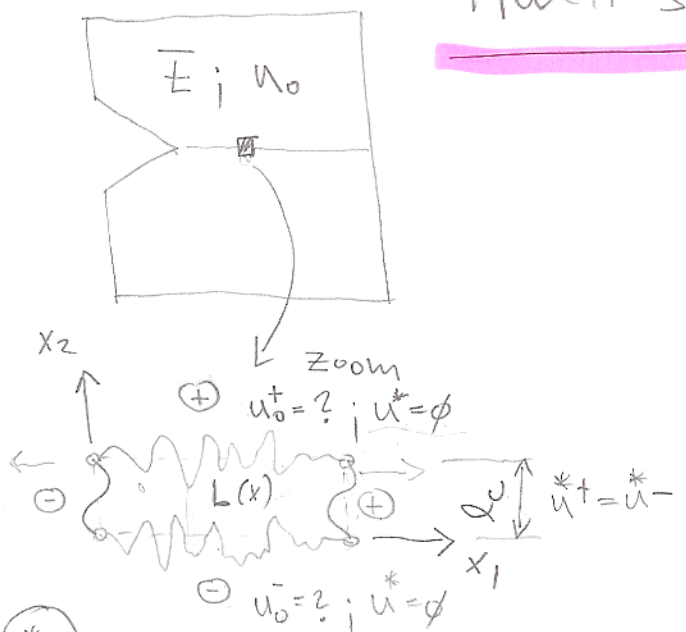
Multiscale Cohesive Model

Derived by Dr. Karel Matouš

These are my original hand written notes, I derived sometimes in June 2004.

2D - plane strain

Multi-scale cohesive law



Displacement decomposition

$$u(x) = u_0 + u^*$$

$$\varepsilon(x) = \varepsilon_0 + \varepsilon^* = \nabla^s u_0 + \nabla^s u^*$$

Hill's Lema

(u^*) Periodic in x_1 and $u^* = \phi$ on $\partial\Omega_{x_2}$

$$\delta W = \frac{1}{A}\int_\Omega \sigma \cdot \delta\varepsilon(u)\, d\Omega = \Sigma \cdot [\![\delta u_0]\!]$$

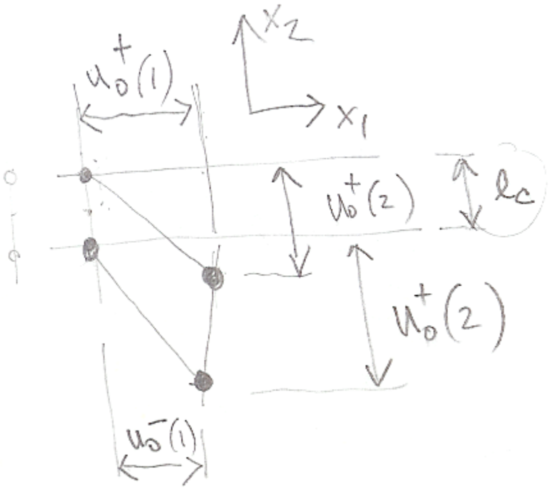
$$[\![u_0]\!] = u_0^+ - u_0^-$$

Average of fluctuation strain

$$\frac{1}{A}\int_\Omega \delta\varepsilon^*\, d\Omega = \frac{1}{A}\int_{\partial\Omega} x \otimes N\, dH = \phi$$

Cohesive layer kinematics

and average strain



$$\varepsilon_{11}^0 = \frac{\partial u_0^A}{\partial x_1} = \frac{1}{2}\left[\frac{\partial u_0^+(1)}{\partial x_1} + \frac{\partial u_0^-(1)}{\partial x_1}\right]$$

$$\varepsilon_{22}^0 = \frac{u_0^+(2) - u_0^-(2)}{\ell_c}$$

$$u_0^{Average} = \frac{1}{2}\left(u_0^+(1) + u_0^-(1)\right)$$

ℓ_c - intrinsic length scale

$$\varepsilon_{12}^0 = \frac{u_0^+(1) - u_0^-(1)}{\ell_c}$$

It can be assumed that $\varepsilon_{11}^{0} = \emptyset$ (B)

$\varepsilon_0 = \frac{1}{l_c} [\![u_0]\!]$ ⇒ ε_0 in term of displacement jump

$\delta\varepsilon_0 = \frac{1}{l_c} [\![\delta u_0]\!]$

δu_0 is an arbitrary function

⇒ $\delta\varepsilon_0 = [\![\delta u_0]\!]$! NOT CORRECT

$[-]$ $[m]$

$\delta\varepsilon_0 = \frac{1}{l_c} N \otimes [\![\delta^0 u]\!]$

Cohesive layer constitutive response

$$\Psi(\omega; \varepsilon(u)) = (1-\omega)\Psi_V = (1-\omega)\bar{\sigma} : \varepsilon \frac{1}{2}$$

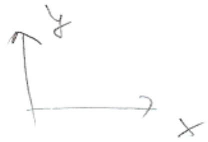
↳ effective stress

Clausius-Duhem inequality

$$-\dot{\Psi} + \sigma : \dot{\varepsilon} \geq \emptyset$$

Ψ_v → virgin free stored energy function

$$\Psi_V = \frac{1}{2} \varepsilon : L : \varepsilon$$



$$\varepsilon_0 = \left\{ \begin{array}{c} \emptyset \\ \frac{1}{l_c} [\![u_0^2]\!] \\ \frac{1}{l_c} [\![u_0^1]\!] \end{array} \right\}$$

$$\sigma = \frac{\partial \Psi}{\partial \varepsilon} = (1-\omega)\frac{\partial \Psi^0}{\partial \varepsilon} = (1-\omega) L : \varepsilon$$ (c)

ω – isotropic damage parameter

$$\dot{\Psi} = -\dot{\omega}\frac{1}{2}\varepsilon : L : \varepsilon + (1-\omega)\frac{1}{2}\left[\dot{\varepsilon} : L : \varepsilon + \varepsilon : L : \dot{\varepsilon}\right]$$

$$\dot{\Psi} = -\dot{\omega}\underbrace{\frac{1}{2}\varepsilon : L : \varepsilon}_{-Y} + \underbrace{(1-\omega)\bar{\sigma}}_{\sigma} \cdot \dot{\varepsilon}$$

C–D $\quad -\dot{\Psi} + \sigma : \dot{\varepsilon} \geq \emptyset$

$$-\dot{\omega} Y - \sigma \cdot \dot{\varepsilon} + \sigma \cdot \dot{\varepsilon} \geq \emptyset$$

$$\mathcal{D} = \dot{\omega}\Psi_0 = -\dot{\omega} Y \Rightarrow \text{Damage dissipation}$$

$$Y = -\frac{1}{2}\varepsilon : L : \varepsilon \Rightarrow \text{Thermodynamic force}$$

$$\dot{\sigma} = (1-\omega) L : \dot{\varepsilon} - \dot{\omega}\underbrace{L : \varepsilon}_{\bar{\sigma}}$$

Damage function

$$g(Y; \chi) = G(Y) - \chi^t \leq \emptyset \quad ; \quad t \in R^+$$

$G(Y)$ – damage function

χ^t – damage treshold

(D)

Damage evolution equation

$$\dot{\omega} = \dot{\kappa}\frac{\partial g}{\partial Y} = \dot{\kappa} H \; ; \quad H = \frac{\partial G(Y)}{\partial Y}$$

we define that $\quad \dot{\kappa} = \mu \langle \phi \rangle$

$$\dot{\chi}^t = \dot{\kappa} H$$

and from consistency condition

$$\dot{\kappa} = \dot{Y}$$

Simo- Ju

$$g(\zeta; \chi) = -Y - \chi^t \le 0$$
$$= \zeta - \chi^t \le 0$$

$$\dot{\kappa} = \dot{\zeta} = \varepsilon : L : \dot{\varepsilon} = \bar{\sigma} \cdot \dot{\varepsilon}$$

$$\dot{\omega} = H \bar{\sigma} \cdot \dot{\varepsilon}$$

Const- equation

$$\dot{\sigma} = (1-\omega) L : \dot{\varepsilon} - H \bar{\sigma} \otimes \bar{\sigma} : \dot{\varepsilon}$$

$$\dot{\sigma} = \underbrace{[(1-\omega) L - H \bar{\sigma} \otimes \bar{\sigma}] : \dot{\varepsilon}}_{\mathcal{L} : \dot{\varepsilon}}$$

Da

Hill's Lema

$$\int_{\Omega} \frac{1}{|\theta|} \int_{\theta} \left[\underline{\delta^{o}\varepsilon} + \delta\varepsilon^{*} \right] : \tau \, d\theta \, d\Omega = \int_{\Gamma_c} t \cdot [\delta^{o}u] \, d\Gamma$$

$$Pa \cdot m \cdot m^2 = Pa\, m^3$$

$$Pa\, m^3$$

$$\frac{N}{m^2} m^3 = \underline{\underline{Nm}}$$

$$l_c \int_{\Gamma_c} \frac{1}{|\theta|} \int_{\theta} \delta^{o}\varepsilon : \tau \, d\theta \, d\Gamma = \int_{\Gamma_c} t \cdot [\delta^{o}u] \, d\Gamma$$

$$\frac{1}{l_c} N \otimes [\delta^{o}u]$$

$$l_c \frac{1}{|\theta|} \int_{\theta} \delta^{o}\varepsilon \cdot \tau \, d\theta = t \cdot [\delta^{o}u]$$

$$m\, Pa = Pa\, m$$

$$[\delta^{o}u] \cdot \left[\frac{1}{|\theta|} \int \tau \, d\theta \right] \cdot N = [\delta^{o}u] \cdot t$$

m · Pa

Hill's Lema

(E)

$$\ell_c \frac{1}{A} \int_\Omega \left[\frac{1}{\ell_c} [\![\delta u_0]\!] + \delta \overset{*}{\varepsilon} \right] \cdot \left[\underbrace{(1-\omega) L(x) : \left(\frac{1}{\ell_c} [\![u_0]\!] + \overset{*}{\varepsilon} \right)}_{\sigma} \right] d\Omega = \underset{m \cdot Pa}{[\![\delta u_0]\!] \cdot \bar{t}}$$

$$\frac{1}{A} \int_\Omega \left\{ (1-\omega) [\![\delta u_0]\!] : L(x) : \left(\frac{1}{\ell_c} [\![u_0]\!] + \overset{*}{\varepsilon} \right) \right\} d\Omega + \frac{1}{A} \int_\Omega \left\{ \frac{(1-\omega)}{\ell_c} \delta \overset{*}{\varepsilon} : L(x) : [\![u_0]\!] + \delta \overset{*}{\varepsilon} \underbrace{(1-\omega) L(x)}_{\text{secant stiffness}} \overset{*}{\varepsilon} \right\} d\Omega$$

δu_0 and $\delta \overset{*}{\varepsilon}$ are independent

(1) $$\frac{1}{A} \frac{1}{\ell_c} [\![\delta u_0]\!] : \left\{ \int_\Omega (1-\omega(x)) L(x) \, d\Omega : [\![u_0]\!] + \ell_c \int_\Omega (1-\omega) L(x) : \overset{*}{\varepsilon} \, d\Omega \right\} = [\![\delta u_0]\!] \cdot \bar{t}$$

$$\Rightarrow \bar{t} = \frac{1}{\ell_c} \frac{1}{A} \int_\Omega (1-\omega) L \, d\Omega : [\![u_0]\!] + \frac{1}{A} \int_\Omega (1-\omega) L : \overset{*}{\varepsilon} \, d\Omega$$

(2) $$\frac{1}{\ell_c} \frac{1}{A} \int_\Omega \delta \overset{*}{\varepsilon} : (1-\omega) L \, d\Omega : [\![u_0]\!] + \frac{1}{A} \int_\Omega \delta \overset{*}{\varepsilon} : (1-\omega) L \overset{*}{\varepsilon} \, d\Omega = \emptyset$$

Ⓕ

From ① for $L(x) = L$ and $\omega = \phi \Rightarrow$ no damage

$$\frac{1}{l_c} [\![\delta u_0]\!] : \left\{ L : [\![u_0]\!] + l_c L \underbrace{\frac{1}{A} \int_\Omega \overset{*}{\varepsilon} \, d\Omega}_{= \phi} \right\} = [\![\delta u_0]\!] \cdot \bar{t}$$

$$\Rightarrow \quad \bar{t} = \frac{1}{l_c} L : [\![u_0]\!] \rightarrow \text{initial cohesive tractions}$$

$\bar{t} = \frac{1}{l_c} L : [\![u_0]\!]$

limit case $l_c \to \phi$ $\quad \bar{t} \to \infty$

$\bar{t} = \sigma_0 \nearrow$ $\bar{t}_2 = \sigma_{22}$ $\quad \bar{t}_1 = \phi$

$\bar{t}_{12} = \sigma_{12}$

FE implementation

$\overset{*}{\varepsilon} = B \overset{*}{u}$ (B: grad operator)

First solve ②

$$\frac{1}{A} \int_\Omega B^T (1-\omega) L B \overset{*}{u} \, d\Omega = - \frac{1}{l_c} \frac{1}{A} \int_\Omega B^T (1-\omega) L \, d\Omega \, [\![u_0]\!]$$

From ① evaulate tractions (macroscopic tractions)

$$\bar{t} = \frac{1}{l_c} \cdot \frac{1}{A} \int_\Omega (1-\omega) L \, d\Omega : [\![u_0]\!] + \frac{1}{A} \int_\Omega (1-\omega) L : (B \overset{*}{u}) \, d\Omega$$