1. Jackson Prob. 2.1: Charge above a grounded plane

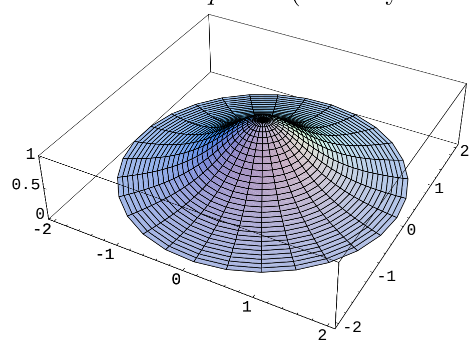
   (a) Surface charge density

   $$E_z(\rho, z) = -\frac{\partial \phi}{\partial z} = \frac{q}{4\pi\epsilon_0}\left[\frac{(z-d)}{[\rho^2 + (z-d)^2]^{3/2}} - \frac{(z+d)}{[\rho^2 + (z+d)^2]^{3/2}}\right]$$

   Evaluate E_z at $z = 0$ and multiply by ϵ_0 to find σ.

   $$\sigma(\rho) = -\frac{qd}{2\pi}\frac{1}{[\rho^2 + d^2]^{3/2}}$$

   Plot of σ with $d = 1$ and $q = 2\pi$ (Actually we show -σ)



   (b) The direction of the force on the plane is along z axis. Its magnitude is

   $$F_z = \int_0^\infty 2\pi\rho d\rho \frac{1}{4\pi\epsilon_0}\frac{q\sigma(\rho)}{\rho^2 + d^2}\frac{d}{\sqrt{\rho^2 + d^2}} = \frac{q^2 d^2}{4\pi\epsilon_0}\int_0^\infty \frac{\rho d\rho}{(\rho^2 + d^2)^3} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4d^2}.$$

   This is precisely the force on the image charge predicted by Coulomb's law.

   (c) The force obtained by integrating $\sigma^2/(2\epsilon_0)$ is

   $$F_z = \frac{1}{2\epsilon_0}\frac{q^2 d^2}{(2\pi)^2}\int_0^\infty \frac{2\pi\rho d\rho}{(\rho^2 + d^2)^3} = \frac{q^2 d^2}{4\pi\epsilon_0}\int_0^\infty \frac{\rho d\rho}{(\rho^2 + d^2)^3} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4d^2}.$$

   (d) Work done to move charge to infinity.

   $$W = \int_d^\infty \frac{q^2}{4\pi\epsilon_0}\frac{dz}{4z^2} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{4d}$$

   (e) Potential energy of charge and image.

   $$V = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{2d}.$$

This is twice the value naively expected. The work-energy relation for a pair of charges discussed in Sec. 1.11 assumed that one charge was fixed! In this case the both charges move as work is done on the system.

(f) W for an electron at d= 1Å from surface.

$$\begin{aligned} W &= \frac{q^2}{4\pi\epsilon_0}\frac{1}{4d} \\ &= \frac{(1.60217653\times 10^{-19})^2}{4\times\pi\times 8.854187817\times 10^{-12}}\ \frac{1}{4\times 10^{-10}} \\ &= 5.767693\times 10^{-19}\ \mathrm{J} \\ &= 3.6\mathrm{eV} \end{aligned}$$

2. Jackson Prob 2.2:

(a) Assuming that the charge is on the z axis at distance d from the origin, the potential at points inside the sphere is

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{\sqrt{r^2-2rd\cos\theta+d^2}} - \frac{q'}{\sqrt{r^2-2rd'\cos\theta+d'^2}}\right\},$$

where $q' = qa/d$ and $d' = a^2/d$.

(b) Induced charge density: First, we determine the radial electric field

$$\begin{aligned} E_r &= -\frac{\partial\Phi}{\partial r} \\ &= \frac{1}{4\pi\epsilon_0}\left\{\frac{q(r-d\cos\theta)}{[r^2-2rd\cos\theta+d^2]^{3/2}} - \frac{q'(r-d'\cos\theta)}{[r^2-2rd'\cos\theta+d'^2]^{3/2}}\right\} \end{aligned}$$

The surface charge density σ is $-\epsilon_0 E_r$, evaluated at $r=a$. (In this case, the normal points inward!). After simplification, this becomes

$$\sigma(\theta) = -\frac{q}{4\pi a^2}\frac{1-d^2/a^2}{[1-2(d/a)\cos\theta+d^2/a^2]^{3/2}}$$

(c) Magnitude and direction of force on q:

$$\begin{aligned} F_z =& \frac{q}{4\pi\epsilon_0}2\pi a^2\int_{-1}^{1}d\mu\frac{|\sigma(\mu)|\,(a\mu-d)}{[a^2-2ad\mu+d^2]^{3/2}} \\ =& \frac{q^2}{4\pi\epsilon_0}\frac{a(a^2-d^2)}{2}\int_{-1}^{1}d\mu\frac{(a\mu-d)}{[a^2-2ad\mu+d^2]^3} \\ =& \frac{q^2}{4\pi\epsilon_0}\frac{ad}{(a^2-d^2)^2}. \end{aligned}$$

The force on q is in the $+z$ direction. This is precisely the force exerted on q by the image charge q'.

(d) Changes in the solution:

i. Sphere at a fixed potential V. In this case,

$$\begin{aligned}\Phi(r,\theta) &\to \Phi(r,\theta)+V, && r<a\\ &\to \frac{aV}{r}, && r\geq a\end{aligned}$$

There is an additional uniformly distributed charge $Q = q + 4\pi\epsilon_0 V$ on the sphere. Thus

$$\sigma \to \sigma + \frac{q+4\pi\epsilon_0 V}{4\pi a^2}.$$

The uniformly distributed charge exerts no additional force of q since

$$\int_{-1}^{1} d\mu \frac{(a\mu - d)}{[a^2 - 2ad\mu + d^2]^{3/2}} = 0$$

ii. Sphere has a fixed charge Q. In this case, an additional charge $Q+q$ is again uniformly distributed over the surface. Therefore

$$\begin{aligned}\Phi(r,\theta) &\to \Phi(r,\theta)+\frac{Q+q}{a}, && r<a\\ &\to \frac{1}{4\pi\epsilon_0}\frac{Q+q}{r}, && r\geq a\end{aligned}$$

The additional uniformly distributed charge is

$$\sigma \to \sigma + \frac{Q+q}{4\pi a^2},$$

and, as above, there is no added force on q.

3. Jackson Prob 2.5:

(a) Quasistatic force needed to balance charge q above a grounded sphere.

$$F_y = \frac{q^2}{4\pi\epsilon_0}\frac{ay}{y^2-a^2}$$

Work done to remove charge to infinity

$$W = \int_d^\infty F_y dy = \frac{q^2}{8\pi\epsilon_0}\frac{a}{d^2-a^2} = \frac{qq'}{8\pi\epsilon_0}\frac{1}{d-d'}$$

This is 1/2 of the (negative of) the potential energy of the charge and its image. Here again the image is not fixed as the charge moves out, so the work-energy theorem, in its usual form, is not valid.

(b) Quasistatic force needed to balance charge q above an isolated sphere carrying charge Q.

$$F_y = -\frac{q}{4\pi\epsilon_0}\left[\frac{Q}{y^2} - \frac{qa^3(2y^2-a^2)}{y^3(y^2-a^2)^2}\right]$$

Work needed charge to remove charge to infinity

$$\begin{aligned} W = \int_d^\infty F_y dy &= \frac{1}{4\pi\epsilon_0}\left[-\frac{Qq}{d} - \frac{q^2 a}{d^2} + \frac{q^2 a}{d^2-a^2}\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[-\frac{q(Q+q')}{d} + \frac{qq'}{2(d-d')}\right] \end{aligned}$$

The first term is the negative of the potential energy of the added charge $Q+q'$ and the charge q. This term has the correct sign as the uniformly distributed added charge charge is effectively at the origin. The second term is the negative of the charge-image potential with the factor of 1/2 associated with the fact that the image moves along with the original charge.

Addendum on the work-energy theorem:

In the plane and spherical image problems worked out above, we found that the work needed to bring the charge q in from infinity was 1/2 the potential energy of the charge and its image. To explain this factor 1/2, let us examine the general expression for energy of a charge distribution

$$W = \frac{1}{8\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}\, d\tau_1 d\tau_2 = \frac{1}{2}\int \Phi(\boldsymbol{r})\rho(\boldsymbol{r})\, d\tau$$

- For two fixed charges, q and q_i, we have $\Phi = \Phi_q + \Phi_i$ and $\rho = q\delta(\boldsymbol{r}-\boldsymbol{r}_q) + q_i\delta(\boldsymbol{r}-\boldsymbol{r}_i)$. Here,

$$\Phi_q(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_q|},$$

with a similar expression for Φ_i. One finds

$$W = \frac{1}{2}\, q\, \Phi_q(\boldsymbol{r}_q) + \frac{1}{2}\, q\, \Phi_i(\boldsymbol{r}_q) + \frac{1}{2}\, q_i\, \Phi_q(\boldsymbol{r}_i) + \frac{1}{2}\, q_i\, \Phi_i(\boldsymbol{r}_i)$$

The first and fourth terms are (infinite) "self-energy" terms and must be excluded from the sum. The second and third terms have identical values and lead to the well-known expression for the interaction energy between two charges

$$W = \frac{qq_i}{4\pi\epsilon_0}\frac{1}{|\boldsymbol{r}_i-\boldsymbol{r}_q|},$$

- For a charge q and a surface distribution σ such as we have in the present case, the energy expression becomes

$$W = \frac{1}{2}\, q\, \Phi_i(\boldsymbol{r}_q) + \frac{1}{2}\int_S \Phi_q(\boldsymbol{r})\sigma(\boldsymbol{r}) da + \frac{1}{2}\int_S \Phi_i(\boldsymbol{r})\sigma(\boldsymbol{r}) da,$$

where we have omitted the self-energy of q. Since the two contributions to the potential Φ_q and Φ_i precisely cancel on the surface, the second and third terms above cancel and we are left with

$$W = \frac{1}{2}\, q\, \Phi_i(\boldsymbol{r}_q) = \frac{q q_i}{8\pi\epsilon_0} \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_q|}.$$

This is, as expected, just 1/2 of the charge-image interaction energy.