1. Jackson Prob. 1.11:

   Choose the coordinate system so that the conductor at the point of observation is tangent to the $x - y$ plane and the normal is in the z direction. Choose the x and y axes to lie along the principal axes of curvature. Near the origin, the equation of the conducting surface, which is a surface of constant potential, is

$$z = -\frac{x^2}{2R_1} - \frac{y^2}{2R_2}$$

Method 1: The potential just outside the surface may be written abstractly as

$$\phi(x,y,z) = F\left(z + \frac{x^2}{2R_1} + \frac{y^2}{2R_2}\right).$$

As a specific example, the potential just outside a spherical conductor of radius R with center at $z = -R$ is

$$\phi_{\mathrm{S}}(x,y,z) = \frac{Q}{4\pi\epsilon_0}\,\frac{1}{R}\,\frac{1}{\sqrt{1 + \frac{1}{R}\left(z + \frac{x^2+y^2}{2R}\right)}}.$$

Here, we neglect terms of order z^2 compared with $2zR$. Note that $R_1 = R_2 = R$ in this example. From Gauss's law $\boldsymbol{\nabla}\cdot\boldsymbol{E} = 0$ one can write

$$\frac{\partial E}{\partial n} \equiv \frac{\partial E_z}{\partial z} = -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

From the functional dependence of $\phi(x,y,z)$ it follows

$$\frac{\partial\phi}{\partial x} = \frac{x}{R_1}\frac{\partial\phi}{\partial z}$$

Furthermore

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{R_1}\frac{\partial\phi}{\partial z} + \frac{x^2}{R_1^2}\frac{\partial^2\phi}{\partial z^2}$$

The second term above vanishes on the axis. We find, therefore, on the z axis,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{\partial\phi}{\partial z}$$

It follows that on the axis just above the surface

$$\frac{1}{E}\frac{dE}{dn} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Method 2: A second approach to the problem uses the integral form of Gauss's law. Arrange a "pillbox" above the surface. This pillbox has its lower surface at the conductor surface and its upper surface displaced by dz along the normal. The boundaries of the pillbox are defined by the arcs swept out by the two principal radii when they are displaced from the vertical by angles $d\theta_1$ and $d\theta_2$. The surface area of the lower face of the pillbox is $da_l = R_1 d\theta_1\, R_2 d\theta_1$ the contribution to Gauss's law is $-E\, R_1 d\theta_1\, R_2 d\theta_1$. The contribution to Gauss's law from the upper face is $(E+dE)(R_1+dz)d\theta_1\, (R_2+dz)d\theta_1$. The net contribution is

$$\begin{aligned}\int \boldsymbol{E}\cdot\boldsymbol{n}\, da =& (E+dE)(R_1+dz)d\theta_1\,(R_2+dz)d\theta_1 - E\, R_1 d\theta_1\, R_2 d\theta_1 \\ =& \left[(R_1+R_2)E dz + R_1 R_2 dE\right] d\theta_1 d\theta_2 = 0\end{aligned}$$

Since no charge is enclosed by the pillbox, the terms in the square bracket above $(R_1+R_2)Edz + R_1R_2dE$ vanish. Therefore

$$\frac{1}{E}\frac{dE}{dn} = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

2. Application of the "Reciprocity Theorem":

$$\int_V \rho\Phi' d\tau + \int_S \sigma\Phi' da = \int_V \rho'\Phi d\tau + \int_S \sigma'\Phi da$$

where Φ is the potential created by volume and surface charge distributions ρ and σ and Φ' is the potential creates by distributions ρ' and σ'. Both problems have the same bounding surfaces. Apply the theorem to obtain the charge on the upper plate of a grounded parallel plate capacitor with a charge q located inside. In this case, σ and Φ are unknowns, and

$$\rho = q\delta(x)\delta(y)\delta(z-z_0).$$

Choose a comparison parallel plate capacitor with lower plate grounded and upper plate at potential V. With this choice $\Phi' = Vx/d$ and $\rho' = 0$, but $\sigma' \neq 0$. The reciprocity theorem gives

$$Vq\frac{z_0}{d} + V\int \sigma da = 0 + 0.$$

It follows that the induced charge on the upper plate $q_{\rm ind} = \int \sigma da$ is

$$q_{\rm ind} = -q\,\frac{z_0}{d}.$$

3. Problems 1.23 and 1.24. These two problems are addressed in detail in the MATHEMATICA notebook Prob1.23-24.nb.