CSE 30151 Theory of Computing: Homework 1
Math and Proof Techniques

Version 1: Jan. 16, 2018

Instructions

- Unless otherwise specified, all problems from "the book" are from Version 3. When a problem in the International Edition is different from Version 3, the problem will be listed as V3:x.yy/IE:x.zz, where x.zz is the equivalent number. When Version 2 is different, it will be listed as V3:x.yy/V2:x.zz. If either IE or V2 do not have a matching number, the problem text will be duplicated.
- You can prepare your solutions however you like (handwriting, LaTeX, etc.), but you must submit them in **legible PDF**. You can scan written solutions on the printer in the back of the classroom, or using a smartphone (with a scanner app like CamScanner). It is up to you to ensure that submissions are legible. REMEMBER THAT IF WE CAN"T READ IT OR SCAN IS CUT OFF¡ YOU DON"T GET A GRADE FOR IT.
- The problems marked as "TEAM" may be solved in a collaborative fashion with up to 2 other students. In such cases, your submission should have the word "TEAM" at the start of the problem, followed by the names of your collaborators (must be other students in this class). When such problems are graded, the first submission encountered by the grader for the team will be used for a common grade for all identified team members.
- Please give every PDF file a unique filename.
  - If you're making a complete submission (all problems), name your PDF file `netid-hw5.pdf`, where `netid` is replaced with your `NetID`.
  - If you're submitting some problems now and other problems later, name your file `netid-hw5-1-2-3.pdf`, where `1-2-3` is replaced with just the problems you are submitting now. This may be useful for team submissions.
  - If you use the same filename twice, only the most recent version will be graded.
  - The time of submission is the time the most recent file was uploaded.
- If you use LaTeX and want to draw something like a state diagram, consider using the `tikz` package. A reference document is on the website under "Assignments".
- You may also find the website http://madebyevan.com/fsm/ a useful tool for drawing state diagrams via drop and drag. It will output both .png image files and latex in the tikz format.
- Submit your PDF file in Sakai to the appropriate directory. Don't forget to click the Submit (or Resubmit) button!

Practice Problems

These problems are from the book, and most have solutions listed for them. They are listed here for you to practice on as needed and any answers you generate **should not** be submitted. You are free to discuss these with others, but you are not allowed to post solutions to any public forum.

1. Describing and working with sets: 0.1, 0.2, 0.3
2. Graphs: 0.9
3. Proofs: V3:0.10/IE:0.13, V3:0.14/IE:0.15, V3:0.15/IE:0.14

Book Exercises

These problems are found in the text book and are to be answered and submitted by each student. **If they are not marked as "TEAM," you are to solve them individually.** If they are marked as "TEAM" you may submit the same answer as the others in your team. In any case, use of solution manuals from any source or shared solutions is a violation of the ND Honor Code. You are also not allowed to show your solutions to another student not part of your TEAM.

1. (5pt) Book 0.6

   Solution:

   (a) $f(2) = 7$
   (b) Range of f: $\{6, 7\}$, domain of f: $\{1, 2, 3, 4, 5\}$
   (c) $g(2, 10) = 6$
   (d) Range of g: $\{6, 7, 8, 9, 10\}$, domain of g: $\{(x, y) \mid x \leq 5,\ 6 \leq y \leq 10,\ x, y \in Z^+\}$
   (e) $g(4, f(4)) = g(4, 7) = 8$

   Common Grading comments:

   - -1 for each incorrect part (5 parts total)

2. (5pt) Book 0.7. Assume the relation is built from the set of positive integers N.

3. (5pt) Book 0.8

   Solution: (Satyaki Sikdar)

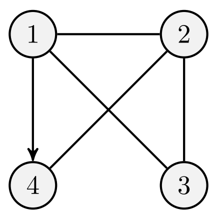
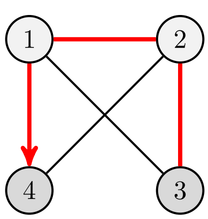
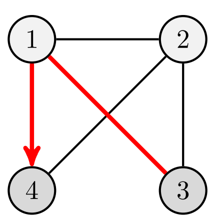


Figure 1: The graph $G(V, E)$, $V = \{1, 2, 3, 4\}$, $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\}$

Node	Degree
1	3
2	3
3	2
4	2

[t]0.3 

[t]0.3 

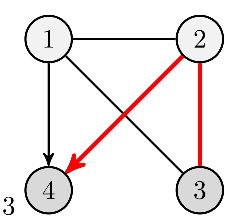
[t]0.3 

Figure 2: All the *simple* paths from node 3 to node 4 in the graph given in Figure 1

Common Grading comments:

- -3 for all degrees incorrect/missing

- -2 for two or more degrees incorrect
- -1 for one degree incorrect
- -2 no graph no path
- -1 for incorrect path/no path
- -1 no graph, but a path is somehow specified

4. (5pt) V3:0.12/V2:0.11/IE:0.11

5. (TEAM-10pt) V3:0.11/IE:0.12 but not in V2: Let $S(n) = 1 + 2 + ...n$ and $C(n) = 1^3 + 2^3 + ... + n^3$. Prove each of the following by induction, then show $C(n) = S^2(n)$ using the two results.

   (a) $S(n) = 0.5 * n(n+1)$

   (b) $C(n) \;=\; 0.25 * (n^4 + 2n^3 + n^2) \;=\; 0.25 * n^2(n+1)^2$

   Solution:

   (a) ***Basis:*** Prove that the $S(n)$ is true for $n = 1$.
   If $n = 1$,
   $$S(n) = S(1) = 0.5 * 1 * 2 = 1.$$
   Therefore, we have proved that the basis of the induction is true.
   Induction step: For each $k \geq 1$, assume that $S(k)$ is true for $n = k$ and show that it is true for $n = k + 1$. The induction hypothesis states that
   $$S(k) = 0.5 * k(k+1).$$
   Our objective is to prove that
   $$S(k+1) = 0.5 * (k+1)(k+2).$$
   Recall that $S(k) = 1 + 2 + ... + k$, therefore, we can write,
   $$\begin{aligned} S(k+1) &= 1 + 2 + ... + k + (k+1) \\ &= S(k) + (k+1) \\ &= 0.5 * k(k+1) + (k+1) \quad &&\text{, using the induction hypothesis} \\ &= (k+1)(0.5 * k + 1) &&\text{, factoring } (k+1) \\ &= 0.5 * (k+1)(k+2). \end{aligned}$$
   Thus the formula is correct for $n = k + 1$, which proves the theorem.

   (b) ***Basis:*** Prove that the $C(n)$ is true for $n = 1$. If $n = 1$,
   $$C(n) = C(1) = 0.25 * (1^4 + 2.1^3 + 1^2) = 0.25 * 1^2(1+1)^2 = 1.$$
   Therefore, we have proved that the basis of the induction is true.
   Induction step: For each $k \geq 0$, assume that $C(k)$ is true for $n = k$ and show that it is true for $n = k + 1$. The induction hypothesis states that
   $$C(k) = 0.25 * (k^4 + 2k^3 + k^2) = 0.25 * k^2(k+1)^2.$$
   Our objective is to prove that
   $$C(k+1) = 0.25 * [(k+1)^4 + 2(k+1)^3 + (k+1)^2] = 0.25 * (k+1)^2(k+2)^2.$$

Recall that $C(k) = 1^3 + 2^3 + ... + k^3$, therefore, we can write,

$$\begin{aligned}
C(k+1) &= 1^3 + 2^3 + ... + k^3 + (k+1)^3 \\
&= C(k) + (k+1)^3 \\
&= 0.25 * k^2(k+1)^2 + (k+1)^3 && \text{, using the induction hypothesis} \\
&= (k+1)^2(0.25 * k^2 + k + 1) && \text{, factoring } (k+1)^2 \\
&= 0.25 * (k+1)^2(k^2 + 4k + 4) && \text{, using } (a+b)^2 = a^2 + 2ab + b^2 \\
&= 0.25 * (k+1)^2(k+2)^2 = 0.25 * [(k+1)^4 + 2(k+1)^3 + (k+1)^2].
\end{aligned}$$

Thus the formula is correct for $n = k + 1$, which proves the theorem.

Common Grading comments:

- 2 points for use of proper language and notation
  - 0 - no or clear lack of explanations / justifications,
  - 1 - some attempt to use formal language notations and justifications,
  - 2 - proper use of terms and sound justification of statements in the proof
- 1 point for correct basis
- 2 points for induction hypothesis and steps
  - 0 - didn't attempt / grossly incorrect,
  - 1 - attempted, but partially correct,
  - 2 - perfect or nearly perfect solution

Non-book Problems

The following problems are not found in the text book. **If they are not marked as "TEAM," you are to solve them individually.** If they are marked as "TEAM" you may submit the same answer as the others in your team. Use of any resource you used other than the text book or class notes must be cited. You are also not allowed to show your solutions to another student.

6. (5pt) Include the following statement as your answer: "I have read and understand both the ND Honor Code policy and the CSE Guide to the Honor Code as posted on the class web site. From the latter, I understand that the color for using an on-line solution manual for book problems is": (fill in color red, green, or yellow).

   Solution: Red

7. (5pt) Prove by construction that between any two consecutive powers of 2 2^n and 2^{n+1} where $n > 0$ there is at least one odd number.

   Solution: if $n > 0$ then 2^n is both even and greater than 1. 2^{n+1} is twice as big, and there are $2^{n+1} - 2^n - 1 = 2^n - 1$ numbers between 2^n and 2^{n+1}. But $2^n - 1 > 0$ for $n > 0$ so there is at least the number $2^n + 1$ between them, which is odd.

8. (TEAM-10pt) Assume a bipartite graph such as introduced in the first day's lecture. Assume we have two sets of N vertexes, $M = \{m_1, ...m_N\}$ and $W = \{w_1, ...w_N\}$, and there are E edges $= \{(m_i, w_j)|m_i$ in M and w_j in W$\}$. Define at a high level (in pseudo code is fine) an algorithm that determines if there is a subset of MxW where each tuple (m_i, w_j) in the subset is an edge in E. (Hint: how many possible matchings are there, and if you picked one of them, how would you "verify" that the matching was valid). In big O notation, what is the complexity of your algorithm.

   In addition, define a heuristic that you might use that might improve performance. (Hint: consider the case where a vertex has degree 1).

Solution: A bipartite matching is a set of N pairs (m_i, w_j) where each m and w appear in exactly one pair, and each pair corresponds to an edge in the graph. The problem is to just determine if a matching exists.

There are $N!$ possible distinct matchings, each with N pairs. Thus a brute force algorithm would be to take these matchings one at a time, and check if each of the N pairs in the current set is a valid edge. The worst case algorithm for this check is to search all E edges each time, so the time for checking one matching of N pairs is at worst $O(NE)$. If we have to check all $N!$ matchings, the max time is thus $O(NN!E)$. This is exponential.

A heuristic to speed up many cases is to first scan the graph for any vertex that has degree one. For each such vertex we know that if a matching exists it must have one of the pairs the two vertexes touched by the single edge that touches the degree 1 vertex. We can then remove from the graph all vertices making up such pairs, and all edges touching them. We can then repeat the process, looking for more degree one vertices. After removing all possible cases like this, we could run the brute force algorithm on the remaining graph.

If we ever find a degree zero vertex we can stop and declare no solution.

Note: the first solution used to grade this problem assumed N^2 pairings, not $N!$. If you said something like $N!$ and lost some points, see me.