (p. 174 Ex. 3.11) C = {$a^i b^j c^k$ | ixj=k, i,j,k ≥1}
  - (p.175 Ex. 3.12} E = {$\#x_1\#x_2\# \ldots \#x_l$ | no two x's are equal}

- (p. 194) **Acceptance problem**: does some DFA accept some string?
  - Can we build a TM that:
    - given a representation for some FA and some string,
    - tell us if that FA accepts the string, or not
    - and do so in finite time
    - and never loop

- Define **A_{DFA}** = {<B,w>| B is a DFA that accepts w}
  - <B,w> is "encoding" of DFA B and string w in a way that a TM can "interpret" B's processing of w
    - E.g. <B> is a list of B's 5 components
  - A_{DFA} is set of all encoded DFAs & the strings they accept
- **Is A_{DFA} decidable**?
  - Does there exist a TM that accepts *all* members of A_{DFA} and rejects all other inputs?
    - I.e. does it always halt
- (p. 194) Theorem 4.1: **A_{DFA} is decidable**
  - Proof: M = "On input <B,w> where B is a DFA & w a string"
    - M receives a tape with <B,w> on it
    - Determine if representation of <B> is formatted ok
    - Simulate DFA B on string w
      - Keep track of B's current state and position into its input w on M's tape
      - Search for correct transition
      - Update state and index
    - If simulated B ends in accept, accept. If it ends in nonaccept, reject.
      - Note: formatted B always stops after finite # of steps
      - Thus so will TM

- Define A_{NFA} = {<B,w>| B is an *NFA* that accepts w}
- (p. 195) Theorem 4.2: **A_{NFA} is decidable**
  - Proof: N = "On input <B,w> where B is NFA & w a string"
    - Convert NFA B into equivalent DFA C
    - Encode C and w on tape as <C,w>
      - Having a multi-tape TM may be useful
    - Run machine M from Theorem 4.1 on <C,w>
    - If M accepts, N accepts, else N rejects
  - Note use of a "subroutine" M

- Define A_{REX} = {<R,w>| R is a regex that generates w}
- (p. 196) Theorem 4.3 **A_{REX} is decidable**
  - Proof: Convert R into an NFA
    - Then run TM N
    - If N accepts, then accept, else reject

- Define **E_{DFA}** = {<A>| A is a DFA where L(A) = Φ}
  - "E" for "empty"
  - I.e. the set of all DFAs that accept no strings
- (p. 196) Theorem 4.4 **E_{DFA} is decidable**
  - Proof: Use the BFS algorithm starting on start state of A
    - Mark states that are reachable from start state
    - If any Final State is marked, reject
    - If not, accept
  - Again will halt since only finite # of states in any DFA

- Define **EQ_{DFA}** = {<A,B>| A,B both DFAs & L(A) = L(B)}
  - "EQ" stands for Equivalent
  - I.e. the set of all pairs of DFAs that are equivalent
- (p. 196) Theorem 4.5 **EQ_{DFA} is decidable**
  - Proof:
    - Construct a new DFA C from A and B that
      - Accepts only those strings that are accepted by either A or B, but not both
        - i.e. L(C) = (L(A) ∩ not(L(B))) U (not(L(A)) ∩ L(B))
        - Called **Symmetric Difference**
        - If L(C) is empty then A & B gen same language
    - Then use machine from Theorem 4.4

- (p. 198) Decidable Problems re CFLs

- Define $\mathbf{A_{CFG}}$ = {<G,w>|G is a CFG that generates w}
- (p. 198) Theorem 4.7 **A_{CFG} is a decidable language**
  - If G is in Chomsky Normal Form, any derivation of w has 2n-1 steps, where |w|=n
  - TM S
    - Convert G to Chomsky
    - List all derivations with 2n-1 steps
    - If any generate w, accept, else reject

- Define $\mathbf{E_{CFG}}$ = {<G>|G is a CFG & L(G) = Φ}
- (p. 199) Theorem 4.8 **E_{CFG} is a decidable language**
  - TM R
    - Mark all terminal symbols in G
    - Repeat until no new variables get marked
      - Mark any variable A where G has a rule $A \rightarrow U_1U_2 \ldots U_k$ and each symbol U_i has already been marked
    - If start variable not marked, accept, else reject

- Define **EQ_{CFG}** = {<G,H>|G & H are CFGs, & L(G)=L(H)}
  - Cannot use DFA approach because CFLs not closed under complement or intersection & this is NOT decidable
- (p. 200) Theorem 4.9 **Every CFL is decidable**
  - Don't want to try converting a PDA into an TM
    - Some branches of PDAs computation may go on forever, so TM can't be a decider
  - Proof: Let G be a CFG for A; TM M_G is to decide A
    - Run TM S on <G,w>
    - If it accepts, then accept, else reject

- Result: p.201 Fig. 4.10. Following are proper subsets of the next one
  - Regular languages
  - Context-Free languages
  - Decidable Languages
  - Turing-recognizable languages