1. Jackson Prob. 5.1: Reformulate the Biot-Savart law in terms of the solid angle subtended at the point of observation by the current-carrying circuit.

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0 I}{2\pi}\oint \frac{d\boldsymbol{l}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\\ &= -\frac{\mu_0 I}{2\pi}\oint d\boldsymbol{l}'\times\boldsymbol{\nabla}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\\ &= \frac{\mu_0 I}{2\pi}\boldsymbol{\nabla}\times\oint\frac{d\boldsymbol{l}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\end{aligned}$$

Let

$$\boldsymbol{V} = \oint \frac{d\boldsymbol{l}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

then

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{2\pi}\boldsymbol{\nabla}\times\boldsymbol{V}$$

The $i$th component of $\boldsymbol{V}$ may be written

$$V_i = \oint \frac{d\boldsymbol{l}'\cdot\hat{\boldsymbol{\imath}}}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

where $\hat{\boldsymbol{\imath}}$ is the unit vector along the $i$th axis. By virtue of Stoke's theorem this can be converted into a surface integral

$$V_i = \int_S da\left[\boldsymbol{\nabla}'\times\frac{\hat{\boldsymbol{\imath}}}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\cdot\boldsymbol{n}' = \int_S da\left[\boldsymbol{n}'\times\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right]\cdot\hat{\boldsymbol{\imath}}$$

where $S$ is a surface bounded by the circuit and where the direction of the surface normal $\boldsymbol{n}'$ is related to the sense of the current $(\boldsymbol{l}')$ by the right-hand rule. The above equation can be rewritten as

$$\boldsymbol{V} = \boldsymbol{\nabla}\times\int_S da\frac{\boldsymbol{n}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

Therefore

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi}\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times\boldsymbol{W}]$$

with

$$\boldsymbol{W} = \int_S da\frac{\boldsymbol{n}'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

Now

$$\begin{aligned}\boldsymbol{\nabla}\times[\boldsymbol{\nabla}\times\boldsymbol{W}] &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{W}) - \nabla^2\boldsymbol{W}\\ &= -\boldsymbol{\nabla}\int_S da\,\frac{\boldsymbol{n}'\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + 4\pi\int_S da\,\boldsymbol{n}'\delta(\boldsymbol{r}-\boldsymbol{r}')\\ &= -\boldsymbol{\nabla}\Omega + 0.\end{aligned}$$

The second integral vanishes since $\boldsymbol{r}'$ is on a surface bounding the circuit, which is away from the observation point $\boldsymbol{r}$. The first integral is, as shown on page 33 in Chap. 1 of the text, the solid angle $\Omega$ subtended at the observation point by the circuit that bounds $S$. Therefore,

$$\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_0 I}{4\pi}\boldsymbol{\nabla}\Omega$$

**Example:** Consider a point of observation on the $z$ axis above a circular loop of radius $a$ in the $xy$ plane that carries current I. The loop subtends a solid angle $\Omega = 2\pi(1-\cos\theta)$, where $\theta$ is the angle between the $z$ axis and a line from the point of observation to the loop. Thus

$$\Omega = 2\pi\left[1 - \frac{z}{\sqrt{a^2+z^2}}\right]$$

and

$$\boldsymbol{\nabla}\Omega = 2\pi\left[-\frac{1}{\sqrt{a^2+z^2}} + \frac{z^2}{(a^2+z^2)^{3/2}}\right]\hat{\boldsymbol{z}} = -\frac{2\pi a^2}{(a^2+z^2)^{3/2}}\hat{\boldsymbol{z}}.$$

Therefore

$$\boldsymbol{B}(z) = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}\hat{\boldsymbol{z}},$$

confirming a result obtained in class directly from the Biot-Savart law.

2. Jackson Prob 5.3: Find $B_z$ inside a uniformly wound solenoid. Use result from Prob. 5.1 to write the contribution from a segment of the solenoid of length $dz$ as

$$dB_z = -\frac{\mu_0 NIdz}{4\pi}\frac{d\Omega}{dz} = -\frac{\mu_o NI}{4\pi}d\Omega.$$

where $\Omega = 2\pi(1-\cos\theta)$ where $\theta$ is the angle that the line from the observation point to the ring at $z$ makes with the axis. Integrate from end 1 to end 2 to find

$$B_z = \frac{\mu_o NI}{4\pi}\left[\Omega_1 - \Omega_2\right] = \frac{\mu_0 NI}{2}\left[\cos\theta_2 - \cos\theta_1\right].$$

In terms of the angles shown in the figure in the text this becomes

$$B_z = \frac{\mu_0 NI}{2}\left[\cos\theta_1 + \cos\theta_2\right]$$

3. Jackson Prob 5.7:

   (a) As shown in class and in example with Prob. 5.1, the field of a single loop in the $xy$ plane at a distance $z$ from its center on the axis is

$$B_z = \frac{\mu_0 I}{2}\frac{a^2}{[a^2+z^2]^{3/2}}$$

(b) The field near the center of a Helmholtz pair is, therefore,

$$\begin{aligned} B_z &= \frac{\mu_0 I}{2}\left[\frac{a^2}{[a^2+(z-b/2)^2]^{3/2}} + \frac{a^2}{[a^2+(z-b/2)^2]^{3/2}}\right] \\ &= \frac{\mu_0 I a^2}{d^3}\left[1 + \frac{3\left(b^2-a^2\right)z^2}{2\,d^4} + \frac{15\left(2a^4-6b^2a^2+b^4\right)z^4}{16\,d^8}\right. \\ &\qquad \left. - \frac{7\left(5a^6-30b^2a^4+15b^4a^2-b^6\right)z^6}{16\,d^{12}} + \cdots\right], \end{aligned}$$

where $d = \sqrt{b^2+4a^2}$.

(c) $\rho$ dependence of of field. On the axis, we may write $B_z = \sigma_0 + \sigma_2 z^2 + \cdots$, where the coefficients $\sigma_k$ can be inferred from the above equation. With the aid of $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ we find that near the origin,

$$\frac{\partial(\rho B_\rho)}{\partial\rho} = -\rho\frac{\partial B_z}{\partial z} = -2\sigma_2\rho\, z + \cdots$$

Solving for $B_\rho$ (taking into account that $B_\rho = 0$ on axis) we find that near the axis,

$$B_\rho(z,\rho) \approx -\sigma_2\rho\, z$$

From $\boldsymbol{\nabla}\times\boldsymbol{B} = 0$, we may write

$$\frac{\partial B_z}{\partial\rho} = \frac{\partial B_\rho}{\partial z} \approx -\sigma_2\rho$$

Solving for $B_z$, we obtain

$$B_z(z,\rho) \approx B_z(z,0) - \sigma_2\frac{\rho^2}{2} = \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right) + \cdots$$

(d) From MATHEMATICA the asymptotic series for $B_z$ is

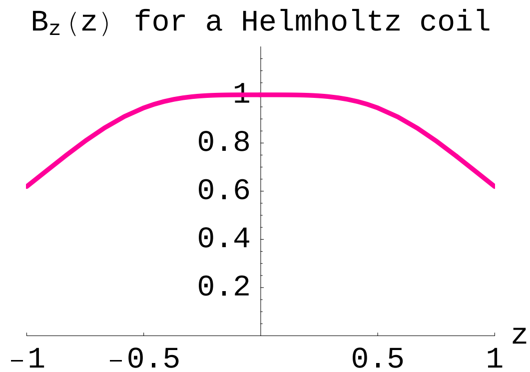
$$\begin{aligned} B_z &== \frac{\mu_0 I a^2}{|z|^3}\left[1 + \frac{3\left(b^2-a^2\right)}{2\,z^2} + \frac{15\left(2a^4-6b^2a^2+b^4\right)}{16\,z^4}\right. \\ &\qquad \left. - \frac{7\left(5a^6-30b^2a^4+15b^4a^2-b^6\right)}{16\,z^6} + \cdots\right], \end{aligned}$$

which can be obtained from the power series by the replacement $d \to |z|$.

(e) For a Helmholtz coil one sets $b = a$. With this choice the terms in the bracket for the small $z$ expansion become

$$[\cdots] \approx 1 - \frac{144 z^4}{125 a^4}$$

Thus, to differ from uniformity by $\leq \epsilon$, the fractional distance must satisfy $z/a \leq [(125/144)\,\epsilon]^{1/4}$. For $\epsilon = 10^{-4}$ the limit is 0.097 and for $\epsilon = 10^{-2}$ the limit is 0.305. Below is a figure showing the variation of $B_z$ on the axis between two coils located at $a = \pm 1$.




4. Jackson Prob 5.13: Find the vector potential and magnetic induction for a uniformly charged sphere of radius $a$ rotating about an axis with angular momentum $\boldsymbol{\omega}$. We orient $\boldsymbol{\omega}$ along the $z$ axis and let $\boldsymbol{r}$ lie in the $xz$ plane. The vector $\boldsymbol{r}'$ is used to locate a point on the sphere. The surface current density at $\boldsymbol{r}'$ is $\boldsymbol{K}(\boldsymbol{r}') = [\boldsymbol{\omega} \times \boldsymbol{r}']\,\sigma$. The corresponding vector potential is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')d^3r'}{|\boldsymbol{r}-\boldsymbol{r}'|} \rightarrow \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{K}(\boldsymbol{r}')da'}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{\mu_0\sigma}{4\pi}\int \frac{[\boldsymbol{\omega}\times\boldsymbol{r}']\,da'}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

We write

$$[\boldsymbol{\omega}\times\boldsymbol{r}'] = a\omega\sin\theta'(-\sin\phi'\,\hat{\boldsymbol{x}} + \cos\phi'\,\hat{\boldsymbol{y}})$$

As in the example worked out in Sec. 5.5 of the text, only the $y$ component can contribute to the integral. Therefore,

$$A_y(\boldsymbol{r}) = \frac{\mu_0\sigma a^3\omega}{4\pi}\int \frac{\sin\theta'\cos\phi' d\Omega'}{\sqrt{r^2+a^2-2ar\cos\gamma}},$$

where $\cos\gamma$ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. Expanding the denominator in a series of spherical harmonics we obtain

$$A_y(\boldsymbol{r}) = \frac{\mu_0\sigma a^3\omega}{4\pi}\sum_{lm}\frac{r_<^l}{r_>^{l+1}}\frac{4\pi}{2l+1}Y_{lm}^*(\hat{r})\int d\Omega'\sin\theta'\cos\phi' Y_{lm}(\hat{r}')$$

We first carry out the $\phi'$ integral to find

$$\int_0^{2\pi} d\phi'\cos\phi'\,Y_{lm}(\theta',\phi') = \pi\sqrt{\frac{(2l+1)(l-1)!}{4\pi(l+1)!}}\,P_l^1(\cos\theta')\,(\delta_{m1}-\delta_{m-1}).$$

Noting that $\sin\theta' = -P_1^1(\cos\theta')$, we find

$$\int_{-1}^{1}\sin\theta' P_l^1(\mu')d\mu' = -\frac{4}{3}\delta_{l1}$$

Putting the previous two results together, we find

$$\int d\Omega' \sin\theta' \cos\phi' Y_{lm}(\hat{r}') = -\frac{4\pi}{3}\sqrt{\frac{3}{8\pi}}\,(\delta_{m1} - \delta_{m-1})\delta_{l1}$$

The sum over $lm$ above becomes

$$\sum_{lm} \frac{r_<^l}{r_>^{l+1}} \cdots = -\frac{16\pi^2}{9}\sqrt{\frac{3}{8\pi}}\,(Y_{11}^*(\hat{r}) - Y_{1-1}^*(\hat{r}))\,\frac{r_<}{r_>^2} = \frac{4\pi}{3}\sin\theta\frac{r_<}{r_>^2}$$

Finally,

$$\begin{aligned} A_\phi(\boldsymbol{r}) &= \frac{\mu_0\sigma a^4\omega}{3}\sin\theta\frac{1}{r^2} \qquad && r > a \\ &= \frac{\mu_0\sigma a\,\omega}{3} r\sin\theta && r < a \end{aligned}$$

In vector form, this becomes

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}) &= \frac{\mu_0\sigma a^4}{3}\frac{[\boldsymbol{\omega}\times\boldsymbol{r}]}{r^3} \qquad && r > a \\ &= \frac{\mu_0\sigma a}{3}\,[\boldsymbol{\omega}\times\boldsymbol{r}] && r < a \end{aligned}$$

The corresponding formulas for the magnetic induction $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ are

$$\begin{aligned} \boldsymbol{B}(\boldsymbol{r}) &= \frac{\mu_0\sigma a^4}{3}\left(\frac{3(\boldsymbol{\omega}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{\omega}}{r^3}\right) \qquad && r > a \\ &= \frac{2\mu_0\sigma a}{3}\,\boldsymbol{\omega} && r < a \end{aligned}$$

Note: A somewhat different (and simpler) solution to this problem is found in the text by Griffiths. He chooses coordinates with $r$ be along the $z$ axis and $\boldsymbol{\omega}$ in the $xz$ plane at an angle $\theta$ with the $z$ axis.