pp. 275-284. **Complexity** (Sec. 7.1)

- Definition 7.1. **Running Time or Time complexity**
  - M is a deterministic TM that halts on all inputs
  - f_M:N->N maps for machine M length n into max # of steps M takes to solve any string of length n for which it halts
    - Usually drop subscript
  - We say that M is an f(n) time TM
- Definition 7.2 We say **f(n) = O(g(n))** if
  - f,g functions N->R^+
  - there are positive ints c and n_0 such that
  - f(n) ≤ cg(n) for all n>n_0
  - g(n) said to be an **asymptotic upper bound**
- Notes on **"big O"**
  - "O" of polynomials = largest exponent
  - $\log_a(n)$ differs by a constant value from $\log_b(n)$ for any a,b
    - Thus $O(\log_a(n)) = O(\log_b(n)) = O(\log(n))$
  - **polynomial bounds** if $O(n^c)$
  - **exponential bounds** if O(2^n^δ)
- **"Little o"** Def 7.5: f(n) = **o(g(n))** if $\lim_{n\to\infty}(f(n)/g(n)) = 0$
  - for any c, there is some n_0 such that f(n) < cg(n_0) for n>n_0
  - f(n) is asymptotically <u>less</u> than g(n)

- (p. 279) **Time Complexity class**:
  - **TIME(t(n))** = set of all languages decidable by O(t(n)) TM
  - Time(n) called **"Linear Time"**
- Example: A =$\{0^k 1^k | k \geq 0\}$
  - M1 with input w, |w|=n
    1. **O(n)** Scan tape & reject if 0 to right of a 1
    2. **O(n) repetitions:** Repeat if both 0s and 1s on tape
       3. **O(n)** each: Scan tape, crossing off one 0 and one 1
    4. **O(n)** If 0s remain after all 1s, or vv, reject. Else accept
    - Time = O(n) + O(n^2) + O(n) = **O(n^2)**
    - Thus A in TIME(n^2). Anything better?
  - M2 with input w, |w|=n
    1. **O(n)** Scan tape & reject if 0 to right of a 1
    2. **O(log(n)) repetitions** Repeat if both 0s and 1s on tape
       3. **O(n)** each: Scan tape to see if even or odd #s of 0s & 1s. If odd, reject
       4. **O(n) each:** scan tape, crossing off every other 0, then every other 1
    5. **O(n)** if no 0s or 1s, accept, else reject
    - Time = O(n) + O(log(n))*O(n) + O(n) = **O(n log(n))**
    - Thus A now in TIME(nlog(n)). Anything better?

- 2-tape TM M3 can solve in O(n) time!
  - M3 with input w on tape 1, |w|=n
    1. **O(n)** Scan tape & reject if 0 to right of a 1
    2. **O(n)** scan tape 1 to 1^{st} 1, copying all 0s to tape 2
    3. **O(n)** each: Scan tape 1. For each 1, cross off a 0 from tape 2. If all 0s crossed off before all 1s, reject
    4. **O(n):** If all 1s off tape1, & no 0s on tape2, accept, else reject
  - Thus A in $TIME_{2tape}(n)$
- Generalization: (Problem 7.49): a**ny language decidable in o(nlog(n)) on single tape TM is regular**

- (p. 282) **Theorem 2.8. Every O(t(n)) multi-tape TM has an equivalent O(t^2(n)) 1 tape TM**
  - Assume M = k-tape TM with O(t(n)) time
  - Let S = equivalent 1-tape machines
  - S's 1st step: initialize its 1 tape to store k tapes
    - Use "#" to separate and "'" to show tape head
  - For *each of M's O(t(n))* steps, S performs
    - **O(t(n))**: scan tape to find current values under heads
    - **O(t(n))**: scan tape again to update each of k tapes
      - If any of M's tapes writes into blank area, shift rest of simulated tapes 1 cell right
  - Total is O(t(n))*O(t(n)) = O(t^2(n))

- (p. 283) Definition 7.9: **Running time of a NTM** (1-tape) decider is f:N->N where f(n) is max # steps for an input of length n on any branch of computation tree.
  - See Fig. 7.10 on p. 283
- (p. 284) **Theorem 7.11. Let t(n) be a function where t(n) >n. Every t(n) NTM (1-tape) has equivalent $2^{O(t(n))}$ time deterministic 1-tape TM**.
  - Proof: given input of length n
    - Each branch of NTM computation of length t(n)
    - If b=max # of choices in each tree of computation
      - Then # of leaves at most $b^{t(n)}$
    - TM simulator D (Theorem 3.16) uses 3-tapes and visits all choices at depth d before going to depth d+1
    - Total # nodes in tree < 2X # leaves, so bound as $O(b^{t(n)})$
    - Time from root to node is O(t(n))
    - Running time of D is $O(t(n)b^{t(n)}) = 2^{O(t(n))}$
    - Simulating on 1-tape squares time: $(2^{O(t(n))})^2 = 2^{O(t(n))}$
- Sample problems:
  - O notation: 7.1
  - o notation: 7.2