1. Jackson Prob. 4.1: Multipole expansion for various charge distributions

   (a) In the first case, we have 4 charges in the xy plane at distance a from the origin along the $\pm x$ and $\pm y$ axes.

$$\begin{aligned}\Phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\boldsymbol{r}-a\hat{\boldsymbol{x}}|}-\frac{1}{|\boldsymbol{r}+a\hat{\boldsymbol{x}}|}+\frac{1}{|\boldsymbol{r}-a\hat{\boldsymbol{y}}|}-\frac{1}{|\boldsymbol{r}+a\hat{\boldsymbol{y}}|}\right]\\ &= \frac{q}{4\pi\epsilon_0}\sum_l \frac{a^l}{r^{l+1}}\left[P_l(\hat{r}\cdot\hat{x})-P_l(-\hat{r}\cdot\hat{x})+P_l(\hat{r}\cdot\hat{y})-P_l(-\hat{r}\cdot\hat{y})\right]\\ &= \frac{q}{4\pi\epsilon_0}\sum_{l=1,3,\cdots}\frac{2a^l}{r^{l+1}}\left[P_l(\hat{r}\cdot\hat{x})+P_l(\hat{r}\cdot\hat{y})\right]\\ &= \frac{1}{\epsilon_0}\sum_{l=1,3,\cdots}\sum_m \frac{2qa^l}{r^{l+1}}\frac{1}{2l+1}Y_{lm}(\hat{r})\left[Y^*_{lm}(\hat{x})+Y^*_{lm}(\hat{y})\right]\end{aligned}$$

   Compare the last formula with the definition of q_{lm} to find

$$q_{lm} = 2qa^l\left[Y^*_{lm}(\hat{x})+Y^*_{lm}(\hat{y})\right]$$

   The polar angles of $\hat{x}$ and $\hat{y}$ are: $\theta_x=\theta_y=\pi/2$, $\phi_x=0$, and $\phi_y=\pi/2$. Therefore,

$$\begin{aligned}&q_{11} = -(1-i)\sqrt{\tfrac{3}{2\pi}}\,qa\\ &q_{33} = -\tfrac{1}{4}(1+i)\sqrt{\tfrac{35}{\pi}}\,qa^3 \qquad q_{31} = \tfrac{1}{4}(1-i)\sqrt{\tfrac{21}{\pi}}\,qa^3\\ &q_{55} = -\tfrac{3}{16}(1-i)\sqrt{\tfrac{77}{\pi}}\,qa^5 \quad q_{53} = \tfrac{1}{16}(1+i)\sqrt{\tfrac{335}{\pi}}\,qa^5 \quad q_{51} = -\tfrac{1}{8}(1-i)\sqrt{\tfrac{165}{2\pi}}\,qa^5\end{aligned}$$

   All terms q_{lm} with l or m odd vanish. Furthermore $q_{l-m}=(-1)^m q^*_{lm}$.
   Note: From text, one has

$$q_{11} = -\sqrt{\frac{3}{8\pi}}(p_x-ip_y) \qquad q_{10} = \sqrt{\frac{3}{4\pi}}p_z$$

   It follows that $\boldsymbol{p}=(2qa,2qa,0)$.

   (b) In the second case we have charges q at $z=\pm a$ balanced by a charge

$-2q$ at the origin. We find

$$\begin{aligned}
\Phi(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\boldsymbol{r}-a\hat{\boldsymbol{z}}|}+\frac{1}{|\boldsymbol{r}+a\hat{\boldsymbol{z}}|}-\frac{2}{r}\right] \\
&= \frac{q}{4\pi\epsilon_0}\sum_{l>0}\frac{a^l}{r^{l+1}}\left[P_l(\hat{r}\cdot\hat{z})+P_l(-\hat{r}\cdot\hat{x})\right] \\
&= \frac{q}{4\pi\epsilon_0}\sum_{l=2,4,\cdots}\frac{2a^l}{r^{l+1}}P_l(\hat{r}\cdot\hat{z}) \\
&= \frac{1}{\epsilon_0}\sum_{l=2,4,\cdots}\sum_{m}\frac{2qa^l}{r^{l+1}}\frac{1}{2l+1}Y_{lm}(\hat{r})Y^*_{lm}(\hat{z})
\end{aligned}$$

Again compare with definition to find

$$q_{lm} = 2qa^l Y^*_{lm}(\hat{z})$$

As is well known

$$Y^*_{lm}(\hat{z}) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(1) = \sqrt{\frac{2l+1}{4\pi}}$$

Therefore, for $l = 2,\ 4,\ \cdots$

$$q_{l0} = \sqrt{\frac{2l+1}{\pi}}\,qa^l, \qquad q_{lm} = 0 \quad \text{for } m \neq 0.$$

The rectangular components of the quadrupole tensor are found by comparing with the formulas in text

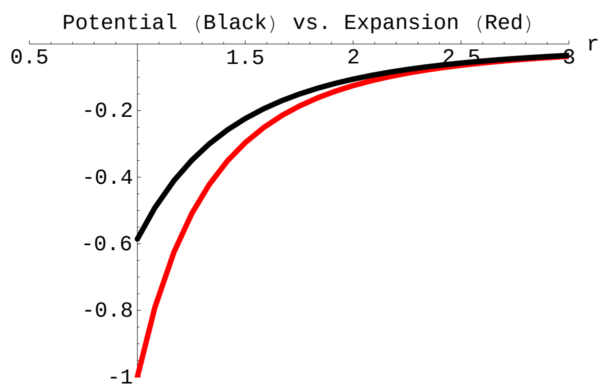
$$Q = \begin{pmatrix} 4qa^2 & 0 & 0 \\ 0 & -2qa^2 & 0 \\ 0 & 0 & -2qa^2 \end{pmatrix}$$

(c) Plot the dominant contribution for second case as a function of r in the $x-y$ plane.

$$\Phi(r,\mu) = \frac{1}{\epsilon_0}\frac{q_{20}}{5r^3}\sqrt{\frac{5}{4\pi}}P_2(\mu) = \frac{1}{4\pi\epsilon_0}\frac{2qa^2}{r^3}P_2(\mu)$$

The plot is shown below along with the plot required for the next item.

(d) Compare the plot required above with a plot of the exact potential. The two plots are shown together below. The distance a is taken to be 1 in this case, and we plot $4\pi\epsilon_0\Phi(r)$. It can be seen that the quadrupole potential substantially overestimates the size of the potential (by 40% in this case) at $r = a$ but comes into close agreement as r increases.



2. Jackson Prob. 4.6: Nucleus in a cylindrically symmetric field.

   (a) Show that

$$W = -\frac{1}{4}\, Q\ \left.\frac{\partial E_z}{\partial z}\right|_0$$

   In the principal axis system of a spheroidal nucleus, $Q_{xx} = Q_{yy} = -Q_{zz}/2$. It follows that

$$W = -\frac{1}{6}\, Q_{zz} \left[\frac{\partial E_z}{\partial z} - \frac{1}{2}\frac{\partial E_x}{\partial x} - \frac{1}{2}\frac{\partial E_y}{\partial y}\right]$$

   With the aid of

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\partial E_z}{\partial z} + \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

   One finds

$$W = -\frac{1}{6}\, Q_{zz} \frac{3}{2}\frac{\partial E_z}{\partial z} \equiv -\frac{1}{4}\, Q\ \left.\frac{\partial E_z}{\partial z}\right|_0$$

   (b) Given that $W = 10$ MHz and $Q = 10^{-28}$ m^2, find the value of

$$\frac{\partial E_z}{\partial z} = -\frac{4eW}{Q}$$

   in units $e/(4\pi\epsilon_0 a_0^3)$.
   We obtain the following

$$\begin{aligned} -\frac{4eW}{Q} &= -8.27133 \times 10^{20}\ \text{MKS} \\ \frac{e}{4\pi\epsilon_0\, a_0^3} &= 9.71758 \times 10^{21}\ \text{MKS} \\ \frac{\partial E_z}{\partial z} &= -0.085117\ \frac{e}{4\pi\epsilon_0\, a_0^3} \end{aligned}$$

(c) Quadrupole moment of a uniformly charged spheroid with semimajor axis a and semiminor axis b:

$$Q = Q_{33} = 4\pi\rho_q \int_0^a dz \int_0^{b\sqrt{1-z^2/a^2}} \rho d\rho (2z^2 - \rho^2) = \frac{8\pi a b^2}{15}(a^2 - b^2)\rho_q,$$

where

$$\rho_q = q\frac{3}{4\pi a b^2}$$

is the charge density. Therefore, in terms of the total charge $q = Ze$,

$$Q = \frac{2}{5}(a^2 - b^2)\, Ze = \frac{4}{5}(a+b)RZe,$$

where $R = (a+b)/2$. It follows that

$$\frac{(a-b)}{R} = \frac{5}{4}\frac{Q'}{R^2 Z} = \frac{1.25 \times 2.5 \times 10^{-28}}{(7\times 10^{-15})^2 63} = 0.1012$$

where $Q' = Q/e$.

3. Jackson Prob. 4.8: A cylindrical shell (outer radius b - inner radius a) is filled with a material with dielectric constant ϵ and placed in an electric field normal to it's axis.

(a) Find the potential and electric field. We expand the potential in a series In the outer region, $r > b$, the potential takes the form

$$\Phi(\rho, \phi) = \sum_n \left[a_n \rho^n + \frac{b_n}{\rho^n}\right] \cos n\phi,$$

where $a_1 = -E0$ and $a_n = 0$, for $n \neq 0$. (Also, $b_0 = 0$.) As in the case of a dielectric sphere in an external field, only terms in the expansion with $n = 1$ will be nonvanishing once the boundary conditions are applied. We therefore assume that the potential takes the form

$$\begin{aligned}
\Phi(\rho, \phi) = &\left[-E_0\rho + \frac{c_1}{\rho}\right]\cos\phi \qquad && b \le \rho \\
= &\left[c_2\rho + \frac{c_3}{\rho}\right]\cos\phi && a \le \rho \le b \\
= &\, c_4\, \rho\cos\phi && 0 \le \rho \le a
\end{aligned}$$

The four equations $\Delta\Phi = 0$ at $\rho = a, b$ and $\Delta D = 0$ at $\rho = a, b$ lead to the following results for the expansion coefficients:

$$
\begin{aligned}
c_1 &= \frac{b^2\left(a^2-b^2\right)\left(\epsilon^2-1\right)}{a^2(\epsilon-1)^2-b^2(\epsilon+1)^2}\,E_0 \\
c_2 &= -\frac{2b^2(\epsilon+1)}{b^2(\epsilon+1)^2-a^2(\epsilon-1)^2}\,E_0 \\
c_3 &= \frac{2a^2b^2(\epsilon-1)}{a^2(\epsilon-1)^2-b^2(\epsilon+1)^2}\,E_0 \\
c_4 &= \frac{4b^2\epsilon}{a^2(\epsilon-1)^2-b^2(\epsilon+1)^2}\,E_0
\end{aligned}
$$

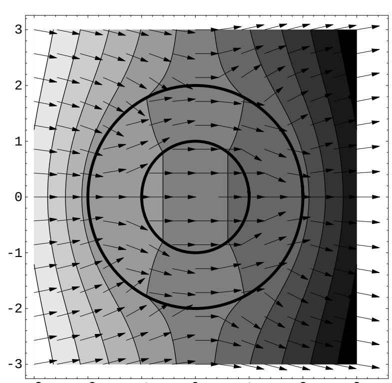
The electric field is as usual $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi$. For the radial component, we have

$$
\begin{aligned}
E_\rho(\rho,\phi) &= \left[E_0 + \frac{c_1}{\rho^2}\right]\cos\phi & b \le \rho \\
&= \left[-c_2 + \frac{c_3}{\rho^2}\right]\cos\phi & a \le \rho \le b \\
&= -\,c_4\,\cos\phi & 0 \le \rho \le a
\end{aligned}
$$

For the angular component, we have

$$
\begin{aligned}
E_\phi(\rho,\phi) &= \left[-E_0 + \frac{c_1}{\rho^2}\right]\sin\phi & b \le \rho \\
&= \left[c_2 + \frac{c_3}{\rho^2}\right]\sin\phi & a \le \rho \le b \\
&= c_4\,\sin\phi & 0 \le \rho \le a
\end{aligned}
$$

(b) Sketch the Field: Here is the case $\epsilon = 10,\ a = 1,\ b = 2,\ E_0 = 1$



(c) Limiting case $a \to 0$:

$$\Phi(\rho,\phi) = \left[-\rho + \frac{\epsilon-1}{\epsilon+1}\frac{b^2}{\rho}\right] E_0 \cos\phi \qquad b \le \rho$$
$$= -\frac{2}{\epsilon+1}\rho\, E_0\, \cos\phi \qquad 0 \le \rho \le b$$

For the case of a hollow cylinder imbedded in a dielectric the potential is given by the above expression with $\epsilon \to 1/\epsilon$.

4. Jackson Prob. 4.13: The energy of a dielectric material in an external electric field is given by Eq. (4.93) in the text:

$$W = -\frac{1}{2}\int \boldsymbol{P}\cdot\boldsymbol{E} d\tau.$$

For the liguid in the capillary tube,

$$\boldsymbol{P} = \epsilon_0 \chi_e \boldsymbol{E}$$

Therefore

$$W = -\frac{\epsilon_0\chi_e}{2}\int E^2 d\tau = -\frac{\epsilon_0\chi_e}{2}\left(\frac{V}{\ln(b/a)}\right)^2 2\pi x \int_a^b \rho d\rho \frac{d\rho}{\rho^2},$$

where we have used the easily established fact that the electric field in the capillary is

$$E_\rho(\rho) = \frac{V}{\ln(b/a)}\frac{1}{\rho}.$$

It follows that

$$W = -\frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}\, x$$

and that the (upward) force on the liquid is

$$F = -\frac{dW}{dx} = \frac{\pi\epsilon_0\chi_e V^2}{\ln(b/a)}$$

This force balances the downward weight of column of liquid $\rho g\, h\pi(b^2 - a^2)$. Therefore,

$$\chi_e = \frac{\rho g h(b^2-a^2)\ln(b/a)}{\epsilon_0 V^2}$$

Here is an alternative solution mentioned in class: Let the capacitance/length of the capillary tube be

$$C_0 = \frac{2\pi\epsilon_0}{\ln(b/a)}$$

and the capacitance/length of the tube filled with liquid be

$$C = \frac{2\pi\epsilon}{\ln(b/a)}$$

If the tube is filled to height x with liquid, an excess charge

$$\Delta Q = (C - C_0)xV = \frac{2\pi\epsilon_0\chi_e xV}{\ln(b/a)}$$

will be drawn from the battery and appear on the surface of the capillary. The battery gives up energy

$$W_B = V\Delta Q = (C - C_0)xV^2$$

Part of this energy is the increased energy stored in the capacitor part is avalible to do work. Assuming that the capillary is filled to height h, the available energy is

$$W_a(x) = \frac{1}{2}(C - C_0)xV^2 - W_B = -\frac{1}{2}\frac{2\pi\epsilon_0\chi_e xV^2}{\ln(b/a)}$$

The force on the (liquid) dielectric is

$$F = -\frac{dW_a}{dx} = \frac{\pi\epsilon_0\chi_e xV^2}{\ln(b/a)},$$

which agrees with the previously obtained result.