1. Jackson 2.22: Study the potential inside two hemispheres with $\Phi(a,\theta) = V$ for $\theta < \pi/2$ and $\Phi(a,\theta) = -V$ for $\theta > \pi/2$.

   (a) The interior solution is obtained from Eq. (2.19) in the text by changing sign. (Why?)

   $$\Phi(r,\theta) = \frac{a(a^2-r^2)}{4\pi}\int \frac{\Phi(a,\theta',\phi')}{(r^2+a^2-2ar\cos\gamma)^{3/2}}\,d\Omega',$$

   where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta\cos(\phi-\phi')$. Along the z axis, this reduces to

   $$\begin{aligned}\Phi(z) &= \frac{a(a^2-z^2)}{2}\left[\int_0^1 \frac{V\,d\mu'}{(z^2+a^2-2az\mu')^{3/2}} + \int_{-1}^0 \frac{-V\,d\mu'}{(z^2+a^2-2az\mu')^{3/2}}\right]\\ &= \frac{a(a^2-z^2)}{2}\frac{V}{az}\left[\frac{2a}{(a^2-z^2)} - \frac{2}{\sqrt{z^2+a^2}}\right]\\ &= V\frac{a}{z}\left[1 - \frac{a^2-z^2}{a\sqrt{a^2+z^2}}\right]\\ &= V\frac{3z}{2a}\left(1 - \frac{7z^2}{12a^2} + \frac{11z^4}{24a^4} - \frac{25z^6}{64a^6} + \frac{133z^8}{384a^8} + \cdots\right)\end{aligned}$$

   The counterpart of Eq. (2.27) for $r < a$ is

   $$\Phi(r,\theta) = \frac{3Vr}{2a}\left[P_1(\cos\theta) - \frac{7r^2}{12a^2}P_3(\cos\theta) + \frac{11r^2}{24a^2}P_5(\cos\theta) + \cdots\right]$$

   Since $P_l(1) = 1$, we see that, for $\theta = 0$ and $r = z$, the first three terms in the two expansions agree.

   (b) Field along the axis. For $z > a$

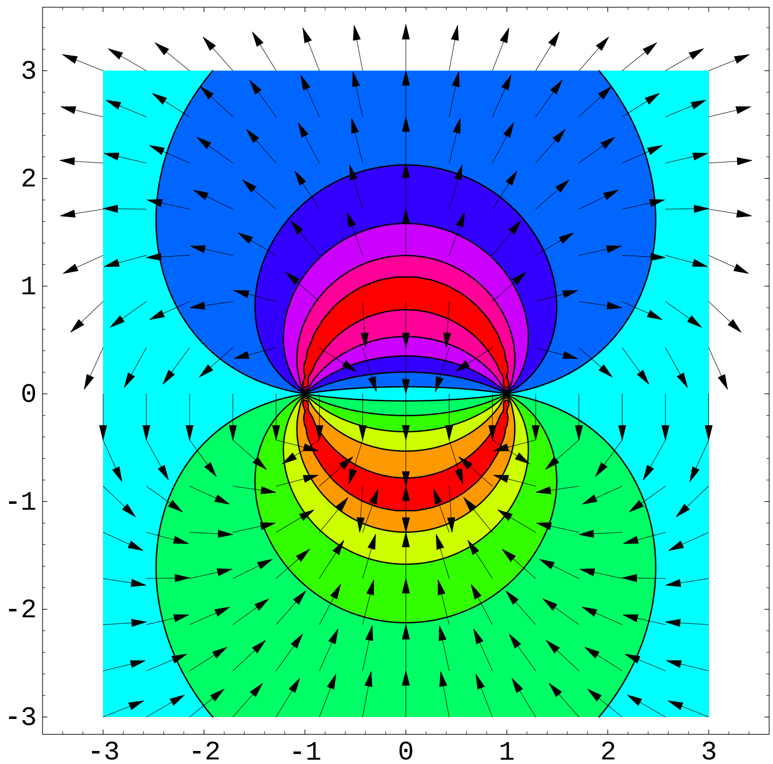
   $$E_z = -V\frac{d}{dz}\left[1 - \frac{z^2-a^2}{z\sqrt{z^2+a^2}}\right] = \frac{Va^2}{(z^2+a^2)^{3/2}}\left[3 + \frac{a^2}{z^2}\right]$$

   and for $z < a$

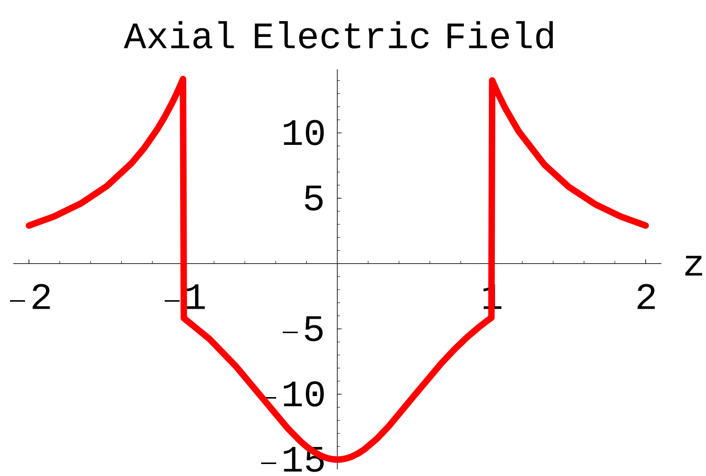
   $$E_z = -V\frac{d}{dz}\frac{a}{z}\left[1 - \frac{a^2-z^2}{za\sqrt{z^2+a^2}}\right] = -\frac{V}{a}\left[\frac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} - \frac{a^2}{z^2}\right]$$

   The leading term in powers of z in an expansion of this expression is $-3V/2a$.
   Therefore, $E_z(0) = -3V/a$. Similarly, $E_z(a) = -(\sqrt{2}-1)V/a$ inside and $E_z(a) = \sqrt{2}\,V/a$ outside.

(c) **Sketch of field lines**



Plot of $E_z(z)$



2. Jackson 2.23:

   (a) Potential inside cube of side a subject to boundary conditions $\Phi = 0$ on surfaces $x = 0,\ a$ and $y = 0,\ a$, and $\Phi = V$ on surfaces $z = 0,\ a$. A solution that satisfies the x and y boundary conditions is

$$\Phi(x, y, z) = \sum_{m,n} \sin\frac{m\pi x}{a}\ \sin\frac{m\pi y}{a}\left[a_{mn}e^{k_{mn}z} + b_{mn}e^{-k_{mn}z}\right]$$

   where

$$k_{mn} = \sqrt{m^2 + n^2}\,\frac{\pi}{a}$$

   At $z = 0$ this reduces to

$$\Phi(x, y, 0) = \sum_{m,n} c_{mn}\sin\frac{m\pi x}{a}\ \sin\frac{n\pi y}{a}$$

   where

$$c_{mn} = a_{mn} + b_{mn}.$$

   At $z = a$ we have $\Phi(x, y, a) = \Phi(x, y, 0)$, from which it follows

$$c_{mn} = a_{mn}e^{k_{mn}a} + b_{mn}e^{-k_{mn}a}$$

   With a little algebra one obtains

$$a_{mn}e^{k_{mn}z} + b_{mn}e^{-k_{mn}z} = \frac{\cosh k_{mn}(z - a/2)}{\cosh k_{mn}a/2}\ c_{mn}$$

   where c_{mn} is determined from

$$\sum_{m,n} c_{mn}\sin\frac{m\pi x}{a}\ \sin\frac{m\pi y}{a} = V$$

   This problem has been previously solved for x and y separately. We find:

$$\Phi(x, y, z) =$$
$$\frac{16V}{\pi^2}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\sin\left[(2m+1)\pi x/a\right]\,\sin\left[(2n+1)\pi y/a\right]}{(2m+1)(2n+1)}\ \frac{\cosh k_{mn}(z - a/2)}{\cosh k_{mn}a/2}$$

   where $k_{mn} = \sqrt{(2m+1)^2 + (2n+1)^2}\,\pi/a$

   (b) Average at center $\Phi = 0.3329\,V$ including only 4 terms ($m = 0$, 1 and $n = 0$, 1). The result compares well with average value of $V/3$.

   (c) Surface charge density at $z = a$. One can obtain a formal expression for the surface charge, but the sum does not converge! The corresponding situation for the two-dimensional case was discussed in class.

3. Jackson Prob. 2.26

(a) Solution in wedge shaped region.

$$\Phi(\rho,\phi) = \sum_n a_n \left[\rho^{n\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{n\pi/\beta}\right] \sin(n\pi\phi/\beta)$$

is a solution to Laplace's equation satisfying all 3 boundary conditions.

(b) The lowest term above is

$$\Phi(\rho,\phi) \approx a_1 \left[\rho^{\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right] \sin(\pi\phi/\beta)$$

$$\begin{aligned} E_\rho =& -a_1 \frac{\pi}{\beta\rho} \left[\rho^{\pi/\beta} + \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right] \sin(\pi\phi/\beta) \\ E_\phi =& -a_1 \frac{\pi}{\beta\rho} \left[\rho^{\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right] \cos(\pi\phi/\beta) \end{aligned}$$

It follows that

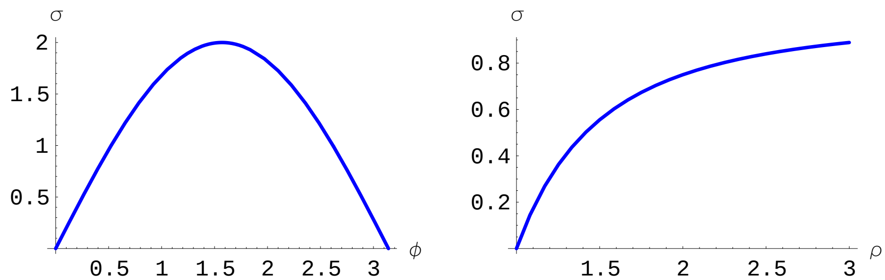
$$\begin{aligned} \sigma(\phi=0) =& \epsilon_0 E_\phi = -a_1\epsilon_0 \frac{\pi}{\beta\rho} \left[\rho^{\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right] \\ \sigma(\phi=\beta) =& -\epsilon_0 E_\phi = -a_1\epsilon_0 \frac{\pi}{\beta\rho} \left[\rho^{\pi/\beta} - \left(\frac{a^2}{\rho}\right)^{\pi/\beta}\right] \\ \sigma(\rho=a) =& -\epsilon_0 E_\rho = -2\epsilon_0 a_1 \frac{\pi}{\beta} a^{\pi/\beta-1} \sin(\pi\phi/\beta) \end{aligned}$$

(c) For the case $\beta = \pi$,

$$\begin{aligned} E_\rho =& -a_1 \left[1 + \left(\frac{a}{\rho}\right)^2\right] \sin\phi \to -a_1 \sin\phi \\ E_\phi =& -a_1 \left[1 - \left(\frac{a}{\rho}\right)^2\right] \cos\phi \to -a_1 \cos\phi \end{aligned}$$

Thus $E_x = \cos\phi E_\rho - \sin\phi E_\phi = 0$ and $E_y = \sin\phi E_\rho + \cos\phi E_\phi = -a_1$. The field far away is uniform and in the y direction and has magnitude $E = -a_1$.

Plot of charge density on plane and on cylinder ($a_1 = -1$).



The charge on the cylindrical boss is

$$Q_a = -2a_1\epsilon_0 \int_)^{\pi} a \sin\phi d\phi = -4a_1 a\epsilon_0$$

This is just twice the charge on a uniformly charged strip of width $2a$ with charge density $\sigma = -a_1\epsilon_0$.

Now, consider the total charge in the interval $[0,L]$. From the right half of the boss, we have $Q_b = -2a_1 a\epsilon_0$. From the section of the plane $[a.\ L]$, we have

$$Q_p = -a_1\epsilon_0 \int_a^L \left[1 - \frac{a^2}{\rho^2}\right] d\rho = -a_1\epsilon_0 \left[L - a - a + \frac{a^2}{L}\right]$$

In the limit as $L \to \infty$, one finds

$$Q_b + Q_p \to -a_1\epsilon_0 L,$$

independent of the boss!