1. Jackson 2.7: Green function for a plane.

   (a) Green function: Let $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{r'} = (x', y', z')$, then

$$G(\boldsymbol{r}, \boldsymbol{r'}) = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

   (b) Solution for $\Phi = V$ inside a circle of radius a on $x - y$ plane. First we evaluate

$$\frac{\partial G}{\partial n'} = -\left.\frac{\partial G}{\partial x'}\right|_{z'=0} = -\frac{2z}{[z^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi' - \phi)]^{3/2}}$$

   From azimuthal symmetry Φ is independent of ϕ. It follows that

$$\Phi(\rho, z) = \frac{zV}{2\pi} \int_0^a \rho' d\rho' \int_0^{2\pi} \frac{d\phi'}{[z^2 + \rho^2 + \rho'^2 - 2\rho\rho' \cos\phi']^{3/2}}$$

   (c) For $\rho = 0$, we find

$$\begin{aligned} \Phi(0, z) =& \frac{zV}{2\pi} \int_0^a \rho' d\rho' \int_0^{2\pi} \frac{d\phi'}{[z^2 + \rho'^2]^{3/2}} \\ =& \frac{zV}{2} \int_0^{a^2} \frac{d\rho'^2}{[z^2 + \rho'^2]^{3/2}} \\ =& V \left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right] \end{aligned}$$

   (d) Asymptotic expansion for $z^2 + \rho^2 \gg a^2$. Let $x^2 = z^2 + \rho^2$, then

$$\frac{1}{[x^2 + \rho'^2 - 2\rho\rho' \cos\phi']^{3/2}} = \frac{1}{x^3}\left[1 - \frac{3}{2}\frac{\rho'^2 - 2\rho\rho'\cos\phi'}{x^2} + \frac{15}{8}\frac{(\rho'^2 - 2\rho\rho'\cos\phi')^2}{x^4} + \cdots\right]$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\phi'}{[x^2 + \rho'^2 - 2\rho\rho' \cos\phi']^{3/2}} = \frac{1}{x^3}\left[1 - \frac{3}{2}\frac{\rho'^2}{x^2} + \frac{15}{8}\frac{2\rho^2\rho'^2 + \rho'^4}{x^4} + \cdots\right]$$

$$\frac{1}{2\pi}\int_0^a \rho' d\rho' \int_0^{2\pi} \frac{d\phi'}{[x^2 + \rho'^2 - 2\rho\rho' \cos\phi']^{3/2}} = \frac{a^2}{2x^3}\left[1 - \frac{3}{4}\frac{a^2}{x^2} + \frac{5}{8}\frac{3\rho^2 a^2 + a^4}{x^4} + \cdots\right]$$

Substituting $x^2 = z^2 + \rho^2$, we obtain

$$\Phi(\rho, z) = \frac{Vza^2}{2(z^2+\rho^2)^{3/2}} \left[1 - \frac{3}{4}\frac{a^2}{z^2+\rho^2} + \frac{5}{8}\frac{3\rho^2 a^2 + a^4}{(z^2+\rho^2)^2} + \cdots \right]$$

On the axis, $\Phi(\rho, z)$ reduces to

$$\Phi(0, z) = \frac{Va^2}{2z^2} \left[1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \cdots \right]$$

This is identical to the expansion for $z \gg a$ of the result from (c) above:

$$V \left[1 - \frac{z}{\sqrt{z^2+a^2}} \right] = \frac{Va^2}{2z^2} \left[1 - \frac{3}{4}\frac{a^2}{z^2} + \frac{5}{8}\frac{a^4}{z^4} + \cdots \right]$$

2. Jackson 2.11: Image potential for charged wire at $x = R$ parallel to a cylinder of radius b centered at the origin.

   (a) The potential in cylindrical coordinates is

   $$\Phi(\rho, \phi) = \frac{1}{2\pi\epsilon_0} \Big[-\tau \ln \sqrt{\rho^2 + R^2 - 2R\rho\cos\phi} + \tau' \ln \sqrt{\rho^2 + r^2 - 2r\rho\cos\phi} \Big]$$

   where r is the distance of the image from the axis of the cylinder. To achieve $\Phi = V$ on surface of the cylinder and $\lim_{\rho\to\infty} \Phi(\rho, \phi) = 0$, we choose $r = b^2/R$ and $\tau' = \tau$.

   (b) With the above conditions, we find

   $$\Phi(\rho, \phi) = \frac{\tau}{4\pi\epsilon_0} \ln \left[\frac{\rho^2 + r^2 - 2r\rho\cos\phi}{\rho^2 + R^2 - 2R\rho\cos\phi} \right]$$

   Note that the potential at the cylindrical surface is

   $$V = \Phi(b, \phi) = \frac{\tau}{2\pi\epsilon_0} \ln \left[\frac{b}{R} \right]$$

   This equation relates the potential on the cylinder to the other parameters of the problem. For large ρ, we find

   $$\Phi(\rho, \phi) = \frac{\tau}{2\pi\epsilon_0} \left[\frac{R - r}{\rho} \cos\phi + \frac{R^2 - r^2}{2\rho^2} \cos 2\phi + \frac{R^3 - r^3}{3\rho^3} \cos 3\phi + \cdots \right]$$

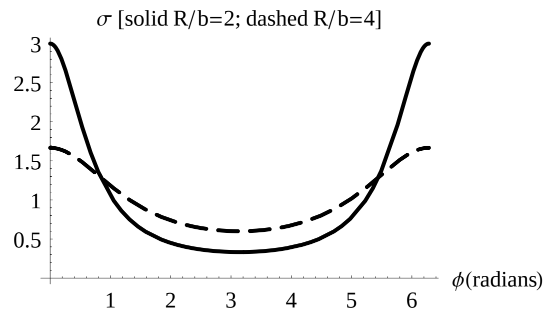
   (c) Induced charge density. The radial electric field at the surface is

   $$E_\rho = -\left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b} = -\frac{\tau}{2\pi\epsilon_0 b} \frac{R^2 - b^2}{R^2 + b^2 - 2Rb\cos\phi}$$

Therefore

$$\sigma = -\frac{\tau}{2\pi b}\ \frac{R^2 - b^2}{R^2 + b^2 - 2Rb\cos\phi}$$

Below is a graph of the negative of the induced charge -σ



(d) The force on the charged wire: We first evaluate E_ρ at the wire.

$$E_\rho(\rho,\phi) = -\frac{\partial\Phi}{\partial\rho} = -\frac{\tau}{2\pi\epsilon_0}\left[\frac{\rho - r\cos\phi}{\rho^2 + r^2 - 2r\rho\cos\phi} - \frac{\rho - R\cos\phi}{\rho^2 + R^2 - 2R\rho\cos\phi}\right]$$

From this, it follows that at the wire

$$E_\rho(R,0) = -\frac{\tau}{2\pi\epsilon_0}\ \frac{R-r}{R^2 + r^2 - 2rR} = -\frac{\tau}{2\pi\epsilon_0}\ \frac{1}{R-r}$$

The force/length on the charged wire is, therefore,

$$F_\rho/L = \tau E_\rho(R,0) = -\frac{\tau^2}{2\pi\epsilon_0}\ \frac{1}{R-r} = -\frac{\tau^2}{2\pi\epsilon_0}\ \frac{R}{R^2 - b^2}$$

3. Jackson 2.13; "Cracking an interesting integral"

(a) Let $\Phi(b,\phi') = V_1$ for $0 < \phi' < \pi$ and V_2 for $\pi < \phi' < 2\pi$. It follows from the Green function given in Prob. 2.12 that

$$\begin{aligned}\Phi(\rho,\phi) &= \frac{V_1}{2\pi}(1-\xi^2)\int_0^{\pi}\frac{d\phi'}{1+\xi^2-2\xi\cos(\phi'-\phi)} \\ &\quad + \frac{V_2}{2\pi}(1-\xi^2)\int_{\pi}^{2\pi}\frac{d\phi'}{1+\xi^2-2\xi\cos(\phi'-\phi)}\end{aligned}$$

where $\xi = \rho/b$. Notice that the second integral can be obtained from the first by the transformation $\xi \to -\xi$. It follows that

$$\Phi(\rho,\phi) = (1-\xi^2)\frac{1}{2\pi}\left[V_1 I(\xi,\phi) + V_2 I(-\xi,\phi)\right]$$

where

$$I(\xi,\phi) = -\int_{\phi}^{\phi+\pi}\frac{d\psi}{1+\xi^2-2\xi\cos\psi}$$

With a change of variables to $x = e^{i\psi}$, we can then rewrite the integral as

$$I(\xi,\phi) = \frac{i}{\xi}\int_{-x_0}^{x_0} \frac{dx}{(x-\xi)(x-1/\xi)}$$

where $x_0 = e^{i\phi}$. In this later form, the integral may be done using a partial fraction decomposition. One finds

$$\begin{aligned} I(\xi,\phi) &= \frac{i}{1-\xi^2}\ln\left[\frac{(1-x_0\xi)(x_0^{-1}\xi+1)}{(1+x_0\xi)(x_0^{-1}\xi-1)}\right] \\ &= \frac{i}{1-\xi^2}\ln\left[\frac{1-\xi^2-2i\xi\sin\phi}{-1+\xi^2+2i\xi\sin\phi}\right] \end{aligned}$$

The absolute values of the numerator and denominator in the above fraction are equal. The phase of the numerator is

$$-\arctan(\xi\sin\phi/(1-\xi^2))$$

and the phase of the denominator is

$$\pi + \arctan(\xi\sin\phi/(1-\xi^2))$$

. Therefore

$$I(\xi,\phi) = \frac{1}{1-\xi^2}\left[\pi + 2\arctan\left(\frac{2\xi\sin\phi}{1-\xi^2}\right)\right]$$

(n.b. This integral is given in Eq. (47a) on p. 100 of *Integraltafel, teil 2, Bestimmte Integrale*, W. Gröbner & N. Hofreiter, Springer (1958).) Combining terms, we find

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\arctan\left(\frac{2b\rho\sin\phi}{b^2-\rho^2}\right)$$

(n.b. Jackson's cylinder is rotated by $\pi/2$ with respect to our's)

(b) Surface charge density:

$$\sigma = -\epsilon_0 E_r = \epsilon_0 \left.\frac{\partial\Phi}{\partial\rho}\right|_{\rho=b} = \epsilon_0\frac{V_1-V_2}{\pi b}\csc\phi.$$

Equal and opposite charges accumulate on the two halves and the charge density diverges at the gap!

(c) Jackson 2.13; **Alternative solution.**

Expand $\Phi(\rho,\phi)$ in a series:

$$\Phi(\rho,\phi) = a_0 + \sum_{n=1}^{\infty} a_n\rho^n\cos n\phi + \sum_{n=1}^{\infty} b_n\rho^n\sin n\phi$$

The expansion coefficients are easily found in terms of the potentials on the surface.

$$\begin{aligned} a_0 &= \frac{V_1+V_2}{2} \\ a_n &= 0, \quad n>0 \\ b_n &= \frac{2(V_1-V_2)}{b^n\pi} \qquad n=1,\ 3,\ \ldots \end{aligned}$$

Therefore

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{(V_1-V_2)}{\pi}\ 2\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\left(\frac{\rho}{b}\right)^{2n+1}\sin{(2m+1)\phi}$$

Now, let $z=\rho e^{i\phi}/b$ and note that (twice) the sum becomes

$$S = 2\sum_{m=0}^{\infty}\frac{1}{(2m+1)}\left(\frac{\rho}{b}\right)^{2n+1}\sin{(2m+1)\phi} = -i\left[\sum_{m=0}^{\infty}\frac{z^{2m+1}}{(2m+1)} - \text{c.c.}\right]$$

Make a second transformation $z=i\xi$ to obtain

$$S = \left[\sum_{m=0}^{\infty}\frac{(-1)^m\xi^{2m+1}}{(2m+1)} + \text{c.c.}\right] = \arctan\xi + \arctan\xi^*$$

From the rule $\tan{(A+B)} = (\tan A + \tan B)/(1-\tan A\,\tan B)$, it follows

$$\tan S = \frac{\xi+\xi^*}{1-\xi\xi^*} = -i\frac{z-z^*}{1-zz^*} = 2\frac{(\rho/b)\sin\phi}{1-(\rho/b)^2} = \frac{2b\rho\sin\phi}{b^2-\rho^2}$$

The sum S is therefore

$$S = \arctan\left[\frac{2b\rho\sin\phi}{b^2-\rho^2}\right]$$

and

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\arctan\left(\frac{2b\rho\sin\phi}{b^2-\rho^2}\right)$$

4. Jackson 2.16: Green Function for rectangular region. The potential is given by

$$\begin{aligned} \Phi(x,y) =& \frac{1}{4\pi\epsilon_0}\int_0^1 dx'\int_0^1 dy' G(x,y;x'y')\rho(x',y') \\ =& \frac{2}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n\sinh n\pi}\int_0^{\infty}\sin n\pi x' dx'\ \left[\sinh n\pi(1-y)\int_0^y \sinh n\pi y'\,dy'\right. \\ & \left.\sinh n\pi y\int_y^1 \sinh n\pi(1-y')\,dy'\right]\rho(x',y') \end{aligned}$$

For the case of a uniform charge distribution $\rho(x,y)=1$, we can carry out the integrals easily:

$$\begin{aligned}
\int_0^1 \sin n\pi x'\,dx' &= \frac{2}{n\pi}\text{for odd } n \text{ and } 0 \text{ for even } n \\
\int_0^y \sinh n\pi y'\,dy' &= \frac{1}{n\pi}[\cosh n\pi y - 1] \\
\int_y^1 \sinh n\pi(1-y')\,dy' &= \frac{1}{n\pi}[\cosh n\pi(1-y) - 1]
\end{aligned}$$

The combination arising from the y' integration can be simplified:

$$\begin{aligned}
&(\cosh n\pi y - 1)\sinh n\pi(1-y) + (\cosh n\pi(1-y) - 1)\sinh n\pi y \\
&\quad = \sinh n\pi - \sinh n\pi y - \sinh n\pi(1-y) \\
&\quad = \sinh n\pi \left[1 - \frac{2\sinh\frac{n\pi}{2}\cosh n\pi(y-\frac{1}{2})}{\sinh n\pi}\right] \\
&\quad = \sinh n\pi \left[1 - \frac{\cosh n\pi(y-\frac{1}{2})}{\cosh\frac{n\pi}{2}}\right]
\end{aligned}$$

Collecting terms and introducing m through $n = 2m+1$, we obtain

$$\Phi(x,y) = \frac{4}{\pi^3\epsilon_0}\sum_{m=0}^{\infty}\frac{\sin(2m+1)\pi x}{(2m+1)^3}\left[1 - \frac{\cosh(2m+1)\pi(y-\frac{1}{2})}{\cosh\frac{(2m+1)\pi}{2}}\right]$$

Here is a plot of $4\pi\epsilon_0\Phi(x,y)$ obtained by summing terms up to $m = 10$