1. Jackson Prob. 3.12. A circular hole of radius a in a conducting plane is held at potential V.

   (a) Find an integral expression for the potential above the plane.a For a cylindrically symmetric case, the solution to the BV problem takes the form

$$\Phi(\rho, z) = \int_0^\infty A(k) J_0(k\rho) e^{-kz} dk$$

With the aid of the orthogonality relation:

$$\int_0^\infty J_0(k\rho) J_0(k'\rho) \rho d\rho = \frac{1}{k} \delta(k - k')$$

we find

$$A(k) = k \int_0^\infty J_0(k\rho) \Phi(\rho, 0) \rho d\rho = kV \int_0^a J_0(k\rho) \rho d\rho = aV J_1(ka),$$

where we have used the fact

$$\int_0^a J_0(k\rho) \rho d\rho = \frac{a}{k} J_1(ka).$$

It follows that

$$\Phi(\rho, z) = Va \int_0^\infty J_1(ka) J_0(k\rho) e^{-kz} dk.$$

   (b) Find the potential at a distance z above the center of the hole. From Mathematica 5.2

$$\Phi(0, z) = V \int_0^\infty J_1(ka) e^{-kz} dk = V \left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right]$$

   (c) Find the potential at a distance z above the edge of the hole. From Mathematica 5.2, we find

$$\Phi(a, z) = \frac{V}{2} \left[1 - \frac{2}{\pi} \text{EllipticK}\left(-\frac{4a^2}{z^2}\right)\right]$$

This result can be transformed into the result given in the text: Note, Mathematica's EllipticK$(k^2) \equiv K(k)$. The above result in standard notation is

$$\Phi(a, z) = \frac{V}{2} \left[1 - \frac{2}{\pi} K\left(i\frac{2a}{z}\right)\right]$$

From `mathworld.wolfram.com`, we have the identity

$$K(ik) = \frac{1}{\sqrt{1+k^2}} K\left(\sqrt{\frac{k^2}{1+k^2}}\right).$$

Substituting, we find

$$\Phi(a, z) = \frac{V}{2}\left[1 - \frac{k' z}{\pi a} K(k')\right]$$

where $k' = 2a/\sqrt{4a^2 + z^2}$, which is the result given in the text.

2. Jackson Prob. 3.13: Solve Prob. 3.1 using the Green Function method. Find the potential in the region between two concentric spheres. The upper half of the inner sphere (radius a) and the lower half of the outer sphere (radius b) are at potential V. The lower half of the inner and upper half of outer sphere are at potential 0. Owing to azimuthal symmetry, we can write

$$\Phi(r, \mu) = \sum_{l=0}^{\infty} P_l(\mu)\left[\frac{V}{4\pi}\int_{S_a} P_l(\mu')\, a^2\, d\Omega' \left.\frac{\partial g_l(r, r')}{\partial r'}\right|_{r'=a} - \frac{V}{4\pi}\int_{S_b} P_l(\mu')\, b^2\, d\Omega' \left.\frac{\partial g_l(r, r')}{\partial r'}\right|_{r'=b}\right]$$

Where the integrations are over the S_a, the upper half of the inner sphere, and S_b, the lower half of the outer sphere. The integral over ϕ' gives a factor of 2π in each term. As in Prob. 3.1, we write the integral over μ' in terms of the parameters A_l and B_l defined as

$$A_l = \frac{2l+1}{2} V \int_0^1 P_l(\mu) d\mu$$
$$B_l = \frac{2l+1}{2} V \int_{-1}^0 P_l(\mu) d\mu,$$

and find

$$A_0 = B_0 = \frac{1}{2}V$$
$$A_l = -B_l = V\left\{\frac{3}{4},\, 0 - \frac{7}{16},\, 0,\, \frac{11}{32},\, \cdots\right\} \quad \text{for} \quad l = 1, 2, 3, 4, 5, \cdots$$

Note that

$$\left.\frac{\partial g_l(r, r')}{\partial r'}\right|_{r'=a} = \frac{2l+1}{D_l}\, a^{l-1}\left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}}\right)$$

and

$$\left.\frac{\partial g_l(r, r')}{\partial r'}\right|_{r'=b} = -\frac{(2l+1)}{D_l}\,\frac{1}{b^{l+2}}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)$$

Here

$$D_l = 1 - (a/b)^{2l+1}.$$

Substituting into the formula for Φ, we find

$$\Phi(r,\mu) = \sum_l P_l(\mu) \frac{1}{D_l} \left[A_l\, a^{l+1} \left(\frac{1}{r^{l+1}} - \frac{r^l}{b^{2l+1}} \right) + B_l \frac{1}{b^l} \left(r^l - \frac{a^{2l+1}}{r^{l+1}} \right) \right]$$

This can be rewritten

$$\begin{aligned} &\Phi(r,\mu) \\ &\quad = \frac{V}{2} + \sum_{l=1,3,\cdots} P_l(\mu)\, \frac{A_l}{D'_l} \left[\left(\frac{b^l}{r^{l+1}} - \frac{r^l}{b^{l+1}} \right) - \left(\frac{r^l}{a^{l+1}} - \frac{a^l}{r^{l+1}} \right) \right] \\ &\quad = \frac{V}{2} + \sum_{l=1,3,\cdots} P_l(\mu)\, \frac{A_l}{D'_l} \left[\left(a^l + b^l \right) \frac{1}{r^{l+1}} - \left(\frac{1}{a^{l+1}} + \frac{1}{b^{l+1}} \right) r^l \right] \end{aligned}$$

where

$$D'_l = \left(\frac{b^l}{a^{l+1}} - \frac{a^l}{b^{l+1}} \right)$$

is the denominator factor defined in the previous solution. This result is seen to agree with the previous solution to Prob. 3.1.

3. Jackson Prob. 3.14. Given that the linear charge density is proportional to $d^2 - r^2$, we may write

$$\rho(\boldsymbol{r}) = \kappa \frac{1 - r^2/d^2}{r^2} \left[\delta(\mu - 1) + \delta(\mu + 1) \right] \qquad r < d$$

Integrate to find

$$Q = \int d^3r\ \rho(r\boldsymbol{r}) = \frac{8\pi}{3}\ \kappa$$

(a) Find the potential:
With the aid of the above result and azimuthal symmetry, we may write

$$\Phi(r,\mu) = \frac{1}{4\pi\epsilon_0} \frac{3Q}{8\pi d} \sum_k P_k(\mu) \int_0^{2\pi} d\phi'$$

$$\int_{-1}^{1} d\mu' \left[\delta(\mu' - 1) + \delta(\mu' + 1) \right] P_k(\mu') \int_0^b dr' \lambda(r', d) \left(\frac{r_<^k}{r_>^{k+1}} - \frac{r^k r'^k}{b^{2k+1}} \right)$$

where

$$\lambda(r,d) = \begin{cases} (1 - r^2/d^2) & r \le d \\ 0 & r > d \end{cases}$$

Carrying out the angular integrations leads to

$$\Phi(r,\mu) = \frac{3Q}{8\pi\epsilon_0\, d} \sum_{k=0,2,\cdots} P_k(\mu) \int_0^b dr' \lambda(r', d) \left(\frac{r_<^k}{r_>^{k+1}} - \frac{r^k r'^k}{b^{2k+1}} \right)$$

Set

$$c_k[r] = \int_0^b dr' \lambda(r', d) \left(\frac{r_<^k}{r_>^{k+1}} - \frac{r^k r'^k}{b^{2k+1}} \right)$$

and we may write

$$\Phi(r, \mu) = \frac{3Q}{8\pi\epsilon_0\, d} \sum_{k=0,2,\cdots} c_k(r)\, P_k(\mu)$$

Two cases must be distinguished:

i. In the region $r < d$ (except when $k = 0$ or $k = 2$) we find

$$c_k(r) = \frac{2k+1}{k(k+1)} + \frac{2}{k(k-2)} \frac{r^k}{d^k} - \frac{2k+1}{(k-2)(k+3)} \frac{r^2}{d^2} - \frac{2}{(k+3)(k+1)} \frac{d^{k+1} r^k}{b^{2k+1}}$$

For the special cases $k = 0$ and $k = 2$, $c_k(r)$ has the following values

$$c_0(r) = \frac{1}{2} + \frac{1}{6} \frac{r^2}{d^2} + \log\left(\frac{d}{r}\right) - \frac{2}{3} \frac{d}{b}$$

$$c_2(r) = \frac{5}{6} - \frac{7}{10} \frac{r^2}{d^2} - \frac{r^2}{d^2} \log\left(\frac{d}{r}\right) - \frac{2}{15} \frac{d^3 r^2}{b^5}$$

ii. In the region $d < r \leq b$, we find

$$c_k(r) = \frac{2}{(k+1)(k+3)} \left[\frac{d^{k+1}}{r^{k+1}} - \frac{d^{k+1} r^k}{b^{2k+1}} \right]$$

(b) Find the surface charge density:

$$\sigma(\mu) = \epsilon_0 \frac{\partial \Phi}{\partial r} = \frac{3Q}{8\pi\, d} \sum_{k=0,2,\cdots} d_k\, P_k(\mu),$$

where

$$d_k = \left. \frac{dc_k(r)}{dr} \right|_{r=b} = -\frac{2(2k+1)}{(k+1)(k+3)} \frac{d^{k+1}}{b^{k+2}}$$

Note that

$$d_0 = -\frac{2}{3} \frac{d}{b^2}.$$

The total induced charge is

$$Q_{\text{ind}} = \frac{3Q}{8\pi\, d} 2\pi b^2 \sum_{k=0,2,\cdots} d_k \int_{-1}^{1} d\mu P_k(\mu)$$

By the orthogonality theorem for Legendre polynomials, only the $k = 0$ term contributes. Therefore,

$$Q_{\text{ind}} = \frac{3Q}{2} \frac{b^2}{d} d_0 = -Q$$

(c) Discuss the limiting case $d \ll b$. The region near the origin shrinks to negligible size. In the outer region, the expansion coefficients c_k are proportional to d^{k+1} and dominated by the term with $k = 0$. Therefore the limiting potential is

$$\Phi(r) \to \frac{3Q}{8\pi\epsilon_0 d}\ c_0(r) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$$

This is just the potential of a point charge at the center of a grounded sphere. The surface charge density is isotropic and has the limiting value

$$\sigma \to \frac{3Q}{8\pi d}\ d_0 = -\frac{Q}{4\pi b^2}$$