1. Jackson 3.1: Between two spheres: Upper half of inner (radius a) and lower half of outer (radius b) are at potential V. Lower half of inner and upper half of outer are at potential 0.

   (a) Find potential in region between spheres.

$$\Phi(r,\mu) = \sum_{l=0}^{\infty}\left[a_l\, r^l + \frac{b_l}{r^{l+1}}\right] P_l(\mu)$$

at the spherical surfaces,

$$\Phi(a,\mu) = \sum_{l=0}^{\infty}\left[a_l\, a^l + \frac{b_l}{a^{l+1}}\right] P_l(\mu) = \sum_{l=0}^{\infty} A_l P_l(\mu)$$
$$\Phi(b,\mu) = \sum_{l=0}^{\infty}\left[a_l\, b^l + \frac{b_l}{b^{l+1}}\right] P_l(\mu) = \sum_{l=0}^{\infty} B_l P_l(\mu)$$

Boundary conditions give

$$A_l = \frac{2l+1}{2} V \int_0^1 P_l(\mu) d\mu$$
$$B_l = \frac{2l+1}{2} V \int_{-1}^0 P_l(\mu) d\mu$$

We find:

$$A_0 = B_0 = \frac{1}{2} V$$
$$A_l = -B_l = V\left\{\frac{3}{4},\, 0 - \frac{7}{16},\, 0,\, \frac{11}{32},\, \cdots\right\} \quad \text{for} \quad l = 1,\, 2,\, 3,\, 4,\, 5,\, \cdots$$

Solve for a_l and b_l in terms of A_l and B_l. We find

$$a_0 = A_0 = \frac{1}{2} V, \qquad b_0 = 0.$$

For even values of l, $a_l = b_l = 0$. For odd values of l

$$a_l = -\left[\frac{1}{a^{l+1}} + \frac{1}{b^{l+1}}\right] \frac{A_l}{D_l}$$
$$b_l = \left[a^l + b^l\right] \frac{A_l}{D_l},$$

where

$$D_l = \frac{b^l}{a^{l+1}} - \frac{a^l}{b^{l+1}}$$

Putting this together, we find:

$$\begin{aligned}\Phi(r,\mu) &= \frac{V}{2} + \frac{3V}{4}\left[-\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\frac{r}{D_1} + \frac{a+b}{r^2 D_1}\right] P_1(\mu) \\ &\quad -\frac{7V}{16}\left[-\left(\frac{1}{a^4}+\frac{1}{b^4}\right)\frac{r^3}{D_3} + \frac{a^3+b^3}{r^4 D_3}\right] P_3(\mu) \\ &\quad +\frac{11V}{32}\left[-\left(\frac{1}{a^6}+\frac{1}{b^6}\right)\frac{r^5}{D_5} + \frac{a^5+b^5}{r^6 D_5}\right] P_5(\mu) + \cdots\end{aligned}$$

(b) Check the limits: As $b \to \infty$, $D_l \to b^l/a^{l+1}$

$$\lim_{b\to\infty} \Phi(r,\mu) = V\left[\frac{1}{2} + \frac{3}{4}\frac{a^2}{r^2}P_1(\mu) - \frac{7}{16}\frac{a^4}{r^4}P_3(\mu) + \frac{11}{32}\frac{a^6}{r^6}P_5(\mu) + \cdots\right].$$

This is the potential outside an upper hemisphere of radius a. Similarly, as $a \to 0$, $D_l \to b^l/a^{l+1}$ and

$$\lim_{a\to 0} \Phi(r,\mu) = V\left[\frac{1}{2} - \frac{3}{4}\frac{r}{b}P_1(\mu) + \frac{7}{16}\frac{r^3}{b^3}P_3(\mu) - \frac{11}{32}\frac{r^5}{a^5}P_5(\mu) + \cdots\right],$$

which is the potential inside a lower hemisphere of radius b.

2. Jackson 3.2: Potential for a uniformly charged sphere with an omitted cap of angle α at the north pole.

(a) The potentials inside and outside the sphere are

$$\begin{aligned}\Phi_{\text{in}}(r,\mu) &= \sum_{l=0}^{\infty} a_l\, r^l P_l(\mu) \\ \Phi_{\text{out}}(r,\mu) &= \sum_{l=0}^{\infty} b_l\, r^{-l-1} P_l(\mu)\end{aligned}$$

The potential is continuous at $r = a$; therefore, $b_l = a^{2l+1}\, a_l$ The radial electric field inside and outside the sphere at the surface is

$$\begin{aligned}E_r(\text{in}) &= -\sum_{l=0}^{\infty} l\, a_l\, a^{l-1} P_l(\mu) \\ E_r(\text{out}) &= \sum_{l=0}^{\infty} (l+1)\, b_l\, a^{-l-2} P_l(\mu) = \sum_{l=0}^{\infty} (l+1)\, a_l\, a^{l-1} P_l(\mu)\end{aligned}$$

The surface charge density is, therefore,

$$\sigma = \epsilon_0 \Delta E_r = \sum_{l=0}^{\infty} (2l+1)\, a_l\, a^{l-1} P_l(\mu) = \begin{cases} 0 & \theta < \alpha \\ \dfrac{Q}{4\pi a^2} & \text{otherwise} \end{cases}$$

It follows that

$$a_l = \frac{Q}{8\pi\epsilon_0 a^{l+1}} \int_{-1}^{\cos\alpha} P_l(\mu)\, d\mu$$

For $l = 0$, we find

$$\int_{-1}^{\cos\alpha} P_l(\mu)\, d\mu = \cos\alpha + 1$$

For $l > 0$, we make use of the identity

$$(2l+1)P_l(\mu) = \frac{dP_{l+1}(\mu)}{d\mu} - \frac{dP_{l-1}(\mu)}{d\mu}$$

to show

$$\int_{-1}^{\cos\alpha} P_l(\mu)\, d\mu = \frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right].$$

In the above, we use $P_{l-1}(-1) = P_{l+1}(-1)$. It follows that

$$\begin{aligned}
\Phi_{\rm in}(r,\mu) = & \; \frac{Q}{8\pi\epsilon_0 a}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\left(\frac{r}{a}\right)^l P_l(\mu) \\
\Phi_{\rm out}(r,\mu) = & \; \frac{Q}{8\pi\epsilon_0 a}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)\right]\left(\frac{a}{r}\right)^{l+1} P_l(\mu)
\end{aligned}$$

where we define $P_{-1}(\cos\alpha) = -1$.

(b) Only the $l = 1$ term contributes to the electric field at the origin. We find

$$\begin{aligned}
E_r(0) = & \; -\frac{Q}{8\pi\epsilon_0 a^2}\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]\cos\theta \\
E_\theta(0) = & \; \frac{Q}{8\pi\epsilon_0 a^2}\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]\sin\theta
\end{aligned}$$

Note that $E_z = E_r\cos\theta - E_\theta\sin\theta = E_z$ and $E_x = E_r\sin\theta + E_\theta\cos\theta = E_x$. Therefore, $\boldsymbol{E}$ is along the z axis and has magnitude

$$E = -\frac{Q}{8\pi\epsilon_0 a^2}\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right]$$

This cam be simplified using the fact

$$\frac{1}{3}\left[P_2(\cos\alpha) - P_0(\cos\alpha)\right] = \frac{1}{3}\left[\frac{3}{2}\cos^2\alpha - \frac{1}{2} - 1\right] = -\frac{1}{2}\sin^2\alpha$$

Therefore

$$E = \frac{Q\sin^2\alpha}{16\pi\epsilon_0 a^2}$$

(c) Limiting case $\alpha \to 0$: In this limit, only the $l = 0$ term contributes. The potential reduces to that of a uniformly charged sphere and the field at the origin vanishes. For small but finite α the total charge excluded by the cap is $Q_{\text{cap}} \approx Q\alpha^2/4$. By superposition, the field at the origin should be the field of a negative charge Q_{cap} located at the north pole. Indeed, we find

$$E = \frac{Q \sin^2 \alpha}{16\pi\epsilon_0 a^2} \to \frac{Q_{\text{cap}}}{4\pi\epsilon_0 a^2}$$

(d) Limiting case $\alpha \to \pi$: In this limit the potential vanishes. Setting $\beta = \pi - \alpha$, the total charge on the cap near the south pole is $Q_{\text{cap}} = Q\beta^2/4$. Again the field at the origin reduces to the field of the positively cap at the south pole.

$$E = \frac{Q \sin^2(\pi - \beta)}{16\pi\epsilon_0 a^2} \to \frac{Q_{\text{cap}}}{4\pi\epsilon_0 a^2}.$$

3. Jackson 3.3: A circular conducting disc of radius R in the $x-y$ plane with center at the origin is at potential V.

(a) Given that $\sigma = \kappa/\sqrt{R^2 - \rho^2}$, find the potential for $r > R$. First, note that the total charge Q on the disc is

$$Q = \kappa \int_0^R \frac{2\pi\rho d\rho}{\sqrt{R^2 - \rho^2}} = 2\pi\kappa R$$

Thus, we can rewrite

$$\sigma(\rho) = \frac{Q}{2\pi R} \frac{1}{\sqrt{R^2 - \rho^2}}.$$

The potential on the z axis is

$$\Phi(z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{2\pi R} \int_0^R \frac{2\pi\rho d\rho}{\sqrt{\rho^2 + z^2}\sqrt{R^2 - \rho^2}}$$

Changing independent variable to $\xi = \sqrt{R^2 - \rho^2}$, we find

$$\Phi(z) = \frac{Q}{4\pi\epsilon_0 R} \int_0^R \frac{d\xi}{\sqrt{z^2 + R^2 - \xi^2}} = \frac{Q}{4\pi\epsilon_0 R} \arctan\left(\frac{R}{z}\right)$$

Since $\Phi(0) = V$, it follows

$$V = \frac{Q}{4\pi\epsilon_0 R} \frac{\pi}{2} = \frac{Q}{8\epsilon_0 R},$$

and

$$\Phi(z) = \frac{2V}{\pi} \arctan\left(\frac{R}{z}\right) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{z}\right)^{2l+1}.$$

Comparing this expansion with the general expansion for azimuthal symmetry, we infer

$$\Phi(r,\theta) = \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{R}{r}\right)^{2l+1} P_{2l}(\cos\theta)$$

(b) for $r < R$, we have

$$\begin{aligned}\Phi(z) &= \frac{2V}{\pi} \arctan\left(\frac{R}{z}\right) = \frac{2V}{\pi}\left[\frac{\pi}{2} - \arctan\left(\frac{z}{R}\right)\right] \\ &= V - \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{z}{R}\right)^{2l+1}\end{aligned}$$

Thus we have for $r < R$,

$$\Phi(r,\theta) = V - \frac{2V}{\pi} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{r}{R}\right)^{2l+1} P_{2l+1}(\cos\theta).$$

(c) The capacitance of the disc is

$$C = \frac{Q}{V} = 8\epsilon_0 R$$